Exact Functions and Integrating Factors

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Problems are from Kreyzig's *Advanced Engineering Mathematics*.

1 Find the general solution for the following: $2xy\,dx + x^2\,dy = 0$

Looking at this form of a differential equation, consider that the differential for a function $f(x, y)$ is described as:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{1}$$

Fitting the two together, we can ascertain that the term in front of the dx is the partial derivative with respect to x, and the term in front fo the dy is the partial derivative with respect to y. This means that at any (x, y) coordinate, f is not changing ($f(x, y) = C$— this equation is *exact*). But, we first must ensure that this can be the case, namely by checking that the mixed partial derivatives of f are equal, so we know that the two components in our differential equation are indeed f_x and f_y.

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}\,(2xy) = 2x \tag{2}$$

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}\,(x^2) = 2x \tag{3}$$

Now, we can manipulate these partial derivatives to get back to our function f.

$$f(x, y) = \int (f_x)\,dx = \int 2xy\,dx = x^2y + G(y) \tag{4}$$

Note that as we integrate a multivariable function with respect to x, there will be an arbitrary function of solely y, because the partial derivative (with respect to x) of our integral result wouldn't include any terms with just y. Now that we have some components of f, we can partially differentiate with respect to y and compare with the $\frac{\partial f}{\partial y}$ to find $G(y)$.

$$\frac{\partial f}{\partial y} = x^2 + \frac{dG}{dy} = x^2 \implies G(y) = 0 \tag{5}$$

Thus, we have our final result:

$$f(x, y) = x^2y = C \tag{6}$$

2 Find the general solution for the following non-exact differential equation: $2x\tan(y)\,dx + \sec^2(y)\,dy = 0$

We know that this is not exact, because the partial derivative with respect to x of $2x\tan(y)$ is not equal to the partial derivative with respect to y of $\sec^2(y)$; these derivatives being unequal makes this non-exact because these terms simply aren't partial derivatives of the same function, and we can't use the relationship in (1) to solve for f. Therefore, we can multiply this equation by an integrating factor, $F(x)$, to make it so this equation is exact.

$$\frac{\partial}{\partial y}(F(x)(2x\tan(y))) = \frac{\partial}{\partial x}(F(x)\sec^2(y)) \tag{7}$$

$$(F)(2x)(\sec^2(y)) = \frac{dF}{dx}(\sec^2(y)) \tag{8}$$

$$\frac{dF}{F} = 2x\,dx \tag{9}$$

Integrating both sides:

$$\ln(F) = x^2 + C \implies F = Ce^{x^2} \tag{10}$$

This expression, when multiplied into our differential equation, should make it exact.

$$Ce^{x^2}2x\tan(y)\,dx + Ce^{x^2}\sec^2(y)\,dy = 0 \tag{11}$$

Checking for exactness:

$$\frac{\partial}{\partial y}(Ce^{x^2}2x\tan(y)) = Ce^{x^2}2x\sec^2(y) = Ce^{x^2}x\sec^2(y) \tag{12}$$

$$\frac{\partial}{\partial x}(Ce^{x^2}\sec^2(y)) = 2x(Ce^{x^2})\sec^2(y) = Ce^{x^2}x\sec^2(y) \tag{13}$$

Now we can solve like we did in our first exercise, knowing that our differential equation is represented by (1).

$$f(x,y) = \int Ce^{x^2}2x\tan(y)\,dx = C\tan(y)(e^{x^2}) + G(y) \tag{14}$$

$$\frac{\partial f}{\partial y} = Ce^{x^2}\sec^2(y) = Ce^{x^2}\sec^2(y) + G'(y) \implies G(y) = 0 \tag{15}$$

We have our answer:

$$f(x,y) = C_1e^{x^2}\tan(y) = C_2 \implies f(x,y) = e^{x^2}\tan(y) = C \tag{16}$$

One might be worried that this is a solution to a differential equation other than what our original problem was because we introduced a new factor. However, since we multiplied our integrating factor on both sides on our equation, the differential equation given was simply scaled (albeit by a function) to make it easier to solve.