fined by $\hat{v}(x) = a^2/2b(1 - \delta)$, all $x \in \mathbf{R}$, satisfied (1)–(3). Moreover, it follows by induction that the functions $T^n\hat{v}$ take the form:

$$(T^n\hat{v})(x) = \alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n,$$

where the coefficients of these quadratic functions are given recursively by $\alpha_0 = \beta_0 = 0$, $\gamma_0 = a^2/2b(1 - \delta)$, and

$$\beta_{n+1} = b + \frac{\delta\beta_n c}{\delta\beta_n + c}, \tag{6}$$

$$\alpha_{n+1} = a + \frac{\delta\alpha_n c}{\delta\beta_n + c}, \tag{7}$$

$$\gamma_{n+1} = \delta\gamma_n + \frac{1}{2}\frac{(\delta\alpha_n)^2}{\delta\beta_n + c}, \quad n = 0, 1, \ldots. \tag{8}$$

It is a simple exercise to verify from (6) that $\beta_n \to \beta$, where $b < \beta < b + c$, and then from (7) and (8) that $\alpha_n \to \alpha$ and $\gamma_n \to \gamma$. The limit function $v(x) = \alpha x - \beta x^2/2 + \gamma$ clearly satisfies the functional equation, and hence Theorem 4.14 implies that it is the supremum function v^*. The associated policy function is $g(x) = (\delta\alpha + cx)/(\delta\beta + c)$, and it follows from Theorem 4.5 that any sequence $\{x_t\}$ generated from it is optimal.

In this particular example, it would make economic sense to restrict $\{x_t\}$ to the interval $X' = [0, a/b]$, since negative capital has no interpretation and accumulating more capital than a/b is costly and *de*creases revenues. F is bounded on $X' \times X'$, so with this restriction the theory of Section 4.2 would apply. But the computational advantage of quadratic returns stems from the fact that marginal returns are linear in the state variable(s). Thus if all maxima are described by first-order conditions, the optimal policy function is also linear in the state variable(s). Hence the convenience of the quadratic form is realized only if maxima are attained at interior points of the feasible set. Setting $X = \mathbf{R}$ and $\Gamma(x) = \mathbf{R}$, all $x \in X$, ensures that this is the case. After obtaining a solution, we can always check to see if it satisfies economically reasonable restrictions. [Note that in the example above, if x_0 is in the interval $[0, a/b]$, then the optimal sequence $x_{t+1} = g(x_t)$, $t = 0, 1, \ldots$, remains in this interval for all t.]

Theorem 4.14 is also useful in dealing with many-dimensional quadratic problems. An upper bound $\hat{v}$ satisfying (1)–(3) is easy to calculate, since any concave quadratic is bounded above. The iterates $T^n\hat{v}$ are readily computed, since they are defined by a finite number of parameters. If the sequence converges, Theorem 4.14 implies that the limit function is the supremum function and Theorem 4.5 implies that the linear policy that attains it is optimal. If the problem is strictly concave, there are no other optimal policies.

4.5 Euler Equations

There is a classical (eighteenth-century) mode of attack on the sequence problem

$$\text{(SP)} \qquad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots,$$

$$x_0 \in X \text{ given},$$

that involves treating it as straightforward programming problem in the decision variables $\{x_{t+1}\}_{t=0}^{\infty}$. Necessary conditions for an optimal program can be developed from the observation that if $\{x_{t+1}^*\}_{t=0}^{\infty}$ solves the problem (SP), given x_0, then for $t = 0, 1, \ldots, x_{t+1}^*$ must solve

$$\max_y \, [F(x_t^*, y) + \beta F(y, x_{t+2}^*)] \tag{1}$$

$$\text{s.t.} \quad y \in \Gamma(x_t^*) \text{ and } x_{t+2}^* \in \Gamma(y).$$

That is, a feasible variation on the sequence $\{x_{t+1}^*\}$ at one date t cannot lead to an improvement on an optimal policy. (A derivation of necessary conditions by this kind of argument is called a *variational* approach. In the present context the conditions so derived are called Euler equations, since Euler first obtained them from the continuous-time analogue to this problem.)

Let Assumptions 4.3–4.5, 4.7, and 4.9 hold; let F_x denote the l-vector consisting of the partial derivatives $(F_1, \ldots, F_l)$ of F with respect to its first l arguments, and F_y denote the vector $(F_{l+1}, \ldots, F_{2l})$. Since F is continuously differentiable and strictly concave, if x_{t+1}^* is in the interior of the set $\Gamma(x_t^*)$ for all t, the first-order conditions for (1) are

$$(2) \qquad 0 = F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*), \quad t = 0, 1, 2, \ldots .$$

This is a system of l second-order difference equations in the vector x_t of state variables. With the l-vector x_0 given, its solutions form an l-parameter family, and l additional boundary conditions are needed to single out the one solution that is in fact optimal.

These additional boundary conditions are supplied by the *transversality condition*

$$(3) \qquad \lim_{t\to\infty} \beta^t F_x(x_t^*, x_{t+1}^*) \cdot x_t^* = 0.$$

This condition has the following interpretation. Since the vector of derivatives F_x is the vector of marginal returns from increases in the current state variables, the inner product $F_x \cdot x$ is a kind of total value in period t of the vector of state variables. For example, in the many-sector growth model, F_x is the vector of capital goods prices. In this case (3) requires that the present discounted value of the capital stock in period t, evaluated using period t market prices, tends to zero as t tends to infinity. Whether or not one finds these market interpretations helpful, we have the following result.

THEOREM 4.15 (*Sufficiency of the Euler and transversality conditions*) *Let $X \subset \mathbf{R}_+^l$, and let F satisfy Assumptions 4.3–4.5, 4.7, and 4.9. Then the sequence $\{x_{t+1}^*\}_{t=0}^{\infty}$, with $x_{t+1}^* \in \operatorname{int} \Gamma(x_t^*)$, $t = 0, 1, \ldots$, is optimal for the problem (SP), given x_0, if it satisfies (2) and (3).*

Proof. Let x_0 be given; let $\{x_t^*\} \in \Pi(x_0)$ satisfy (2) and (3); and let $\{x_t\} \in \Pi(x_0)$ be any feasible sequence. It is sufficient to show that the difference, call it D, between the objective function in (SP) evaluated at $\{x_t^*\}$ and at $\{x_t\}$ is nonnegative.

Since F is continuous, concave, and differentiable (Assumptions 4.4, 4.7, and 4.9),

$$D = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1})]$$

$$\geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_x(x_t^*, x_{t+1}^*) \cdot (x_t^* - x_t) + F_y(x_t^*, x_{t+1}^*) \cdot (x_{t+1}^* - x_{t+1})].$$

Since $x_0^* - x_0 = 0$, rearranging terms gives

$$D \geq \lim_{T\to\infty} \left\{ \sum_{t=0}^{T-1} \beta^t [F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*)] \cdot (x_{t+1}^* - x_{t+1}) + \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1}^* - x_{T+1}) \right\}.$$

Since $\{x_t^*\}$ satisfies (2), the terms in the summation are all zero. Therefore, substituting from (2) into the last term as well and then using (3) gives

$$D \geq -\lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot (x_T^* - x_T)$$

$$\geq -\lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot x_T^*,$$

where the last line uses the fact that $F_x \geq 0$ (Assumption 4.5) and $x_t \geq 0$, all t. It then follows from (3) that $D \geq 0$, establishing the desired result. ■

(Note that Theorem 4.15 does not require any restrictions on Γ or β, because the theorem applies only if a sequence satisfying (2) and (3) has already been found. Restrictions on Γ and β are needed to ensure that such a sequence can be located.)

Exercise 4.9 a. Use Theorem 4.15 to obtain an alternative proof that the policy function $g(k) = \alpha\beta k^\alpha$ is optimal for the unit-elastic optimal growth model of Section 4.4.

b. Use Theorem 4.15 to obtain an alternative proof that the policy function $g(x) = (\delta\alpha + cx)/(\delta\beta + c)$ is optimal for the quadratic investment model of Section 4.4.

The Euler equations can also be derived directly from the functional equation

$$\text{(FE)} \qquad v(x) = \max_{y\in\Gamma(x)} [F(x, y) + \beta v(y)].$$

Suppose the value function v is differentiable; suppose, as above, that the right side of (FE) is always attained in the interior of $\Gamma(x)$; and let $v'(y)$ denote the vector $[v_1(y), \ldots, v_l(y)]$ of partial derivatives of v. Then the first-order conditions for the maximum problem (FE) are

$$(4) \qquad 0 = F_y[x, g(x)] + \beta v'[g(x)].$$

The envelope condition for this same maximum problem is

$$(5) \qquad v'(x) = F_x[x, g(x)].$$

Now set $x = x_t$ and $g(x) = g(x_t) = x_{t+1}$ in (4) to get

$$0 = F_y(x_t, x_{t+1}) + \beta v'(x_{t+1}),$$

and set $x = x_{t+1}$ and $g(x) = g(x_{t+1}) = x_{t+2}$ in (5) to get

$$v'(x_{t+1}) = F_x(x_{t+1}, x_{t+2}).$$

Eliminating $v'(x_{t+1})$ between these two equations then gives the Euler equations (2).

Implicitly, (4) is a system of l first-order difference equations in x_t, and the l initial values x_0 are sufficient to select a unique solution. No boundary conditions are missing from the problem, viewed in this way. Using (5) to eliminate $v'[g(x)]$ from (4) reproduces the Euler equation (2), but this step also discards useful information, so it is not surprising that many sequences $\{x_t\}$ satisfy (2) but not (4).

4.6 Bibliographic Notes

The terms *dynamic programming* and *Principle of Optimality* were introduced by Richard Bellman. Bellman's (1957) monograph is still useful and entertaining reading, full of ideas, applications, and problems from operations research and economics. There are also a number of more recent treatments of dynamic programming. Denardo (1967), Bertsekas (1976), and Harris (1987) are three that provide much material complementary to ours. See Sargent (1987) and Manuelli and Sargent (1987) for an alternative discussion of some of the material here and for many applications of these methods to macroeconomic problems.

An existence theorem for a problem very close to the one in Section 4.1 was proved by Karlin (1955). Our treatment is adapted from Blackwell (1965) and Strauch (1966), whose approaches carry over to—indeed, were designed for—stochastic problems as well, and hence will serve us again in Chapter 9.

Two recent papers discuss issues related to the material in Section 4.2. Blume, Easley, and O'Hara (1982) studied the differentiability of the approximations $v_n = T^n v_0$, $n = 0, 1, \ldots$, and the associated approximations g_n to the optimal policy function. Their Theorem 2.2, specialized to the deterministic case, provides conditions under which each function v_n is p times differentiable and each g_n is $(p - 1)$ times differentiable. In conjunction with Theorems 4.6 and 4.9, these facts can be useful in establishing properties of the limiting functions v and g.

Easley and Spulber (1981) studied the properties of rolling plans, policies for an infinite-horizon setting that are generated by solving, in each period t, a problem with finite horizon of $t + T$. They showed that for T sufficiently large, the plans so generated and the associated discounted returns are arbitrarily close to those generated by the optimal policy function.

Pablo Werning provided a most helpful counterexample to a claim about differentiability that appeared in class notes of Lucas, which were a predecessor to parts of Section 4.2.

Our treatment of the constant-returns-to-scale case discussed in Section 4.3 is based on Song (1986).

Our treatment of the unbounded-returns case in Section 4.4 draws on Prescott (1975) and Hansen (1985). Theorem 4.14 is adapted from