Double Integral Practice

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November 13th, 2021

These problems are taken from Paul's Online Notes, under their *Multiple Integrals* section.

1 Evaluate $\iint\limits_D 9 - \frac{6y^2}{x^2}\, dA$, where D is the region in the first quadrant bounded by the curves: $y = x^3$ and $y = 4x$.

We can choose our outer integral to integrate along the x-axis (this would mean summing up all the dy integral sums, which are functions of x).

$$\int_0^2 \int_{x^3}^{4x} (9 - \frac{6y^2}{x^2})\, dy\, dx \tag{1}$$

$$\int_0^2 \left[9y - \frac{6y^3}{3x^2}\right]_{x^3}^{4x} dx = \int_0^2 (-92x - 9x^3 - 2x^7)\, dx \tag{2}$$

Now that we have the integrals up the y-axis as functions of x, we can sum them all up in order to get our total volume:

$$\iint\limits_D 9 - \frac{6y^2}{x^2} dA = -156 \tag{3}$$

2 Evaluate $\iint\limits_D e^{y^2+1}\, dA$, where D is the triangle with the vertices: $(0,0)$, $(-2,4)$, and $(8,4)$.

Because the triangle has a side parallel to the x-axis, it might be easier to first integrate with respect to x, and later sum those areas up along the y-axis. The line connecting $(0,0)$ and $(-2,4)$ will be of the equation $x = -\frac{y}{2}$, and the line connecting $(0,0)$ and $(8,4)$ will be of the equation $x = 2y$.

$$\int_0^4 \int_{-\frac{y}{2}}^{2y} e^{y^2+1}\, dx\, dy \tag{4}$$

Note that the bounds in the inner integral give us x coordinate values, as the inner integral treats y as constant, so manipulating y will give us varying x values.

$$\int_0^4 \left[e^{y^2+1} x\right]_{-\frac{y}{2}}^{2y} dy = \frac{5}{2} \int_0^4 y e^{y^2+1}\, dy \tag{5}$$

$$\iint\limits_D e^{y^2+1} dA = 3.019 \times 10^7 \tag{6}$$

Note that after (4), we could have moved e^{y^2+1} to the outer integral, because it was a constant as we integrate with respect with x.

3 Evaluate $\int_{-4}^{0}\int_{\sqrt{-x}}^{2} x^{-\frac{2}{3}}\sqrt{y^{\frac{5}{3}}+1}\,dy\,dx$, but by first switching the order of integration.

Looking at the graph of $y = -\sqrt{x}$, we can see that it has the point $(-4, 2)$. So to switch the order, our outer integral must run from $y = 0$ to $y = 2$:

$$\int_{0}^{2}\int_{-y^2}^{0} x^{-\frac{2}{3}}\sqrt{y^{\frac{5}{3}}+1}\,dx\,dy = \int_{0}^{2}\sqrt{y^{\frac{5}{3}}+1}\int_{-y^2}^{0} x^{-\frac{2}{3}}\,dx\,dy \tag{7}$$

Notice that the limits on the outer integral takes care of any domain issues for the $-y^2$ limit on the inner integral.

$$\int_{0}^{2}(\sqrt{y^{\frac{5}{3}}+1}\left[-\frac{3}{2}x^{\frac{1}{3}}\right]_{-y^2}^{0})\,dy = \int_{0}^{2}-\frac{3}{2}y^{\frac{2}{3}}\sqrt{y^{\frac{5}{3}}+1}\,dy \tag{8}$$

$$\int_{-4}^{0}\int_{\sqrt{-x}}^{2} x^{-\frac{2}{3}}\sqrt{y^{\frac{5}{3}}+1}\,dy\,dx = -4.518 \tag{9}$$

4 Use a double integral to determine the formula for the area of a right triangle with base b and height h.

Finding the volume a triangular prism with a depth value of 1 unit would yield us the same value as the area of a corresponding 2-dimensional triangle. So, we can set up our double integral accordingly:

$$V = \iint_D 1\,dA \tag{10}$$

Where D is a region on the xy-plane bounded by the triangle with vertices $(0,0)$, $(b,0)$, and $(0,h)$.

$$V = \int_{0}^{b}\int_{0}^{-\frac{h}{b}x+h} dy\,dx = \int_{0}^{b}(-\frac{h}{b}x+h)\,dx \tag{11}$$

This gives us the triangular area formula that we know and love:

$$V = \left[-\frac{h}{2b}x^2 + hx\right]_{0}^{b} = \frac{hb}{2} \tag{12}$$