Triple Integral Practice

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These problems are taken from Paul's Online Notes, under their *Triple Integrals* section and *Spherical Coordinates* section.

1 Use a triple integral to determine the volume of the region below $z = 8 - y$ and above the region in the xy-plane bounded by $y = 2x$, $x = 3$, $y = 0$.

Because we're using a triple integral to find volume, we can say that the function that we're integrating will always be equal to 1: $f(x, y, z) = 1$. This means that as we keep taking $f(x, y, z) \cdot dV$, we're just adding pieces of volume:

$$V = \iiint\limits_E dV \tag{1}$$

Where E is the solid bounded by the plane $z = 8 - y$, and the lines $y = 2x$ an $x = 3$. We can choose our outermost integral to go from $x = 0$ to $x = 3$, and our innermost integral from the xy-plane up to $z- = 8 - y$.

$$V = \int_0^3 \int_0^{2x} \int_0^{8-y} dz\, dy\, dx \tag{2}$$

Physically, ths means that we're accumulating volume going up in the z dimension (while we're at some y coordinate, so we accumulate up until $8 - y$)—we're now left with area density. This area density is a function of x and y, so if we were to draw a 3-dimensional plane, then this area density could be the heights over our xy-plane. We can integrate with respect to y to find the second dimension of our volume; afterwards we will have linear density. These pieces will now be a function of x, leaving us to integrate these portions with respect to x and get a final numeric answer for our total volume (technically mass, but we have uniform density of 1 mass unit per unit volume).

$$V = \int_0^3 \int_0^{2x} (8 - y)\, dy = \int_0^3 \left[8y - \frac{y^2}{2}\right]_0^{2x} dx \tag{3}$$

$$V = \int_0^3 (16x - 2x^2)\, dx = \left[8x^2 - \frac{2}{3}x^3\right]_0^3 \implies V = \frac{162}{3} \tag{4}$$

2 Evaluate $\iiint\limits_E 12y\,dV$, where E is the region below $6x+4y+3z=12$ in the first octant:

First, we need to find our bounds of integration, as this is the first step (and frankly the only step) to finding our integral. Finding the line which the plane crosses the yz-plane:

$$6(0)+4y+3z=12 \implies 4y+3z=12 \tag{5}$$

Doing this for the xy-plane and xz-plane:

$$6x+4y+3(0)=12 \implies 6x+4y=12 \tag{6}$$

$$6x+4(0)+3z=12 \implies 6x+3z=12 \tag{7}$$

We can choose to first integrate up the z-axis, but this choice is arbitrary:

$$\int_0^2 \int_0^{6x-3} \int_0^{\frac{12-6x-4y}{3}} 12y\,dz\,dy\,dx \tag{8}$$

Note that as we integrate with respect to z, we can't simply take the expression for z against either of the xz or yz planes, because we the result to be applicable all around the xy-plane. This means that we want to double integrate the result of the first integral all around the xy-plane, and thus the result must be a function of x and y.

$$\int_0^2 \int_0^{6-3x} 4y(12-6x-4y)\,dy\,dx = \int_0^2 \int_0^{6-3x} (48y-24xy-16y^2)\,dy\,dx \tag{9}$$

$$\int_0^2 \left[24y^2-12xy^2-\frac{16y^3}{3}\right]_0^{6-3x} dx \tag{10}$$

$$\int_0^2 (216(2-x)^2-108x(2-x)^2-144(2-x)^3)\,dx = -144 \tag{11}$$

3 Evaluate $\iiint\limits_E 2yz\,dV$, where E is the region inside both $x^2+y^2+z^2=16$ and $z=\sqrt{3x^2+3y^2}$, in the 1st octant.

Because we're integrating through the (partial) region of a sphere, this problem lends itself well to a conversion from Cartesian coordinates to spherical coordinates. Because of the hyperbolic cone shape that $z=\sqrt{3x^2+3y^2}$ gives us, the elevating angle (ϕ) is the tough one to describe. However, we can quite simply find the angle that the cone makes with the positive z axis, by setting y (or x) to be 0. We then find that the ratio of z to x is $\sqrt{3}$. Thus, we know that the angle ϕ that the cone makes with the positive z-axis is $\arctan(\frac{1}{\sqrt{3}})=\frac{\pi}{6}$.

$$\iiint\limits_E 2yz\,dV = \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \int_0^4 2(\rho\sin(\phi)\sin(\theta))(\rho\cos(\phi))\cdot\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi \tag{12}$$

$$2\int_0^{\frac{\pi}{6}} \sin^2(\phi)\cos(\phi) \int_0^{\frac{\pi}{2}} \sin(\theta) \int_0^4 \rho^4\,d\rho\,d\theta\,d\phi = 17.067 \tag{13}$$