202A Question Bank

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1. (Comp 2017) Consider an economy with a representative household with N_t identical members. The household's preferences are given by,

$$\sum_{t=0}^{\infty} \beta^t N_t[\log c_t + Alog(1-h_t)]$$

Each member of the household is endowed with one unit of labor each period. The number of members evolves over time according to the law of motion, $N_{t+1} = \eta N_t, \eta > 1$.
Output is produced using the following technology:

$$Y_t = \gamma^t e^{z_t} K_t^{\mu} (N_t h_t)^{\phi} L_t^{1-\mu-\phi}$$

Here $\gamma > 1$ is the gross rate of exogeneous total factor productivity growth, K_t is the total (not per capita) capital, Y_t is the total output, and L_t is the total stock of land. Land is assumed to be a fixed factor; it cannot be produced and does not depreciate. To simplify without loss of generality, assume that $L_t = 1$ for all t.
The variable z_t is a technology shock that follows an autoregeressive process, $z_{t+1} = \rho z_t + \epsilon_{t+1}$, where ϵ is independently and identically distributed over time with mean 0 and standard deviation σ.
The resource constraint, assuming 100 percent depreciation of capital each period, is given by:

$$N_t c_t + K_{t+1} = Y_t$$

(a) Formulate the social planning problem for this economy as a stationary dynamic program. Be clear about the transformation performed such that all the variables are stationary.

Solution:

1. Normalize the population so $N_0 = 1 \implies N_t = \eta^t$.
2. Simplify the total capital equation $K_t = N_t k_t = \eta^t k_t$
3. Substitute for Y_t in the resource constraint: $N_t c_t + K_{t+1} = \gamma^t e^{z_t} K_t^{\mu} (N_t h_t)^{\phi} L_t^{1-\mu-\phi}$
4. Substitute N_t and K_t: $\eta^t c_t + \eta^{t+1} k_{t+1} = \gamma^t e^{z_t} (\eta^t k_t)^{\mu} (\eta_t h_t)^{\phi} L_t^{1-\mu-\phi}$
5. Since $L_t = 1$: K_t: $\eta^t c_t + \eta^{t+1} k_{t+1} = \gamma^t e^{z_t} (\eta^t k_t)^{\mu} (\eta_t h_t)^{\phi}$
6. (IMPROVE) Define the following terms: $c_t = g_c^t \hat{c}_t,\ k_t = g_k^t \hat{k}_t,\ h_t = g_h^t \hat{h}_t$
7. (improve) $g_h = 1$ as we assume hours worked is constant
8. Substitute into the resource constraint: $\eta^t g_c^t \hat{c}_t + \eta^{t+1} g_k^{t+1} \hat{k}_{t+1} = \gamma^t e^{z_t} (\eta^t g_k^t \hat{k}_t)^{\mu} (\eta_t g_h^t \hat{h}_t)^{\phi}$
9. Group terms by t: $(\eta g_c)^t \hat{c}_t + (\eta g_k)^{t+1} \hat{k}_{t+1} = e^{z_t} \hat{k}_t^{\mu} \hat{h}_t^{\phi} (\gamma \eta^{\mu+\phi} g_k^{\mu})^t$

10. Divide by g_c^t and η^t: $\hat{c}_t + \eta g_k (\frac{g_k}{g_c})^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^{\mu} \hat{h}_t^{\phi} (\frac{\gamma \eta^{\mu+\phi-1} g_k^{\mu}}{g_c})^t$

11. Therefore, to be stationary, $g_k = g_c$ and $g_c = \gamma \eta^{\mu+\phi-1} g_k^{\mu}$

12. Using $g_k = g_c \implies g_k = g_c = (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}}$

13. Substitute g_k into the resource constraint: $\hat{c}_t + \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} (1)^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^{\mu} \hat{h}_t^{\phi} (1)^t$

14. We do the same for preferences

$$\sum_{t=0}^{\infty} \beta^t N_t [\log c_t + Alog(1 - h_t)]$$
$$\sum_{t=0}^{\infty} \beta^t \eta^t [\log g_c^t \hat{c}_t + A \log(1 - g_h^t \hat{h}_t)]$$
$$\sum_{t=0}^{\infty} \beta^t \eta^t [\log g_c^t \hat{c}_t + A \log(1 - \hat{h}_t)]$$

15. The social planning problem is now:

$$\max_{\hat{c}_t, \hat{h}_t, \hat{k}_{t+1}} \sum_{t=0}^{\infty} \beta^t \eta^t [\log g_c^t \hat{c}_t + A \log(1 - g_h^t \hat{h}_t)]$$
$$\text{s.t. } \hat{c}_t + \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} (1)^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^{\mu} \hat{h}_t^{\phi} (1)^t$$
$$\text{and } z_{t+1} = \rho z_t + \epsilon_{t+1}, \text{ where } \epsilon \sim N(0, \sigma^2) \text{ and } \epsilon \text{ is iid}$$
$$\text{given } k_0, z_0$$

16. Or we can use the Dynamic Programming problem:

$$V(\hat{k}, z) = \max_{\hat{c}, \hat{h}, \hat{k}'} \log \hat{c} + A \log(1 - \hat{h}) + \beta \eta E[V(\hat{k}', z'|z)]$$
$$\text{s.t. } \hat{c} + \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \hat{k}' = e^z \hat{k}^{\mu} \hat{h}^{\phi}$$
$$\text{and } z' = \rho z + \epsilon, \text{ where } \epsilon \sim N(0, \sigma^2) \text{ and } \epsilon \text{ is iid}$$