Econ 202A

10/2/2023

Basic Neoclassical Growth Model (Cass-Koopmans)

Economy consists of many identical infinitely lived households—all have the same preferences and endowments.

Several interpretations:

1. Representative Agent (Robinson Crusoe)
2. Social planner
3. Infinitely lived family (dynasty)

One production sector—output produced from capital and labor. Can be consumed or invested.

- Investment becomes productive capital after one period.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1$$

subject to

(resource constraint) $c_t + i_t \leq y_t = F(k_t, n_t)$

$$k_{t+1} \leq (1-\delta) k_t + i_t \quad , \quad 0 < \delta \leq 1$$

$$0 \leq n_t \leq 1$$

k_0 given, $k_0 > 0$

$\longrightarrow$ no prices here!!!

Production Function: $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is

1) Continuously differentiable

2) Homogeneous of degree 1 $\Leftrightarrow$ constant returns to scale

3) strictly quasi concave

4) $F(0,n) = 0$ - capital essential

F_k = marginal product of capital > 0

$F_n > 0$

5) $\lim_{k \to 0} F_k(k,1) = \infty \quad \lim_{k \to \infty} F_k(k,1) = 0$

$u : \mathbb{R}_+ \to \mathbb{R}$ utility function

1) Bounded - needed for dynamic programing
2) Continuously differentiable
3) strictly increasing
4) strictly concave
5) $\lim_{c \to 0} u'(c) = \infty$

Note: We will use functional forms for F & u for most everything we do in this class.

Simplifying the Planner's Problem:

1. $F_N > 0$ and $u'(c) > 0 \Rightarrow n_t = 1$ for all t

   Later will introduce leisure in the utility function.

2. $u'(c) > 0 \Rightarrow$ resource constraint holds with equality - $c_t + i_t = F(k_t, n_t)$

3. $\beta < 1 \Rightarrow$ require positive rate of return to give up a unit of consumption today for consumption tomorrow [actually $MP_K + 1 - \delta$]

   $\Rightarrow k_{t+1} = (1-\delta)k_t + i_t$

4. Let $f(k) \equiv F(k, 1) + (1-\delta)k$

Problem becomes:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$0 \leq k_{t+1} \leq f(k_t)$$

k_0 given

- Called a “sequence problem by Stokey and Lucas.

- Infinite number of choice variables.

$\Rightarrow$ Use recursive methods.

Dynamic Programming

Let $V(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$ given k_0.

$\rightarrow$ maximized discounted utility given k_0.

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right\}$$

$$= \max_{k_1} \left\{ u(f(k_0) - k_1) + \beta V(k_1) \right\}$$

where $V(k_1) = \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})$

$$V(k) = \max_{k'} \{ u(f(k) - k') + \beta V(k') \}$$

- called "Bellman's Equation"
- a functional equation where the unknown is $V(k)$
- $V(k)$ called the "value function"
- $u(f(k) - k')$ called the "return function"

FOC $\quad u'(f(k) - k') = \beta V'(k')$

$\Rightarrow$ solve for $k' = g(k)$ — The "policy function"

Solving for $V(k)$:

- Guess a function $V_0(k)$

- $$T(V_0(k)) = \max_{k'} \{u(f(k) - k') + \beta V_0(k')\}$$

- Let $V_1(k) = T(V_0(k))$

- Repeat forming a sequence of functions where $V_n(k) = T(V_{n-1}(k))$

$\Rightarrow \{V_n\}_{n=0}^{\infty}$ Continue until $V_{n-1}(k)$ & $V_n(k)$ are the same or close.