Note: These are extra problems taken from an old textbook by Tom Sargent. Try these along with Assignment 1, ~~although Exercise 1.3 is best done along with Assignment 2.~~

Exercise 1.3. **Levhari and Srinivasan (1969)**

Assume that

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha}, \qquad \alpha > 0.$$

Assume that R_t is independently and identically distributed and is such that $ER_t^{1-\alpha} < 1/\beta$. Consider the problem

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1,$$

subject to $A_{t+1} \le R_t(A_t - c_t)$, $A_0 > 0$ given. It is assumed that c_t must be chosen before R_t is observed. Show that the optimal policy function takes the form $c_t = \lambda A_t$ and give an explicit formula for λ.

Hint. Consider a value function of the general form $v(A) = BA^{1-\alpha}$, for some constant B.

Exercise 1.4. **Habit Persistence, 1**

Consider the problem of choosing a consumption sequence c_t to maximize

$$\sum_{t=0}^{\infty} \beta^t(\ln c_t + \gamma \ln c_{t-1}), \qquad 0 < \beta < 1, \qquad \gamma > 0,$$

$$\begin{aligned} \text{subject to} \quad & c_t + k_{t+1} \le Ak_t^{\alpha}, \\ & A > 0, \\ & 0 < \alpha < 1, \\ & k_0 > 0, \quad \text{and} \quad c_{-1} \text{ given.} \end{aligned}$$

Here c_t is consumption at t, and k_t is capital stock at the beginning of period t. The current utility function $\ln c_t + \gamma \ln c_{t-1}$ is designed to represent habit persistence in consumption.

a. Let $v(k_0, c_{-1})$ be the value of $\Sigma_{t=0}^{\infty} \beta^t(\ln c_t + \gamma \ln c_{t-1})$ for a consumer who begins time 0 with capital stock k_0 and lagged consumption c_{-1} and behaves optimally. Formulate Bellman's functional equation in $v(k, c_{-1})$.

b. Prove that the solution of Bellman's equation is of the form $v(k, c_{-1}) = E + F \ln k + G \ln c_{-1}$ and that the optimal policy is of the form $\ln k_{t+1} = I + H \ln k_t$ where E, F, G, H, and I are constants. Give explicit formulas for the constants E, F, G, H, and I in terms of the parameters A, β, α, and γ.

Exercise 1.5. **Habit Persistence, 2**

Consider the more general version of the preceding problem, to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1}), \qquad 0 < \beta < 1,$$

subject to $c_t + k_{t+1} \le f(k_t)$, $k_0 > 0$, c_{-1} given, where $u(c_t, c_{t-1})$ is twice continuously differentiable, bounded, increasing in both c_t and c_{t-1}, and concave in (c_t, c_{t-1}), and where $f'(0) = +\infty$, $f' > 0$, $f'' < 0$.

a. Formulate Bellman's functional equation for this problem.

b. Argue that in general, the optimal consumption plan is to set c_t as a function of both k_t and c_{t-1}. What features of the example in the preceding problem combine to make the optimal consumption plan expressible as a function of k_t alone?

Exercise 1.8. Two-Sector Growth Models

a. Consider the following two-sector model of optimal growth. A social planner seeks to maximize the utility of the representative agent given by $\sum_{t=0}^{\infty}\beta^t u(c_t, l_t)$, where c_t is consumption of good 1 at t, whereas l_t is leisure at t. Sector 1 produces consumption goods using capital, k_{1t}, and labor, n_{1t}, according to the production function $c_t \leq f_1(k_{1t}, n_{1t})$. Sector 2 produces the capital good according to the production function $k_{t+1} \leq f_2(k_{2t}, n_{2t})$. Total employment, $n_t = n_{1t} + n_{2t}$, and leisure, l_t, is constrained by the endowment of time, $\bar{l}$, and satisfies $l_t + n_t \leq \bar{l}$. The sum of the amounts of capital used in each sector cannot exceed the initial capital in the economy, that is, $k_{1t} + k_{2t} \leq k_t$, $k_0 > 0$ given. Formulate this problem as a dynamic programming problem. Display the functional equation that the value function satisfies, and clearly specify the state and control variables.

b. Consider another economy that is similar to the previous one except for the fact that capital is sector specific. The economy starts period t with given amounts of capital k_{1t} and k_{2t} that must be used in sectors 1 and 2, respectively. During this period the capital-good sector produces capital that is specific to each sector according to the transformation curve $g(k_{1t+1}, k_{2t+1}) \leq f_2(k_{2t}, n_{2t})$. Display the Bellman's equation associated with the planner's problem. Specify which variables you choose as states and which as controls.

Exercise 1.9. Learning to Enjoy Spare Time

A worker's instantaneous utility, $u(\cdot)$, depends on the amount of market-produced goods consumed, c_{1t}, and also on the amount of home-produced goods, c_{2t} (for example, entertainment, leisure). In order to acquire market-produced goods, the worker must allocate some amount of time, l_{1t}, to market activities that pay a salary of w_t, measured in terms of consumption good. The worker takes wages as given and beyond the worker's control. There is no borrowing or lending. It is known that the market wage evolves according to the law of motion $w_{t+1} = h(w_t)$.

The quantity of home-produced goods depends on the stock of "expertise" that the worker has at the beginning of the period, which we label a_t. This stock of "expertise" depreciates at the rate δ and can be increased by allocating time to nonmarket activities. To summarize the problem, the individual agent maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}), \qquad 0 < \beta < 1,$$

subject to	$c_{1t} \leq w_t l_{1t}$	[budget constraint]
	$c_{2t} \leq f(a_t)$	[production function of the home-produced good]
	$a_{t+1} \leq (1-\delta)a_t + l_{2t}$	[law of motion of the stock of expertise]
	$l_{1t} + l_{2t} \leq \bar{l}$	[restriction on the uses of time]
	$w_{t+1} = h(w_t)$	[law of motion for the wage rate]
	$a_0 > 0$	[given].

It is assumed that $u(\cdot)$ and $f(\cdot)$ are bounded and continuous. Formulate this problem as a dynamic programming problem.

Exercise 1.10. **Investment with Adjustment Costs**

A firm maximizes present value of cash flow, with future earnings discounted at the rate β. Income at time t is given by sales, $p_t \cdot q_t$, where p_t is the price of good, and q_t is the quantity produced. The firm behaves competitively and therefore takes prices as given. It knows that prices evolve according to a law of motion given by $p_{t+1} = f(p_t)$.

Total or gross production depends on the amounts of capital, k_t, and labor, n_t, and on the square of the difference between current ratio of sales to investment, x_t, and the previous-period ratio. This last feature captures the notion that changes in the ratio of sales to investment require some reallocation of resources within the firm and consequently reduce the level of efficiency. It is assumed that the wage rate is constant and equal to w. Capital depreciates at the rate δ. The firm's problem is

$$\max \sum_{t=0}^{\infty} \beta^t (p_t q_t - w n_t), \qquad 0 < \beta < 1,$$

$$\begin{aligned} \text{subject to} \quad & q_t + x_t \leq g\left[k_t, n_t, \left(\frac{q_t}{x_t} - \frac{q_{t-1}}{x_{t-1}}\right)^2\right] \\ & k_{t+1} \leq (1-\delta)k_t + x_t, \qquad 0 < \delta < 1 \\ & p_{t+1} = f(p_t) \\ & k_0 > 0, \ \frac{q_{-1}}{x_{-1}} > 0 \ \text{ given.} \end{aligned}$$

We assume that $g(\cdot)$ is bounded, increasing in the first two arguments and decreasing in the third. Formulate the firm's problem recursively, that is, **formulate Bellman's functional equation for this problem. Identify the state and the controls, and indicate the laws of motion of the state** variables.