Time integration

Step by step based methods are general approaches to obtain the system's response to dynamic loading. In these formulations, both the loading and the response history are divided into a sequence of time steps. Each step constitutes an independent analysis where the dynamic problem is solved based in the solution of the previous step. In the finite element method the time dependent system of equations is written like:

$$MA(t) + CV(t) + KU(t) = P(t)$$

where M, C and K are standard mass, damping and stiffness finite element-like matrices, while $A(t)$, $V(t)$ and $U(t)$ are generalized acceleration, velocity and displacement nodal vectors. These are termed generalized as they are not necessarily mechanical quantities. Based on this formulation, it's posible to consider the nonlinear behavior of the system simply by assumming that the assembled properties remain constant during each step and that the change of those properties only happens from one step to the next. Hence, nonlinear analysis becomes a sequence of linear analysis of a changing system (Clough & Penzien, 2003). Consequently, it is convenient to reformulate the system response in terms of the incremental equation of motion, due to the assumption that in nonlinear analysis the properties of the system remains constant only in short increments of time or deformation.

$$M\Delta A + C\Delta V + K\Delta U = \Delta P$$

In this development, the nonlinear behavior is considered in changes in the stiffness contribution. Time integration is conducted through a "θ Wilson" method.

Algorithm 1: The θ Wilson's method

Initial calculations:

1. Solve $\mathbf{M}\ddot{u}_0 = \mathbf{P}_0 - \mathbf{C}\dot{u}_0 \to \ddot{u}_0$
2. Select θ and Δt
3. $a = \frac{6}{\theta\Delta t}\mathbf{M} + 3\mathbf{C}$
4. $b = 3\mathbf{M} + \frac{\theta\Delta t}{2}\mathbf{C}$

Calculations for each time step, i

for $i \leftarrow 0$ **to** N_{times} **do**

$\delta\widehat{\mathbf{P}}_i = a\dot{u}_i + b\ddot{u}_i + \theta\Delta\mathbf{P}_i$;

Determine tangent stiffness matrix $\mathbf{K}_{T\,i}$;

$\widehat{\mathbf{K}}_{T_i} = \mathbf{K}_{T_i} + \frac{3}{\theta\Delta t}\mathbf{C} + \frac{6}{(\theta\Delta t)^2}\mathbf{M}$;

Solve δu_i from $\widehat{\mathbf{K}}_{T_i}$ and $\delta\widehat{\mathbf{P}}_i$ using **Algorithm 2**;

$\delta\ddot{u}_i = \frac{6}{(\theta\Delta t)^2}\delta u_i - \frac{6}{\theta\Delta t}\dot{u}_i - 3\ddot{u}_i$;

$\Delta\ddot{u}_i = \frac{1}{\theta}\delta\ddot{u}_i$;

$\Delta\dot{u}_i = (\Delta t)\ddot{u}_i + \frac{\Delta t}{2}\Delta\ddot{u}_i$;

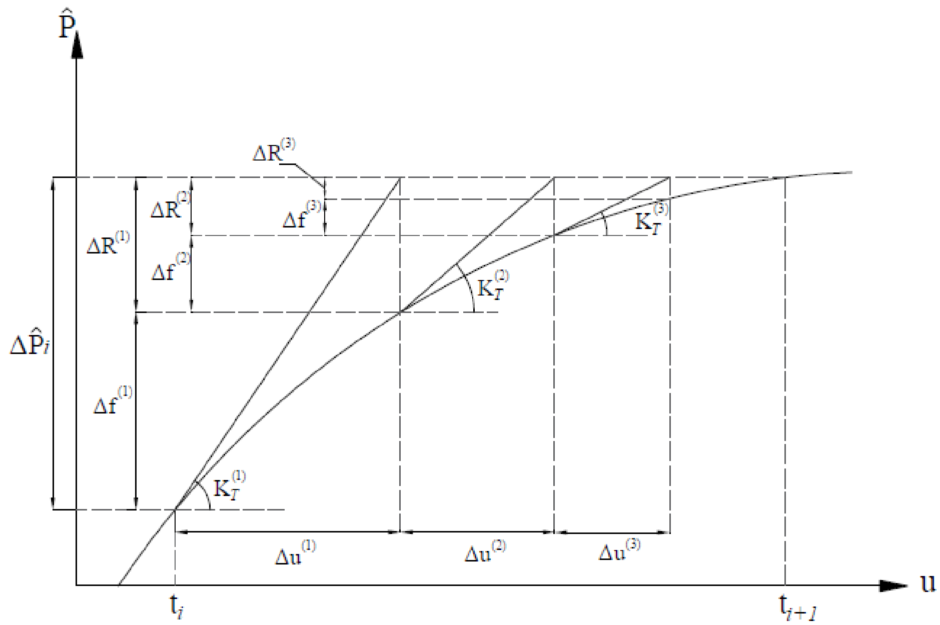
$\Delta u_i = (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \frac{(\Delta t)^2}{6}\Delta\ddot{u}_i$;

$u_{i+1} = u_i + \Delta u$; $\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}$ and $\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}$;

end

Repetition for the next time step (Repeat steps 2.1 to 2.11).

The system's nonlinear response is obtained considering a generalized Newton-Raphson iteration scheme, in which the effective tangent stiffness matrix K_T is calculated at the time i (the beginning of the time step), and that is used through each iteration of changing deformation Δu within that time step. The tangent stiffness matrix K_{T_i} is updated for each iteration until Δu_i it becomes small enough.



Algorithm 2: The Newton-Raphson iterative scheme

Result: $\Delta u_i, \widehat{\mathbf{K}}_{T_i}$

Initialize data:

1. Solve Δu_i from $\widehat{\mathbf{K}}_{T_i} = \widehat{\mathbf{K}}_T^{(0)}$ and $\delta \widehat{\mathbf{P}}_i$
2. Compute the system's internal forces, $\Delta \mathbf{f}^{(0)}$
3. $\Delta R^{(0)} = \delta \widehat{\mathbf{P}}_i - \Delta \mathbf{f}^{(0)}$

if $|\Delta R^{(0)}| > Tol$ **then**
  while $|\Delta R^{(j)}| > Tol$ **do**
    Solve $\Delta u^{(j)}$ from $\widehat{\mathbf{K}}_T^{(j)}$ and $\Delta R^{(j)}$;
    $\Delta u_i = \Delta u_i + \Delta u^{(j)}$;
    Update $\widehat{\mathbf{K}}_T^{(j)}$ from Δu_i;
    Compute $\Delta \mathbf{f}^{(j)}$;
    $\Delta R^{(j)} = \delta \widehat{\mathbf{P}}_i - \Delta \mathbf{f}^{(j)}$;
  end
end

In solving the nonlinear time step, two convergency criteria are considered. The iteration step is deemed to be completed as soon as both the residual forces and the residual deformations (Δf_i and Δu_i) are smaller than the tolerance value stablished by the user.

In the following, it will be exposed a comparison between analytical solution of the linear response of a single degree of freedom system due to an armonic loading excitation and the response that is obtained with "The θ Wilson" step by step procedure: