202A Question Bank

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1. (Final 2016)

   (a) Suppose that z_t can take on one of three possible values, a_1, a_2, a_3. Suppose that the stochastic process for the shock z_t is a Markov chain with transition matrix P.

      1. Provide an example of a transtion matrix P such that there is a unique invariant distribution and one transient state. Describe how to find the invariant distribution.

      Solution:

2. (Comp 2017) Consider an economy with a representative household with N_t identical members. The household's preferences are given by,

$$\sum_{t=0}^{\infty} \beta^t N_t[\log c_t + Alog(1 - h_t)]$$

Each member of the household is endowed with one unit of labor each period. The number of members evolves over time according to the law of motion, $N_{t+1} = \eta N_t, \eta > 1$.
Output is produced using the following technology:

$$Y_t = \gamma^t e^{z_t} K_t^{\mu} (N_t h_t)^{\phi} L_t^{1-\mu-\phi}$$

Here $\gamma > 1$ is the gross rate of exogeneous total factor productivity growth, K_t is the total (not per capita) capital, Y_t is the total output, and L_t is the total stock of land. Land is assumed to be a fixed factor; it cannot be produced and does not depreciate. To simplify without loss of generality, assume that $L_t = 1$ for all t.
The variable z_t is a technology shock that follows an autoregeressive process, $z_{t+1} = \rho z_t + \epsilon_{t+1}$, where ϵ is independently and identically distributed over time with mean 0 and standard deviation σ.
The resource constraint, assuming 100 percent depreciation of capital each period, is given by:

$$N_t c_t + K_{t+1} = Y_t$$

   (a) Formulate the social planning problem for this economy as a stationary dynamic program. Be clear about the transformation performed such that all the variables are stationary.

   Solution:

   1. Normalize the population so $N_0 = 1 \implies N_t = \eta^t$.
   2. Simplify the total capital equation $K_t = N_t k_t = \eta^t k_t$
   3. Substitute for Y_t in the resource constraint: $N_t c_t + K_{t+1} = \gamma^t e^{z_t} K_t^{\mu} (N_t h_t)^{\phi} L_t^{1-\mu-\phi}$
   4. Substitute N_t and K_t: $\eta^t c_t + \eta^{t+1} k_{t+1} = \gamma^t e^{z_t} (\eta^t k_t)^{\mu} (\eta_t h_t)^{\phi} L_t^{1-\mu-\phi}$

5. Since $L_t = 1$: K_t: $\eta^t c_t + \eta^{t+1} k_{t+1} = \gamma^t e^{z_t} (\eta^t k_t)^\mu (\eta_t h_t)^\phi$
6. (IMPROVE) Define the following terms: $c_t = g_c^t \hat{c}_t,\ k_t = g_k^t \hat{k}_t,\ h_t = g_h^t \hat{h}_t$
7. (improve) $g_h = 1$ as we assume hours worked is constant
8. Substitute into the resource constraint: $\eta^t g_c^t \hat{c}_t + \eta^{t+1} g_k^{t+1} \hat{k}_{t+1} = \gamma^t e^{z_t} (\eta^t g_k^t \hat{k}_t)^\mu (\eta_t g_h^t \hat{h}_t)^\phi$
9. Group terms by t: $(\eta g_c)^t \hat{c}_t + (\eta g_k)^{t+1} \hat{k}_{t+1} = e^{z_t} \hat{k}_t^\mu \hat{h}_t^\phi (\gamma \eta^{\mu+\phi} g_k^\mu)^t$
10. Divide by g_c^t and η^t: $\hat{c}_t + \eta g_k (\frac{g_k}{g_c})^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^\mu \hat{h}_t^\phi (\frac{\gamma \eta^{\mu+\phi-1} g_k^\mu}{g_c})^t$
11. Therefore, to be stationary, $g_k = g_c$ and $g_c = \gamma \eta^{\mu+\phi-1} g_k^\mu$
12. Using $g_k = g_c \implies g_k = g_c = (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}}$
13. Substitute g_k into the resource constraint: $\hat{c}_t + \eta (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} (1)^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^\mu \hat{h}_t^\phi (1)^t$
14. We do the same for preferences

$$\sum_{t=0}^{\infty} \beta^t N_t [\log c_t + Alog(1 - h_t)]$$

$$\sum_{t=0}^{\infty} \beta^t \eta^t [\log{(g_c^t \hat{c}_t)} + A \log(1 - g_h^t \hat{h}_t)]$$

$$\sum_{t=0}^{\infty} \beta^t \eta^t [\log{(g_c^t \hat{c}_t)} + A \log(1 - \hat{h}_t)]$$

15. The social planning problem is now:

$$\max_{\hat{c}_t, \hat{h}_t, \hat{k}_{t+1}} \sum_{t=0}^{\infty} \beta^t \eta^t [\log{(g_c^t \hat{c}_t)} + A \log(1 - \hat{h}_t)]$$
$$\text{s.t. } \hat{c}_t + \eta (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} (1)^t \hat{k}_{t+1} = e^{z_t} \hat{k}_t^\mu \hat{h}_t^\phi (1)^t$$
$$\text{and } z_{t+1} = \rho z_t + \epsilon_{t+1}, \text{ where } \epsilon \sim N(0, \sigma^2) \text{ and } \epsilon \text{ is iid}$$
$$\text{given } k_0, z_0$$

16. Or we can use the Dynamic Programming problem:

$$V(\hat{k}, z) = \max_{\hat{c}, \hat{h}, \hat{k}'} \log \hat{c} + A \log(1 - \hat{h}) + \beta \eta E[V(\hat{k}', z' | z)]$$
$$\text{s.t. } \hat{c} + \eta (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \hat{k}' = e^z \hat{k}^\mu \hat{h}^\phi$$
$$\text{and } z' = \rho z + \epsilon, \text{ where } \epsilon \sim N(0, \sigma^2) \text{ and } \epsilon \text{ is iid}$$

(b) Characterize the balanced growth path of this economy. That is, find expressions that determine c_t h_t and K_t along this growth path. In particular, solve explicitly for the growth rate of this set of variables.

Solution:

1. To characterize the balanced growth path of this economy we take the FOC: (IMPROVE: why can we drop subscripts)

$$[c] : \frac{1}{\hat{c}} - \lambda = 0 \tag{1}$$

$$[h] : -\frac{A}{1-\hat{h}} + \lambda \phi e^{z} \hat{k}^{\mu} \hat{h}^{\phi-1} = 0 \tag{2}$$

$$[k] : \mu e^{z} \hat{k}^{\mu-1} \hat{h}^{\phi} = \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \text{ Using the envelope condition} \tag{3}$$

2. We then combine (1) with (2), use the fact that in steady state hat variables are constant, and add in the resource constraint for the steady state:

$$[h] : -\frac{A}{1-\bar{h}} + \frac{1}{\bar{c}} \phi e^{z} \bar{k}^{\mu} \bar{h}^{\phi-1} = 0 \tag{4}$$

$$[k] : \mu e^{z} \bar{k}^{\mu-1} \bar{h}^{\phi} = \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \tag{5}$$

$$\bar{c}_t + \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \bar{k}_{t+1} = e^{z_t} \bar{k}_t^{\mu} \bar{h}_t^{\phi} \tag{6}$$

3. These three equations characterize the balanced growth path for $\bar{c}, \bar{h}, \bar{k}$ that we would then plug in for:

$$c_t = g_c^t \bar{c} \tag{7}$$

$$h_t = g_h^t \bar{h} \tag{8}$$

$$K_t = g_k^t \eta^t \bar{k} \tag{9}$$

4. We found the explicit growth rate of these variables in the previous part:

$$g_k = g_c = (\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}}, g_h = 1$$

(c) Discuss how your answer to part B would change if $\phi = 1 - \mu$. In particular, what is the growth rate of income per capita in the two cases (part A and B)?

Solution: (IMPROVE) We still have $c_t = g_c^t \bar{c}, h_t = g_h^t \bar{h}, K_t = g_k^t \eta^t \bar{k}$, but now $g_k = g_c = \gamma^{\frac{1}{1-\mu}}$. And our charateristic equations are now:

$$[h] : -\frac{A}{1-\bar{h}} + \frac{1}{\bar{c}} \phi e^{z} \bar{k}^{\mu} \bar{h}^{\phi-1} = 0 \tag{10}$$

$$[k] : \mu e^{z} \bar{k}^{\mu-1} \bar{h}^{\phi} = \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \tag{11}$$

$$\bar{c}_t + \eta(\gamma \eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} \bar{k}_{t+1} = e^{z_t} \bar{k}_t^{\mu} \bar{h}_t^{\phi} \tag{12}$$

To find the growth rate per capita of income, let's first simplify the income equation:

$$Y_t = \gamma^t e^{z_t} K_t^{\mu} (N_t h_t)^{\phi} L_t^{1-\mu-\phi}$$

$$\eta^t y_t = \gamma^t e^{z_t} (\eta^t k_t)^{\mu} (\eta^t h_t)^{1-\mu} (1)$$

$$y_t = \gamma^t e^{z_t} k_t^{\mu} h_t^{1-\mu}$$

The change in income between periods in the steady state is

$$\frac{y_{t+1}}{y_t} = \gamma \frac{e^{z_{t+1}} k_{t+1}^{\mu} h_{t+1}^{1-\mu}}{e^{z_t} k_t^{\mu} h_t^{1-\mu}}$$

Since we are in steady state we can simplify capital, leisure, and the technology shock:

$$\frac{y_{t+1}}{y_t} = \gamma$$

Therefore the per capita growth rate of income is y^t

(d) Suppose that a period is one quarter and suppose you are given annual growth rates for population and per capita income. You are also given values for factor shares, the average amount of time spent working and the annual capital-output ratio. Show how these facts can be used to calibrate γ, η, A, β

Solution:

1. Since annual growth rate = (quarterly growth rate)^4
   $\implies \eta =$ (annual growth rate of population)$^{\frac{1}{4}}$
2. TODO

(e) Define a recursive competitive equilibrium for this economy assuming markets for labor, consumption goods, land rental, and capital services.

Solution: Note: that we drop the hats. The household problem is:

$$\begin{aligned} V(k, K, z) = max_{c,h,k',l} \log c + A \log(1-h) + \beta \eta E[V(k', K', z'|z)] \\ s.t.\ c + \eta(\gamma\eta^{\mu+\phi-1})^{\frac{1}{1-\mu}} k' = hw(K, z) + kr(K, z) + t(K, z)l \\ K' = G(K) \\ z_{t+1} = \rho z_t + \epsilon_{t+1}, \text{where } \epsilon \sim N(0, \sigma^2) \text{ and } \epsilon \text{ is iid} \\ \text{given } k_0, z_0 \end{aligned}$$

The firm problem is:

$$\max_{h^f, l^f, k^f} [e^{z_t}(h^f)_t^{\phi}(k_f)_t^{\mu} l_f^{1-\mu-\phi} - t(K, z)l^f - k^f r(K, z) - h^f w(K, z)]$$

A recursive competetive equilibrium is defined by:

- A set of household decision rules $c(k, K, z), h(k, K, z), k'(k, K, z), l(k, K, z)$
- A set of firm decision rules $h^f(K, z), l^f(K, z), k^f(K, z)$
- A set of pricing functions $w(K, z), r(K, z), t(K, z)$ where consumption price is the numeraire
- Perceived law of motion for aggregate capital $G(K)$ such that:
  - Given pricing functions and a percieved law of motion for aggregate capital, the household decision rules solve the household problem
  - Given pricing functions, firm decision rules solve the firm's problem
  - Markets clear
    * $h^f(K, z) = h(K, K, z)$
    * $l^f(K, z) = l(K, K, z)$

* $k^f(K,z) = K$
* The consumption market by Walras' Law

Note that little k is now big K, as the firm optimizes over aggregate capital (improve)

– Rational expectations: $G(K) = k'(K,K,z)$

(f) Add a real estate market to your equilibrium definition in part E. Derive an equation determining the price of land.

Solution: Note: that we drop the hats. The household problem is:

$$V(k,K,l,L,z) = max_{c,h,k',l'} \log c + A\log(1-h) + \beta\eta E[V(k',K',l',L',z'|z)]$$

$$s.t.\ c + \eta(\gamma\eta^{\mu+\phi-1})^{\frac{1}{1-\mu}}k' + s(K,z,L)l' = hw(K,z,L) + kr(K,z,L) + (t(K,z,L)+s(K,z,L))l$$

$$K' = G(K)$$

$$L' = H(L)$$

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \text{where } \epsilon \sim N(0,\sigma^2) \text{ and } \epsilon \text{ is iid}$$

given k_0, z_0

The firm problem is:

$$\max_{h^f,l^f,k^f}[e^{z_t}(h^f)_t^\phi (k_f)_t^\mu l_f^{1-\mu-\phi} - t(K,z,L)l^f - k^f r(K,z,L) - h^f w(K,z,L)]$$

A recursive competetive equilibrium is defined by:

- A set of household decision rules $c(k,K,z,l,L), h(k,K,z,l,L), k'(k,K,z,l,L), l(k,K,z,l,L)$
- A set of firm decision rules $h^f(K,z,L), l^f(K,z,L), k^f(K,z,L)$
- A set of pricing functions $w(K,z,L), r(K,z,L), t(K,z,L)$ where consumption price is the numeraire
- Perceived law of motion for aggregate capital $G(K)$ and land $H(L)$ such that:
  - Given pricing functions and a percieved law of motion for aggregate capital and land, the household decision rules solve the household problem
  - Given pricing functions, firm decision rules solve the firm's problem
  - Markets clear
    * $h^f(K,z,L) = h(K,K,L,L,z)$
    * $l'(K,z,L) = l'(K,K,L,L,z)$
    * $l^f(K,z,L) = l(K,K,L,L,z)$
    * $k^f(K,z,L) = K$
    * The consumption market by Walras' Law
  - Rational expectations: $G(K) = k'(K,K,z,L,L)$ and $H(L) = l'(K,K,z,L,L)$