203A Question Bank

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1. (Final 2023): Let $X_n \sim N(0, 1)$ if n is odd, and $X_n \sim N(0, n)$ if n is even. Is $X_n = O_p(1)$?

> **Solution:** No. The definition of $O_p(1)$ is that $\lim_{n\to\infty} P(|X_n| \leq B) = 1$ for all B. In this case, the limit is not 1 for any B. Note: I'm not sure I like this solution.

2. (Final 2023): Suppose that $X_1, X_2, \ldots$ are iid such that their common MGF is

$$E[exp(tX_i)] = (\frac{1}{1-t})^2$$

Let $F_n(x) \equiv P(\frac{1}{n}\sum_{i=1}^n X_i^2 \leq x)$. What is $\lim_{n\to\infty} F_n(1)$?

> **Solution:** First note that the MGF is that of an exponential distribution with $\lambda = 1$. The mean of an exponential distribution is $\frac{1}{\lambda} = 1$. The variance is $\frac{1}{\lambda^2} = 1$. Since $Var(X) = E[X^2] - E[X]^2$ That implies $E[X^2] = 2$. Now we use the fact that $\lim_{n\to\infty} F_n(1) \equiv \lim_{n\to\infty} P(\frac{1}{n}\sum_{i=1}^n X_i^2 \leq 1) = P(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^2 \leq 1) = P(E[X^2] \leq 1)$. Since $E[X^2] = 2$, the answer is 0. Note: I'm not sure I like this solution.

3. (Final 2023): Suppose that X has the CDF equal to $\Lambda(t) = \frac{exp(t)}{1+exp(t)}$. Let $X_n \equiv cos(X/n)$. What is $\lim_{n\to\infty} E[cos(X_n)]$?

> **Solution:** (CHECK)

4. (Final 2023): Let X_1 denote a random sample of (size 1) from $N(1, \theta)$. We have $H_0 : \theta = 4$ and $H_1 : \theta = 9$. You decided to use the Neyman-Pearson test of size 5%. If you observe $X_1 = 6$, do you reject H_0 or not?

> **Solution:** We do not reject H_0.
>
> 1. identify our test statistic: $Z = (X_1 - \mu)/\sigma = (6 - 1)/4 = 5/4$
> 2. find the critical value: $z_\alpha = 1.645$
> 3. compare the test statistic to the critical value: $5/4 \leq 1.645$. So we do not reject H_0.

5. (Final 2023): Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 1)$ converges in distribution to $N(0,1)$. By the multivariate delta method, we can see that

$$\begin{pmatrix} \sqrt{n}(X_n^3 - \mu_1) \\ \sqrt{n}(\ln(X_n) - \mu_2) \end{pmatrix} \xrightarrow{d} N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \right]$$

for some $\mu_1, \mu_2, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2}$, and $\sigma_{1,2}$. What are their numerical values?

Solution: (TODO) The multivariate delta method:

$$\begin{aligned} \text{if } \sqrt{n}(X_n - \theta) &\xrightarrow{d} N(0, \Sigma) \\ \text{then } \sqrt{n}(h(X_n) - h(\theta)) &\xrightarrow{d} N(0, H\Sigma H') \\ \text{where } H &= \left.\frac{\partial h(t)}{\partial t'}\right|_{t=\theta} \end{aligned}$$

6. (Final 2023): Suppose that a random sample X_1 of size 1 where X_i $N(\mu, 1)$. We want to test the null hypothesis $\mu = 0$ versus $H_1 : \mu > 0$. We will reject the null if $X_1 > c$. Suppose that c was chosen such that the size of the test is 2.5%. For what value of μ is the power of the test 5%?

Solution: The power of a test is the probability of correctly rejecting the null hypothesis. In this case, we know that $c = 1.96$ as the null hypothesis is $X \sim N(0,1)$ and
$Pr(\text{reject null if null is correct}) = 1 - \Phi(1.96) = .025$

$$\begin{aligned} Pr(x + \mu > 1.96) &= Pr(x > 1.96 - \mu) \\ &= 1 - Pr(x \leq 1.96 - \mu) \\ &= 1 - \Phi(1.96 - \mu) = .05 \\ &\implies .95 = \Phi(1.96 - \mu) \\ &= 1.645 = 1.96 - \mu \\ &\implies \mu = 1.96 - 1.645 = .315 \end{aligned}$$

7. (Final 2023): Let X_1 denote a random sample (of size 1) from a Poisson distribution with mean equal to θ. We would like to test $H_0 : \theta = 5$ against $H_1 : \theta \neq 5$.

   (a) Suppose that $X_1 = 25$ What is the value of the LR statistic? Hint 1: The PDF of the Poisson Distribution with mean equal to θ is given by $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$ Hint 2: $\ln 5 = 1.6094$

   Solution: (CHECK)

   1. The MLE of θ is $\hat{\theta} = X_1 = 25$ Note: I forgot how to calculate MLE

2. The Likelihood ratio statistic is:

$$
\begin{aligned}
LR &= -2\ln(\frac{L(\theta_0)}{L(\hat{\theta})}) \\
&= -2\ln(\frac{L(5)}{L(25)}) \\
&= -2\ln(\frac{5^2 5e^{-5}}{25^2 5e^{-25}}) \\
&= -2\ln(\frac{5^2 5}{25^2 5}) \\
&= 50\ln(5) - 50 \\
&= 50(1.6094) - 50 \\
&= 30.47
\end{aligned}
$$

(b) Do you reject or accept the null at the 5% significance level?

Solution: (CHECK) if $LR > \chi^2_\alpha$ then reject the null. In this case, if $X \sim \chi^2(1) \implies P(X > 1.96^2) = 5\%$. Since $30.47 > 1.96^2$, we reject the null.

8. (Final 2023): Suppose that $X_1, ..., X_{10}$ are iid $N(0,1)$. Let $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$. What is $E[\bar{X}^2(\sum_{i=1}^{n}(X_i - \bar{X})^2)]$?

Solution: (CHECK) Note: that I am confused whether it is the $E[X]^2$ or $E[X^2]$ for $\bar{X}^2$.

$$
\begin{aligned}
Var(X) = E[X^2] - E[X]^2 = 1 &\implies E[X^2] = 1 \\
E[\bar{X}^2(\sum_{i=1}^{n}(X_i - \bar{X})^2)] &= \bar{X}^2 E[\sum_{i=1}^{n}(X_i - \bar{X})^2)] \\
&= \bar{X}^2 E[Var(X)] \\
&= \bar{X}^2 E[1] = \bar{X}^2 \\
&= 0
\end{aligned}
$$

9. (Final 2018): Let X have the uniform distribution over the $(0,1)$ interval

(a) Calculate $E[\log(X)]$

Solution: Use integration by parts: $E[\log(X)] = \int_0^1 \log(x)dx = x\log(x) - x|_0^1 = 0 - 1 = -1$.

(b) Calculate $\log E[X]$

Solution: $E[X] = \int_0^1 xdx = 1/2$. So $\log E[X] = \log(1/2) = -\log(2)$.

(c) Which is bigger?

Solution: $\log E[X] = -\log(2) > -1 = E[\log(X)]$. So $\log E[X]$ is bigger.

(d) Now let g denote some positive valued strictly increasing function defined on $(0,1)$. Let $Y = g(X)$. Between $E[\log(Y)]$ and $\log(E[Y])$, which one is bigger? Or does the answer depend on g? Note that there are only three choices.

> **Solution:** Jensen's inequality says that $\phi(E[X]) \leq E[\phi(X)]$ for a convex function ϕ. Since log is concave, we have that $\log(E[Y]) \geq E[\log(Y)]$. So $\log(E[Y])$ is bigger.
>
> Note: Concavity is the tricky part here. You can convert log into a convex function by multiplying the log by -1, which makes it convex. Then you can apply Jensen's inequality.

10. (Final 2018): Suppose that the support of Y is $\{0,1\}$, and $Pr[Y=1|X] = \Phi(X'\beta)$ where Φ is the standard normal cdf. We know that $Pr[Y=1|(X_1,X_2)=(2,1)] = .5$ and $Pr[Y=1|(X_1,X_2)=(2,2)] = .975$. What is the value of $\beta = (\beta_1,\beta_2)'$?

The Phi scores you may want to use are:

$\Phi(x)$	Value
$\Phi(0)$	0.5
$\Phi(0.253)$	0.6
$\Phi(0.534)$	0.7
$\Phi(0.842)$	0.8
$\Phi(1.282)$	0.9
$\Phi(1.645)$	0.95
$\Phi(1.960)$	0.975
$\Phi(2.576)$	0.995

> **Solution:** $\beta_1 = -.98$ and $\beta_2 = 1.96$
>
> We have $Pr[Y=1|(2,1)] = \Phi(2\beta_1+\beta_2) = .5$ and $Pr[Y=1|(2,2)] = \Phi(2\beta_1+2\beta_2) = .975$. We can take the inverse of Φ using the table above. So we have $2\beta_1+\beta_2 = 0$ and $2\beta_1+2\beta_2 = 1.960$. By subtraction, we get $\beta_2 = 1.96$. We then solve for β_1 using the first equation to get $\beta_1 = -.98$.

11. (Final 2018): Let $X_1,...,X_n$ be a random sample of size n from $N(3,2)$, and let $\bar{X}$ denote the sample average. what is the asymptotic distribution of $\sqrt{n}(\bar{X}^2 - 9)$? Your answer should be numerical.

> **Solution:** We use the Delta method: $\sqrt{n}[g(\bar{X}) - g(a)] \xrightarrow{d} N(0, g'(a)^2\sigma^2)$ where $g(x) = x^2$ and $a = 3$. So $g'(a) = 6$ and $\sigma^2 = 2$. So the answer is $N(0,72)$.

12. (Final 2018): Let X_n denote a sequence of random variables such that the PDF f_n of X_n is given by

$$f_n(x) = \frac{1}{2n}1(|X| \leq n)$$

(a) Let $0 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \leq B)$?

> **Solution:** $\lim_{n\to\infty} P(|X_n| \leq B) = \lim_{n\to\infty} \int_{-B}^{B} \frac{1}{2n}dx = \lim_{n\to\infty} \frac{B}{n} = 0$
> Note: I'm not sure I like this solution.

(b) Is $X_n = O_p(1)$?

Solution: X_n is not bounded in probability. The definition of bounded in probability is $\lim_{n\to\infty} P(|X_n| \leq B) = 1$, which is not the case here.

13. (Final 2018, 2019) Suppose that X and ϵ are independent $N(0,1)$ variables. Let $Y = X + \epsilon$ What is the correlation between X^2 and Y

Solution: The correlation is zero. I'll write the algebra down later.

14. (Final 2018, 2019): True or False

(a) If $X_n = a + o_p(1)$, then $E[X_n] = a + o(1)$

Solution: (ASK)

(b) If $X_n = a + o_p(1)$, then $X_n^2 = a^2 + o_p(1)$

Solution: True

$$\begin{aligned} X_n^2 &= (a + o_p(1))^2 \\ &= a^2 + (2a)o_p(1) + o_p(1) \\ &= a^2 + o_p(1) \end{aligned}$$

Since a constant times $o_p(1)$ is $o_p(1)$, and since $o_p(1) + o_p(1) = o_p(1)$.

(c) If $X_n \xrightarrow{d} N(0,1)$, then $E[X_n] = o(1)$

Solution: (ASK)

(d) If $X_n \xrightarrow{d} N(0,1)$, then $E[X_n^2] = 1 + o(1)$

Solution: (ASK)

(e) If $X_n \xrightarrow{d} N(0,1)$, $\chi_n^2 \xrightarrow{d} \chi^2(1)$

Solution: True. Using the continuous mapping theorem we know that if $X_n \xrightarrow{d} X$ and g is a continuous function, then $g(X_n) \xrightarrow{d} g(X)$. In this case, $g(x) = x^2$ is continuous, so $X_n^2 \xrightarrow{d} \chi^2(1)$.

15. (Final 2019): Suppose that $Z \sim N(0,1)$, and let $X_n = Z^2 1(|Z| \geq \frac{1}{n})$. What is $lim_{n\to\infty} E[X_n]$?

Solution: (Check)

$$\lim_{n\to\infty} E[X_n] = E[\lim_{n\to\infty} X_n] = E[\lim_{n\to\infty} Z^2 1(|Z| \geq \frac{1}{n})] = E[Z =^2] = \chi^2(1) = 1$$

16. (Final 2019): Let $X_1, ..., X_n$ be a random sample of size n from $N(\mu, 1)$, and let $\bar{X}$ denote the sample average.

(a) Compute $sup_{\mu<0} \lim_{n\to\infty} P(\sqrt{n}\bar{X} > 1.645)$

Solution:

$$\begin{aligned} sup_{\mu<0} \lim_{n\to\infty} P(\sqrt{n}\bar{X} > 1.645) &= sup_{\mu<0} \lim_{n\to\infty} P(\sqrt{n}(\bar{X}-\mu) > 1.645 - \sqrt{n}\mu) \\ &= sup_{\mu<0} \lim_{n\to\infty} P(Z > 1.645 - \sqrt{n}\mu) \\ &= 0 \end{aligned}$$

(b) $\lim_{n\to\infty} sup_{\mu<0} P(\sqrt{n}\bar{X} > 1.645)$

Solution:

$$\begin{aligned} &\lim_{n\to\infty} sup_{\mu<0} P(\sqrt{n}(\bar{X}-\mu) > 1.645 - \sqrt{n}\mu) \\ =&\lim_{n\to\infty} sup_{\mu<0} P(Z > 1.645 - \sqrt{n}\mu) \\ =&\lim_{n\to\infty} P(Z > 1.645) \\ =&1-\Phi(1.645) \\ =&1-.95 = .05 \end{aligned}$$

17. (Final 2019) Let X_n denote a sequence of random variables such that the PDF f_n of X_n is given by

$$f_n(x) = \frac{1}{4}1(n \le |x| \le n+1) + \frac{1}{4}1(|x| \le 1)$$

(a) Let $1 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \le B)$?

Solution:
$$\lim_{n\to\infty} P(|X_n| \le B) = \int_{-1}^{1} \frac{1}{4} dx = \frac{1}{2}$$

(b) Is $X_n = O_p(1)$?

Solution: No. The definition of $O_p(1)$ is that $\lim_{n\to\infty} P(|X_n| \le B) = 1$ for all B. In this case, the limit is 1/2.

18. (Final 2019, 2021) Consider $X_1, ..., X_n$ iid $N(\mu, \sigma^2)$ We assume that $\sigma^2 = 1$ We have $H_0 : \mu = 0$ vs $H_1 : \mu > 0$ Suppose that C is a critical region such that (1) the test of the form "Reject H_0 if $(X_1, ..., X_n) \in C$" is the uniformly most powerful test; and (2) the probability of rejecting H_0 when $\mu = 0$ is 5%

(a) Provide a mathematical characterization of rejecting H_0 of such a test as a function of n and $\mu > 0$

Solution:

(b) What is the power of the test when $\mu = .1645$ and $n = 100$?

Solution:

19. (Final 2021) Suppose that $X_1, X_2, ...$ are iid such that $X_i = 0$ with probability $1/2$, and $X_i = 2$ with probability $1/2$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, and $F_n(x) \equiv P(\frac{1}{n}\sum_{i=1}^n X_i \leq x)$. What is $lim_{n\to\infty}F(1.1)$?

Solution:

$$E[X_i] = 0(1/2) + 2(1/2) = 1$$

$$\lim_{n\to\infty} F(1.1) = P(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i \leq 1.1)$$

$$= P(1 \leq 1.1) = 1$$

20. (Final 2021) Suppose that X has the CDF equal to

$$Pr[X \leq x] = \frac{exp(x)}{1 + exp(x)} \equiv \Lambda(x)$$

Let $\Phi(.)$ denote the CDF of $N(0, 1)$, and let Φ^-1 denote its inverse. $Y \equiv \Phi^-1(\Lambda(X))$. What is $E[Y^4]$?

Solution:

21. (Final 2021) (Verified) Consider $X_1, ..., X_n$ iid $N(0, 1)$. Let $Y_n = 1(\bar{X}_n \geq 1)$.

(a) What is $E[Y_n]$?

Solution:

$$\bar{X}_n \sim N(0, 1/n) \implies \sqrt{n}\bar{X}_n \sim N(0, 1)$$

$$E[Y_n] = P(\bar{X}_n \geq 1)$$
$$= P(\sqrt{n}\bar{X}_n \geq \sqrt{n})$$
$$= 1 - \Phi(\sqrt{n})$$

(b) What is $Var(Y_n)$?

Solution: Note that Y_n is a Bernoulli random variable.
So $Var(Y_n) = p(1 - p) = (1 - \Phi(\sqrt{n}))(\Phi(\sqrt{n}))$

(c) What is the probability limit of Y_n?

Solution: Markov's inequality is $P(x \geq a) \leq \frac{E[X]}{a}$:

$$P(Y_n \geq 1) \leq \frac{E[Y_n]}{1} = 1 - \Phi(\sqrt{n})$$

As $n \to \infty$, $1 - \Phi(\sqrt{n}) \to 0$. So $Y_n \xrightarrow{p} 0$.

22. (Final 2021) Suppose that $X_1, ..., X_4$ are iid and their common distribution is uniform $(0, \theta)$, i.e. their common PDF $f(x)$ is equal to $1(0 < x < \theta)$. We have $H_0 : \theta = 1$ vs $H_1 : \theta = 2$. Suppose that you decided to reject H_0 if $\max(X_1, ..., X_4) > 1$

   (a) What is the size of the test?

   Solution: Size is the probability of rejecting the null when the null is true. In this case, $P(\max(X_1, ..., X_4) > 1|\theta = 1) = 0$ since under the null hypothesis, the maximum value of X_i is 1.

   (b) What is the power of the test? Hint: $P(\max(X_1, ..., X_4) \leq 1) = P[X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 1]$

   Solution: The power of a test is the probability of correctly rejecting the null hypothesis. In this case, the power is $P(\max(X_1, ..., X_4) > 1|\theta = 2) = 1 - P(\max(X_1, ..., X_4) \leq 1|\theta = 2) = 1 - P[X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 1|\theta = 2] = 1 - (1/2)^4 = 15/16$

23. (Final 2021) Let $X_1, ..., X_n$ be a random sample of size n from $N(0, 1)$. For any positive integer k, let $m_k = E[X_i^k]$ and $\hat{m_k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$. Assuming that $E[|X_i^k|] < \infty$ for all k, derive the asymptotic distribution $\sqrt{n}(\hat{m_k}^{1/2} - m_k^{1/2})$. The asymptotic distribution is normal with mean zero, so your job is to derive the numerical value of the asymptotic variance. Hint: the MGD of $N(0, 1)$ is $exp(t^2/2)$.

   Solution:

24. (Final 2021) Let $X_1, ..., X_5$ denote a random sample from $N(\theta, 1)$. We would like to test $H_0 : \theta = 5$ against $H_1 : \theta \neq 5$. If $X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 5, X_5 = 6$, what is the LR statistic?

   Solution: First, note that the MLE of θ is $\hat{\theta} = \bar{X} = 4$. The likelihood ratio statistic is:

$$
\begin{aligned}
LR &= -2\ln\left(\frac{L(\theta_0)}{L(\theta_1)}\right) \\
&= -2\ln\left(\frac{L(5)}{L(4)}\right)
\end{aligned}
$$

   Note that $L(\theta) = \prod_{i=1}^{n} f(x; \theta)$ so the above equation simplifies to:

$$
\begin{aligned}
&\frac{e^{-.5\sum_{i=1}^{n}(x_i-5)^2}}{e^{-.5\sum_{i=1}^{n}(x_i-4)^2}} \\
&\frac{e^{1/2(-15)}}{e^{1/2(-10)}} \\
&= 5
\end{aligned}
$$

25. (Final 2022) Let $X_n \sim b(n, \frac{q}{n})$ for some $q > 0$. Is $X_n = O_p(1)$

   Solution: Yes. $X_n = O_p(1)$ if $\lim_{n\to\infty} P(|X_n| \leq B) = 1$ for some finite B.

26. (Final 2022) Let Y_n denote the maximum of a random sample of size n from a uniform (0,1) distribution. What is $\lim_{n\to\infty} Pr[Y_n \leq 0.9]$?

> **Solution:** 0. As n approaches infinity, the maximum value of a sample from a uniform distribution approaches 1.

27. (Final 2022) Let $\Lambda(t) \equiv \frac{exp(t)}{1+exp(t)}$ denote the CDF of the logistic distribution (with location and scale parameters equal to 0 and 1, although these particular details are irrelevant for this question). Let X_n denote a sequence of random variables such that the CDF F_n of X_n is given by $F_n(x) = \Lambda(nx)$. What is $\lim_{n\to\infty} E[cos(X_n)]$?

> **Solution:**

28. (Final 2022) Let X_1 denote a random sample (of size 1) from $N(0, \theta)$. We have null hypothesis $H_0 : \theta = 1$ and alternative hypothesis $H_1 : \theta = 9$. You decided to use the Neyman-Pearson test of size 5%. If you observe $X_1 = 2.5$, do you reject H_0 or not? Your answer should be either Reject or Do not reject.

> **Solution:** Note: Neyman Pearson means uniformly most powerful.
>
> *TODO*

29. (Final 2022) Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 1)$ converges in distribution to $N(0, 1)$. By the multivariate delta method, we can see that

$$\begin{pmatrix} \sqrt{n}(X_n^2 - \mu_1) \\ \sqrt{n}(\ln(X_n) - \mu_2) \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix}\right]$$

for some $\mu_1, \mu_2, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2}$, and $\sigma_{1,2}$. What are their numerical values?

> **Solution:**

30. (Final 2022) Let X denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda exp(-\lambda x)1(x > 0)$. We have $H_0 : \lambda = 1$ and $H_1 : \lambda < 1$. Suppose that C is a critical region such that (1) the test of the form "Reject H_0 if $X_1 \in C$" is the uniformly most powerful test; and (2) the probability of rejecting H_0 when $\lambda = 1$ is α. What is the power of your test when $\lambda = 1/2$ and $\alpha = 5\%$

> **Solution:**

31. (Final 2022) Let X denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda \exp(-\lambda x)1(x > 0)$. We would like to test $H_0 : \lambda = 1$ against $H_1 : \lambda \neq 1$. Suppose that $X = e$

   (a) What is the value of the LR statistic?

Solution:

(b) Do you reject or accept the null at the 5% significance level? You maye use the approximation $e = 2.7183$

Solution:

32. (Final 2022) Let (X, Y) be a two dimensional random vector with the joint PDF $f_{X,Y}(x, y) = 4e^-2y1(y > x > 0)$. Let $(\alpha, \beta) \equiv argmin_{a,b}E[(Y - (a + bX))^2]$ What is (α, β)? Hint: a small number of students may find it useful to know that $\int_0^\infty x^m \lambda e^(- \lambda x)dx = \frac{m!}{\lambda^m}$

33. (Final 2015) Suppose that X_n is a sequence of random variables such that $\sqrt{n}(X_n - 3) \xrightarrow{d} N(0, 1)$

(a) What is the asymptotic distribution of $\sqrt{n}(X_n^2 - 9)$?

Solution: (Verified) By the delta method $\sqrt{n}(g(X_n) - g(a)) \xrightarrow{d} N(0, g'(a)^2\sigma^2)$. In this case, $g(x) = x^2$ and $a = 3$. So $g'(a) = 6$ and $\sigma^2 = 1$. So the answer is $N(0, 36)$

(b) What is the asymptotic distribution of $(\sqrt{n}(X_n - 3))^2$?

Solution: (Verified) A standard normal distribution squared is a chi-squared distribution. So the answer is $\chi^2(1)$

34. (Final 2015) Let $F(y|x)$ denote the conditional CDF of Y given X i.e. $F(y|x) = Pr[Y \leq y|X = x]$. Suppose that $F(y|x)$ is continuous and strictly increasing in y for all x in the support of X. Let $V = F(Y|X)$. (It is not $F(Y|x)$). Derive the conditional distribution of V given x. Prove that V and X are independent.

Solution: