Proof of Theorem 1

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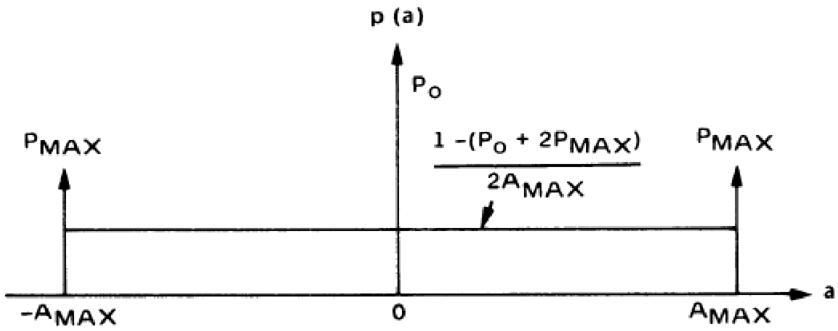
September 23, 2018

Theorem 1.0. Given the following probability density function:



Show that the resulting variance σ_m^2 is given by the formula:

$$\sigma^2 = \frac{A_{max}^2}{3}\Big[1 + 4P_{max} - P_0\Big]$$

Proof. Observing the graph, the total probability density function is given by 3 components:

$$P_{max}\delta(x + A_{max}),$$
$$P_{max}\delta(x - A_{max}),$$
$$(1 - (P_0 + 2P_{max}))\Big[\frac{1}{2A_{max}}\Big]$$

If we apply the variance formula, $\sigma^2 = E[x^2] - (E[x])^2$ and noticing the E[x] for the three distributions is zero, we only need to find the $E[x^2]$ by applyign the formula of second moment, $\int_\infty^\infty x^2 pdf(x)dx$ to each individual pdfs. We have:

$$\sigma_1^2 = \int_\infty^\infty x^2 P_{max}\delta(x + A_{max})dx = P_{max}A_{max}^2$$
$$\sigma_2^2 = \int_\infty^\infty x^2 P_{max}\delta(x - A_{max})dx = P_{max}A_{max}^2$$

$$
\begin{aligned}
\sigma_3^2 &= \int_{\infty}^{\infty} x^2(1-(P_0+2P_{max}))\left[\frac{1}{2A_{max}}\right]dx \\
&= (1-P_0-2P_{max})\int_{-A_{max}}^{A_{max}} x^2\frac{1}{2A_{max}}\bigg]dx \\
&= (1-P_0-2P_{max})\left[\frac{2A_{max}^3}{3}\frac{1}{2A_{max}}\right] \\
&= (1-P_0-2P_{max})\left[\frac{A_{max}^2}{3}\right]
\end{aligned}
$$

The final σ_m^2 is the sum of all 3 σ :

$$
\begin{aligned}
\sigma_m^2 &= (3/3)\sigma_1^2 + (3/3)\sigma_2^2 + \sigma_3^2 \\
&= 6P_{max}\frac{A_{max}^2}{3} + (1-P_0-2P_{max})\left[\frac{A_{max}^2}{3}\right] \\
&= (1-P_0-2P_{max}+6P_{max})\left[\frac{A_{max}^2}{3}\right] \\
&= (1-P_0+4P_{max})\left[\frac{A_{max}^2}{3}\right]
\end{aligned}
$$

The target can accelerate at a maximum rate A_{max} or $-A_{max}$ and will do each with a probability P_{max}. The target undergoes no acceleration with a probability P_0, and will accelerate between the limits $-A_{max}$ and A_{max} according to the uniform distribution shown in the figure. Then, the variance, σ_m^2, of the resulting acceleration probability density model is given by the theorem 1. □

From Estimating Optimal Tracking Filter Performance for Manned Maneuvering Targets, ROBERT A. SINGER, Member, IEEE.