Introduction to Differential Equations

Jackson Goenawan

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Problems are from Kreyzig's *Advanced Engineering Mathematics*— these are introductory ordinary differential equations exercises.

1 Solve the differential equation: $y' + 2\sin(2\pi x) = 0$

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We can simply integrate both sides to find y as a function of x:

$$\frac{dy}{dx} = -2\sin(2\pi x) \implies dy = -2\sin(2\pi x)\,dx \tag{1}$$

$$y = \int -2\sin(2\pi x)\,dx = \frac{1}{\pi}\cos(2\pi x) + C \tag{2}$$

2 Solve the differential equation: $y' = y$

.

You might instinctively want to throw a trig. function into the solution, but remember that $\sin(x)$ and $\cos(x)$ only come back to their opposites every other derivative. So, we're left with $y = Ce^x$.

3 Solve the differential equation: $y'' = -y$

.

We *could* try integrating both sides twice, but that would be pretty difficult, as we'd be integrating y with respect to x. So, we could play fast-and-loose with this solution, and say that both $\sin(x)$ and $\cos(x)$ satisfy this equation. But, it would just be more elegant to get a more general solution.

Let's say $y_1 = k_1\sin(x)$, and $y_2 = k_2\cos(x)$. For both situations, $y'' + y = 0$ must be true. So, we have:

$$y_1'' + y_1 = 0 \tag{3}$$

$$y_2'' + y_2 = 0 \tag{4}$$

Adding the two, we can find a general solution that includes both y_1 and y_2:

$$(y_1'' + y_1) + (y_2'' + y_2) = 0 \tag{5}$$

$$(y_1'' + y_2'') + (y_1 + y_2) = 0 \implies (y_1'' + y_2'') = -(y_1 + y_2) \tag{6}$$

Remember that $y_1'' + y_2'' = (y_1 + y_2)''$. Considering this, recognize that this expression is our original problem; this means that $y_1 + y_2$ our solution (our original $y(x)$. Thus,

$$y = y_1 + y_2 = k_1\sin(x) + k_2\cos(x) \tag{7}$$

Note that this solution includes e^{ix} and e^{-ix} (recall $e^{ix} = \cos(x) + i\sin(x)$).

4 Verify that $y = Ce^{-2.5x^2}$ is the solution to the following: $y' + 5xy = 0$. And given $y(0) = \pi$, find the particular expression for y.

To verify the solution to this differential equation, we can simply differentiate the proposed solution and manipulate to match the equation.

$$y' = \frac{d}{dx} Ce^{-2.5x^2} = -5Cxe^{-2.5x^2} \tag{8}$$

Substituting this into the different equation: we can see that the given solution is correct:

$$-5Cxe^{-2.5x^2} + 5x(Ce^{-2.5x^2}) = 0 \tag{9}$$

Now to solve for the particular solution for y, we simply solve for our constant, given our initial conditions:

$$\pi = Ce^{-2.5(0)^2} = Ce^0 \tag{10}$$

$$C = \pi \implies y = \pi e^{-2.5x^2} \tag{11}$$

5 Verify that $y = \frac{1}{1+Ce^{-x}}$ is the solution to the following: $y' = y - y^2$. And given $y(0) = 0.25$, find the particular expression for y.

$$\frac{d}{dx}\left(\frac{1}{1+Ce^{-x}}\right) = \frac{d}{dx}(1+Ce^{-x})^{-1} = (Ce^{-x})(1+Ce^{-x})^{-2} \tag{12}$$

Now to verify:

$$(Ce^{-x})(1+Ce^{-x})^{-2} = (1+Ce^{-x})^{-1} - (1+Ce^{-x})^{-2} \tag{13}$$

Dividing by $(1+Ce^{-x})^{-2}$ on both sides:

$$Ce^{-x} = (1+Ce^{-x}) - \frac{(1+Ce^{-x})^{-2}}{(1+Ce^{-x})^{-2}} \tag{14}$$

$$Ce^{-x} = (1+Ce^{-x}) - 1 = Ce^{-x} \tag{15}$$

Solving for C:

$$0.25 = \frac{1}{1+Ce^{-(0)}} = \frac{1}{1+C} \implies C = 3 \tag{16}$$

Now that we have that coefficient, we can form our equation:

$$y = \frac{1}{1+4e^{-x}} \tag{17}$$

5.1 Find constant solutions for the differential equation by inspection.

Essentially looking for trivial solutions here, we can see that if y were to just be a constant, then y' would be 0—thus, $y - y^2$ must be 0.

$$y - y^2 = 0 \tag{18}$$

$$y(1-y) = 0 \implies y = 0 \text{ and } y = 1 \tag{19}$$

6 For $(y')^2 - xy' + y = 0$, verify that $y = Cx - C^2$ is the general solution, and that $y = \frac{x^2}{4}$ is a singular solution.

$$\frac{d}{dx}(Cx - C^2) = C \tag{20}$$

Substituting this into the given differential equation, we can verify this general solution:

$$(C)^2 - x(C) + (Cx - C^2) = 0 \tag{21}$$

$$C^2 - Cx = -(Cx - C^2) = C^2 - Cx \tag{22}$$

Now for the singular solution:

$$\frac{d}{dx}(\frac{x^2}{4}) = \frac{1}{2}x \tag{23}$$

Verifying:

$$(\frac{1}{2}x)^2 - x(\frac{3}{4}x) + (\frac{x^2}{4}) = 0 \tag{24}$$

$$\frac{1}{2}x^2 - \frac{3}{4}x^2 + \frac{1}{4}x^2 = \frac{2}{4}x^2 - \frac{3}{4}x^2 + \frac{1}{4}x^2 = 0 \tag{25}$$