3 Mathematical Preliminaries

In Chapter 2 the optimal growth problem

$$\max_{\{(c_t, k_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq f(k_t),$$

$$c_t, k_{t+1} \geq 0, \quad t = 0, 1, \ldots,$$

$$\text{given } k_0,$$

was seen to lead to the functional equation

$$v(k) = \max_{c,y} [U(c) + \beta v(y)] \tag{1}$$

$$\text{s.t.} \quad c + y \leq f(k),$$

$$c, y \geq 0.$$

The purpose of this chapter and the next is to show precisely the relationship between these two problems and others like them and to develop the mathematical methods that have proved useful in studying the latter. In Section 2.1 we argued in an informal way that the solutions to the two problems should be closely connected, and this argument will be made rigorous later. In the rest of this introduction we consider alternative methods for finding solutions to (1), outline the one to be pursued, and describe the mathematical issues it raises. In the remaining sections of the chapter we deal with these issues in turn. We draw upon this material extensively in Chapter 4, where functional equations like (1) are analyzed.

In (1) the functions U and f are given—they take specific forms known to us—and the value function v is unknown. Our task is to prove the existence and uniqueness of a function v satisfying (1) and to deduce its properties, given those of U and f. The classical (nineteenth-century) approach to this problem was the *method of successive approximations,* and it works in the following very commonsensical way. Begin by taking an initial guess that a specific function, call it v_0, satisfies (1). Then define a new function, v_1, by

(2) $$v_1(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v_0(y)\}.$$

If it should happen that $v_1(k) = v_0(k)$, for all $k \geq 0$, then clearly v_0 is a solution to (1). Lucky guessing (cf. Exercise 2.3) is one way to establish the existence of a function satisfying (1), but it is notoriously unreliable. The method of successive approximations proceeds in a more systematic way.

Suppose, as is usually the case, that $v_1 \neq v_0$. Then use v_1 as a new guess and define the sequence of functions $\{v_n\}$ recursively by

(3) $$v_{n+1}(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v_n(y)\}, \quad n = 0, 1, 2, \ldots.$$

The hope behind this iterative process is that as n increases, the successive approximations v_n get closer to a function v that actually satisfies (1). That is, the hope is that the limit of the sequence $\{v_n\}$ is a solution v. Moreover, if it can be shown that $\lim_{n\to\infty} v_n$ is the same for any initial guess v_0, then it will follow that this limit is the only function satisfying (1). (Why?)

Is there any reason to hope for success in this analytical strategy? Recall that our reason for being interested in (1) is to use it to locate the optimal capital accumulation policy for a one-sector economy. Suppose we begin by choosing any feasible capital accumulation policy, that is, any function g_0 satisfying $0 \leq g_0(k) \leq f(k)$, all $k \geq 0$. [An example is the policy of saving a constant fraction of income: $g_0(k) = \theta f(k)$, where $0 < \theta < 1$.] The lifetime utility yielded by this policy, as a function of the

initial capital stock k_0, is

$$w_0(k_0) = \sum_{t=0}^{\infty} \beta^t U[f(k_t) - g_0(k_t)],$$

where

$$k_{t+1} = g_0(k_t),\ t = 0, 1, 2, \ldots.$$

The following exercise develops a result about (g_0, w_0) that is used later.

Exercise 3.1 Show that

$$w_0(k) = U[f(k) - g_0(k)] + \beta w_0[g_0(k)], \quad \text{all } k \geq 0.$$

If the utility from the policy g_0 is used as the initial guess for a value function—that is, if $v_0 = w_0$—then (2) is the problem facing a planner who can choose capital accumulation optimally for one period but must follow the policy g_0 in all subsequent periods. Thus $v_1(k)$ is the level of lifetime utility attained, and the maximizing value of y—call it $g_1(k)$—is the optimal level for end-of-period capital. Both v_1 and g_1 are functions of beginning-of-period capital k.

Notice that since $g_0(k)$ is a feasible choice in the first period, the planner will do no worse than he would by following the policy g_0 from the beginning, and in general he will be able to do better. That is, for any feasible policy g_0 and associated initial value function v_0,

(4) $$\begin{aligned} v_1(k) &= \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v_0(y)\} \\ &\geq \{U[f(k) - g_0(k)] + \beta v_0[g_0(k)]\} \\ &= v_0(k), \end{aligned}$$

where the last line follows from Exercise 3.1.

Now suppose the planner has the option of choosing capital accumulation optimally for two periods but must follow the policy g_0 thereafter. If y is his choice for end-of-period capital in the first period, then from the second period on the best he can do is to choose $g_1(y)$ for end-of-period capital and enjoy total utility $v_1(y)$. His problem in the first period is thus $\max[U(c) + \beta v_1(y)]$, subject to the constraints in (1). The maximized value of this objective function was defined, in (3), as $v_2(k)$. Hence it follows from (4) that

$$\begin{aligned} v_2(k) &= \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v_1(y)\} \\ &\ge \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta v_0(y)\} \\ &= v_1(k). \end{aligned}$$

Continuing in this way, one establishes by induction that $v_{n+1}(k) \ge v_n(k)$, all k, $n = 0, 1, 2, \ldots$. The successive approximations defined in (3) are improvements, reflecting the fact that planning flexibility over longer and longer finite horizons offers new options without taking any other options away. Consequently it seems reasonable to suppose that the sequence of functions $\{v_n\}$ defined in (3) might converge to a solution v to (1). That is, the method of successive approximations seems to be a reasonable way to locate and characterize solutions.

This method can be described in a somewhat different and much more convenient language. As we showed in the discussion above, for any function w: $\mathbf{R}_+ \to \mathbf{R}$, we can define a new function—call it Tw: $\mathbf{R}_+ \to \mathbf{R}$—by

$$(5) \qquad (Tw)(k) = \max_{0 \le y \le f(k)} \{U[f(k) - y] + \beta w(y)\}.$$

When we use this notation, the method of successive approximations amounts to choosing a function v_0 and studying the sequence $\{v_n\}$ defined by $v_{n+1} = Tv_n$, $n = 0, 1, 2, \ldots$. The goal then is to show that this sequence converges and that the limit function v satisfies (1). Alternatively, we can simply view the operator T as a mapping from some set C of functions into itself: T: $C \to C$. In this notation solving (1) is equivalent to locating a *fixed point* of the mapping T, that is, a function $v \in C$ satisfying $v = Tv$, and the method of successive approximations is viewed as a way to construct this fixed point.

To study operators T like the one defined in (5), we need to draw on several basic mathematical results. To show that T maps an appropriate space C of functions into itself, we must decide what spaces of functions are suitable for carrying out our analysis. In general we want to limit attention to continuous functions. This choice raises the issue of whether, given a continuous function w, the function Tw defined by (5) is also continuous. Finally, we need a fixed-point theorem that applies to operators like T on the space C we have selected. The rest of the chapter deals with these issues.

In Section 3.1 we review the basic facts about metric spaces and normed vector spaces and define the space C that will be used repeatedly later. In Section 3.2 we prove the Contraction Mapping Theorem, a fixed-point theorem of vast usefulness. In Section 3.3 we review the main facts we will need about functions, like Tw above, that are defined by maximization problems.

3.1 Metric Spaces and Normed Vector Spaces

The preceding section motivates the study of certain functional equations as a means of finding solutions to problems posed in terms of infinite sequences. To pursue the study of these problems, as we will in Chapter 4, we need to talk about infinite sequences $\{x_t\}_{t=0}^{\infty}$ of states, about candidates for the value function v, and about the convergence of sequences of various sorts. To do this, we will find it convenient to think of both infinite sequences and certain classes of functions as elements of infinite-dimensional normed vector spaces. Accordingly, we begin here with the definitions of vector spaces, metric spaces, and normed vector spaces. We then discuss the notions of convergence and Cauchy convergence, and define the notion of completeness for a metric space. Theorem 3.1 then establishes that the space of bounded, continuous, real-valued functions on a set $X \subseteq \mathbf{R}^l$ is complete.

We begin with the definition of a vector space.

DEFINITION *A (real)* ***vector space*** *X is a set of elements (vectors) together with two operations, addition and scalar multiplication. For any two vectors $x, y \in X$, addition gives a vector $x + y \in X$; and for any vector $x \in X$ and any real number $\alpha \in \mathbf{R}$, scalar multiplication gives a vector $\alpha x \in X$. These operations obey the usual algebraic laws; that is, for all $x, y, z \in X$, and $\alpha, \beta \in \mathbf{R}$:*

a. $x + y = y + x$;

b. $(x + y) + z = x + (y + z)$;