Econ 202A Notes: Macroeconomic Theory I

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Fall 2023

1 Basic Neoclassical Growth Model (Cass-Koopmans)

Economy consists of many identical infinitely lived households – all have the same preferences and endowments.

Several Interpretations

1. Representative Agent (Robinson Crusoe)
2. Social Planner
3. Infinitely Lived Family (dynasty)

One production sector – output produced from capital and labor. Can be consumed or invested. Investment becomes productive capital after one period.

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } \beta \in (0,1) \\ c_t + i_t \leq y_t = f(k_t, n_t) \\ k_{t+1} \leq (1-\delta)k_t + i_t \\ 0 \leq n_t \leq 1 \\ k_0, k_0 > 0 \end{aligned}$$

Note that no prices are given here!

1. β is the amount you value future consumption. Note that we assume you value future consumption less than present consumption, but more than zero.
2. c_t is consumption at time t
3. i_t is investment at time t
4. y_t is output at time t

5. k_t is capital at time t
6. n_t is labor at time t
7. δ is the depreciation rate of capital
8. f is the production function
9. u is the utility function

1.1 Production Function

$F : \mathbb{R}^2_+ \to \mathbb{R}_+$

1. continuously differentiable
2. homogenous of degree 1 $\Leftrightarrow$ constant returns to scale
3. strictly quasi-concave
4. $F(0, n) = 0$
   (a) implies capital is essential
   (b) F_k = marginal product of capital > 0
   (c) F_n = marginal product of labor > 0
5. $\lim_{k \to 0} F(k, 1) = \infty$
6. $\lim_{k \to \infty} F(k, 1) = 0$

1.2 Utility Function

$u : \mathbb{R}_+ \to \mathbb{R}$

1. bounded - needed for dynamic programming
2. continuously differentiable
3. strictly increasing
4. strictly concave
5. $\lim_{c \to 0} u'(c) = \infty$

Note: We will use functional forms for F and N for most everything we do in this class.

1.3 Simplifying the Planner's Problem

1. $F_n > 0$ and $u'(c) > 0 \Rightarrow n_t = 1 \ \forall \ t$
2. $u'(c) > 0 \Rightarrow$ resource constraint holds with equality: $c_t + i_t = F(k_t, n_t)$
3. $\beta < 1 \Rightarrow$ require positive rate of return to give up a unit of consumption today for consumption tomorrow [actually $mp_{k+1} - \delta$]
4. Let $f(k) = F(k, 1) + (1 - \delta)k$

Problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$
$$\text{s.t. } 0 \leq k_{t+1} \leq f(k_t)$$
$$k_0 \text{ given}$$

1. Called a sequences problem by Stokey and Lucas
2. Infinite number of choice variables

$\Rightarrow$ **Use recursive methods.**

Remember $f(k) = F(k, 1) + (1 - \delta)k$

2 Dynamic Programming

$$V(k_0) = \max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$
$$\text{given } k_0$$
$$\rightarrow \text{maximized discounted utility given } k_0$$

$$V(k_0) = \max_{\{k_t\}_{t=0}^{\infty}} u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})$$
$$= \max_{k_1} u(f(k_0) - k_1) + \beta V(k_1)$$
$$\text{where } V(k_1) = \max_{\{k_t+1\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})$$

$V(k) = max_{k'} \ u(f(k) - k') + \beta V(k')$

1. called "Bellman's Equation"
2. a functional equation where the unknown is $V(k)$
3. $V(k)$ is called the value function
4. $u(f(k) - k')$ is called the return function

First order conditions
$u'(f(k)-k') = \beta V'(k')$
$\Rightarrow$ **solve for** $k' = g(k)$ **- The "policy function"**

Solving for $V(k)$

1. Guess a function $V_0(k)$
2. $T(v_0)(k) = \max_{k'} u(f(k)-k') + \beta V_0(k')$
3. Let $v_1(k) = T(v_0)(k)$
4. Repeat forming a sequence of functions where $v_n(k) = T(v_{n-1})(k)$

   V_n FIX
   $V(k) = max_{k'} u(f(k)-k') + \beta V(k')$

Solve by computing a sequence of functions beginning with an arbitrary $V_0(k)$
$V_{n+1}(k) = max_{k'} u(f(k)-k') + \beta V_n(k')$
That is, $V_{n+1}(k) = T(V_n)(k)$
We want to find fixed point: Some $V(k)$ **where** $V(k) = T(V(k))$

Very few cases exist where a fixed point can be found analytically

1. Problem 1 on Assignment 1
2. Return function is quadratic and constraints are linear

If a closed form can be found, we can find it using the "method of undetermined coefficients"

This involves two steps:

1. Find the function form for v
2. Find the parameters of the fixed point in the Bellman mapping.

1. Suppose $V_n(k) = ak^2 + bk + c$ and $T(v_n(k)) = \tilde{a}\tilde{k}\tilde{b}k + \tilde{c}$
2. T transforms a quadratic function into another quadratic function
3. Fixed point is a quadratic function
4. How do we find the fixed point?
5. Must be the case that:

$$\tilde{a} = f_1(a, b, c)$$
$$\tilde{b} = f_2(a, b, c)$$
$$\tilde{c} = f_3(a, b, c)$$

6. Hence, fixed point is a solution to a system of three equations in three unknowns:

$$a = f_1(a, b, c)$$
$$b = f_2(a, b, c)$$
$$c = f_3(a, b, c)$$

7. How do we know if V (the fixed point) exists and is unique?

8. For review of concepts used in stating the following theoreoms, see Stokey and Lucas, Chapter 3, or appendix on Functional Analysis in Ljunqvist and Sargent.

3 Contraction mapping theorem

If (S, P) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then a T has exactly one fixed point $v \in S$ and b for any $v_0 \in S$, $P(T^n V_0, V) \leq \beta^n P(v_0, v)$ for $n = 0, 1, 2, ...$

1. Breaking this down:

2. S is a space of functions

3. S is a measure of distance between two points in S (a metric)

4. Cauchy sequence: A sequence

5. Complete - Every Cauchy sequence in S converges to some element of S

6. A "Contraction mapping" - $T : S \to S$ is a contraction

7. A "fixed point" is some $v \in S$ such that $T(v) = v$

Problem: While the therorem guarantees that iterating on Bellman's equation will provide a sequence of functions that converges to a uniqued fixed point, verifying the conditions of the theorem is hard.

1. Blackwell's sufficient condition for a contraction:

2. Let $X \leq R^n$ and $B(X)$ be a space of $\underline{boundedfunctions}$ $F : X \to R$ with sup norm, $||f|| = \sup_{x \in X} |f(x)|$

3. Let $T : B(X) \to B(X)$ satisfy

4. a) (Monotonocity) $f, g \in B(X)$ and $f(x) \leq g(x) \forall x \in X \Rightarrow T(f)(x) \leq T(g)(x) \forall x \in X$

5. b) (Discounting) $\exists \beta \in (0, 1)$ such that $T(f + a)(x) \leq (T(f)(x)) + \beta a \forall f \in B(X) anda \geq 0$

Exercise Verify $T : B(X) \to B(X)$ Monotonocity and Discounting for the bellman mapping from the Neoclassical Growth model.

4 Envelope condition and Euler equation

1. First order condition for the right hande side of Bellman equation:
2. $u'(f(k) - g(k)) = \beta V'(g(k))$ where $k' = g(k)$
3. Envelope Theorem:
4. Suppose V is concave and W is concave and differentiable with $W(X_0) = V(X_0)$ and $W(X) \leq V(X) \forall x$ in a neighborhood of x_0 Then V is differentiable at X_0 and $V_i(X_0) = W_i(X_0)$

1. In Bellman case, let $W(k) \equiv u(f(k) - g(k_0)) + \beta V(g(k_0))$
2. $\Rightarrow W(k_0) = V(k_0)$ and $W(k) \leq V(k)$ for k near k_0
3. Envelope Conditions:
4. $V'(k_0) = w'(k_0) = u'(f(k_0) - g(k_0))(f'(k_0))$
5. in general, $V'(k) = u'(f(k) - g(k))(f'(k))$
6. $u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k)))f'(g(k))$
7. or $u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1})$
8. $\to$ Euler Equation, one for every $t \geq 0$

5 transversatality condition

$\lim_{t\to\infty} \beta^t u'(f(k_t) - k_{t+1})f'(k_t) * k_t = 0$ See Thm 4.15 on page 98 of Stokey and Lucas

Given k_0 only one k, consistent with transversatality condition

Steady State The steady state stock of capital is some $\bar{k} > 0$ such that $\bar{k} = g(\bar{k})$

This can be easily found from the Euler Equation:

$$\begin{aligned} u'(f(k_t) - k_{t+1}) &= \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) \\ &\text{by setting } k_t = k_{t+1} = k_{t+2} = k \\ &\Rightarrow u'(f(\bar{k}) - \bar{k}) = \beta u'(f(\bar{k}) - \bar{k})f'(\bar{k}) \\ \Rightarrow \frac{1}{\beta} &= f'(\bar{k}), \text{ where } f(k) \equiv F(k, 1) + (1 - \delta)k \end{aligned}$$

Does a solution exist?

$$\lim_{k \to 0} f'(k) = \infty$$
$$\lim_{k \to \infty} f'(k) = 1 - \delta < 1$$
$$f(k) \text{ is continuously differentiable}$$

Is $k > 0$ unique?

$f(k)$ is concave - See Exercise 4.8 of Stokey and Lucas. Follows from quasiconcavity and CRS

$\Rightarrow f''(k) < 0$

$\Rightarrow \bar{k}$ unique

But, $\bar{k} = 0$ is also a steady state since $f(0) = 0$

Maximum Sustainable Capital Stock k^*, the maximum sustainable capital stock is the largest value of k such that $f(k) \geq k$

1. for k to ever be above $k^*, k_0 > k^*$ (perhaps due to shift in production function)
2. If so, k must decrease over time until it is less than or equal to k^*
3. can effectively ignore capital stocks above k^*
4. can ignore consumption above $\bar{c} = f(k^*)$
5. can bound utility by $u(\bar{c})$
6. can apply contraction mapping theorem

Dynamics of Growth Model How k_t evolves from $k_0 > 0$ depends on the policy function, $g(k)$. What do we know about this function:

1. $g(0) = 0$ and increasing
2. Must cross 45 degree line only once for $k > 0$ since $\bar{k}$ is unique.

The key to dynamics is whether or not $g(k) > k$ for arbitrarily small $k > 0$
$\Rightarrow lim_{t \to \infty} k_t = 0$ if $k_0 < \bar{k} \Rightarrow$ this contradicts optimization Hence, $g(k) > k$ for arbitrarily small k

Adding Labor-Leisure Tradeoff • key to real business cycle literature

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$
$$\text{s.t. } c_t + k_{t+1} = F(k_t, n_t) + (1 - \delta) k_t$$
$$k_0 \text{ given}$$

Can make assumptions about u to guarantee an interior solution where:
$n \in (0,1)$

Writing as a dynamic programming problem

$$V(k) = \max_{k',h} u(c, 1-h) + \beta V(k')$$

$$s.t. c + k' = F(k,h) + (1-\delta)k$$

why is h not included as a state variable?

because h_0 not needed to solve sequence problem!

h_0 determined in period 0 and has no impact on problem solved in period 1

In fact, first order condition for h (static):

$$u_2(c, 1-h) = u_1(c, 1-h)F_2(k,h)$$

can be solved for: $h = H(k,k')$

$\Rightarrow$ can write Bellman equation: $V(k) = max_{k'} u(c, 1-H(k,k')) + \beta V(k')$

$$s.t. c + k' = F(k, H(k,k')) + (1-\delta)k$$

$\rightarrow$ Fits into structure of Stokey & Lucas!