Multivariable Intro

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This problem set is taken from Mr. Deruiter's Monta Vista High School AP Calculus BC from 2013. This is good practice for students beginning to learn about multivariable functions— enjoy!

1 Find the equation of the plane that passes through the points $A = (3, 4, -2)$, $B = (1, 1, -1)$ and $C = (-1, 2, 2)$.

Visually, we can see that for a plane through these three points, a cross product between any two vectors (formed by the three points) will be normal to the plane:

$$\vec{AB} = \langle 1-3, 1-4, (-1)-(-2)\rangle = \langle -2, -3, 1\rangle \tag{1}$$

$$\vec{AC} = \langle (-1)-3, 2-4, 2-(-2)\rangle = \langle -4, -2, 4\rangle \tag{2}$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -3 & 1 \\ -4 & -2 & 4 \end{vmatrix} = \langle -10, 4, -8\rangle \tag{3}$$

Because this vector is normal to the plane in question, any other vector lying on the plane should be orthogonal to it. Using this idea, we can find a tri-variate relationship to describe this plane. Call this fourth arbitrary point $D = (x, y, z)$:

$$\vec{AD} = \langle x-3, y-4, z+2\rangle \tag{4}$$

Knowing that this vector is orthogonal to our normal vector, we can use a dot-product to find our plane equation:

$$\vec{AD} \cdot (\vec{AB} \times \vec{BC}) = \langle x-3, y-4, z+2\rangle \cdot \langle -10, 4, -8\rangle = 0 \tag{5}$$

$$-10(x-3) + 4(y-4) - 8(z+2) = 0 \tag{6}$$

Simplifying, we get our final answer:

$$-5x + 2y - 4z = 1 \tag{7}$$

2 Evaluate the limit:

$$\lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1}$$

Direct substitution gives us an indeterminate result:

$$\frac{(1)^3(1)^3-1}{(1)(1)-1} = \frac{0}{0} \tag{8}$$

This means that the surface likely isn't approaching any asymptote, as an asymptote would result in a nonzero number divided by 0 (a number very big in magnitude). We can't use L'Hospital's Rule here, because the derivative involved in that would have to extend into all directions, not just one. $xy-1$ is a factor of x^3y^3-1, however, so maybe we can conclude that there is a hole (isolated undefined area) in the surface at $(x,y)=(1,1)$.

$$\lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = \lim_{(x,y)\to(1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} \tag{9}$$

$$\lim_{(x,y)\to(1,1)} [x^2y^2+xy+1] = 3 \implies \lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = 3 \tag{10}$$

As we approach $(x,y)=(1,1)$, the function will still approach 3, because the factor of x^2y^2+xy+1 keeps the numerator three times as large as the denominator, even as each tends to 0.

3 Find the first partial derivatives for the function:

$$f(x,y) = x\,\mathrm{arcsec}(\frac{y}{x+y})$$

Recall that $\frac{d}{dx}[\mathrm{arcsec}(x)] = \frac{1}{|x|\sqrt{x^2-1}}$:

$$\frac{\partial f}{\partial x} = \mathrm{arcsec}(\frac{y}{x+y}) + (\frac{x}{|\frac{y}{x+y}|\sqrt{(\frac{y}{x+y})^2-1}})(\frac{y}{(x+y)^2}) \tag{11}$$

$$\frac{\partial f}{\partial y} = (\frac{x}{|\frac{y}{x+y}|\sqrt{(\frac{y}{x+y})^2-1}})(\frac{(x+y)-(y)}{(x+y)^2}) \tag{12}$$

4 Find all second partial derivatives for:

$$f(x,y) = \cos^2(5x+2y)$$

Recall that the chain rule is conceptually similar to that for single-variable differentiation: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{du}{dx}$

$$\frac{\partial f}{\partial x} = 2\cos(5x+2y)(-\sin(5x+2y))(5) = -5\sin(10x+4y) \tag{13}$$

$$\frac{\partial f}{\partial y} = 2\cos(5x+2y)(-\sin(5x+2y))(2) = -2\sin(10x+4y) \tag{14}$$

Now that we have our first partial derivatives, we can find our second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -5\cos(10x+4y)(10) = -50\cos(10x+4y) \tag{15}$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cos(10x+4y)(4) = -8\cos(10x+4y) \tag{16}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -5\sin(10x+4y)(4) = -20\cos(10x+4y) \tag{17}$$

5 Find ∇f at the point $(4, 2, 1)$ for the function:

$$f(x,y,z) = \tfrac{x}{y} + \tfrac{y}{z}$$

We can simply find the partial derivatives with respect to each variable, and those will be the components of our gradient vector:

$$\frac{\partial f}{\partial x} = \frac{1}{y} \tag{18}$$

$$\frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{z} \tag{19}$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2} \tag{20}$$

Now that we have our partial derivatives, we can form our gradient vector:

$$\nabla f(4,2,1) = \langle F_x(4,2,1), F_y(4,2,1), F_z(4,2,1) \rangle \tag{21}$$

$$\nabla f(4,2,1) = \langle \frac{1}{2}, 0, -2 \rangle \tag{22}$$

Conceptual note for gradients: we can have an infinite amount of lines tangent to this function at this point, but because the gradient takes into account the different "weights" of each dimension, the gradient is a vector tangent to the graph, pointing in the direction of steepest ascent.

6 Find the directional derivative at the point $(1, 6, 2)$ in the direction $\vec{u} = \langle 3, 4, 12 \rangle$ for the function:

$$f(x,y,z) = z^3 - x^2 y$$

We can define this directional derivative to be the rate of change of f as we move in the direction of $\vec{u}$. This means that we take the component of the gradient that's in the same direction as our $\vec{u}$ vector. This is because each component in $\vec{u}$ will scale the corresponding partial derivative, as we take separate "steps" through the function in the direction of $\vec{u}$, although our changes in the x, y, and z directions are still infinitesimally small.

$$\nabla f = \langle -2yx, -x^2, 3z^2 \rangle \tag{23}$$

Now, we want to scale $\vec{u}$ down to a unit vector, so that the magnitude of our directional derivative is solely dependent on the orientation of our vector, not its magnitude. Having this vector reduced to magnitude 1 ensures that we're simply multiplying our gradient by 1 (and taking the appropriate component) and not scaling it at all. Alternatively, we can think of this as taking steps where the total scaling factors of our steps along our partial derivatives adds up to 1, in order not to scale up or down our gradient.

$$|u| = \sqrt{3^2 + 4^2 + 12^2} = 13 \implies \frac{\vec{u}}{|u|} = \frac{1}{13}\langle 3, 4, 12 \rangle \tag{24}$$

$$\nabla_{\vec{u}} f(1,6,2) = \nabla f(1,6,2) \cdot \frac{\vec{u}}{|u|} \tag{25}$$

Substituting all of our values, we get:

$$\nabla_{\vec{u}} f(1,6,2) = 8 \tag{26}$$

7 Use partial derivatives to find $\frac{dy}{dx}$, where $y = f(x)$ is determined implicitly by the equation:

$$x^3 - 4xy^3 - 3y + x - 2 = 0$$

In order to find $\frac{dy}{dx}$ this way, we can use the relationship for our total differential of $f(x, y)$— we can just express the left-hand side of the given equation as $f(x, y) = 0$:

$$f(x, y) = x^3 - 4xy^3 - 3y + x - 2 = 0 \tag{27}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{28}$$

Diving by a differential in x to get the rate of change of f with respect to x:

$$\frac{df}{dx} = (\frac{\partial f}{\partial x})(\frac{dx}{dx}) + (\frac{\partial f}{\partial y})(\frac{dy}{dx}) = 0 \tag{29}$$

Note that $\frac{df}{dx} = 0$ because the function itself is constant, so a change in x won't result in any change in f, as the total differential takes into account any consequent changes in y. Now we can solve for $\frac{dy}{dx}$:

$$(\frac{\partial f}{\partial y})(\frac{dy}{dx}) = -\frac{\partial f}{\partial x} \implies \frac{dy}{dx} = -\frac{F_x}{F_y} \tag{30}$$

Although $\frac{dy}{dx} = 0$, the partial derivatives aren't necessarily 0, because the partial derivative denotes a change in the function given *only* a change in the appropriate dimension. Now, we can just find our two partial derivatives:

$$\frac{\partial f}{\partial x} = 3x^2 - 4y^3 + 1 \tag{31}$$

$$\frac{\partial f}{\partial y} = -12xy^2 - 3 \tag{32}$$

Using (30), we can finally find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{3x^2 - 4y^3 + 1}{12xy^2 + 3} \tag{33}$$

Intuitively, we can understand (30) by seeing that the ratio of change in y to change in x will be higher when a sole change in x leads to a considerable change in f (which is represented in $\frac{\partial f}{\partial x}$). This is because y will have to compensate more, so to speak, in order to make sure the total derivative with respect to x, $\frac{df}{dx}$, is still 0.

8 Find the maximum rate of change of f at the given point, and the direction in which it occurs at $(1,0)$, if:

$$f(x,y) = xe^{-xy} + 3xy - 2x$$

A key feature of the gradient vector is that it points in the direction of greatest rate of ascent in the function. So, we can simply find ∇f, and this will represent the rate of change of f as we move along f in the direction of this vector.

$$\frac{\partial f}{\partial x} = e^{-xy} - xye^{-xy} + 3y - 2 \tag{34}$$

$$\frac{\partial f}{\partial y} = -x^2 e^{-xy} + 3x \tag{35}$$

$$\nabla f(1,0) = \langle \frac{\partial f}{\partial x}\bigg|_{(1,0)}, \frac{\partial f}{\partial y}\bigg|_{(1,0)} \rangle = \langle -1, 2 \rangle \tag{36}$$

The maximum rate of change of this function is the magnitude of this vector, as the total change in the function will take into account the partial derivatives in both dimensions. Let r_{max} denote the maximum rate of change of f at (1,0):

$$r_{max} = \sqrt{(-1)^2 + (2)^2} = \sqrt{5} \tag{37}$$

The direction of this maximum rate—meaning the direction we travel in order for f to be increasing the fastest—already lies within the gradient vector. If we wanted to unitize this direction, then we could describe the direction as $\frac{1}{\sqrt{5}}\langle 1, 2 \rangle$.

9 Find the equation of the tangent plane and the parametric description of the normal line at $(-1,2,2)$, given the equation:

$$f(x,y,z) = xyz - 4xz^2 + y^3 = 20$$

Because our surface is implicitly defined here, finding the gradient of f holds a different visual meaning. Although f is constant, moving solely in one direction will clearly cause a change in f; thus, putting together our partial derivatives into one vector will give us a vector (with components in the x, y, and z dimensions) pointing in the direction that will move us away from that constant value f the fastest. Therefore, we can imagine this direction of greatest departure from f as being the direction where we move away from the surface (which is defined by our constant f value) the fastest.

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle yz - 4z^2, xz + 3y^2, xy - 8xz \rangle \tag{38}$$

$$\nabla f(-1,2,2) = \langle -12, 10, 14 \rangle \tag{39}$$

Because this vector is perpendicular to the surface, any vectors on the tangent plane should be orthogonal to it:

$$\langle x+1, y-2, z-2 \rangle \cdot \langle -12, 10, 14 \rangle = 0 \tag{40}$$

From this expression we can get our tangent plane equation:

$$-12(x+1) + 10(y-2) + 14(z-2) = 0 \implies -12x + 10y + 14x = 60 \tag{41}$$

And to our normal line can easily be derived from our gradient vector, because as our parametric variable increases by one unit, each dimension of the normal line will change with its corresponding dimension's partial derivative, and thus the relative movements of the line in each dimension match those described in the gradient vector.

$$\vec{N}(t) = \langle -1 - 12t, 2 + 10t, 2 + 14t \rangle \tag{42}$$

10 Find the extrema and saddle points of the function:

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{3}{2}x^2 - 4y$$

To start, we should find the x and y coordinates where $\nabla f = \vec{0}$, because this signifies that the function is not changing as we move in either dimension, and thus the surface is flat at that point.

$$\frac{\partial f}{\partial x} = x^2 - 3x = 0 \implies x = 0, 3 \tag{43}$$

$$\frac{\partial f}{\partial y} = y^2 - 4 = 0 \implies x = \pm 2 \tag{44}$$

Next, we can find the discriminant of the function, which helps us determine concavity and is described by the following equation: $D = f_{xx}f_{yy} - f_{xy}^2$.

$$\frac{\partial^2 f}{\partial x^2} = 2x - 3 \tag{45}$$

$$\frac{\partial^2 f}{\partial y^2} = 2y \tag{46}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \tag{47}$$

With these second partial derivatives, we can now find D at each critical point:

$$D(0, -2) = (2(0) - 3)(2(-2)) = 12 \tag{48}$$

$$D(0, 2) = (2(0) - 3)(2(2)) = -12 \tag{49}$$

$$D(3, -2) = (2(3) - 3)(2(-2)) = -12 \tag{50}$$

$$D(3, 2) = (2(3) - 3)(2(2)) = 12 \tag{51}$$

Because $D < 0$ at $(0, 2)$ and $(3, -2)$, $(0, 2, f(0, 2))$ and $(3, -2, f(3, -2))$ are saddle points. But for $f(0, -2)$ and $f(3, 2)$, we need to look at the individual second partial derivatives and see whether they're positive or negative—this will tell us whether the surface is concave up or down. Note that because $D > 0$ in these cases, f_{xx} and f_{yy} have the same sign.

$$f_{xx}(0, -2) = 2(0) - 3 < 0 \tag{52}$$

$$f_{xx}(3, 2) = 2(3) - 3 > 0 \tag{53}$$

Thus, $(0, -2, f(0, -2))$ is a relative maximum and $(3, 2, f(3, 2))$ is a relative minimum.

The expression for the discriminant can seem arbitrary, so here's some (vague) intuition behind the second partial derivative test. If $D > 0$, then we know that f_{xx} and f_{yy} have the same sign. Visually, this is easier to understand than the $D < 0$ case, because if the surface is curving the same way in both directions, then the surface definitely has a bowl shape, and therefore gives us an extreme point. D being negative is a little bit harder to reason through. First, let's say that f_{xx} and f_{yy} have a different sign; then we know that one dimension is concave upwards while the other is concave downwards. Consequently, this point simply cannot be an extreme point, because the function changes in both directions (up and down) surrounding the point. However, if f_{xx} and f_{yy} have the same sign but D is still negative, then we can conclude that f_{yx}^2 is large relative to $f_{xx}f_{yy}$. In this case, as you move along the y-axis, then the slope as you move along the x-axis changes considerably, which means you can't have a minimum or maximum, because the surface is twisting (as you move along the y-axis), so to speak. Therefore, there is still duality in movement surrounding the point, and it cannot be a local maximum or a minimum.