Econ 202A

10/4/2023

Problem becomes:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$0 \leq k_{t+1} \leq f(k_t)$$

k_0 given

- Called a “sequence problem by Stokey and Lucas.

- Infinite number of choice variables.

$\Rightarrow$ Use recursive methods.

Remember: f(k) = F(k,1)

Dynamic Programming

Let $$V(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

→ maximized discounted utility given k_0

$$\Rightarrow v(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right\}$$

$$\Rightarrow v(k_0) = \max_{k_1} \Big\{ u(f(k_0) - k_1)$$

$$+ \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \Big\}$$

$$= \max_{k_1} \left\{ u(f(k_0) - k_1) + \beta v(k_1) \right\}$$

$$V(k) = \max_{k'} \{ u(f(k) - k') + \beta V(k') \}$$

- called "Bellman's Equation"
- a functional equation where the unknown is $V(k)$
- $V(k)$ called the "value function"
- $u(f(k) - k')$ called the "return function"

FOC $\quad u'(f(k) - k') = \beta V'(k')$

$\Rightarrow$ solve for $k' = g(k)$ – The "policy function"

Solving for $V(k)$:

- Guess a function $V_0(k)$

- $$T(V_0(k)) = \max_{k'} \{u(f(k)-k') + \beta V_0(k')\}$$

- Let $V_1(k) = T(V_0(k))$

- Repeat forming a sequence of functions where $V_n(k) = T(V_{n-1}(k))$

$\Rightarrow \{V_n\}_{n=0}^{\infty}$ Continue until $V_{n-1}(k)$ & $V_n(k)$ are the same or close.

$$V(k) = \max_{k'} \{ u(f(k) - k') + \beta V(k') \}$$

Solve by computing a sequence of functions beginning with an arbitrary $V_0(k)$:

$$V_{n+1}(k) = \max_{k'} \{ u(f(k) - k') + \beta V_n(k') \}$$

That is,

$$V_{n+1}(k) = T(V_n(k))$$

Want to find fixed point:

Some $V(k)$ where

$$V(k) = T(V(k))$$

Very few case exist where a fixed point can be found analytically.

1. Problem 1 on Assignment 1.

2. Return function is quadratic and constraints linear.

If a closed form can be found, we can find it using the "method of undetermined coefficients."

This involves two steps:

1. Find the functional form for $\boldsymbol{v}$.

2. Find the parameters of the the fixed point in the Bellman mapping.

Example: Suppose $V_n(k) = ak^2 + bk + c$

and $T(V_n(k)) = \tilde{a} k^2 + \tilde{b} k + \tilde{c}$

$\Rightarrow$ T transforms a quadratic function into another quadratic function

$\Rightarrow$ fixed point is a quadratic function.

How do we find the fixed point?

Must be the case that

$$\tilde{a} = f_1(a, b, c)$$
$$\tilde{b} = f_2(a, b, c)$$
$$\tilde{c} = f_3(a, b, c)$$

Hence, fixed point is solution to 3 equations in 3 unknowns:

$$a = f_1(a, b, c)$$
$$b = f_2(a, b, c)$$
$$c = f_3(a, b, c)$$

How do we know if v (the fixed point) exists and is unique?

For review of concepts used in stating the following theorems, see Stokey & Lucas, Chapter 3, or Appendix on Functional Analysis in Ljungqvist and Sargent.

Contraction Mapping Theorem

If (S, ρ) is a complete metric space and $T: S \to S$ is a contraction mapping with modulus β, then

a. T has exactly one fixed point $v \in S$, and

b. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$ for $n = 0, 1, 2 \ldots$

Breaking this down:

- S is a space of functions
- ρ is a measure of distance between two points in S (a metric)
- Cauchy sequence: A sequence $\{v_n\}_{n=1}^{\infty}$ of elements of S where for all $\varepsilon > 0$, there exists N_ε such that $\rho(v_n, v_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$.
- Complete – Every Cauchy sequence in S converges to some element of S.

- A "contraction mapping" - $T: S \to S$ is a contraction if for some $\beta \in (0,1)$,
  $$\rho(Tx, Ty) \leq \beta \rho(x, y) \text{ for any } x, y \in S.$$
- A "fixed point" is some $v \in S$ such that $Tv = v$.

Problem: While the theorem guarantees that iterating on Bellman's equation will provide a sequence of functions that converge to a unique fixed point, verifying the conditions of the theorem is hard.

Blackwell's sufficient condition for a contraction:

Let $X \subseteq \mathbb{R}^n$ and $B(X)$ be a space of bounded functions $f: X \to \mathbb{R}$ with sup norm, $\|f\| = \sup_{x \in X} |f(x)|$.

Let $T: B(X) \to B(X)$ satisfy

a) (monotonicity) $f, g \in B(X)$ & $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$

b) (discounting) There exists some $\beta \in (0,1)$ such that $[T(f+a)](x) \leq (Tf)(x) + \beta a$ for all $f \in B(X)$ and $a \geq 0$.

Exercise: Verify $T: B(X) \to B(X)$, monotonicity and discounting for the Bellman mapping from the neoclassical growth model.

Envelope condition and Euler Equation

First order condition for r.h.s. of Bellman equation:

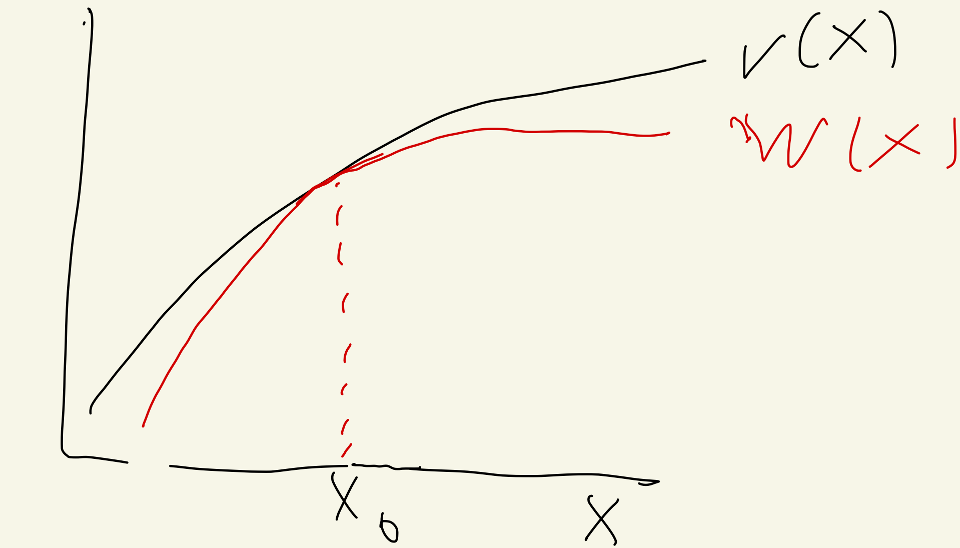
$$(*) \qquad u'(f(k) - g(k)) = \beta v'(g(k))$$

where $k' = g(k)$.

Envelope Theorem:

Suppose V is concave and W is concave and differentiable with $W(X_0) = V(X_0)$ and $W(X) \leq V(X)$ for all X in a neighborhood of X_0. Then V is differentiable at X_0 and

$$V_i(X_0) = W_i(X_0)$$



In Bellman case, let $W(k) \equiv u(f(k) - g(k_0)) + \beta V(g(k_0))$

$\Rightarrow \quad W(k_0) = V(k_0)$

and $W(k) \le V(k)$ for k near k_0

$\Rightarrow$ Envelope condition:

$$V'(k_0) = W'(k_0) = u'(f(k_0) - g(k_0)) f'(k_0)$$

$\Rightarrow$ in general, $V'(k) = u'(f(k) - g(k)) f'(k)$ $(**)$

Combine $(*)$ and $(**)$:

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k))) f'(g(k))$$

<u>or</u> $u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$

$\longrightarrow$ Euler Equation. One for every $t \geq 0$

Transversality Condition

$$\lim_{t\to\infty} \beta^t U'\left(f(k_t) - k_{t+1}\right) f'(k_t) \cdot k_t = 0$$

See Thm 4.15 on page 98 of Stokey and Lucas.