Differential Equations - Darcy's Law

Jackson Goenawan

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Darcy's Law, developed by Henry Darcy, describes the rate of fluid flow through a porous medium (sand, for example). The law states the the flow rate of a laminar fluid (in volume per unit time) is directly proportional to the rate of change of fluid pressure with respect to time, medium permeability, the ratio of cross-sectional area to length travelled, and inversely proportional to the viscosity of the fluid. Understand that the fluid will flow from higher pressure to lower pressure within whatever pipe this fluid is flowing through.

Our problem is to solve for the pressure of the fluid as a function of time. Starting our problem let's name our variables: Q will be our fluid flow rate, p will be our fluid pressure, k is our permeability, A is our cross-sectional area (the area that the fluid is flowing through), L is the length that the fluid travels, and μ is our fluid viscosity. Let's form our differential equation:

$$Q = -\frac{p'(t)kA}{l\mu} \tag{1}$$

Note that we don't need an arbitrary constant in front of the fraction, because the scaling of the fraction is already taken care of in the permeability constant.

Now, let's solve this differential equation:

$$\frac{dp}{dt} = -\frac{Ql\mu}{kA} \tag{2}$$

This is simple separation of variables:

$$dp = -\frac{Ql\mu}{kA}\,dt \implies p(t) = -\frac{Ql\mu}{kA}t + C \tag{3}$$

Just looking at this equation, we can tell that the pressure grows linearly with time, and the arbitrary constant in the solution is simply the starting point of our pressure at time t_0.

This differential equation (and solution) models a laminar fluid's flow through porous media. The differential equation for a turbulent fluid (characterized by a more chaotic flow) is forced by a term proportional to Q^2.

$$\frac{dp}{dt} = -\frac{Ql\mu}{kA} + \lambda Q^2 \tag{4}$$

Technically the whole righthand side of the equation is a forcing term, because they both make this nonhomogeneous. Also, note that we could've isolated $\frac{dp}{dt}$ "less"— this would give us the same result, but λ would just be scaled differently. Of course, finding this actual value will depend on conditions of the system.

$$dp_t = (-\frac{Ql\mu}{kA} + \lambda Q^2)\,dt \tag{5}$$

$$\int dp_t = \int (-\frac{Ql\mu}{kA} + \lambda Q^2)\,dt \implies p_t(t) = (-\frac{Ql\mu}{kA} + \lambda Q^2)t + C \tag{6}$$

Again, the C constant is the initial pressure, and would be our initial condition, if we were given one.