Discretization of an RC circuit state space*

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Abstract. The discretization of a RC circuit is presented. The literature does not address clear that problem. Therefore a proper article is required.

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1 Problem Description

To be efficiently computable by the BMS, we will consider a discrete-time version of the cell dynamics. Each measurement interval, indexed by integer valued time index k (e.g., perhaps once per second) the model updates its state and output values based on its input. A very general framework that we may use is a state-space model of discrete-time lumped linear dynamic systems.

A simple example of linear Kalman filtering, We consider the system defined by the linear circuit in Fig. 1. We find the continuous-time state-space model of the model is :

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{x}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1}$$

Using Kirchoff's law to determine the dynamic of the circuit, we have (See reference [3]):

$$\begin{aligned} \dot{\boldsymbol{v}}_c &= -\frac{1}{R_2 C} v_c(t) - \frac{1}{C} i(t) \\ \boldsymbol{v}_t(t) &= v_c(t) - R_1 i(t) + \nu(t) \end{aligned} \tag{2}$$

where v_c(t) is the capacitor voltage as a function of time, i(t) the current exciting the circuit, and v_t(t) the terminal voltage, as indicated in the figure 1. This circuit is a crude linear model of a battery cell if both C and R_2 are large and R_1 is small. R_2 is the resistor governing self-discharge and R_1 is the internal resistance of the cell.

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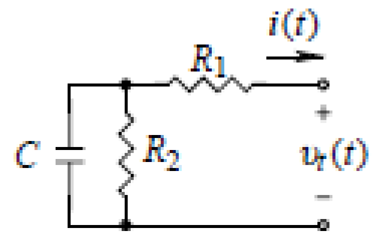

Fig. 1. Simple linear circuit.

Kalman filtering requires to discretize the continuous time system. There are good references in the literature from where the discretization of a time continues linear system can be follow. In this work, reference [1] will be followed. Needless to say, reference [2] is faster but less clear.

Following the steps in reference [1] , equation 1 will be solved using traditional solution of a linear equation, but observing that the variables are not real values but matrices.

$$\frac{d}{dt}\mathrm{e}^{\mathbf{A}t} = \mathbf{A}\mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{\mathbf{A}t}\mathbf{A} \tag{3}$$

Multiplying both sides of equation (1) with $\mathrm{e}^{-\mathbf{A}t}$

$$\mathrm{e}^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - \mathrm{e}^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = \mathrm{e}^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{4}$$

which implies,

$$\frac{d}{dt}\big(\mathrm{e}^{-\mathbf{A}t}\mathbf{x}(t)\big) = \mathrm{e}^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{5}$$

Its integration from 0 to t yields,

$$\mathrm{e}^{-\mathbf{A}t}\mathbf{x}(t)|_0^t = \int_0^t \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{6}$$

Thus, we have

$$\mathrm{e}^{-\mathbf{A}t}\mathbf{x}(t) - \mathrm{e}^{0}\mathbf{x}(0) = \int_0^t \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{7}$$

But the inverse of $\mathrm{e}^{-\mathbf{A}t}$ is $\mathrm{e}^{\mathbf{A}t}$ and e^{0} is $\mathbf{I}$, then we have the final solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{8}$$

A useful equation to prove that the (8) is indeed the solution is:

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)d\tau = \int_{t_0}^{t}\Big(\frac{\partial}{\partial t}f(t,\tau)\Big)d\tau + f(t,\tau)|_{t=\tau} \tag{9}$$

Theorem 1. *Consider the system of equations given by (1), where* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ *are* $n \times n, n \times p, q \times n$ *and* $q \times p$ *respectively constant matrices. The solution of this system is given by the application of equation (8).*

Proof. See from equation (3) to equation (8).

We need to transform the system of equations (2) into a state-form. This yield the following:

$$\begin{aligned} A &= -\frac{1}{R_2 C} \\ B &= -\frac{1}{C} \end{aligned} \tag{10}$$

If we apply Theorem 1 to the system of equations presented in (2)

$$\begin{aligned} \mathbf{v}(t) &= \mathrm{e}^{(-\frac{1}{R_2 C}t)}v(0) + \int_0^t \mathrm{e}^{-(\frac{1}{R_2 C}\tau)}\frac{-R_2}{R_2 C}i(\tau)d\tau \\ &= \mathrm{e}^{(-\frac{1}{R_2 C}t)}v(0) - R_2 i\int_0^t \mathrm{e}^{-(\frac{1}{R_2 C}(t-\tau))}\frac{1}{R_2 C}(\tau)d\tau \\ &= \mathrm{e}^{(-\frac{1}{R_2 C}t)}v(0) - R_2 i \mathrm{e}^{-(\frac{1}{R_2 C}(t-\tau))}|_0^t \\ &= \mathrm{e}^{-t/R_2 C}v(0) - R_2 i(1-\mathrm{e}^{-t/R_2 C}) \end{aligned} \tag{11}$$

If now, use equation (11) and discrete it, we have:

$$\begin{aligned} v_{c,k} &= \mathrm{e}^{-T_s/R_2 C}v_{c,k} - R_2 i_k(1-\mathrm{e}^{-T_s/R_2 C}) + \omega_k \\ v_{t,k} &= v_{c,k} - R_1 i_k + \nu_k \end{aligned} \tag{12}$$

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