Line Integral Practice (I)

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These exercises are from Paul's Online Math Notes, from their *Line Integrals, Part I* section.

1 Evaluate $\int\limits_C (3x^2 - 2y)\, ds$, where C is the path along the line segment between points $(3, 6)$ and $(1, -1)$.

Integrating f along this path, we could describe each differential in path traversal as $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$, which we can intuitively understand, as the total rate will just be made up of the components of this rate in each dimension. If a particle moves right 3 units in a second and up 4 units in a second, then our total rate will be the vector sum of the two velocities. However, this expression for ds would be difficult to manage. So, we can parameterize f, giving us an easier way to describe our differentials. If we say that $\vec{r}(t) = \langle x(t), y(t)\rangle$, then $|\vec{r}'(t)| = \frac{ds}{dt}$, and thus our differential is $ds = |\vec{r}'(t)|\, dt$.

Now, we must find a parameterized description for our bounds of integration. Because the line between the two points can be described simply with the equation $y = \frac{7}{2}x - \frac{9}{2}$:

$$\vec{r}(t) = \langle t, \frac{7}{2}t - \frac{9}{2}\rangle\ ,\ 1 \le t \le 3 \tag{1}$$

To find how fast the vector change (not the change in the magnitude of the vector, but rather the magnitude of the change) as t changes (and we go along the line), take the derivative with respect to t:

$$\vec{r}'(t) = \langle 1, \frac{7}{2}\rangle \implies |\vec{r}'(t)| = \sqrt{1 + (\frac{7}{2})^2} = \frac{\sqrt{53}}{2} \tag{2}$$

Thus, the difference between two consecutive vectors will be:

$$ds = |\vec{r}'(t)|\, dt = \frac{\sqrt{53}}{2}\, dt \tag{3}$$

Now that we have the points in our path laid out in terms of t, we need to find our integrand in terms of t. As we defined the x and y coordinates along our path in (1), we can use these to redefine f:

$$3x^2 - 2y = 3(t)^2 - 2(\frac{7}{2}t - \frac{9}{2}) = 3t^2 - 7t + 9 \tag{4}$$

Now, our integral is:

$$\int\limits_1^3 (3t^2 - 7t + 9) \cdot \frac{\sqrt{53}}{2}\, dt = 8\sqrt{53} \tag{5}$$

2 Evaluate $\int_C (2yx^2 - 4x)\, ds$ where C is the lower half of the circle centered at the origin with radius 3, with clockwise rotation.

Our parameterized vectors that draw out the path of our circle will be described as:

$$\vec{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle \ , \ \pi \leq t \leq 2\pi \tag{6}$$

Because we only want points from the bottom half of the circle, t can only be angles from π to 2π, to ensure that our sine and cosine expressions reflect that.

$$\vec{r}'(t) = \langle -3\sin(t), 3\cos(t) \rangle \implies ds = 3\, dt \tag{7}$$

Now to redefine our integral:

$$\int_{2\pi}^{\pi} [2(3\sin(t))(9\cos^2(t)) - 4(3\cos(t))] \cdot 3\,(-dt) \tag{8}$$

$$\int_{2\pi}^{\pi} (162\sin(t)\cos^2(t) - 36\cos(t))\,(-dt) = -108 \tag{9}$$

Note that we need a negative sign in front of our ds, because as t advances, our expressions for x and y will traverse our path counterclockwise. Thus, we need a negative movement in t to move clockwise.

3 Evaluate $\int_C 6x\, ds$ where C is the portion of $y = x^2$ from $x = -1$ tp $x = 2$. The path of C is in the direction of increasing x.

$$\vec{r}(t) = \langle t, t^2 \rangle \ , \ -1 \leq t \leq 2 \tag{10}$$

$$\vec{r}'(t) = \langle 1, 2t \rangle \implies ds = \sqrt{4t^2 + 1}\, dt \tag{11}$$

Now that our x and y values throughout the path are defined as functions of t, the x and y values can be converted into t values in our integrand. This takes care of the actual f value, as the x coordinate we would have evaluated f at is now just in terms of t.

$$\int_{-1}^{2} 6t \cdot \sqrt{4t^2 + 1}\, dt \tag{12}$$

$$\frac{3}{4}\int_{-1}^{2} 8t\sqrt{4t^2 + 1}\, dt = \frac{3}{4}\left[\frac{2}{3}(4t^2 + 1)^{\frac{3}{2}}\right]_{-1}^{2} = 29.46 \tag{13}$$