# 203A Question Bank

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April 4, 2024

1. (Final 2018): Let $X$ have the uniform distribution over the $(0, 1)$ interval

   (a) Calculate $E[\log(X)]$

   **Solution:** Use integration by parts: $E[\log(X)] = \int_0^1 \log(x)dx = x\log(x) - x|_0^1 = 0 - 1 = -1$.

   (b) Calculate $\log E[X]$

   **Solution:** $E[X] = \int_0^1 xdx = 1/2$. So $\log E[X] = \log(1/2) = -\log(2)$.

   (c) Which is bigger?

   **Solution:** $\log E[X] = -\log(2) > -1 = E[\log(X)]$. So $\log E[X]$ is bigger.

   (d) Now let $g$ denote some positive valued strictly increasing function defined on $(0, 1)$. Let $Y = g(X)$. Between $E[\log(Y)]$ and $\log(E[Y])$, which one is bigger? Or does the answer depend on $g$? Note that there are only three choices.

   **Solution:** Jensen's inequality says that $\phi(E[X]) \leq E[\phi(X)]$ for a convex function $\phi$. Since log is concave, we have that $\log(E[Y]) \geq E[\log(Y)]$. So $\log(E[Y])$ is bigger.

   Note: Concavity is the tricky part here. You can convert log into a convex function by multiplying the log by -1, which makes it convex. Then you can apply Jensen's inequality.

2. (Final 2018): Suppose that the support of $Y$ is $\{0, 1\}$, and $Pr[Y = 1|X] = \Phi(X'\beta)$ where $\Phi$ is the standard normal cdf. We know that $Pr[Y = 1|(X_1, X_2) = (2, 1)] = .5$ and $Pr[Y = 1|(X_1, X_2) = (2, 2)] = .975$. What is the value of $\beta = (\beta_1, \beta_2)'$?

   The Phi scores you may want to use are:

   | $\Phi(x)$ | Value |
   |---|---|
   | $\Phi(0)$ | 0.5 |
   | $\Phi(0.253)$ | 0.6 |
   | $\Phi(0.534)$ | 0.7 |
   | $\Phi(0.842)$ | 0.8 |
   | $\Phi(1.282)$ | 0.9 |
   | $\Phi(1.645)$ | 0.95 |
   | $\Phi(1.960)$ | 0.975 |
   | $\Phi(2.576)$ | 0.995 |

**Solution:** $\beta_1 = -.98$ and $\beta_2 = 1.96$

We have $Pr[Y = 1|(2,1)] = \Phi(2\beta_1 + \beta_2) = .5$ and $Pr[Y = 1|(2,2)] = \Phi(2\beta_1 + 2\beta_2) = .975$. We can take the inverse of $\Phi$ using the table above. So we have $2\beta_1 + \beta_2 = 0$ and $2\beta_1 + 2\beta_2 = 1.960$. By subtraction, we get $\beta_2 = 1.96$. We then solve for $\beta_1$ using the first equation to get $\beta_1 = -.98$.

3. (Final 2018): Let $X_1, ..., X_n$ be a random sample of size $n$ from $N(3, 2)$, and let $\bar{X}$ denote the sample average. what is the asymptotic distribution of $\sqrt{n}(\bar{X}^2 - 9)$? Your answer should be numerical.

**Solution:** We use the Delta method: $\sqrt{n}[g(\bar{X}) - g(a)] \xrightarrow{d} N(0, g'(a)^2\sigma^2)$ where $g(x) = x^2$ and $a = 3$. So $g'(a) = 6$ and $\sigma^2 = 2$. So the answer is $N(0, 72)$.

4. (Final 2018): Let $X_n$ denote a sequence of random variables such that the PDF $f_n$ of $X_n$ is given by

$$f_n(x) = \frac{1}{2n}1(|X| \leq n)$$

(a) Let $0 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \leq B)$?

**Solution:** $\lim_{n\to\infty} P(|X_n| \leq B) = \lim_{n\to\infty} \int_{-B}^{B} \frac{1}{2n} dx = \lim_{n\to\infty} \frac{B}{n} = 0$
Note: I'm not sure I like this solution.

(b) Is $X_n = O_p(1)$?

**Solution:** $X_n$ is not bounded in probability. The definition of bounded in probability is $\lim_{n\to\infty} P(|X_n| \leq B) = 1$, which is not the case here.

5. (Final 2018, 2019) Suppose that $X$ and $\epsilon$ are independent $N(0, 1)$ variables. Let $Y = X + \epsilon$ What is the correlation between $X^2$ and $Y$

**Solution:** The correlation is zero. I'll write the algebra down later.

6. (Final 2018, 2019): True or False

(a) If $X_n = a + o_p(1)$, then $E[X_n] = a + o(1)$

**Solution:**

(b) If $X_n = a + o_p(1)$, then $X_n^2 = a^2 + o_p(1)$

**Solution:**

(c) If $X_n \xrightarrow{d} N(0, 1)$, then $E[X_n] = o(1)$

**Solution:**

(d) If $X_n \xrightarrow{d} N(0, 1)$, then $E[X_n^2] = 1 + o(1)$

**Solution:**

(e) If $X_n \xrightarrow{d} N(0,1)$, $\chi_n^2 \xrightarrow{d} \chi^2(1)$

**Solution:**

7. (Final 2019): Suppose that $Z \sim N(0,1)$, and let $X_n = Z^2 1(|Z| \geq \frac{1}{n})$. What is $lim_{n\to\infty} E[X_n]$?

**Solution:** Not finished. I think there's some law that let's us move the limit inside the expectation.

$$lim_{n\to\infty} E[X_n] = E[lim_{n\to\infty} X_n] = E[lim_{n\to\infty} Z^2 1(|Z| \geq \frac{1}{n})] = E[Z =^2] = \chi^2(1) = 1$$

8. (Final 2019): Let $X_1, ..., X_n$ be a random sample of size $n$ from $N(\mu, 1)$, and let $\bar{X}$ denote the sample average.

(a) Compute $sup_{\mu<0} lim_{n\to\infty} P(\sqrt{n}\bar{X} > 1.645)$

**Solution:**

(b) $lim_{n\to\infty} sup_{\mu<0} P(\sqrt{n}(\bar{X} - \mu) > 1.645)$

**Solution:**

9. (Final 2019) Let $X_n$ denote a sequence of random variables such that the PDF $f_n$ of $X_n$ is given by

$$f_n(x) = \frac{1}{4} 1(n \leq |x| \leq n+1) + \frac{1}{4} 1(|x| \leq 1)$$

(a) Let $1 < B < \infty$ be given what is $\lim_{n\to\infty} P(|X_n| \leq B)$?

**Solution:**

(b) Is $X_n = O_p(1)$?

**Solution:**

10. (Final 2019, 2021) Consider $X_1, ..., X_n$ iid $N(\mu, \sigma^2)$ We assume that $\sigma^2 = 1$ We have $H_0 : \mu = 0$ vs $H_1 : \mu > 0$ Suppose that C is a critical region such that (1) the test of the form "Reject $H_0$ if $(X_1, ..., X_n) \in C$" is the uniformly most powerful test; and (2) the probability of rejecting $H_0$ when $\mu = 0$ is 5%

(a) Provide a mathematical characterization of rejecting $H_0$ of such a test as a function of $n$ and $\mu > 0$

**Solution:**

(b) What is the power of the test when $\mu = .1645$ and $n = 100$?

**Solution:**

11. (Final 2021) Suppose that $X_1, X_2, ...$ are iid such that $X_i = 0$ with probability $1/2$, and $X_i = 2$ with probability $1/2$. Let $\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $F_n(x) \equiv P(\frac{1}{n}\sum i = 1^n X_i \leq x)$. What is $lim_{n\to\infty} F(1.1)$?

**Solution:**

12. (Final 2021) Suppose that $X$ has the CDF equal to

$$Pr[X \leq x] = \frac{exp(x)}{1 + exp(x)} \equiv \Lambda(x)$$

Let $\Phi(.)$ denote the CDF of $N(0,1)$, and let $\Phi^-1$ denote its inverse. $Y \equiv \Phi^-1(\Lambda(X))$. What is $E[Y^4]$?

**Solution:**

13. (Final 2021) Consider $X_1, ..., X_n$ iid $N(0,1)$. Let $Y_n = 1(\bar{X_n} \geq 1)$.
    (a) What is $E[Y_n]$?

    **Solution:**

    (b) What is $Var(Y_n)$?

    **Solution:**

    (c) What is the probability limit of $Y_n$?

    **Solution:**

14. (Final 2021) Suppose that $X_1, ..., X_4$ are iid and their common distribution is uniform $(0, \theta)$, i.e. their common PDF $f(x)$ is equal to $1(0 < x < \theta)$. We have $H_0 : \theta = 1$ vs $H_1 : \theta = 2$. Suppose that you decided to reject $H_0$ if $\max(X_1, ..., X_4) > 1$
    (a) What is the size of the test?

    **Solution:**

    (b) What is the power of the test? Hint: $P(\max(X_1, ..., X_4) \leq 1) = P[X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 1]$

    **Solution:**

15. (Final 2021) Let $X_1, ..., X_n$ be a random sample of size $n$ from $N(0,1)$. For any positive integer $k$, let $m_k = E[X_i^k]$ and $\hat{m_k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$. Assuming that $E[|X_i^k|] < \infty$ for all $k$, derive the asymptotic distribution $\sqrt{n}(\hat{m_k}^{1/2} - m_k^{1/2})$. The asymptotic distribution is normal with mean zero, so your job is to derive the numerical value of the asymptotic variance. Hint: the MGD of $N(0,1)$ is $exp(t^2/2)$.

**Solution:**

16. (Final 2021) Let $X_1, ..., X_5$ denote a random sample from $N(\theta, 1)$. We would like to test $H_0 : \theta = 5$ against $H_1 : \theta! = 5$. If $X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 5, X_5 = 6$, what is the LR statistic?

**Solution:**

17. (Final 2022) Let $X_n$ $b(n, \frac{q}{n})$ for some $q > 0$. Is $X_n = Op(1)$

**Solution:**

18. (Final 2022) Let $Y_n$ denote the maximum of a random sample of size $n$ from a uniform (0,1) distribution. What is $lim_{n\to\infty} Pr[Y_n \; leq 0.9]$?

**Solution:**

19. (Final 2022) Let $\Lambda(t) \equiv \frac{exp(t)}{1+exp(t)}$ denote the CDF of the logistic distribution (with location and scale parameters equal to 0 and 1, although these particular details are irrelevant for this question). Let $X_n$ denote a sequence of random variables such that the CDF $F_n$ of $X_n$ is given by $F_n(x) = \lambda(nx)$. What is $\lim_{n\to\infty} E[cos(X_n)]$?

**Solution:**

20. (Final 2022) Let $X_1$ denote a random sample (of size 1) from $\mathcal{N}(0, \sigma^2)$. We have null hypothesis $H_0 : \mu = 1$ and alternative hypothesis $H_1 : \mu = 9$.

**Solution:**

21. (Final 2022) Suppose that $X_n$ is a sequence of random variables such that $\sqrt{n}(X_n - 1)$ converges in distribution to $N(0, 1)$. By the multivariate delta method, we can see that

$$\begin{pmatrix} \sqrt{n}(X_n^2 - \mu_1) \\ \sqrt{n}(\ln(X_n) - \mu_2) \end{pmatrix} \xrightarrow{d} N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \right]$$

for some $\mu_1, \mu_2, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2}$, and $\sigma_{1,2}$. What are their numerical values?

**Solution:**

22. (Final 2022) Let $X$ denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda exp(-\lambda x)1(x > 0)$. We have $H_0 L\lambda = 1$ and $H_1 : \lambda < 1$. Suppose that C is a critical region such that (1) the test of the form "Reject $H_0$ if $X_1 \in C$" is the uniformly most powerful test; and (2) the probability of rejecting $H_0$ when $\lambda = 1$ is $\alpha$. What is the power of your test when $\lambda = 1/2$ and $\alpha = 5\%$

**Solution:**

23. (Final 2022) Let $X$ denote a random sample (of size 1) from a distribution with the PDF equal to $\lambda exp(-\lambda x)1(x > 0)$. We would like to test $H_0 : \lambda = 1$ against $H_1 : \lambda! = 1$. Suppose that $X = e$

(a) What is the value of the LR statistic?

**Solution:**

(b) Do you reject or accept the null at the 5% significance level? You maye use the approximation $e = 2.7183$

**Solution:**

24. (Final 2022) Let $(X, Y)$ be a two dimensional random vector with the joint PDF $f_{X,Y}(x, y) = 4e^-2y1(y > x > 0)$. Let $(\alpha, \beta) \equiv argmin_{a,b} E[(Y - (a + bX))^2]$ What is $(\alpha, \beta)$? Hint: a small number of students may find it useful to know that $\int_0^\infty x^m \lambda e^( - \lambda x)dx = \frac{m!}{\lambda^m}$

25. (Final 2015) Suppose that $X_n$ is a sequence of random variables such that $\sqrt{n}(X_n - 3) \xrightarrow{d} N(0, 1)$

(a) What is the asymptotic distribution of $\sqrt{n}(X_n^2 - 9)$?

**Solution:**

(b) What is the asymptotic distribution of $(\sqrt{n}(X_n^2 - 3))^2$?

**Solution:**

26. (Final 2015) Let $F(y|x)$ denote the conditional CDF of $Y$ given $X$ i.e. $F(y|x) = Pr[Y \leq y|X = x]$. Suppose that $F(y|x)$ is continuous and strictly increasing in y for all x in the support of $X$. Let $V = F(Y|X)$. (It is not $F(Y|x)$). Derive the conditional distribution of $V$ given $x$. Prove that $V$ and $X$ are independent.

**Solution:**