Econ 202A

10/09/23

Characterizing Dynamics of Growth Model

$$V(k) = \max_{k'} \{ u(f(k) - k') + \beta V(k') \}$$

$$\Rightarrow \quad k' = g(k)$$

Given k_0, can obtain sequences

$$\{k_{t+1}, y_t, c_t, i_t\}_{t=0}^{\infty}$$

where $y_t = f(k_t) - (1-\delta) k_t = c_t + i_t$

$c_t = f(k_t) - k_{t+1}$

$i_t = k_{t+1} - (1-\delta) k_t$

Envelope condition and Euler Equation

First order condition for r.h.s. of Bellman equation:

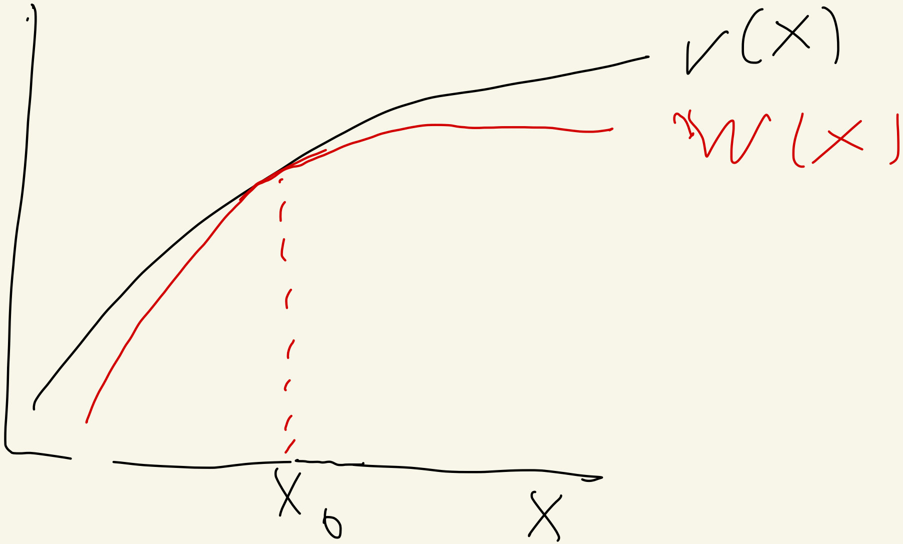
$$(*) \qquad u'(f(k)-g(k)) = \beta V'(g(k))$$

where $k' = g(k)$.

Envelope Theorem:

Suppose V is concave and W is concave and differentiable with $W(X_0) = V(X_0)$ and $W(X) \leq V(X)$ for all X in a neighborhood of X_0. Then V is differentiable at X_0 and

$$V_i(X_0) = W_i(X_0)$$



In Bellman case, let $w(k) \equiv u(f(k) - g(k_0)) + \beta v(g(k_0))$

$\Rightarrow \quad w(k_0) = v(k_0)$

and $w(k) \leq v(k)$ for k near k_0

$\Rightarrow$ Envelope condition:

$$V'(k_0) = W'(k_0) = u'(f(k_0) - g(k_0))\, f'(k_0)$$

$\Rightarrow$ in general, $V'(k) = u'(f(k) - g(k))\, f'(k)$ $(**)$

Combine $(*)$ and $(**)$:

$$u'(f(k) - g(k)) = \beta\, u'(f(g(k)) - g(g(k)))\, f'(g(k))$$

or $u'(f(k_t) - k_{t+1}) = \beta\, u'(f(k_{t+1}) - k_{t+2})\, f'(k_{t+1})$

$\longrightarrow$ Euler Equation. One for every $t \geq 0$

Transversality Condition

$$\lim_{t\to\infty} \beta^t u'\left(f(k_t)-k_{t+1}\right)f'(k_t) \cdot k_t = 0$$

See Thm 4.15 on page 98 of Stokey and Lucas.

Given k_0, only one k_1 consistent with transversality condition.

Steady State

The steady state stock of capital is some $\bar{k} > 0$ such that

$$\bar{k} = g(\bar{k})$$

This can be easily found from the Euler equation:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1})$$

by setting $k_t = k_{t+1} = k_{t+2} = \bar{k}$

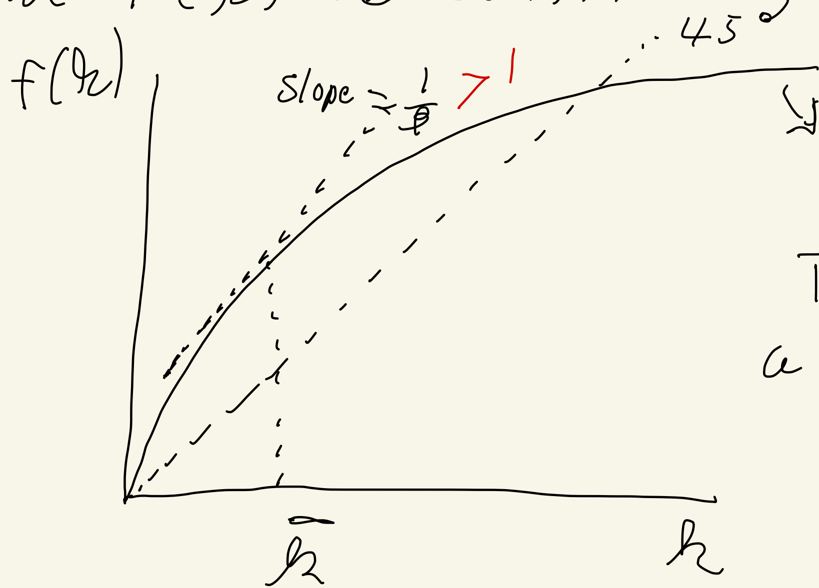
$$\Rightarrow \quad u'(f(\bar{k}) - \bar{k}) = \beta u'(f(\bar{k}) - \bar{k}) f'(\bar{k})$$

$$\Rightarrow \quad \frac{1}{\beta} = f'(\bar{k}), \quad \text{where } f(k) \equiv F(k,1)+(1-\delta)k$$

Does a solution exist?

$$\lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) = 1-\delta < 1$$

and $f(k)$ is continuously differentiable.



There must be a point where $f'(k) = \frac{1}{\beta}$.

Is $\bar{k} > 0$ unique?

$f(k)$ is concave - see Exercise 4.8 of Stokey & Lucas. Follows from quasi concavity and CRS.

$\Rightarrow \quad f''(k) < 0$

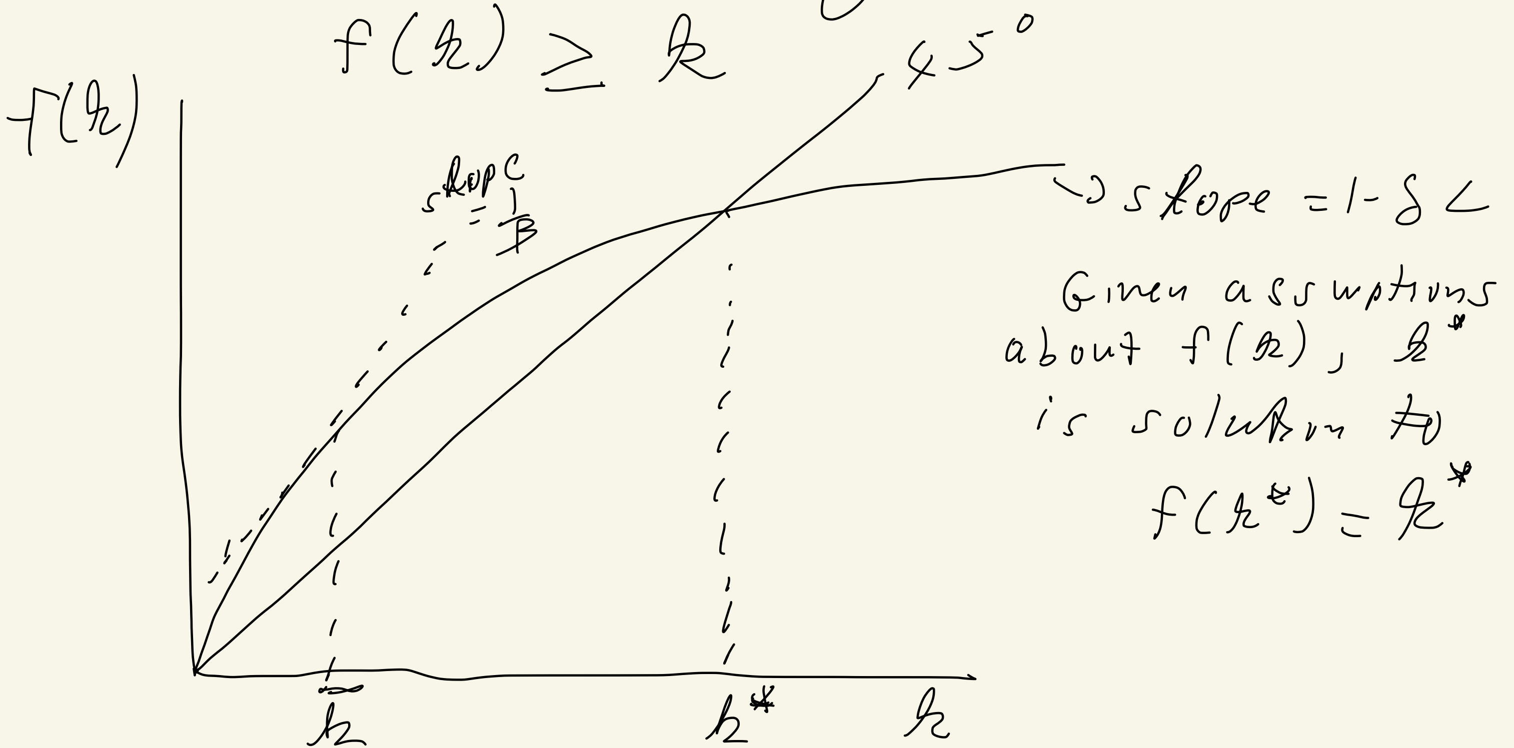
$\Rightarrow \quad \bar{k}$ unique

<u>But</u>, $\bar{k} = 0$ is also a steady state since $f(0) = 0$.

Maximum sustainable Capital Stock

k^*, the maximum sustainable capital stock, is the largest value of k such that

$$f(k) \geq k$$



Given assumptions about $f(k)$, k^* is solution to

$$f(k^*) = k^*$$

For k to <u>ever</u> be above k^*, $k_0 > k^*$.
(perhaps due to shift in production function).

If so, k <u>must</u> decrease over time until it is less than or equal to k^*.

$\Rightarrow$ can effectively ignore capital stocks above k^*.

$\Rightarrow$ can ignore consumption above $\bar{c} = f(k^*)$

$\Rightarrow$ can bound utility by $u(\bar{c})$.

$\Rightarrow$ can apply contraction mapping thm.

Dynamics of Growth Model

How k_t evolves from $k_0 > 0$ depends on the policy function, $g(k)$. What do we know about this function:

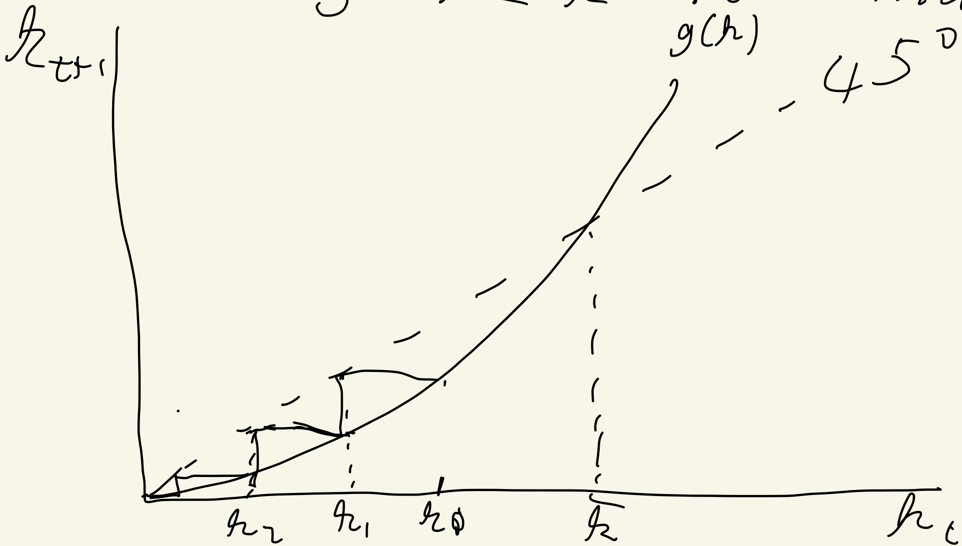
1. $g(0) = 0$ and increasing

2. Must cross 45° line only once for $k > 0$ since $\bar{k}$ is unique.

The key to dynamics is whether or not

$g(k) > k$ for arbitrarily small $k > 0$.

Suppose $g(k) < k$ for small k:

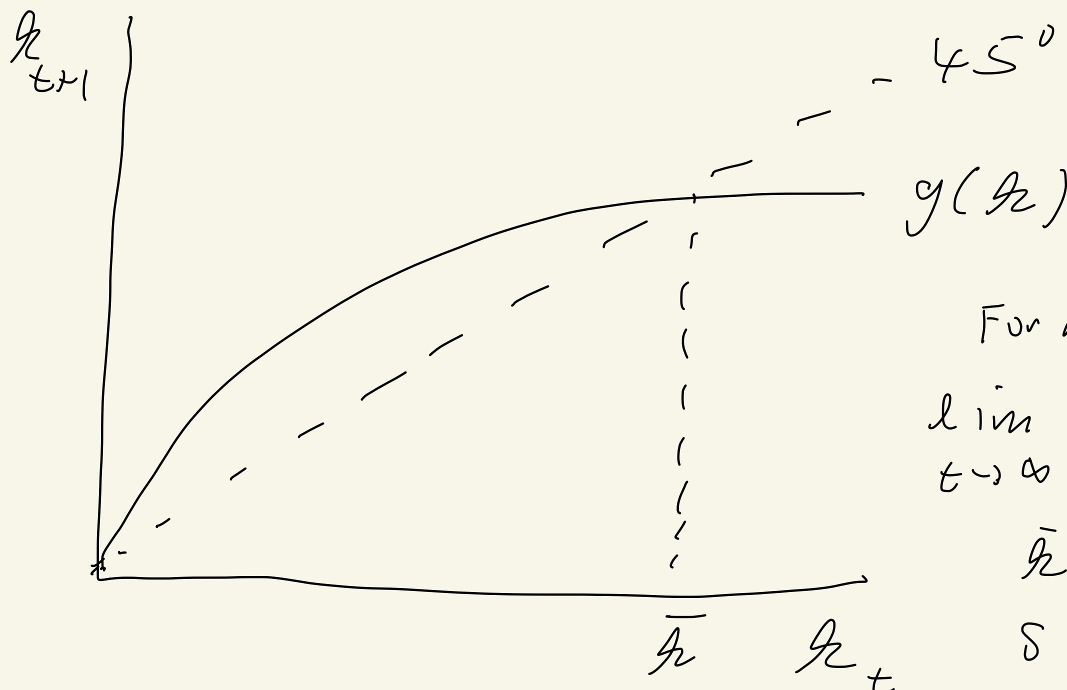


$\bar{k}$ exists, so must cross 45^0 line.

$$\Rightarrow \lim_{t \to \infty} k_t = 0 \quad \text{if } k_0 < \bar{k}$$

$\Rightarrow$ this contradicts optimization

Hence, $g(k) > k$ for arbitrarily small k:



For any $k_0 > 0$, $\lim_{t \to \infty} k_t = \bar{k}$.

$\bar{k}$ is a stable steady state

Key takeaway - $g(k)$ must cross 45^0 line from above!

Adding Labor-Leisure Tradeoff

- key to real business cycle literature.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t)$$

subject to $c_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t$

k_0 given

Can make assumptions about u to guarantee an interior solution where $0 < h < 1$.

writing as dynamic programming problem:

$$V(k) = \max_{k', h} \{ u(c, 1-h) + \beta V(k') \}$$

$$\text{s.t.} \quad c + k' = F(k, h) + (1-\delta) k$$

Why is h not included as a state variable?

Because h_0 not needed to solve sequence problem!

h_0 determined in period 0 and has no impact on problem solved in period 1

In fact, first order condition for h: static

$$u_2(c, 1-h) = u_1(c, 1-h) F_2(k, h)$$

can be solved for

$$h = \mathcal{H}(k, k')$$

$\Rightarrow$ can write Bellman equation

$$V(k) = \max_{k'} \{ u(c, 1-\mathcal{H}(k,k')) + \beta V(k') \}$$

$$\text{s.t.} \quad c + k' = F(k, \mathcal{H}(k,k')) + (1-\delta)k$$

$\longrightarrow$ Fits into structure of Stokey & Lucas!