203A Question Bank

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1. (Final 2023): Let $X_n \sim N(0,1)$ if n is odd, and $X_n \sim N(0,n)$ if n is even. Is $X_n = O_p(1)$?

> **Solution:** No. The definition of $O_p(1)$ is that $\lim_{n\to\infty} P(|X_n| \leq B) = 1$ for all B. In this case, the limit is not 1 for any B. Note: I'm not sure I like this solution.

2. (Final 2023): Suppose that $X_1, X_2, ...$ are iid such that their common MGF is

$$E[exp(tX_i)] = (\frac{1}{1-t})^2$$

Let $F_n(x) \equiv P(\frac{1}{n}\sum_{i=1}^n X_i^2 \leq x)$. What is $\lim_{n\to\infty} F_n(1)$?

> **Solution:** First note that the MGF is that of an exponential distribution with $\lambda = 1$. The mean of an exponential distribution is $\frac{1}{\lambda} = 1$. The variance is $\frac{1}{\lambda^2} = 1$. Since $Var(X) = E[X^2] - E[X]^2$ That implies $E[X^2] = 2$. Now we use the fact that $\lim_{n\to\infty} F_n(1) \equiv \lim_{n\to\infty} P(\frac{1}{n}\sum_{i=1}^n X_i^2 \leq 1) = P(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^2 \leq 1) = P(E[X^2] \leq 1)$. Since $E[X^2] = 2$, the answer is 0. Note: I'm not sure I like this solution.

3. (Final 2023): Suppose that X has the CDF equal to $\Lambda(t) = \frac{exp(t)}{1+exp(t)}$. Let $X_n \equiv cos(X/n)$. What is $\lim_{n\to\infty} E[cos(X_n)]$?

> **Solution:** (CHECK)

4. (Final 2023): Let X_1 denote a random sample of (size 1) from $N(1,\theta)$. We have $H_0 : \theta = 4$ and $H_1 : \theta = 9$. You decided to use the Neyman-Pearson test of size 5%. If you observe $X_1 = 6$, do you reject H_0 or not?

> **Solution:** We do not reject H_0.
>
> 1. identify our test statistic: $Z = (X_1 - \mu)/\sigma = (6-1)/4 = 5/4$
> 2. find the critical value: $z_\alpha = 1.645$
> 3. compare the test statistic to the critical value: $5/4 \leq 1.645$. So we do not reject H_0.

5. (Final 2023): Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 1)$ converges in distribution to $N(0, 1)$. By the multivariate delta method, we can see that

$$\begin{pmatrix} \sqrt{n}(X_n^3 - \mu_1) \\ \sqrt{n}(\ln(X_n) - \mu_2) \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix}\right]$$

for some $\mu_1, \mu_2, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2}$, and $\sigma_{1,2}$. What are their numerical values?

> **Solution:** (TODO) The multivariate delta method:
>
> $$\begin{aligned} \text{if } \sqrt{n}(X_n - \theta) &\xrightarrow{d} N(0, \Sigma) \\ \text{then } \sqrt{n}(h(X_n) - h(\theta)) &\xrightarrow{d} N(0, H\Sigma H') \\ \text{where } H &= \left.\frac{\partial h(t)}{\partial t'}\right|_{t=\theta} \end{aligned}$$

6. (Final 2023): Suppose that a random sample X_1 of size 1 where X_i $N(\mu, 1)$. We want to test the null hypothesis $\mu = 0$ versus $H_1 : \mu > 0$. We will reject the null if $X_1 > c$. Suppose that c was chosen such that the size of the test is 2.5%. For what value of μ is the power of the test 5%?

> **Solution:** The power of a test is the probability of correctly rejecting the null hypothesis. In this case, we know that $c = 1.96$ as the null hypothesis is $X \sim N(0, 1)$ and
> $Pr(\text{reject null if null is correct}) = 1 - \Phi(1.96) = .025$
>
> $$\begin{aligned} Pr(x + \mu > 1.96) &= Pr(x > 1.96 - \mu) \\ &= 1 - Pr(x \leq 1.96 - \mu) \\ &= 1 - \Phi(1.96 - \mu) = .05 \\ &\implies .95 = \Phi(1.96 - \mu) \\ &= 1.645 = 1.96 - \mu \\ &\implies \mu = 1.96 - 1.645 = .315 \end{aligned}$$

7. (Final 2023): Let X_1 denote a random sample (of size 1) from a Poisson distribution with mean equal to θ. We would like to test $H_0 : \theta = 5$ against $H_1 : \theta \neq 5$.
   (a) Suppose that $X_1 = 25$ What is the value of the LR statistic? Hint 1: The PDF of the Poisson Distribution with mean equal to θ is given by $f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}$ Hint 2: $\ln 5 = 1.6094$

   > **Solution:** (CHECK)
   >
   > 1. The MLE of θ is $\hat{\theta} = X_1 = 25$ Note: I forgot how to calculate MLE

2. The Likelihood ratio statistic is:

$$\begin{aligned} LR &= -2\ln(\frac{L(\theta_0)}{L(\hat{\theta})}) \\ &= -2\ln(\frac{L(5)}{L(25)}) \\ &= -2\ln(\frac{5^2 5e^{-5}}{25^2 5e^{-25}}) \\ &= -2\ln(\frac{5^2 5}{25^2 5}) \\ &= 50\ln(5) - 50 \\ &= 50(1.6094) - 50 \\ &= 30.47 \end{aligned}$$

(b) Do you reject or accept the null at the 5% significance level?

Solution: (CHECK) if $LR > \chi^2_\alpha$ then reject the null. In this case, if $X \sim \chi^2(1) \implies P(X > 1.96^2) = 5\%$. Since $30.47 > 1.96^2$, we reject the null.

8. (Final 2023): Suppose that $X_1, ..., X_{10}$ are iid $N(0,1)$. Let $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$. What is $E[\bar{X}^2(\sum_{i=1}^{n}(X_i - \bar{X})^2)]$?

Solution: (CHECK) Note: that I am confused whether it is the $E[X]^2$ or $E[X^2]$ for $\bar{X}^2$.

$$\begin{aligned} Var(X) = E[X^2] - E[X]^2 = 1 &\implies E[X^2] = 1 \\ E[\bar{X}^2(\sum_{i=1}^{n}(X_i - \bar{X})^2)] &= \bar{X}^2 E[\sum_{i=1}^{n}(X_i - \bar{X})^2)] \\ &= \bar{X}^2 E[Var(X)] \\ &= \bar{X}^2 E[1] = \bar{X}^2 \\ &= 0 \end{aligned}$$

9. (Final 2018): Let X have the uniform distribution over the $(0,1)$ interval

(a) Calculate $E[\log(X)]$

Solution: Use integration by parts: $E[\log(X)] = \int_0^1 \log(x)dx = x\log(x) - x|_0^1 = 0 - 1 = -1$.

(b) Calculate $\log E[X]$

Solution: $E[X] = \int_0^1 x dx = 1/2$. So $\log E[X] = \log(1/2) = -\log(2)$.

(c) Which is bigger?

Solution: $\log E[X] = -\log(2) > -1 = E[\log(X)]$. So $\log E[X]$ is bigger.

(d) Now let g denote some positive valued strictly increasing function defined on $(0,1)$. Let $Y = g(X)$. Between $E[\log(Y)]$ and $\log(E[Y])$, which one is bigger? Or does the answer depend on g? Note that there are only three choices.

> **Solution:** Jensen's inequality says that $\phi(E[X]) \leq E[\phi(X)]$ for a convex function ϕ. Since log is concave, we have that $\log(E[Y]) \geq E[\log(Y)]$. So $\log(E[Y])$ is bigger.
>
> Note: Concavity is the tricky part here. You can convert log into a convex function by multiplying the log by -1, which makes it convex. Then you can apply Jensen's inequality.

10. (Final 2018): Suppose that the support of Y is $\{0,1\}$, and $Pr[Y = 1|X] = \Phi(X'\beta)$ where Φ is the standard normal cdf. We know that $Pr[Y = 1|(X_1, X_2) = (2,1)] = .5$ and $Pr[Y = 1|(X_1, X_2) = (2,2)] = .975$. What is the value of $\beta = (\beta_1, \beta_2)'$?

The Phi scores you may want to use are:

$\Phi(x)$	Value
$\Phi(0)$	0.5
$\Phi(0.253)$	0.6
$\Phi(0.534)$	0.7
$\Phi(0.842)$	0.8
$\Phi(1.282)$	0.9
$\Phi(1.645)$	0.95
$\Phi(1.960)$	0.975
$\Phi(2.576)$	0.995

> **Solution:** $\beta_1 = -.98$ and $\beta_2 = 1.96$
>
> We have $Pr[Y = 1|(2,1)] = \Phi(2\beta_1 + \beta_2) = .5$ and $Pr[Y = 1|(2,2)] = \Phi(2\beta_1 + 2\beta_2) = .975$. We can take the inverse of Φ using the table above. So we have $2\beta_1 + \beta_2 = 0$ and $2\beta_1 + 2\beta_2 = 1.960$. By subtraction, we get $\beta_2 = 1.96$. We then solve for β_1 using the first equation to get $\beta_1 = -.98$.

11. (Final 2018): Let $X_1, ..., X_n$ be a random sample of size n from $N(3,2)$, and let $\bar{X}$ denote the sample average. what is the asymptotic distribution of $\sqrt{n}(\bar{X}^2 - 9)$? Your answer should be numerical.

> **Solution:** We use the Delta method: $\sqrt{n}[g(\bar{X}) - g(a)] \xrightarrow{d} N(0, g'(a)^2\sigma^2)$ where $g(x) = x^2$ and $a = 3$. So $g'(a) = 6$ and $\sigma^2 = 2$. So the answer is $N(0, 72)$.

12. (Final 2018): Let X_n denote a sequence of random variables such that the PDF f_n of X_n is given by

$$f_n(x) = \frac{1}{2n} 1(|X| \leq n)$$

(a) Let $0 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \leq B)$?

> **Solution:** $\lim_{n\to\infty} P(|X_n| \leq B) = \lim_{n\to\infty} \int_{-B}^{B} \frac{1}{2n} dx = \lim_{n\to\infty} \frac{B}{n} = 0$
> Note: I'm not sure I like this solution.

(b) Is $X_n = O_p(1)$?

> **Solution:** X_n is not bounded in probability. The definition of bounded in probability is $\lim_{n\to\infty} P(|X_n| \leq B) = 1$, which is not the case here.

13. (Final 2018, 2019) Suppose that X and ϵ are independent $N(0,1)$ variables. Let $Y = X + \epsilon$ What is the correlation between X^2 and Y

> **Solution:** The correlation is zero. I'll write the algebra down later.

14. (Final 2018, 2019): True or False
    (a) If $X_n = a + o_p(1)$, then $E[X_n] = a + o(1)$

    > **Solution:**

    (b) If $X_n = a + o_p(1)$, then $X_n^2 = a^2 + o_p(1)$

    > **Solution:**

    (c) If $X_n \xrightarrow{d} N(0,1)$, then $E[X_n] = o(1)$

    > **Solution:**

    (d) If $X_n \xrightarrow{d} N(0,1)$, then $E[X_n^2] = 1 + o(1)$

    > **Solution:**

    (e) If $X_n \xrightarrow{d} N(0,1)$, $\chi_n^2 \xrightarrow{d} \chi^2(1)$

    > **Solution:**

15. (Final 2019): Suppose that $Z \sim N(0,1)$, and let $X_n = Z^2 1(|Z| \geq \frac{1}{n})$. What is $lim_{n\to\infty} E[X_n]$?

> **Solution:** Not finished. I think there's some law that let's us move the limit inside the expectation.
>
> $$lim_{n\to\infty} E[X_n] = E[lim_{n\to\infty} X_n] = E[lim_{n\to\infty} Z^2 1(|Z| \geq \frac{1}{n})] = E[Z =^2] = \chi^2(1) = 1$$

16. (Final 2019): Let $X_1, ..., X_n$ be a random sample of size n from $N(\mu, 1)$, and let $\bar{X}$ denote the sample average.
    (a) Compute $sup_{\mu<0} lim_{n\to\infty} P(\sqrt{n}\bar{X} > 1.645)$

    > **Solution:**

    (b) $lim_{n\to\infty} sup_{\mu<0} P(\sqrt{n}(\bar{X} - \mu) > 1.645)$

    > **Solution:**

17. (Final 2019) Let X_n denote a sequence of random variables such that the PDF f_n of X_n is given by

$$f_n(x) = \frac{1}{4}1(n \leq |x| \leq n+1) + \frac{1}{4}1(|x| \leq 1)$$

(a) Let $1 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \leq B)$?

Solution:

(b) Is $X_n = O_p(1)$?

Solution:

18. (Final 2019, 2021) Consider $X_1, ..., X_n$ iid $N(\mu, \sigma^2)$ We assume that $\sigma^2 = 1$ We have $H_0 : \mu = 0$ vs $H_1 : \mu > 0$ Suppose that C is a critical region such that (1) the test of the form "Reject H_0 if $(X_1, ..., X_n) \in C$" is the uniformly most powerful test; and (2) the probability of rejecting H_0 when $\mu = 0$ is 5%

(a) Provide a mathematical characterization of rejecting H_0 of such a test as a function of n and $\mu > 0$

Solution:

(b) What is the power of the test when $\mu = .1645$ and $n = 100$?

Solution:

19. (Final 2021) Suppose that $X_1, X_2, ...$ are iid such that $X_i = 0$ with probability 1/2, and $X_i = 2$ with probability 1/2. Let $\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $F_n(x) \equiv P(\frac{1}{n}\sum i = 1^n X_i \leq x)$. What is $lim_{n\to\infty} F(1.1)$?

Solution:

20. (Final 2021) Suppose that X has the CDF equal to

$$Pr[X \leq x] = \frac{exp(x)}{1 + exp(x)} \equiv \Lambda(x)$$

Let $\Phi(.)$ denote the CDF of $N(0,1)$, and let Φ^-1 denote its inverse. $Y \equiv \Phi^-1(\Lambda(X))$. What is $E[Y^4]$?

Solution:

21. (Final 2021) Consider $X_1, ..., X_n$ iid $N(0,1)$. Let $Y_n = 1(\bar{X_n} \geq 1)$.

(a) What is $E[Y_n]$?

Solution:

(b) What is $Var(Y_n)$?

Solution:

(c) What is the probability limit of Y_n?

Solution:

22. (Final 2021) Suppose that $X_1, ..., X_4$ are iid and their common distribution is uniform $(0, \theta)$, i.e. their common PDF $f(x)$ is equal to $1(0 < x < \theta)$. We have $H_0 : \theta = 1$ vs $H_1 : \theta = 2$. Suppose that you decided to reject H_0 if $\max(X_1, ..., X_4) > 1$

    (a) What is the size of the test?

    Solution:

    (b) What is the power of the test? Hint: $P(\max(X_1, ..., X_4) \leq 1) = P[X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 1]$

    Solution:

23. (Final 2021) Let $X_1, ..., X_n$ be a random sample of size n from $N(0,1)$. For any positive integer k, let $m_k = E[X_i^k]$ and $\hat{m_k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$. Assuming that $E[|X_i^k|] < \infty$ for all k, derive the asymptotic distribution $\sqrt{n}(\hat{m_k}^{1/2} - m_k^{1/2})$. The asymptotic distribution is normal with mean zero, so your job is to derive the numerical value of the asymptotic variance. Hint: the MGD of $N(0,1)$ is $exp(t^2/2)$.

    Solution:

24. (Final 2021) Let $X_1, ..., X_5$ denote a random sample from $N(\theta, 1)$. We would like to test $H_0 : \theta = 5$ against $H_1 : \theta ! = 5$. If $X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 5, X_5 = 6$, what is the LR statistic?

    Solution:

25. (Final 2022) Let X_n $b(n, \frac{q}{n})$ for some $q > 0$. Is $X_n = Op(1)$

    Solution:

26. (Final 2022) Let Y_n denote the maximum of a random sample of size n from a uniform (0,1) distribution. What is $lim_{n\to\infty} Pr[Y_n \ leq 0.9]$?

    Solution:

27. (Final 2022) Let $\Lambda(t) \equiv \frac{exp(t)}{1+exp(t)}$ denote the CDF of the logistic distribution (with location and scale parameters equal to 0 and 1, although these particular details are irrelevant for this question). Let X_n denote a sequence of random variables such that the CDF F_n of X_n is given by $F_n(x) = \lambda(nx)$. What is $\lim_{n\to\infty} E[cos(X_n)]$?

    Solution:

28. (Final 2022) Let X_1 denote a random sample (of size 1) from $\mathcal{N}(0, \sigma^2)$. We have null hypothesis $H_0 : \mu = 1$ and alternative hypothesis $H_1 : \mu = 9$.

Solution:

29. (Final 2022) Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 1)$ converges in distribution to $N(0, 1)$. By the multivariate delta method, we can see that

$$\begin{pmatrix} \sqrt{n}(X_n^2 - \mu_1) \\ \sqrt{n}(\ln(X_n) - \mu_2) \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix}\right]$$

for some $\mu_1, \mu_2, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2}$, and $\sigma_{1,2}$. What are their numerical values?

Solution:

30. (Final 2022) Let X denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda exp(-\lambda x)1(x > 0)$. We have $H_0 L\lambda = 1$ and $H_1 : \lambda < 1$. Suppose that C is a critical region such that (1) the test of the form "Reject H_0 if $X_1 \in C$" is the uniformly most powerful test; and (2) the probability of rejecting H_0 when $\lambda = 1$ is α. What is the power of your test when $\lambda = 1/2$ and $\alpha = 5\%$

Solution:

31. (Final 2022) Let X denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda exp(-\lambda x)1(x > 0)$. We would like to test $H_0 : \lambda = 1$ against $H_1 : \lambda! = 1$. Suppose that $X = e$
    (a) What is the value of the LR statistic?

Solution:

    (b) Do you reject or accept the null at the 5% significance level? You maye use the approximation $e = 2.7183$

Solution:

32. (Final 2022) Let (X, Y) be a two dimensional random vector with the joint PDF $f_{X,Y}(x, y) = 4e^-2y1(y > x > 0)$. Let $(\alpha, \beta) \equiv argmin_{a,b} E[(Y - (a + bX))^2]$ What is (α, β)? Hint: a small number of students may find it useful to know that $\int_0^\infty x^m \lambda e^(- \lambda x)dx = \frac{m!}{\lambda^m}$

33. (Final 2015) Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 3) \xrightarrow{d} N(0, 1)$
    (a) What is the asymptotic distribution of $\sqrt{n}(X_n^2 - 9)$?

Solution:

    (b) What is the asymptotic distribution of $(\sqrt{n}(X_n^2 - 3))^2$?

Solution:

34. (Final 2015) Let $F(y|x)$ denote the conditional CDF of Y given X i.e. $F(y|x) = Pr[Y \leq y|X = x]$. Suppose that $F(y|x)$ is continuous and strictly increasing in y for all x in the support of X. Let $V = F(Y|X)$. (It is not $F(Y|x)$). Derive the conditional distribution of V given x. Prove that V and X are independent.

Solution: