1. **Confusing customers.** Two firms i, j with "frames" $f_i, f_j \in \{A, B\}$ charge prices $p_i \in [0,1]$. If both firms have the same frame, consumers buy from the cheaper firm. If firms have different frames, fraction $\alpha \in (0,1)$ of consumers get confused and purchase from a random firm, while the remaining $(1-\alpha)$ purchase from the cheaper firm.

For parts (a) and (b), assume that frames are exogenous, and differ across firms.

(a) Argue that there is no pure NE.

(b) Solve for the unique symmetric mixed NE.

Now assume that firms choose frames; specifically both firms simultaneously choose frames and prices.

(c) Show that there is no symmetric NE where both firms choose frames deterministically.

1. **Confusing Consumers:**

(a) Since the two firms have different frames, firm i's profit is

$$u_i(p_i, p_{-i}) = \begin{cases} \frac{1}{2}\alpha p_i & p_{-i} < p_i \\ \frac{1}{2}p_i & p_{-i} = p_i \\ (1-\alpha+\frac{1}{2}\alpha)p_i = (1-\frac{1}{2}\alpha)p_i & p_{-i} > p_i \end{cases}$$

If $p_i > p_{-i}$, then it has to be $p_i = 1$, but $-i$ will want to deviate to $1-\epsilon$. Therefore, it cannot be an NE. The same argument for $p_i < p_{-i}$.

If $p_i = p_{-i}$, then both firm wants to undercut each other a bit, or deviate to setting price to 1: it cannot be an NE.

Therefore, there is no pure NE.

(b) Suppose both firms use a mixed strategy: $F(\cdot) \sim [\underline{p}, \bar{p}]$. We can use the usual argument for no atoms and no gaps: if there's an atom, then the probability of tie is non-trivial, and firms will want to undercut each other's atom or set price to 1; if there's a gap on $[a,b]$, then b is strictly better since the probability of getting consumers are the same, but b generates higher payoff.

When charging price p, seller i sells to $\alpha/2$ confused consumers and $(1-\alpha)(1-F(p))$ unconfused consumers. So the expected profit is

$$\pi(p) = \frac{\alpha}{2}p + (1-F(p))(1-\alpha)p$$

- By setting $p = \bar{p}$, $\pi(\bar{p}) = \frac{\alpha}{2}\bar{p}$ increases in $\bar{p}$, therefore $\bar{p} = 1$; $\pi(1) = \frac{\alpha}{2} = \pi(p)$
- By $\pi(1) = \frac{\alpha}{2} = \pi(p)$, we can solve for $F(p)$:

$$\frac{\alpha}{2}p + (1-F(p))(1-\alpha)p = \frac{\alpha}{2}$$
$$F(p) = 1 - \frac{\alpha(1-p)}{2(1-\alpha)p}$$

- By setting $p = \underline{p}$, $\pi(\underline{p}) = (1-\alpha/2)\underline{p} = \alpha/2$, so $\underline{p} = \frac{\alpha}{2-\alpha}$.

(c) Now the two firms simultaneously choose frames and prices. First, there cannot be an equilibrium where both firms deterministically choose the same frame: in such an equilibrium Bertrand price competition would imply zero-profits, so firms can profitably deviate to a different frame and charging $p_i = 1$.

Next we show that there can be no NE where both firms deterministically choose different frames: Equilibrium prices in such an equilibrium would have to be as in part b. But then, firm i has an incentive to switch frame and price at $\underline{p}$ to capture the entire market at this price, rather than leaving $\alpha/2$ to firm j at this price.

2. **All-pay Auction:** N workers are competing for a promotion by simultaneously exerting effort e_i. Worker i values the promotion at v_i, which is private information to i and i.i.d. across workers with (smooth) cdf F and support $[\underline{v}, \bar{v}]$. Effort has marginal cost one, and the worker with the highest effort wins the promotion.

(a) What is the expected payoff of effort b when others are using symmetric, strictly increasing, continuous strategies $e(v)$.

(b) Characterize the pure, symmetric BNE in strictly increasing, continuous strategies $e(v)$.

(c) How does the equilibrium effort compare to the equilibrium bid in a first-price auction $b(v) = v - \int_{\underline{v}}^{v} \frac{F(\hat{v})^{N-1}}{F(v)^{N-1}} d\hat{v}$?

(a) If others are playing $e(v)$, effort/bid b wins with probability $F(e^{-1}(b))^{N-1}$. Thus, the expected payoff of v bidding b equals

$$vF(e^{-1}(b))^{N-1} - b.$$

(b) We characterize the pure, symmetric BNE in strictly increasing, continuous strategies $e(v)$. Marginal payoffs wrt b

$$v\frac{F(v')^{N-1}}{dv'}|_{v'=e^{-1}(b)}\frac{de^{-1}(b)}{db} - 1.$$

For the equilibrium bid $b = e(v)$ this must equal zero. Using $\frac{de^{-1}(b)}{db} = \frac{1}{e'(e^{-1}(b))} = \frac{1}{e'(v)}$, we get

$$e'(v) = v\frac{dF(v)^{N-1}}{dv}$$

Note that $e(\underline{v}) = 0$ as $F(\underline{v}) = 0$. Integrating $(*)$ from $\underline{v}$ to v yields

$$\begin{aligned} e(v) &= \int_{\underline{v}}^{v} x dF(x)^{N-1} \\ &= xF(x)^{N-1}|_{\underline{v}}^{v} - \int_{\underline{v}}^{v} F(x)^{N-1}dx \\ &= F(v)^{N-1}\left[v - \frac{\int_{\underline{v}}^{v} F(x)^{N-1}dx}{F(v)^{N-1}}\right]. \end{aligned}$$

(c) Note that the "bidding function" is the winning probability times the "bidding function" of a first price auction. That is, the bidders bid less aggressively since they pay their bids in any case.

3. **Jobs:** A firm and chooses a salary $s \in \mathbb{R}_+$ and simultaneously a worker chooses effort $e \in \mathbb{R}_+$. The production function $\psi(e)$ is differentiable, strictly concave with $\psi(0) = 0$, $\lim_{e \to 0} \psi'(e) = \infty$, and $\psi'(e) < 1$ for large enough e. The firm's payoff is $\psi(e) - s$ and the worker's payoff equals $s - e$.

(a) Show that $e^* = s^* = 0$ is the unique Nash equilibrium of this game.

Now suppose that this game is repeated infinitely often, and future payoffs are discounted at rate $\delta \in (0,1)$. Consider the following strategy profile $\langle s^*, e^* \rangle$. Initially, in stage I, the firm pays s^* and the worker exerts effort e^*. If any player ever deviates from these strategies, players enter stage II, where they play the static Nash equilibrium calculated in part (a) forever after. We say that wage s^* *supports* effort e^* if $\langle s^*, e^* \rangle$ is a SPE, and we say that effort e^* is *supportable* if there exists s^* that supports e^*.

(b) Show that e^* is supportable if and only if

$$e^* \leq \delta^2 \psi(e^*) \qquad (1)$$

(c) Show that for sufficiently high discount factor $\delta < 1$, first-best effort e^{FB}, defined by $\psi'(e^{FB}) = 1$, is supportable.[1]

(d) Now assume that the worker can at any point quit the firm for an outside job with fixed payoff $\underline{s}$ per period. Show that the set of supportable effort levels is decreasing in $\underline{s}$. Give an economic interpretation of this effect.

[1]Hint: Sketch the function $\delta^2\psi(e)$, the region of e where $(*)$ holds and argue geometrically that e^{FB} must lie in this region for large enough $\delta < 1$.

(a) Given s, the worker's payoff equals $s - e$, so his best response is $e = 0$. Given e, the firm's payoff equals $\psi(e) - s$, so the firm's best response is $s = 0$. Thus, the unique Nash equilibrium is $s^* = 0, e^* = 0$.

(b) After a deviation, firm and worker repeat the static Nash equilibrium. Thus it suffices to check whether either of them wants to deviate in period I.

The worker compares an infinite flow of $s^* - e^*$ to a one-time payoff of s^* (by deviating to $e = 0$), so we need

$$s^* - e^* \geq (1-\delta)s^*$$

The firm compares an infinite flow of $\psi(e^*) - s^*$ to a one-time payoff of $\psi(e^*)$ (by deviating to $s = 0$), so we need

$$\psi(e^*) - s^* \geq (1-\delta)\psi(e^*)$$

Combining these constraints, e^* is supportable if and only if

$$e^* \leq \delta^2 \psi(e^*)$$

(c) Given the functional properties of $\psi(e)$, $\delta^2\psi(e)$ intersects the 45^o line at 0 and some $\bar{e} = \bar{e}(\delta) > 0$. This is illustrated in Figure 1. For any $e \in [0, \bar{e}]$, $e \leq \delta^2\psi(e)$, thus such an e is supportable. By the mean-value theorem, $\psi'(e^{FB}) = 1$ implies that $e^{FB} \in (0, \bar{e}(1))$; by continuity $e^{FB} < \bar{e}(\delta)$ for large enough $\delta < 1$, thus e^{FB} is supportable.

(d) Now the worker's constraint changes to

$$s^* - e^* \geq (1-\delta)s^* + \delta\underline{s}$$

Combining this with the firm's constraing, e^* is supportable if and only if

$$e^* + \delta\underline{s} \leq \delta^2\psi(e^*)$$

Clearly, the set of supportable effort levels is decreasing in $\underline{s}$, and it is smaller than that in (b) if $\underline{s} > 0$. This is illustrated in Figure 2.

The outside option of working for $\underline{s}$ after losing the job undermines the worker's incentives in its relational contract with the original firm.

4. **The Power of Nash Equilibrium:** Consider the following game.

		2	
		L	R
1	U	3 , 0	1 , 1
	D	2 , 1	0 , 0

(a) Suppose the players move simultaneously. What are the Nash equilibria?

(b) Suppose Player 1 first chooses $a_1 \in \{U, D\}$, Player 2 sees 1's action and then chooses $a_2 \in \{L, R\}$. What are the SPE?

Suppose Player 1 moves first, but that Player 2 observes 1's action with noise. In particular, Player 2 sees signal $\phi \in \{u, d\}$ such that

$$\Pr(\phi = u | a_1 = U) = 1 - \delta \qquad \text{and} \qquad \Pr(\phi = d | a_1 = D) = 1 - \delta.$$

(c) Draw the extensive form of the game.

(d) Suppose $\delta = 0$. What are the pure strategy weak-PBE? Which of these are sequential equilibria?[2]

(e) Suppose $\delta > 0$. What are the pure strategy weak-PBEs? Which of these are sequential equilibria? Explain the difference between this and the last answer.

(f) Suppose $\delta \in (0, 1/4)$. Now consider mixed strategy equilibria. Suppose Player 1 plays $a_1 = D$ with probability $\lambda \in (0,1)$. Suppose Player 2 plays L with probability $\eta \in (0,1)$ after signal $\phi = u$, and plays L with probability 1 after signal $\phi = d$. Let μ be Player 2's belief that $a_1 = D$ after signal $\phi = u$. Find the mixed strategy equilibrium η^*, λ^*, μ^*.

[2]When considering sequential equilibria, the strict mixing of actions in the approximating strategy profiles only pertains to players' actions, not to nature's realization of the state, which is fixed to $\varepsilon = 0$.

(a) Player 1 has a strictly dominant strategy so (U, R) is the unique NE. (b) $(D, (R, L))$ is the unique SPE.

(c) A game tree can be depicted as follows:

(d) $(D, (R, L))$ with player 2 trusting the signal, i.e. beliefs $\mu(a_1 = \phi) = 1$ after observing signal $\phi \in \{u, d\}$, is a wPBE.

$(U, (R, R))$ with 2 trusting the equilibrium strategy, i.e. beliefs $\mu(a_1 = U) = 1$ after observing either signal $\phi \in \{u, d\}$, is also a wPBE.

Only $(D, (R, L))$ is sequential. For totally mixed strategies of player 1, Bayes' rule implies that player 2 trusts the signal, so $\mu(a_1 = D) = 1$ after observing $\phi = d$; it is then sequentially rational to play L.

(e) In any pure equilibrium, player 2 knows player 1's action in equilibrium, and thus ignores the signal. Since player 2's response is thus determined by her equilibrium beliefs about a_1 and does not depend on 1's actual action choice, player 1 must play her statically dominant action U. Thus, $(U, (R, R))$ with beliefs $\mu(a_1 = U) = 1$ after observing either signal $\phi \in \{d, u\}$ is the only pure weak-PBE (indeed the only pure NE). These beliefs are consistent since for mixed strategies very close to the pure $a_1 = U$ the fixed noise δ in the signal induces player 2 to ignore the signal and stick to her prior equilibrium belief that $a_1 = U$. Thus, the equilibrium is sequential.

In the sequential equilibrium in part (d) we trembled the 1's strategy, so player 2 trusts the signal off the equilibrium path and equilibrium is as in the sequential game in part (b). Here we tremble 2's signal, so in a pure strategy equilibrium player 2 ignores the signal and equilibrium is as in the static game in part (a).

(f) Player 1's indifference condition is

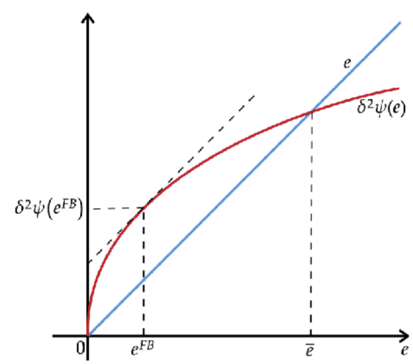
$$\begin{aligned} 3\delta + [3\eta + (1-\eta)](1-\delta) &= 2(1-\delta) + 2\delta\eta \\ (2 - 4\delta)\eta &= 1 - 4\delta \\ \eta^* &= \frac{1-4\delta}{2-4\delta} \end{aligned}$$

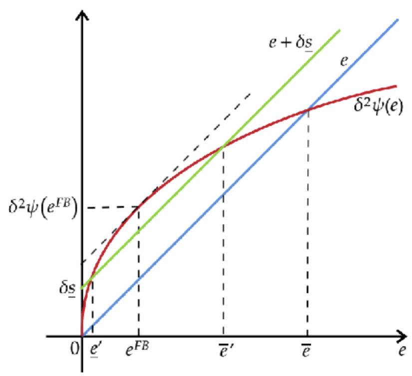
Player 2's belief μ after signal $\phi = u$ is given by Bayes' rule

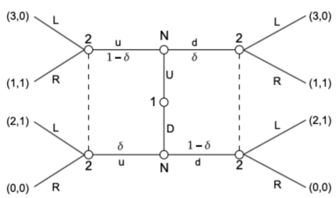
$$\mu = \frac{\delta\lambda}{\delta\lambda + (1-\delta)(1-\lambda)}$$

Player 2's indifference condition after $\phi = u$ is $\mu = 1 - \mu$ yielding $\mu^* = 1/2$. Thus

$$\begin{aligned} \frac{\delta\lambda}{\delta\lambda + (1-\delta)(1-\lambda)} &= \frac{1}{2} \\ 2\delta\lambda &= 2\delta\lambda + 1 - \delta - \lambda \\ \lambda^* &= 1 - \delta \end{aligned}$$







1. **Mergers in a Cournot Model (20 points):** There are $N = 5$ firms competing in quantities q_i of a homogeneous good. Inverse demand is linear, $p(Q) = a - Q$ where $Q = \sum_i q_i$, and unit costs are $c < a$.

(a) Compute quantities and profits in the unique Nash equilibrium.

(b) Now assume that two of the firms can merge before the quantity competition. If they decide to merge, the competition effectively reduces to four identical competing firms and the profits of the merged firms is split evenly between the previous owners. Would the firms decide to merge? Discuss.

(c) How does your answer to part (b) change when three of the firms can merge into a single firm? What if four of the firms can merge?

(a) (6 points) As always, we first characterize firm i's best response. $\max_{q_i} \ (a - q_i - Q_{-i} - c)q_i$

where $Q_{-i} = \sum_{i \neq j} q_j$. FOC yields $BR_i(Q_{-i}) = \dfrac{a - c - Q_{-i}}{2}$ Thus the unique NE is given by $q_i^* = \dfrac{a-c}{N+1} = \dfrac{a-c}{6}$ The equilibrium profit for firm i is thus equal to $\pi_i^* = \dfrac{(a-c)^2}{(N+1)^2} = \dfrac{(a-c)^2}{36}$

(b) (7 points) If two firms decide to merge, the problem is the same as part (a) except that $N = 4$ now. Thus the new equilibrium quantity for each firm is equal to $q_i^* = \dfrac{a-c}{5}$

and the new equilibrium profit for each firm is equal to $\pi_i^* = \dfrac{(a-c)^2}{25}$

Each non-merging firm is strictly better off in part (b) than in part (a). However for the two merged firms, each earns a profit $(a-c)^2/50 < (a-c)^2/36$, so they are strictly worse off in part (b). Thus the firms will choose not to merge.

Merging is usually profitable because it allows the merged firms to reduce quantity. However, in a Cournot model, those non-merging firms exploit the merger to expand quantity at the expense of the merged firms. Thus staying separate effectively acts as a commitment device to produce high quantity to prevent other firms from expanding.

(c) (7 points) If three firms merge, the competition effectively reduces to $N = 3$ identical competing firms. It is easy to see that the new equilibrium profit for each firm is equal to $\pi_i^* = \dfrac{(a-c)^2}{(N+1)^2} = \dfrac{(a-c)^2}{16}$

Thus each merged firm gets $(a-c)^2/48 < (a-c)^2/36$, so they will still not merge.

Similarly, if four of the firms can merge, then after merging, each merged firm gets $\dfrac{1}{4}\dfrac{(a-c)^2}{9} = \dfrac{(a-c)^2}{36}$

Thus these firms weakly prefer to merge. As the portion of merged firms increases, the market becomes more concentrated. The benefit of merger ultimately outweighs the negative externality imposed by the non-merging firms.

2. **Overselling (20 points):** An expert (she) is selling a service to a customer (he). The customer has a problem that may be severe or minor, $i \in \{s, m\}$ with probabilities $\Pr(i = s) = \Pr(i = m) = 1/2$, which if untreated causes the customer a loss of $\ell_i > 0$. The expert observes the type of the problem i, and recommends a treatment $j \in \{s, m\}$ at an exogenous price, equal to ℓ_j. The customer only observes the recommendation j but not the type of the problem i, and decides whether or not to accept the recommendation. The expert incurs a cost of $c_j > 0$ for providing treatment j, and maximizes her revenue, i.e. price minus cost. When the customer rejects the recommendation, the expert gets zero. The customer pays for the treatment ℓ_j when he accepts it, and additionally incurs the loss ℓ_i unless he receives the appropriate treatment, i.e. he accepts treatment and $j = i$. Assume that $0 < \ell_m - c_m < \ell_s - c_s$.

(a) Draw the extensive form game for this interaction.

(b) Describe the strategy sets, and the set of behavioral strategies for each player.

(c) Show that it is not a Bayes-Nash equilibrium for the expert to always recommend the right treatment, $j = i$, and for the customer to always accept the recommendation.

(d) Solve for a Bayes-Nash equilibrium where the expert always recommends the right treatment, $j = i$, and the customer accepts the recommendation $j = m$.

(a) (4 points) Typical "Beer-Quiche" tree

(b) (6 points) Strategies:

i. Expert: $R : \{s, m\} \to \{s, m\}$ where $R(i) \in \{s, m\}$ is the recommendation when the actual problem is i

ii. Customer: $A : \{s, m\} \to \{yes, no\}$ where $A(j)$ is the customer's acceptance decision when recommended treatment j

Behavioral strategies

i. Expert: (p_s, p_m) where $p_i \in [0, 1]$ is the probability of recommending treatment s when the actual problem is i

ii. Customer: (q_s, q_m) where $q_j \in [0, 1]$ is the probability of accepting treatment j when it is recommended

(c) (4 points) When the customer always accepts the expert's recommendation, $A(j) \equiv yes$, the expert would always recommend the more profitable, severe treatment $R(i) \equiv s$, even when the actual problem is moderate $i = m$.

(d) (6 points) The only action that is not yet specified is the customer's acceptance decision when the severe treatment is recommended, q_s. In equilibrium, to keep the expert honest when facing either kind of problem, she must be indifferent between the two recommendations since her preferences do not depend on the actual problem.

So $q_m(\ell_m - c_m) = q_s(\ell_s - c_s)$, or $q_s = \frac{\ell_m - c_m}{\ell_s - c_s} \in (0, 1)$. Since the expert is honest and the price of treatment equals its value to the customer, the customer is indifferent between accepting and rejecting either recommendation.

3. **Sequential Vote Buying (30 points).** An oil lobbyist is buying votes from three legislators $i \in \{A, B, C\}$ to pass a policy to drill in the Arctic Ocean. The policy passes if the lobbyist buys at least two votes. Passing the policy is worth y to the lobbyist, and -1 to each legislator. The timing is as follows

- The lobbyist makes a take-it-or-leave-it (TIOLI) offer $t_A \geq 0$ to legislator A.
- Legislator A chooses whether to accept or reject t_A.
- The lobbyist makes a TIOLI offer $t_B \geq 0$ to legislator B.
- Legislator B chooses whether to accept or reject t_B.
- The lobbyist makes a TIOLI offer $t_C \geq 0$ to legislator C.
- Legislator C chooses whether to accept or reject t_C.

The game has perfect information and the solution concept is SPNE.[1] Assume first that payments are conditional on the policy passing.

(a) Assume that $y > 2$. Show that the policy passes in equilibrium for a total payment of 0.

(b) Assume that $y < 2$. Show that the policy does not pass in equilibrium.

(c) Now assume that payments are up-front, irrespective of whether the policy passes or not. Also assume $1 < y < 2$. Show that the policy passes in equilibrium, and characterize the equilibrium payments. Discuss intuitively why up-front payments help the lobbyist pass her policy.

First note that it is weakly dominant for every legislator i to accept offers $t_i \geq 1$; thus the lobbyist offers at most $t_i = 1$.

(10 points) $y > 2$

- If A rejects t_A, the lobbyist needs to win both remaining votes, and will buy them by setting $t_B = t_C = 1$ since $y - 2 > 0$. Thus, A has utility 0 from rejecting and will accept any offer $t_A \geq 0$.
- If B rejects t_B after A has accepted $t_A \leq 1$, the lobbyist will buy C's vote for $t_C = 1$ (again, we use that $y - t_A - t_C > 2 - t_A - 1 \geq 0$), C will accept, and B's utility is 0. So B accepts any $t_B \geq 0$.
- Thus, on path the lobbyist offers $t_A = t_B = 0$ and legislators A and B accept.

(10 points) $y < 2$

- If A rejects t_A, the lobbyist needs to win both remaining votes, but doing so would cost him $t_B = t_C = 1$ which is not worth it since $y < 2$. Thus, A has utility 1 from rejecting and the lobbyist must offer $t_A = 1$ to make A accept.
- After A accepts $t_A = 1$, B knows that if he rejects t_B, the lobbyist would need to offer $t_C = 1$ to buy C's vote, which is not worth it to the lobbyist, so B's utility from rejecting is 1 and the lobbyist must offer $t_B = 1$ to make B accept.
- Thus, buying two votes is too expensive for the lobbyist.

10 points) Up-front payments

- Assume first A has rejected t_A.
  - If B also rejects, C's vote is useless to the lobbyist and she has utility 0.
  - If B accepts t_B, C's vote is worth 1 to the lobbyist and she buys it for $t_C = 1$.
  - B thus only accepts when $t_B = 1$; but this is overall too expensive for the lobbyist, so she offers $t_B = t_C = 0$ and the policy fails.
- Now assume A has accepted t_A
  - If B rejects t_B, C's vote is worth 1 to the lobbyist and she buys it for $t_C = 1$.
  - Thus, the policy passes independently of B's acceptance, so he accepts any $t_B \geq 0$.
- Thus, the lobbyist offers $t_A = 1$ to win A's vote, and then gets B's vote for free.
- Paying A up-front is a commitment device for the lobbyist that strengthens her bargaining position vis-a-vis B and C.

4. **Public goods (30 points):** Two players $i = 1, 2$ with quasi-linear utilities simultaneously contribute $x_i \in [0, 1]$ towards a discrete public good, which costs $c = 1$. If $x_1 + x_2 \geq 1$, the public good is bought and the residual $x_1 + x_2 - 1$ is divided evenly between the players. If $x_1 + x_2 < 1$, the public good is not bought and the contribution x_i is returned to player i. Player i's value for the public good $\theta_i \sim U[0, 1]$ is private information of player i and independent across i. Players have quasi-linear utility, equal to $\theta_i - x_i + \frac{x_1 + x_2 - 1}{2}$ if the good is provided, and 0 otherwise.

(a) What is the Pareto-efficient provision of the public good as a function of θ_1, θ_2?

Assume from now on that the other player plays a linear strategy $x_{-i}(\theta_{-i}) = \mu + \lambda\theta_{-i}$.

(b) What is the associated distribution of the bids? State support $[\underline{x}, \overline{x}]$ pdf $f(x)$, and cdf $F(x)$.

(c) Argue that expected payoffs when type θ_i contributes x_i equal

$$\pi_i(\theta_i, x_i) = \int_{1-x_i}^{\overline{x}} \left(\theta_i + \frac{x_{-i} - x_i - 1}{2}\right) dF(x_{-i}).$$

when $x_i \geq 1 - \overline{x}$, and $\pi_i(\theta_i, x_i) = 0$ when $x_i \leq 1 - \overline{x}$.

(d) Assume that the other player sets contributions equal to her value $x_i = \theta_i$, i.e. $\mu = 0, \lambda = 1$. What is the best response to this strategy? Is the resulting allocation efficient?

(e) Solve for a symmetric linear equilibrium where the good is provided with positive probability.

(a) (5 points) Provide the good if its value exceeds its cost, i.e. iff $\theta_1 + \theta_2 \geq c = 1$.

(b) (5 points) $-i$'s contribution is uniform between $\underline{x} = \mu$ and $\overline{x} = \mu + \lambda$ with cdf $F(x_{-i}) = (x_{-i} - \mu)/\lambda$ and pdf $f(x_{-i}) \equiv 1/\lambda$.

(c) (5 points)

- If $x_i \geq 1 - \overline{x}$, the public good is provided if $x_i + x_{-i} \geq 1$, i.e. for all $x_{-i} \in [1 - x_i, \overline{x}]$, in which case i's payoff equals $\theta_i - x_i + \frac{x_1 + x_2 - 1}{2} = \theta_i + \frac{x_{-i} - x_i - 1}{2}$.
- If $x_i \leq 1 - \overline{x}$, then $x_i + x_{-i} \leq x_i + \overline{x} < 1$, and the good is never provided, so $\pi_i(\theta_i, x_i) = 0$.
- Integrating yields (1).

(d) (10 points) Best response

- Marginal payoff

$$\begin{aligned}\frac{\partial \pi_i}{\partial x_i}(\theta_i, x_i) &= -\frac{1}{2}(1 - F(1 - x_i)) + (\theta_i - x_i)\, f(1 - x_i) \\ &= -\frac{1}{2}\left(1 - \frac{1 - x_i - \mu}{\lambda}\right) + (\theta_i - x_i)\,\frac{1}{\lambda}\end{aligned}$$

- FOC

$$\lambda = 1 - x_i - \mu + 2\theta_i - 2x_i$$

or

$$x_i = \frac{1 - \lambda - \mu + 2\theta_i}{3} = \frac{2}{3}\theta_i$$

- The resulting allocation is not efficient since i shades his true value, and so the public good is underprovided when $\theta_{-i} + \frac{2}{3}\theta_i < 1 < \theta_{-i} + \theta_i$.

(e) (5 points) In a symmetric equilibrium, we must have

$$\mu + \lambda\theta = \frac{1 - \lambda - \mu + 2\theta}{3}$$

for all θ and so $\lambda = \frac{2}{3}$ and $\mu = \frac{1 - \lambda - \mu}{3} = \frac{1}{12}$.

1. **Repeated Cournot (20 points):** Consider the linear Cournot oligopoly model where firm $i = 1, ..., N$ chooses quantity q_i, price $p = a - bQ$ is a linear function of total output $Q = \sum_{i=1}^{n} q_i$, and where output is produced by each firm i at constant marginal cost c (same for all firms).

(a) Solve for the static Nash equilibrium q^c, Nash profits π^c, the collusive quantity q^m (such that $Q = Nq^m$ maximizes total profits), and the optimal deviation quantity q^d when all other firms $j \neq i$ produce q^m.

(b) Suppose now that the stage game will be repeated infinitely, and that firms discount future payoffs at the rate δ. Let δ_N be the minimum discount factor for which it is possible to sustain complete collusion in subgame perfect equilibrium (SPE), using Nash reversion. Solve for δ_N as a function of N. What is δ_2 and $\lim_{N\to\infty} \delta_N$? Is this what you would expect? Why, or why not?

(a) Let (q^m, π^m) and (q^c, π^c) be a firm's quantity and profit in collusion and Cournot competition, respectively. Then,

$$q^m = \underset{q}{argmax} \ (a - bNq - c)q$$

$$q^c = argmax \ [a - b(N-1)q^c - bq - c]\, q$$

By first order conditions, we have $q^m = \frac{a-c}{2Nb}$ and $q^c = \frac{a-c}{(N+1)b}$. Therefore, $\pi^m = \frac{(a-c)^2}{4bN}$ and $\pi^c = \frac{(a-c)^2}{(N+1)^2 b}$. If all the other firms produce q^m, the optimal one-shot deviation is given by

$$q^d = \underset{q}{argmax} \ \ [a - b(N-1)q^m - bq - c]\, q$$

By first order condition, we have $q^d = \frac{(N+1)(a-c)}{4Nb}$, and thus $\pi^d = \frac{(N+1)^2(a-c)^2}{16N^2 b}$.

(b) To sustain complete collusion, one-shot deviation implies that $\pi^m \geq (1-\delta)\pi^d + \delta\pi^c$ or $\delta \geq \delta^N := \frac{\pi^d - \pi^m}{\pi^d - \pi^c}$ Substituting π^m, π^d and π^c yields

$$\delta^N = \frac{\frac{(N+1)^2}{16N^2} - \frac{1}{4N}}{\frac{(N+1)^2}{16N^2} - \frac{1}{(N+1)^2}} = \frac{(N+1)^2 - 4N}{(N+1)^2 - \frac{16N^2}{(N+1)^2}} = \frac{(N+1)^2(N-1)^2}{(N+1)^4 - 16N^2}$$
$$= \frac{(N+1)^2(N-1)^2}{((N+1)^2 - 4N)((N+1)^2 + 4N)} = \frac{(N+1)^2}{(N+1)^2 + 4N}.$$

Particularly, if $N = 2$, $\delta = \frac{9}{17}$. Note that δ_N is increasing for $N \geq 2$ and $\delta_N \to 1$ as $N \to \infty$. Hence with more firms, it becomes harder to sustain collusion.

Intuitively, the benefit from collusion π^m converges to 0 as $N \to \infty$, while the short-term temptation to deviate π^d is boundedly positive, converging to $\frac{(a-c)^2}{16b} > 0$.

2. **Simultaneous Search (30 points):** There are two firms, A, B, who produce a homogenous good at zero cost. There is mass 1 of consumers who have willingness to pay 1. Fraction $\alpha \in [0,1]$ of consumers know of both firms, fraction $(1-\alpha)/2$ only knows firm A, and fraction $(1-\alpha)/2$ only knows firm B. Firms know α and simultaneously choose prices p_A and p_B. Consumers that know of both firms buy from the cheapest; consumers that know of only one firm buy from that firm (if the price is less than 1).

(a) Solve for the Nash equilibrium of this game if all consumers know both firms, $\alpha = 1$

(b) Solve for the Nash equilibrium of this game if no consumer knows both firms, $\alpha = 0$.

(c) Now assume that some, but not all consumers know both firms, $\alpha \in (0,1)$. Show that there is no Nash equilibrium in pure strategies. Solve for the symmetric Nash equilibrium in mixed strategies.[1] What is the support of the prices in this equilibrium?

Now suppose that α is endogenous: Initially, half of consumers know only of firm A and the other half know only of firm B (but not their prices, which firms have not yet chosen). Then consumers simultaneously decide whether to search for the other firm (and find it) at cost $c > 0$. Then firms observe how many consumers know of each firm, and then choose prices.

(d) Are there SPE in which all consumers search, as in part (a), and in which no consumers search, as in part (b)?

(e) Now assume that some, but not all consumers search, i.e., $\alpha \in (0,1)$ as in part (c). Under what condition on prices, i.e. the distribution of p_A and p_B in the pricing stage, are consumers indifferent between searching and not searching? (You do not need to plug in the price distribution from part (c) and solve for the equilibrium value of α. Just state the indifference condition generically in terms of expectations over p_A and p_B.)

(a) If all consumers are informed, we are in a Bertrand world. Hence $p_A = p_B = 0$.

(b) If no customer is informed, we are in a monopoly world. Hence $p_A = p_B = 1$.

(c) If firms charge different prices, the firm with the lower price would prefer to raise its price. If firms charge the same price $p > 0$, either firm would prefer to slightly undercut its competitor. If both firms charge $p = 0$, profits are zero, and either firm would want to deviate to $p_i = 1$ for a profit of $(1-\alpha)/2$. Hence there cannot be a pure strategy equilibrium.

In a mixed strategy equilibrium, the firm with the highest price should charge the monopoly price 1 and hence has profits $(1-\alpha)/2$. Since firms are indifferent:

$$\frac{1-\alpha}{2} = F_\alpha(p_1)p\frac{1-\alpha}{2} + [1 - F_\alpha(p_1)]p\frac{1+\alpha}{2} \qquad F_\alpha(p) = \frac{1}{2p\alpha}[(1+\alpha)p - (1-\alpha)$$

The lowest price is

$$p = \frac{1-\alpha}{1+\alpha}$$

which wins with probabilty 1 for a total demand of $(1+\alpha)/2$ and profits $(1-\alpha)/2$.

(d) If all consumers search, then both firms set the same price $p = 0$, as in part (a), so searching is not optimal for consumers. Hence, this is not an equilibrium. If no consumers search, then both firms set the same price $p = 1$, as in part (b), so searching is not optimal for consumers. Hence, neither of these strategies is an equilibrium.

(e) Now, consumers must be indifferent

$$E[p_i] = E[\min\{p_A, p_B\}] + c$$

If F_α satisfies this equation, this yields a SPNE of the two-stage game.

3. **A War of Attrition (20 points):** A large and a small firm with respective capacities $\overline{k} = 20$ and $\underline{k} = 10$ are deciding when to exit a declining industry. Inverse demand evolves as $p_t = \max\{100 - K_t - t, 0\}$ where K_t is the aggregate capacity of firms remaining in period $t = 1, 2, 3, ..., 100$ and per period production costs are $c = 10$ for either firm. In every period t, both firms simultaneously choose whether to stay or exit. Exit is irreversible. When one firm has exited, the other one decides in every period whether to stay or exit. The game ends when both firms have exited, or after the last period $t = 100$. Each firm maximizes the undiscounted sum of per-period profits $\pi_t = (p_t - c)k$ over all periods $t < T$ before it exits at time T.

(a) Argue that when both firms remain in period $t \in (100 - \overline{k} - c, 100 - \underline{k} - c)$ the large firm exits and the small firms stays in any SPE.

(b) Argue inductively that the same behavior obtains in all periods $t \in (100 - \overline{k} - \underline{k} - c, 100 - \overline{k} - c]$ in any SPE.

(c) Now assume that each firm is credit-constrained and must exit after any period with negative per-period profit π_t. Show that in contrast to parts (a) and (b) there is a SPE where the small firm exits first on path.

(a) In any period $t > 100 - \overline{k} - c = 70$, if the large firm stays we have $K_t \geq \overline{k}$ so the price $p_t = 100 - K_t - t$ is less than c, so the large firm strictly prefers to exit, irrespective of the behavior of the small firm. Thus, the small firm prefers to stay as long as $p_t = 100 - \underline{k} - t > c$, that is for all $t < 100 - \underline{k} - c = 80$.

(b) By part (a), in any SPE the small firm gains monopoly profits $\pi_t = (100 - \underline{k} - t - c)\underline{k} = (80 - t)\underline{k}$ in all periods $t \in (100 - \overline{k} - c, 100 - \underline{k} - c) = (70, 80)$, so upon reaching period 71, it has continuation payoffs $\Pi = (9 + 8 + ... + 1)\underline{k} = 45\underline{k} > c\underline{k}$.

Thus, in period $t = 70$, the small firm strictly prefers to stay for any period $t = 70$ behavior of the large firm, since any one-period loss, which is at most $c\underline{k}$ is outweighed by the continuation payoff Π.

Since the small firm stays, the large firms prefers to exit in period $t = 70$ as flow profits from staying $(100 - \overline{k} - \underline{k} - c)\overline{k}$ are negative.

Arguing inductively over t from $t = 70$ to $t = 61$ the small firm stays in all these periods (recognizing that its continuation payoffs Π are higher in earlier periods) and the large firm exits.

(c) Consider the strategy profile where as long as both firms remain in the market the small firm exits in all periods $t \geq 100 - \overline{k} - \underline{k} - c = 60$, and the large firm exits in all periods $t \geq 100 - \overline{k} - c = 70$; once only one firm with capacity k remains it exits at $t \geq 100 - k - c$. Given the exit behavior of the small firm, the large firm is profitable until period 70 and hence stays until then. Given the exit behavior of the large firm, the small firm starts losing money by staying past period $t = 60$ and is then forced to exit because of the credit constraint. Thus exiting by period $t = 60$ is optimal.

4. **Reputational Cheap Talk (30 points):** The President is deciding between two alternatives: enact a new educational program (E) or not enact and stay with the status quo (N). The status quo will give the president a (commonly known) payoff of 0. The payoff from the new education program depends on the state of the world. With probability 1/4 the new program will be good (G) and give the President a payoff of 1. With probability 3/4, it will be bad (B) and provide a payoff of -1.

Hoping to obtain sage advice, the President hires an Economist. He is uncertain, however, of the Economist's competence. The Economist is knowledgeable with probability 1/4, and ignorant with probability 3/4. If the Economist is knowledgeable (K), he is aware of the state of the world. If the Economist is ignorant (I), he does not know the state of the world (and therefore believes that the program will be good with probability 1/4). The Economist knows his own type.

The game proceeds as follows:

- Stage 1: Nature chooses the state of the world (G or B) and the type of Economist (K or I). The knowledgeable Economist observes the state of the world.
- Stage 2: The Economist advises the President about the state of the world. Specifically, he sends one of two messages, either g (for good) or b (for bad). The Economist cannot profess ignorance, and is constrained to send one of those two messages.
- Stage 3: The President, viewing the advice, chooses either E or N.

After viewing his payoff (either -1, 0, or $+1$), the President, who is a good Bayesian, infers that the Economist is knowledgable with probability μ. The Economist's payoff is simply μ (he cares about his future reputation). All players are risk-neutral. Throughout, apply the (weak) PBE equilibrium concept.

(a) Describe each player's strategy set.

(b) Now you will explore the possibility that there is an equilibrium in which the knowledgeable Economist gives informative advice based on the state of the world (g when G and b when B), and the ignorant Economist always sends the message b.

i. In such an equilibrium, what would be the probability that the President receives the message g? What would be his belief about the state of the world given he receives g? What will the President do when he receives the message g?

ii. In such an equilibrium, what would be the probability that the President receives the message b? What is his belief about the state of the world given he receives the message b? What will the President do when he receives the message b?

iii. In such an equilibrium, what would be the President's Bayesian beliefs about the probability that the Economist is knowledgeable given the combinations of message and outcome $(g, 1)$ and $(b, 0)$? What is range of beliefs after $(g, -1)$?

iv. Is there an equilibrium of the form we have been examining? If not, explain why not. (Remember that the President's final beliefs about the Economist's type determine the Economist's payoff.)

(c) How do your answers to part (b) change if the probability of the Economist being knowledgeable is 1/2? Interpret your results.

(a) Economist $s_E : \{I, G, B\} \to \{g, b\}$, President $s_P : \{g, b\} \to \{E, N\}$

(b) i. $\Pr(g) = \Pr(K) \times \Pr(G) = 1/4 \times 1/4 = 1/16$. When the president receives g he knows the state is G, so he enacts the educational program.

ii. $\Pr(b) = \Pr(K) \times \Pr(B) + \Pr(I) = 1/4 \times 3/4 + 3/4 = 15/16$. By Bayes' rule, the President's belief after message b is $\Pr(G|b) = \frac{\Pr(I) \times \Pr(G)}{\Pr(b)} = \frac{3/4 \times 1/4}{15/16} = 3/15 = 1/5$. The President then passes on the eduational program.

iii. After a bad message: $\Pr(K|b) = \frac{\Pr(K) \times \Pr(B)}{\Pr(b)} = \frac{1/4 \times 3/4}{15/16} = 3/15 = 1/5$
After a good message and a good outcome: $\Pr(K|g, 1) = 1$
After a good message and a bad outcome (off-path): $\Pr(K|g, -1) \in [0, 1]$

iv. There cannot be such an equilibrium since the ignorant Economist has utility 1/5 when he sends signal b, while sending signal g yields non-negative payoff, equal to 1 with probability 1/4, so expected payoff of at least 1/4.

(c) The Economist sends message g with probability $\Pr(K) \times \Pr(G) = 1/8$ and message b with the residual probability 7/8.

After message g, the President infers $\Pr(G|g) = 1$ and enacts the program.
After message b, the President infers $\Pr(G|b) = \frac{\Pr(I) \times \Pr(G)}{\Pr(b)} = \frac{1/2 \times 1/4}{7/8} = 1/7$ and hence passes on the eduational program.

After a bad message: $\Pr(K|b) = \frac{\Pr(K) \times \Pr(B)}{\Pr(b)} = \frac{1/2 \times 3/4}{7/8} = 3/7$

After a good message: $\Pr(K|g, 1) = 1$

After a good message and a bad outcome (off-path): $\Pr(K|g, -1) \in [0, 1]$

If the President infers $\Pr(K|g, -1) = 0$, then the ignorant Economist would send signal b for a payoff of $\mu = 3/7$, rather than gamble by sending signal b and receiving payoff $\mu = 1$ with probability $\Pr(G) = 1/4$ but $\mu = 0$ with probability $\Pr(B) = 3/4$.

In part (c), the Economist has a greater reputation to protect than in part (b), and hence doesn't dare to guess.

1. **Salop Circle (20 points):** Imagine that consumers are located uniformly around a circle of unit circumference. There are J firms at locations that are equally spaced around the circumference, each of which sells beanie babies. All firms have the same constant production technology, and produce beanie babies at constant unit cost c. Each consumer wants at most 1 beanie baby and derives a gross benefit of v from its consumption. The total cost of buying a beanie baby from firm i for a consumer located a distance d from firm i is $p_i + td^2$, where t is a transportation cost parameter. Assume that firms compete by simultaneously naming prices p_i.

(a) Determine the symmetric Nash equilibrium prices p^*. (You may assume that v is sufficiently large to assure that all customers purchase a beanie baby in equilibrium.) (12 points)

Answer: Suppose there exists a symmetric Nash equilibrium in which every firm names price p^*. We first derive the necessary conditions for the equilibrium. Consider firm i's incentive to deviate. Denote by $\hat{d}$ the distance from firm i to the marginal consumer between firm i and firm $i+1$. Given that firm $i+1$ names p^*, then $\hat{d}$ is given by

$$v - p - t\hat{d}^2 = v - p^* - t(\frac{1}{J} - \hat{d})^2$$

where p is the price chosen by firm i. Due to the symmetry, firm i's total market share is thus given by

$$Q(p, p^*) = 2\hat{d} = \frac{J(p^* - p)}{t} + \frac{1}{J}$$

Thus, firm i's profit is given by

$$\Pi_i(p, p^*) = (p - c)\left[\frac{J(p^* - p)}{t} + \frac{1}{J}\right]$$

By FOC, the optimal price is equal to

$$p = \frac{J^2(p^* + c) + t}{2J^2}$$

In equilibrium, we have

$$p^* = \frac{t}{J^2} + c$$

We now show that p^* is indeed a Nash equilibrium. Note that for $\hat{d} \leq 1/J$, or $p \geq c$ equivalently, the profit function is differentiable and concave. Thus, FOC is sufficient for $p \geq c$. If $p < c$, then profit is negative. Hence, p^* is the unique best response given all the other firms naming p^*. This give us the symmetric Nash equilibrium.

(b) How does the equilibrium price p^* depend on J? What happens as $J \to \infty$? Discuss. (4 points)

Answer: According to (1), p^* is decreasing in J. In particular, $p^* \to c$, as $J \to \infty$. Intuitively, if there are infinitely many firms, firms become perfect substitutes and competition drives prices to marginal costs.

(c) How does the equilibrium price p^* depend on t? What happens as $t \to 0$? Discuss. (4 points)

Answer: According to (1), p^* is increasing in t. In particular, $p^* \to c$, as $t \to 0$. Intuitively, if transportation costs fall to zero, firms become perfect substitutes and competition drives prices to marginal costs.

2. **Cournot with Entry (30 points):** Suppose that there are I consumers, each with demand given by $q_i(p) = a - bp$ (where $a, b > 0$). Firms incur a cost $K > 0$ when they enter the industry, and a variable cost $c > 0$ for each unit produced. There are infinitely many (identical) potential firms.

(a) Suppose that all firms first make entry decisions simultaneously, and then, having observed each others' entry decisions, choose quantities (assuming that the entry cost, K, is sunk). How many firms N enter in SPNE? What happens to the number of entering firms N, equilibrium prices, output per firm, and consumer surplus per capita as I goes to infinity? (18 points)

Answer: Consider first the subgame in which $N < \infty$ firms have entered the market. As always in Cournot competition, we solved for a symmetric equilibrium with quantity q^*. Given others' strategies, firm i chooses a quantity to maximizes its profit

$$\Pi_i(q, q^*) = \left[\frac{a}{b} - \frac{(N-1)q^* + q}{bI} - c\right] q - K$$

where $[(N-1)q^* + q]/I$ is the quantity that each consumer is supplied, which in turn determines the market price. By FOC, we have

$$q^* = \frac{(a - bc)I}{N+1}$$

Plugging q^* into Π_i, and given the non-negative profit condition, we have ex-ante profits

$$\Pi_i^*(N) = \frac{(a-bc)^2 I}{b(N+1)^2} - K$$

To have N firms enter in equilibrium, entrants' profits $\Pi_i^*(N)$ must be non-negative. On the other hand, it must be impossible for outside to enter profitably. All told, $\Pi_i^*(N+1) \le 0 \le \Pi_i^*(N)$ which implies

$$\sqrt{\frac{(a-bc)^2 I}{bK}} - 2 \le N \le \sqrt{\frac{(a-bc)^2 I}{bK}} - 1$$

Given the equilibrium quantity q^*, the equilibrium price p^* equals

$$p^* = \frac{a+Nbc}{b(N+1)},$$

and the equilibrium consumer surplus is equal to

$$CS^* \equiv I\int_0^{Nq^*/I}\left(\frac{a-q}{b} - p^*\right)dq = \frac{(a-bc)^2 N^2 I}{2b(N+1)^2}.$$

From 2, as $I \to \infty$, we have $N \to \infty$ and $I/N \to \infty$. Thus, $p^* \to c$ as $I \to \infty$, that is, the market price converges to the marginal cost as the market size increases and many firms enter. Then the consumer surplus per capita CS^*/I converges to the first-best level $(a-bc)^2/2b$. Moreover, the quantity $q^* \to \infty$ as $I \to \infty$. This is because firms' margin $p^* - c$ converges to zero while they still incur a fixed cost $K > 0$, therefore firms have to produce a large number of goods to cover the fixed cost.

(b) Now assume that firms make entry and production choices (quantities) simultaneously in a single stage. How many firms enter in NE? Discuss the relationship to part (a). (12 points)

Answer:

At the equilibrium, no firms in the market has an incentive to exit market or change the output level, and no firms out of the market has incentive to enter the merket and produce at any level of quantity, given the others' strategies. Firms in the market does not have incentive to deviate if and only if $\Pi_i^*(N) \ge 0$. Suppose now an outside firm wants to make a deviation by entering the market. Given the choices of those N incumbent firms, the deviating firm's profit is given by

$$\Pi_{N+1}(q, q^*) = \left[\frac{a}{b} - \frac{Nq^*+q}{bI} - c\right]q - K$$

where $[Nq^*+q]/I$ is the new quantity that each consumer is supplied, which in turn determines the new market price. Recalling $q^* = \frac{(a-bc)I}{N+1}$, the FOC wrt q yields

$$q = \frac{bI}{2}\left[\frac{a}{b} - \frac{Nq^*}{bI} - c\right] = \frac{(a-bc)I}{2(N+1)} = \frac{q^*}{2}$$

Plugging q and q^* into Π_{N+1}, we have

$$\Pi_{N+1}^*(q, q^*) = \left[\frac{a}{b} - \frac{Nq^* + \frac{q^*}{2}}{bI} - c\right]\frac{q^*}{2} - K$$

$$= \frac{1}{b}\left[a - \left(N+\frac{1}{2}\right)\frac{(a-bc)}{N+1} - bc\right]\frac{(a-bc)I}{2(N+1)} - K$$

$$= \frac{(a-bc)^2 I}{4b(N+1)^2} - K \le 0.$$

By the above inequality and $\Pi_i^*(N) \ge 0$, we have

$$\frac{(a-bc)^2 I}{4bK} \le (N+1)^2 \le \frac{(a-bc)^2 I}{bK}$$

$$\Leftrightarrow \frac{1}{2}\sqrt{\frac{(a-bc)^2 I}{bK}} - 1 \le N \le \sqrt{\frac{(a-bc)^2 I}{bK}} - 1$$

Equilibrium entry in the sequential game in part (a) corresponds to the highest equilibrium entry in the simultaneous game in part (b). Both are determined by the constraint that the entering firms are barely profitable. But the simultaneous game in part (b) admits other equilibria with less entry. In part (a), when an additional firm enters, incumbents observe entry and reduce their quantity. In contrast, in part (b), incumbent firms do not observe entry and are committed to their own quantity. This makes entry less attractive and since entrant profits are (four times) smaller than incumbent profits, equilibrium does not pin down the number of entrants.

3. **Messing up Collusive Equilibria (20 points):** Two firms are engaging in Bertrand competition for a perfectly substitutable good repeatedly in periods $t = 1, 2, 3....$ Demand is $Q(p) = (20-p)^+$ (where p is the lower price, demand is split evenly in case of a tie, and p^m is the monopoly price $\arg\max pQ(p)$), costs are zero, and future profits are discounted at rate $\delta < 1$.

(a) For which values of δ does there exist a strongly symmetric SPNE with constant price $p^c > 0$ on path? Briefly explain why the equilibrium strategy you suggest gives the lower bound of δ. (6 points)

Answer: The minmax payoff is the static Nash payoff, namely 0. Collusion on price $p^c \in (0, p^m]$ can be supported if net present on path payoffs $\frac{1}{2}\frac{1}{1-\delta}p^cQ(p^c)$

3. **Messing up Collusive Equilibria (20 points):** Two firms are engaging in Bertrand competition for a perfectly substitutable good repeatedly in periods $t = 1, 2, 3....$ Demand is $Q(p) = (20-p)^+$ (where p is the lower price, demand is split evenly in case of a tie, and p^m is the monopoly price $\arg\max pQ(p)$), costs are zero, and future profits are discounted at rate $\delta < 1$.

(a) For which values of δ does there exist a strongly symmetric SPNE with constant price $p^c > 0$ on path? Briefly explain why the equilibrium strategy you suggest gives the lower bound of δ. (6 points)

Answer: The minmax payoff is the static Nash payoff, namely 0. Collusion on price $p^c \in (0, p^m]$ can be supported if net present on path payoffs $\frac{1}{2}\frac{1}{1-\delta}p^cQ(p^c)$
exceed deviation payoffs $(p^c - \varepsilon)Q(p^c - \varepsilon)$ for any ε, i.e. if $\frac{1}{2(1-\delta)} \ge 1$, i.e. if $\delta \ge 1/2$. Collusion on higher prices $p^c > p^m$ is more difficult (and so δ must be higher) since a deviation to the monopoly price $p^m = 10$ is more attractive than barely undercutting p^c.

(b) Now assume that regulations impose a minimum price $\underline{p}$. How does your answer to part (a) change? How does the difficulty of collusion (as measured by the lower bound on the discount rate $\underline{\delta}$) vary in the minimum price $\underline{p}$, and the collusive price p^c (assuming $p^c \le p^m$)? Interpret your result. (14 points)

Answer: The minmax payoff is the static Nash payoff, but it now is higher, equal to $\frac{1}{2}\underline{p}Q(\underline{p})$. Thus colluding on price p^c is feasible in SSSPNE iff

$$\frac{p^c}{2}Q(p^c) \ \ge\ (1-\delta)pQ(p^c) + \delta\frac{1}{2}\underline{p}Q(\underline{p})$$

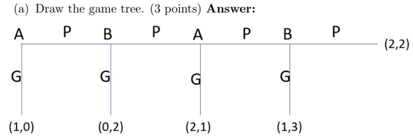
$$\delta \ \ge\ \frac{p^cQ(p^c)}{2p^cQ(p^c) - \underline{p}Q(\underline{p})} = \frac{1}{2 - \frac{\underline{p}Q(\underline{p})}{p^cQ(p^c)}} =: \underline{\delta}$$

Collusion is more difficult for higher minimum prices $\underline{p}$, i.e. the lower bound $\underline{\delta}$ rises in $\underline{p}$, because a minimum price dampens the threat of a price war.

Collusion is less difficult for higher collusive prices p^c, i.e. the lower bound $\underline{\delta}$ falls in p^c, because the gains from collusion $p^c - \underline{p}$ (i.e. the reward from sticking to the collusion) are more sensitive to p^c, than the deviation gain p^c: $\frac{d\log(p^c-\underline{p})}{dp^c} = \frac{1}{p^c-\underline{p}} > \frac{1}{p^c} = \frac{d\log p^c}{dp^c}$.

4. **A Crazy Quadruped (30 points):** Consider a trust-game between two players A, B. In period $t = 1$ a dollar falls from the sky and player A can either pass the dollar to B, or grab it himself. Grabbing ends the game, while passing continues the game to period $t = 2$, where a new dollar falls from the sky, and B can either pass the dollar to A, or grab it himself, and so on. The game also ends exogenously after period $T = 4$. Players maximize their own expected monetary outcome (with no discounting). Denote A's actions by $a_1, a_3 \in \{P, G\}$ and B's actions by $b_2, b_4 \in \{P, G\}$.

(a) Draw the game tree. (3 points) **Answer:**

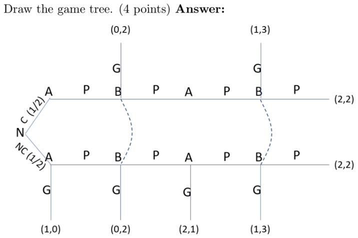


b) What is the SPE of this game? (3 points)

Answer: By backward induction, both players grab the dollar at every occasion.

Now assume that with (prior) probability $p = 20\%$ player A is crazy, in that he always passes the dollar to B when he gets the chance (and write p' for the posterior probability of A being crazy conditional on reaching period 2).[1]

(c) Draw the game tree. (4 points) **Answer:**



(d) Show that SPE from part (b) (i.e. the strategy profile in which B and the non-crazy type of A play as in (b), and the crazy type of A always passes) is not a PBE of this game. (4 points)

Answer: In the SPE (b), the non-crazy A grabs the dollar in period 1, so if period 2 is reached, B infers that A is crazy, $p' = 1$, and will pass in period 3. Thus, passing in period 2 and grabbing in period 4 is one dollar better for B than grabbing in period 2.

(e) Show that this game does not have a PBE in pure strategies. (4 points)

Answer: In period 4, B clearly grabs the dollar. Anticipating this, the non-crazy type of A grabs the dollar in period 3. Thus, B should grab in period 2 iff $p' \ge 50\%$.

- If A grabs in period 1, we have $p' = 1$, so B will pass in period 2 (as in part d), but then A should pass in period 1 (and grab in period 3).
- If A passes in period 1, we have $p' = p = 20\%$, so B will grab in period 2, but then A should grab in period 1.

(f) Solve for the (mixed strategy) PBE of this game. (12 points)

Answer: To make B indifferent in period 2, his posterior belief must equal $p' = 50\%$. Writing q for the probability with which (the non-crazy) A passes in period 1, we have $p' = \frac{p}{p+(1-p)q}$, or $q = \frac{p(1-p')}{p'(1-p)} = \frac{1}{4}$.

To make A indifferent in period 1, his expected payoff from passing must equal his payoff from grabbing 1. Since A grabs in period 3, his payoff equals 2 if B passes in period 2, and 0 if B grabs in period 2. Hence, B must grab in period 2 with probabilitiy 1/2.

2. **Fighting an Unknown Opponent (15 points):** Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack"or "notattack." Each army is either "strong"or "weak" with equal probability (the draws for each army are independent), and an army's type is known only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its opponent does not, or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is s if it is strong and w if it is weak, where $s < w$. There is no cost of attacking if its rival does not.

(a) Assume $2s < M < w$. Identify all pure strategy Bayesian Nash equilibria. (6 points)

(b) Assume $w < M < 2s$. Identify all pure strategy Bayesian Nash equilibria. (9 points)

(a) Since $2s < M$, attacking is a weakly dominant strategy for strong types. Thus strong types should always attack. Given that strong types always attack, weak types should not attack since $M < w$. Thus the unique BNE is symmetric such that strong types attack and weak types do not attack.

(b) Consider the strong type of army 1. If it does not attack, then it must be that both types of army 2 attack since $M > w > s$. Given that both types of army 2 attack, the weak type of army 1 should not attack. Then both types of army 2 would like to attack indeed.

If the strong type of army 1 attacks, then it cannot be that both types of army 2 attack since $M < 2s$. Then the weak type of army 1 should also attack since it can at least get $(M-w)/2 > 0$. Given that both types of army 1 attack, neither type of army 2 attacks. Then both types of army 1 would like to attack indeed.

In summary, there are two asymmetric equilibria: in the first, both types of army 1 attack and neither type of army 2 attacks; in the second, both types of army 2 attack and neither type of army 1 attacks.

3. **Herding (20 points):** N players sequentially decide whether to go to restaurant A or B, exactly one of which is good (utility 1) and one of which is bad (utility 0). The ex-ante probability that A is good is $q < 1/2$. Before choosing, a player observes a signal about which restaurant is good; this signal is correct with probability $p > 1 - q$ and the signals are conditionally independent. He also observes his predecessors' actions, but not their signals. Assume that players play a (weak) PBE. [Hint: In analyzing this question, it is useful to keep track of the likelihood ratio of the state $\theta = A, B$, i.e. $\ell = \frac{\Pr(A)}{\Pr(B)}$, and observe that if a player holds a prior likelihood ratio ℓ and receives a signal s about state θ, the posterior likelihood ratio is given by $\ell' = \frac{\Pr(s|A)}{\Pr(s|B)}\ell$.]

(a) Show that the first player follows his signal. (4 points)

(b) Assume the first player chose A. Does the second player follow his signal? (5 points)

(c) Now assume the first player chose B. Does the second player follow his signal? (5 points)

(d) What is the probability that restaurant A is good, and yet all players end up going to restaurant B? (6 points)

(a) Note that player 1's prior likelihood ratio is given by

$$l_1 = \frac{Pr(A)}{Pr(B)} = \frac{q}{1-q} < 1$$

After receiving a signal s, his posterior likelihood ratio is updated by

$$l_1' = \frac{Pr(A|s)}{Pr(B|s)} = \frac{Pr(A,s)}{Pr(s)}\frac{Pr(s)}{Pr(B,s)} = \frac{Pr(s|A)Pr(A)}{Pr(s|B)Pr(B)} = \frac{Pr(s|A)}{Pr(s|B)}l_1$$

If player 1 receives $s = A$, his posterior likelihood ratio is updated by

$$l_1' = \frac{Pr(s|A)}{Pr(s|B)}l_1 = \frac{p}{1-p}\frac{q}{1-q} > \frac{1-q}{q}\frac{q}{1-q} = 1$$

Thus player 1 should follow his signal.

If player 1 receives $s = B$, his posterior likelihood ratio is updated by

$$l_1' = \frac{Pr(s|A)}{Pr(s|B)}l_1 = \frac{1-p}{p}\frac{q}{1-q} < \left(\frac{q}{1-q}\right)^2 < 1$$

Thus player 1 should follow his signal as well. In summary, player 1 follows his signal.

(b) From part (a), we know that if player 1 chooses A then his signal is A. Then player 2's prior (after observing player 1's choice) likelihood ratio l_2 equals player 1's posterior likelihood ratio l_1'.

If player 2 receives $s = A$, his posterior likelihood ratio is updated by

$$l_2' = \frac{Pr(s|A)}{Pr(s|B)}l_2 = \frac{p}{1-p}\left(\frac{p}{1-p}\frac{q}{1-q}\right) > \left(\frac{1-q}{q}\right)^2\frac{q}{1-q} > 1$$

Thus player 2 should follow his signal.

If player 2 receives $s = B$, his posterior likelihood ratio is updated by

$$l_2' = \frac{Pr(s|A)}{Pr(s|B)} l_2 = \frac{1-p}{p}\left(\frac{p}{1-p}\frac{q}{1-q}\right) = \frac{q}{1-q} < 1$$

Thus player 2 should follow his signal.

In summary, player 2 should follow his signal if player 1 chooses A.

(c) If player 1 chooses B, then his signal is B. Thus $l_2 = l_1'$.

If player 2 receives $s = A$, his posterior likelihood ratio is updated by

$$l_2' = \frac{Pr(s|A)}{Pr(s|B)} l_2 = \frac{p}{1-p}\left(\frac{1-p}{p}\frac{q}{1-q}\right) = \frac{q}{1-q} < 1$$

Thus player 2 should choose B and not follow his signal.

If player 2 receives $s = B$, his posterior likelihood ratio is updated by

$$l_2' = \frac{Pr(s|A)}{Pr(s|B)} l_2 = \frac{1-p}{p}\left(\frac{1-p}{p}\frac{q}{1-q}\right) < \left(\frac{q}{1-q}\right)^3 < 1$$

Thus player 2 should choose B as well.

In summary, player 2 should not follow his signal but follow player 1's action if player 1 chooses B.

(d) This event is possible only if player 1 receives $s = B$. After receiving $s = B$, player 1 chooses B. Then observing player 1 choosing B, player 2 ignores own signal and follows player 1's action. This means that player 3 cannot learn anything about player 2's signal from his action. Thus player 3 has the same prior and posterior likelihood ratio as player 2. As argued in part (c), player 3 should ignore own signal and follow player 1's action. Then by induction, everyone else should ignore own signal and follow player 1's action. In words, once player 1 receives $s = B$, all players choose B in the WPBE. Thus the probability that restaurant A is good, and yet all players end up going to restaurant B is equal to $q(1-p) > 0$, which means that there is a positive probability that everybody makes the wrong choice.

4. **Cheap Talk (20 points):** A pure communication game is a signaling game in which the sender's actions are payoff irrelevant (in contrast to Spence's education game where they are costly). Specifically consider a sender S with type θ uniformly distributed on $[0, 1]$, who sends a message m from some arbitrary message space M to a receiver R, who then takes an action $a \in [0, 1]$. The receiver wants to match his action to the state, $u_R(a, \theta) = -(a-\theta)^2$, while the sender is biased towards higher actions, $u_S(a, \theta) = -(a-(\theta+b))^2$, where $b \geq 0$.

(a) Assume that the sender "babbles", i.e. all types θ send the same message m. Specify a best response by the receiver (both on path and off path) that gives rise to a (weak) PBE. (3 points)

(b) Assume $b > 0$. Show that truthtelling, i.e. $m(\theta) = \theta$ (this assumes that the message set M includes $[0, 1]$) is not part of an equilibrium. (5 points)

(c) Assume that b is small. Construct an equilibrium where some types $\theta \leq \theta^*$ send one message m and others $\theta \geq \theta^*$ send message $m' \neq m$. How small must b be for such an equilibrium to exist? (12 points)

(a) If all types send the same message in equilibrium, then the receiver's on-path belief coincides with his prior. His best response on-path is given by

$$\max_a \; -\int_0^1 (a-\theta)^2 d\theta$$

FOC yields $a^* = 1/2$. Thus $u_S(1/2, \theta) = -(1/2 - (\theta + b))^2$.

For any off-path message, we assume that the receiver ignores the message and assigns uniform beliefs over $[0, 1]$. Then the receiver's off path best response is still $1/2$. Thus the sender has no incentive to deviate, leading to a WPBE.

(b) Suppose in equilibrium $m(\theta) = \theta$, then the receiver's best response is $a^* = \theta$. But then type θ's best response is to report $\theta + b$ if $\theta + b \leq 1$; and to report 1 if $\theta + b > 1$. This contradicts the equilibrium condition.

(c) Suppose such a WPBE exists. Then upon receiving m, the receiver assigns uniform beliefs over $[0, \theta^*]$. Thus the receiver's best response is $a^*(m) = \theta^*/2$. Upon receiving m', the receiver assigns uniform beliefs over $[\theta^*, 1]$. Thus the receiver's best response is $a^*(m') = (\theta^* + 1)/2$. To make the cutoff type θ^* indifferent, we must have

$u_S(\theta^*/2, \theta^*) = u_S((\theta^*+1)/2, \theta^*)$ Substituting and rearranging yields

$[(\theta^*+1)/2 - (\theta^*+b)]^2 - [\theta^*/2 - (\theta^*+b)]^2 = 0$

Solving for θ^* yields $\theta^* = 1/2 - 2b$. Then $u_S((\theta^*+1)/2, \theta) - u_S(\theta^*/2, \theta)$ is increasing in θ on $[\theta^*, 1]$ and decreasing in θ on $[0, \theta^*]$. This implies that all the other types' IC constraints hold as well. For any off-path message, we assume that the receiver ignores the message and assigns uniform beliefs over $[0, \theta^*]$. Then the receiver's off path best response is still $\theta^*/2$, and thus the sender will not deviate to off-path. Therefore, such a WPBE exists if and only if θ^* exits, that is, if and only if $b < 1/4$.

5. **Hold-up with Unobservable Investment (30 points):** A buyer and seller would like to trade. Before they do, the buyer can make an investment that increases the value he or she puts on the object to be traded. This investment cannot be observed by the seller, and does not affect the value the seller puts on the object, which we normalize to zero. (As an example, think of one firm buying another. Some time before the merger, the acquirer could take steps to change the products it plans to introduce, so that they mesh with the acquired firm's products after the merger. If product development takes time and product life cycles are short, there is not enough time for this investment by the acquirer to occur after the merger.) The buyer's initial value for the object is $v > 0$; an investment of I increases the buyer's value to $v + I$ but costs $I^2/2$. The timing of the game is as follows: First, the buyer chooses an investment level I and incurs the cost $I^2/2$. Second, the seller does not observe I but offers to sell the object for the price p. Third, the buyer accepts or rejects the seller's offer: if the buyer accepts, then the buyer's payoff is $v + I - p - I^2/2$, and the seller's payoff is p; if the buyer rejects, then these payoffs are $-I^2/2$ and zero, respectively.

(a) Sketch the game tree for this game and state the (pure) strategy sets for the buyer and the seller. (4 points)

(b) Show that there is no pure strategy subgame-perfect Nash equilibrium of this game. (4 points)

(c) Solve for a mixed strategy equilibrium where the buyer mixes over all investment levels $I \in [0, 1]$ and the seller mixes over all prices $p \in [v, v+1]$. Does the equilibrium distribution of I have an atom? How about the equilibrium distribution of p? (10 points)

(d) Now assume that the buyer's investment is observable and solve for the subgame perfect equilibrium. (5 points)

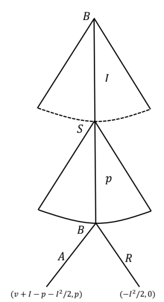
(e) Solve for the first best investment that maximizes surplus, i.e. the sum of utilties. (2 points)

(f) Compare the equilibrium payoffs in part (c) and part (d). Discuss the different inefficiencies (compared to the first best in part (e)) that arise in either case. (5 points)

(a) The game tree is depicted in Figure 1.

The buyer's strategy consists of an investment level $I \geq 0$ and conditional on I and the seller's offer p, either accept (A) or reject (R) the seller's offer.

The seller's strategy is a price $p \geq 0$.



(b) Suppose there exists a pure strategy SPNE such that the buyer chooses $I^* > 0$, then the seller's best response is to offer $v + I^*$, since given $p = v + I^*$ the buyer weakly prefers accepting to rejecting and gets a utility $-I^{*2}/2$. But then the buyer would rather choose $I = 0$ in the first stage so that he can secure a utility 0 by rejecting the seller's offer. This contradicts the equilibrium condition.

Suppose there exists a pure strategy SPNE such that the buyer chooses $I^* = 0$, then the seller's best response is to offer v, since given $p = v$ the buyer weakly prefers accepting to rejecting and gets a utility 0. But then the buyer would rather choose $I = 1$ in the first stage so that he can get a utility 1/2 by accepting the seller's offer. This contradicts the equilibrium condition.

In summary, there is no pure strategy SPNE of this game.

(c) Let $F(I)$ denote the C.D.F. of buyer's investment level with support $[0, 1]$. Let $G(p)$ denote the C.D.F. of seller's price with support $[v, v+1]$. Given I, the buyer accepts an offer if and only if $p \leq v + I$. Since $I \in [0, 1]$, $p = v$ will be accepted for sure, and thus, the seller's expected payoff equals v. Suppose $F(I)$ has an atom at $I^* \in [0, 1)$, then the right limit of $p^* \equiv v + I^*$, p^{*+}, yields a strictly lower expected payoff than p^*:

$E[u_S(p^{*+})] = [1 - F(I^*)]p^{*+} < [1 - F(I^{*-})]p^* = E[u_S(p^*)]$

where I^{*-} is the left limit of I^*, a contradiction. Hence $F(I)$ has no atom on $[0, 1)$.

Then by the seller's indifference condition, we have for any $p \in [v, v+1)$, $\quad v = [1 - F(p-v)]p$

Thus $F(I) = F(p - v) = I/(v + I)$ on $[0, 1)$. Then $Pr(1) = 1 - F(1^-) = v/(v + 1)$.

Thus $F(I)$ has an atom at $I = 1$, with a mass $1/(v + 1)$.

Since $p \in [v, v+1]$, if the buyer chooses $I = 0$, then he accepts the seller's offer if and only if $p = v$, and thus, his expected payoff equals 0. Suppose $G(p)$ has an atom at v, then by choosing investment $I = \varepsilon > 0$ with ε small enough, the buyer can get an expected payoff equal to $\quad G(v)(\varepsilon - \varepsilon^2/2) > 0$

This is because $G(v) > 0$ and $(\varepsilon - \varepsilon^2/2) > 0$ for small ε. Thus the buyer strictly prefers ε to 0, a contradiction. Suppose $G(p)$ has an atom at $p^* \in (v, v+1)$. Note that the buyer must be indifferent between $I^* \equiv p^* - v$ and its left and right limits I^{*-} and I^{*+}. The marginal utility of increasing I equals $G(p^*) - I^*$, since it increases the buyer's gain from trade by $G(p^*)dI$ but incurs an extra investment cost I^*dI. The marginal utility of decreasing I equals $G(p^{*-}) - I^*$, since it saves the buyer an investment cost I^*dI but reduces the the buyer's gain from trade by $G(p^{*-})dI$. By the indifference condition, both directions' marginal utility must equal 0. But since $G(p^*) > G(p^{*-})$, this cannot be true, a contradiction. Thus $G(p)$ has no atom on $[v, v+1)$.

As argued above, the marginal utility of increasing I must be 0 on $[v, v+1)$. It follows that $G(p) = I = p - v$ for any $p \in [v, v+1)$. This implies that $G(v+1) = 1$, hence $G(p) = p - v$ and has no atom on $[v, v+1]$.

(d) We use backward induction. Since the buyer's investment I is observed, the seller's best response is $p = v + I$. Then the buyer is willing to accept p and gets a payoff $-I^2/2$, so he should choose $I = 0$ in the first stage. Thus there exists a unique SPNE such that $I^* = 0$, $p^* = v$ and the buyer accepts any price $p \leq v$.

(e) It is socially optimal to reach an agreement. Then the total surplus equals $v + I - I^2/2$. It follows that the efficient investment level equals 1. The first best outcome is that the buyer chooses investment $I = 1$ and the seller charges a price $p \in [v, v+1/2]$.

(f) The equilibrium payoffs are $(0, v)$ in both part (c) and (d). Since the first-best total surplus equals $v + 1/2$, part (c) and (d) have the same extent of inefficiency. However the source of inefficiency is different. In contrast to the observable case (part (d)), in part (c), unobservability reduces underinvestment caused by the hold-up problem, but it increases the possibility of disagreement. The amount of inefficiency is not changed.

1. **Too good to fire:** A firm employs a worker for potentially infinitely many periods $t = 0, 1, 2, 3....$ In every period t, first the worker chooses effort $e_t \in \{0, 1\}$, and then the firm chooses whether to fire the worker, and thereby ending the game, or to retain him, in which case play proceeds to period $t + 1$. We assume complete and perfect information. Per-period payoffs are $w - e_t$ for the worker and $r(e_t) - w$ for the firm, where $w > 1$ is an exogenous wage level, and $r(e)$ an exogenous revenue function that satisfies $r(0) \leq w < r(1)$. Overall payoffs are additive across periods, discounted at a common rate $\delta < 1$: So if the firm fires the worker in period $T \in \mathbb{N} \cup \{\infty\}$, payoffs are $\sum_{t=0}^{T} \delta^t (w - e_t)$ for the worker, and $\sum_{t=0}^{T} \delta^t (r(e_t) - w)$ for the firm; i.e. payoffs are still collected in the firing period T. The solution concept is SPE, which we'll simply call "equilibrium".

(a) Find an equilibrium where the worker never exerts effort, $e_t = 0$.

(b) For what discount factors δ is there another equilibrium, in which the worker always exerts effort, $e_t = 1$, on the equilibrium path? (Make sure to specify the worker's off-path actions in this equilibrium and argue why the firm's strategy in your equilibrium is optimal for the firm)

Assume from now on that the employee is so awesome that it is worthwhile to employ him even when he shirks, i.e. $r(0) > w$.

(c) Argue that the strategy profiles you constructed in parts (a) and (b) are no longer equilibria.

(d) Show that there is a unique equilibrium and describe this equilibrium.

(e) Now assume that before playing the game, the firm can raise the wage once-and-for-all, i.e. choose a wage level $w' \geq w$, and then the above game is played with wage w' instead of w. Also assume $r(1) - r(0) > r(0) - w$, and that $\delta \geq \delta^*$, where δ^* is the lower bound on the discount factor from part (b). Show that there is an equilibrium where the firm strictly prefers to raise the wage.

(a) (1 point) Given this strategy of the worker, firm payoffs $r(e_t) - w = r(0) - w$ are always non-positive, and it (weakly) prefers to fire the worker at every history. And if the worker anticipates to be fired after any effort choice at any history, shirking $e_t = 0$ is indeed optimal.

(b) (3 points)

- Consider the standard trigger-strategy of the firm, whereby it fires the worker in period t if he has shirked at any point in the past, $e_s = 0$ for any $s \leq t$, and retains the worker otherwise. Also assume that off the equilibrium path, the worker always shirks, $e_t = 0$.
- Off the equilibrium path, this is the strategy profile from part (a), so clearly there are no profitable one-step deviations.

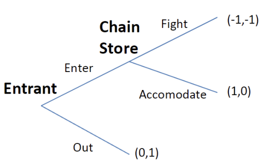
- On the equilibrium path, a one step-deviation by the worker (namely $e_t = 0$ instead of 1) has a gain of 1 today at the cost of losing the job, for a net benefit of
$$1 - \frac{\delta}{1-\delta}(w-1)$$
which is non-positive as long as $\delta \geq \delta^* := 1/w$.
- A one-step deviation by the firm, fire the worker instead of retaining him, throws away the continuation value
$$\frac{\delta}{1-\delta}(r(1) - w)$$
which we assumed to be positive.

(c) (1 point) Since $r(0) > w$, the firm's period payoffs are always positive, so it is never optimal to fire the worker, as the firm was doing in both equilibria in parts (a) and (b).

(d) (2 points) We argued in part (c) that the firm always retains the worker. Hence the worker has no incentives to exert effort, and therefore always chooses $e_t = 0$.

(e) (3 points) If the firm does not raise the wage, so $w' = w$, or raises the wage to some level $w' < r(0)$ the unique continuation equilibrium is the one in part (d), which yields $\frac{1}{1-\delta}(r(0) - w)$ to the firm. But if it raises the wage to $w' = r(0)$, the continuation equilibrium of part (b) becomes feasible which yields the firm $\frac{1}{1-\delta}(r(1) - w') = \frac{1}{1-\delta}(r(1) - r(0)) > \frac{1}{1-\delta}(r(0) - w)$. Raising w' strictly above $r(0)$ also enables the continuation equilibrium from part (b), but is less profitable.

Game Tree: Chain Store



2. **Chain store game with short memory:** (Bhaskar, Mailath, Morris, REStud 2013) A chain store is facing an infinite sequence of potential, short-lived entrants $t = 1, 2, 3....$ The period-t stage game is given by the game tree above.[1] Entrant-t maximizes his period-t payoff; the chain store maximizes long-term payoffs, discounted at rate $\delta < 1$.

(a) What is the SPE of the stage game?

Answer: (1 point) By backward induction, the chain store will accomodate entry, and hence the entrant enters.

Now consider the dynamic game.

(b) Assume first that the game has perfect information and consider the following "grim-trigger" strategy profile: If the chain store has always fought entry in the past (or no entrant has ever entered), then the entrant stays out and (if he enters) the chain store fights; otherwise the entrant enters and the chain store accomodates. For which levels of δ is this a SPE?

Answer: (2 points) The entrants' strategy is clearly a best response to the chain store. As for the chain store, the critical question is whether he is willing to fight entry today (cost $u_F = -1$), to secure a monopoly position for the future (benefit $u_O = 1$): $(1-\delta)u_F \leq \delta u_O$, i.e. $\delta \geq u_F/(u_O + u_F) = 1/2$.

Now assume that entrant-t observes only the outcome, i.e. O,F, or A, of period $t-1$ (and assume for convenience that the outcome in period $t = 0$ was O). Also assume that $\delta \neq 1/2$.

(c) Consider the following "short memory grim-trigger" strategy profile: If period $t-1$'s outcome was O or F, entrant-t stays out and (if he enters) the chain store fights; if period $t-1$'s outcome was A, entrant-t enters and the chain store accomodates.[2] Show that this strategy profile is not a SPE! Discuss!

Answer: (3 points) The entrants' strategy is clearly a best response to the chain store. As for the chain store:

- If last period's outcome was O or F, and the entrant (unexpectedly) enters the condition for the chain store to follow his strategy and fight is as in part (a): $(1-\delta)u_F \leq \delta u_O$, or $\delta \geq u_F/(u_O + u_F) = 1/2$. The chain store must be sufficiently patient to "defend his reputation by fighting entry".
- But if last period's outcome was A, and the entrant enters, the condition for the chain store to follow is strategy and accomodate is the reverse: $(1-\delta)u_F \geq \delta u_O$, or $\delta \leq u_F/(u_O + u_F) = 1/2$. The chain store must be sufficiently impatient to "accept the punishment phase rather than fighting to restore his reputation".
- These conditions cannot both hold since we ruled out the knife-edge case $\delta = u_F/(u_O + u_F) = 1/2$.

(d) Assume that the chain store is sufficiently patient, that is $\delta > 1/2$, and solve for a mixed strategy equilibrium of the following kind: If last period's outcome was O or F, the entrant stays out and (if he enters) the chain store fights; if last period's outcome was A, the entrant enters with probability q and the chain store fights with probability p.

Answer: (4 points) For the entrant (after any history) the expected cost of being fought (cost $v_F = -1$) must equal the expected benefit of being accomodated (benefit $v_A = 1$), i.e. $pv_F + (1-p)v_A = 0$, i.e. $p = v_A/(-v_F + v_A) = 1/2$. The chain store (after entry in period t) must be indifferent between (i) fighting in t and having entrant $t+1$ stay out, and (ii) accomodating in t, having entrant $t+1$ enter with probability q, and if so fight entry; both plans have entrants $t' \geq t+2$ stay out. Plan (i) yields utility $-u_F + \delta u_O$ (in periods t and $t+1$). Plan (ii) yields utilty $\delta(-qu_F + (1-q)u_O)$. Equating yields $u_F = \delta q(u_F + u_O)$, i.e. $q = u_F/(\delta(u_O + u_F)) = 1/(2\delta)$.

1. **Pig in a poke** (Fleckinger, Glachant, Moineville AEJ Micro, 2017)**:** A worker is producing and selling a widget of low or high quality $\theta = L, H$ where we normalize $L = 0$ and $H = 1$. Producing low quality is costless to the worker while high quality costs him $c > 0$, which is distributed with (smooth) cdf $F(c)$ and known only to the worker. A competitive market of buyers observes an imperfect signal $s = \ell, h$ of the quality, where $\Pr(\ell|L) = 1$ and $\Pr(h|H) = \pi < 1$; that is upon observing h, buyers know that $\theta = H$, but upon observing ℓ uncertainty remains. The timing is as follows. First the worker chooses θ; we denote his strategy by $x(c) = \theta$. Next buyers observe s. Finally, buyers bid the price up to the expected value $p_s = E[\theta|s, x(\cdot)]$, conditional on the signal s and the worker's equilibrium strategy $x(\cdot)$. Worker c's utility from producing quality θ and selling it for p equals $p - c\theta$.

(a) First assume that the signal is uninformative, i.e. $\pi = \Pr(h|H) = 0$. What is the equilibrium of this game?

Answer: (2 points) Since the signal is ℓ for either quality level, and hence the price $p = E[\theta|\ell, \widetilde{x}(\cdot)]$ does not depend on actual quality (but only on buyers' beliefs over quality) the worker chooses low quality for any $c > 0$. Thus, $p_\ell = p_h = 0$.

From now on assume $\pi > 0$.

(b) Assume that low-cost types $c \leq c^*$ produce high quality, and high-cost types $c > c^*$ produce low quality for some arbitrary threshold c^* (we simply denote this strategy $x(\cdot)$ by c^*). What is buyers' expectation of quality, $p_0 = E[\theta|c^*]$, i.e. before observing the signal s? Use Bayes' rule to calculate the resulting prices $p_s = E[\theta|s, c^*]$ after signals $s = \ell, h$! Show that p_ℓ increases in c^*! Discuss!

Answer: (3 points) Before observing s, buyers believe that quality is high with probability
$$p_0 = F(c^*).$$

After observing $s = H$, they know that quality is high,
$$p_h = E[\theta|h, c^*] = \frac{p_0 \Pr(h|H)}{p_0 \Pr(h|H) + (1-p_0)\underbrace{\Pr(h|L)}_{=0}} = 1.$$

After observing $s = L$, Bayes' rule implies
$$\begin{aligned} p_\ell &= E[\theta|\ell, c^*] = \frac{p_0 \Pr(\ell|H)}{p_0 \Pr(\ell|H) + (1-p_0)\Pr(\ell|L)} = \\ &= \frac{F(c^*)(1-\pi)}{F(c^*)(1-\pi) + (1-F(c^*))} = \frac{(1-\pi)F(c^*)}{1-\pi F(c^*)} = \frac{1-\pi}{\frac{1}{F(c^*)} - \pi}, \end{aligned}$$
which increases in c^*. Intuitively, the higher the prior belief of high quality, the higher the posterior belief after signal ℓ, since the market is now attributing this bad signal more to bad luck and less to low quality.

(c) Argue that in equilibrium, the threshold c^* must satisfy
$$c^* = \pi(E[\theta|h, c^*] - E[\theta|\ell, c^*]), \tag{1}$$
and argue that this equation defines a unique value for c^*!

Answer: (3 points) The benefit of producing high quality is that it increases the probability of a high signal from 0 to π, and a high signal raises the price by $E[\theta|h, c^*] - E[\theta|\ell, c^*]$. Thus the benefit of a high signal is given by the RHS of (1). In equilibrium, this must equal the cost of producing high quality for the threshold type c^*, the LHS.

Since p_ℓ increases in c^* while p_h is constant, equal to 1, the RHS decreases in c^*. The LHS clearly increases in c^*. Thus they cross at most once, and hence the equilibrium threshold c^* is unique.

(d) Now consider an increase in the signal quality $\pi = \Pr(h|H)$. Does the threshold c^* rise or fall? Discuss!

Answer: (2 points) The RHS of (1)
$$\pi(E[\theta|h, c^*] - E[\theta|\ell, c^*]) = \pi\left(1 - \frac{(1-\pi)F(c^*)}{1-\pi F(c^*)}\right) = \pi\frac{1-F(c^*)}{1-\pi F(c^*)}$$

rises in π; intuitively, a rise in π increases both the likelihood (π) that producing high quality results in the h signal, and the informativeness (and hence the value) of this signal $p_h - p_\ell$. To restore equality in (1), c^* has to rise, increasing the LHS and decreasing the RHS. Intuitively, when signal quality rises, more types (with higher costs) can be incentivized to produce high quality.