normal distribution from a uniform distribution

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1 Normal distribution from a uniform

We know that:

$$\sigma^2(X) = E[X^2] - (E[X])^2$$

where:

$E[X] = \int_{-\infty}^{\infty} Xp(X)dX$ and $E[X^2] = \int_{-\infty}^{\infty} X^2p(X)dX$ with p(x) is the density probability function (pdf). In the case of a uniform distribution: $p(x) = \frac{1}{(b-a)}$.

The expectation of X (average) is:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} X\frac{1}{(b-a)}dX \\
&= \int_{b}^{a} \frac{1}{(b-a)}XdX \\
&= \frac{1}{(b-a)}\frac{1}{2}X^2\Big|_b^a \\
&= \frac{1}{(b-a)}\frac{1}{2}(b^2-a^2) \\
&= \frac{1}{2}(a+b)
\end{aligned}$$

if $a = 0.5$ and $b = -0.5$, then, $E[X] = 0$

The next step is to calculate the second moment of the pdf: $\int_{-\infty}^{\infty} x^2p(x)dx$.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} X^2 \frac{1}{(b-a)} dX \\
&= \int_{b}^{a} \frac{1}{(b-a)} X^2 dX \\
&= \frac{1}{(b-a)} \frac{1}{3} X^3 \Big|_{b}^{a} \\
&= \frac{1}{(b-a)} \frac{1}{3} (a^3 - b^3) \\
&= \frac{1}{3} \frac{1}{(b-a)} (b-a)((a^2 + ab + b^2) \\
&= \frac{1}{3}(a^2 + ab + b^2)
\end{aligned}
$$

With this eqautions, we can calcualte the σ^2

$$
\begin{aligned}
\sigma^2(X) &= E[X^2] - [E(X)]^2 \\
&= \frac{1}{3}(a^2 + ab + b^2) - \left[\frac{1}{2}(a+b)\right]^2 \\
&= \frac{4}{12}(a^2 + ab + b^2) - \frac{3}{12}\left[(a+b)\right]^2 \\
&= \frac{1}{12}(4a^2 + 4ab + 4b^2) - \frac{1}{12}(3a^2 + 6ab + 3b^2) \\
&= \frac{1}{12}(4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2) \\
&= \frac{1}{12}(a^2 - 2ab + b^2) \\
&= \frac{1}{12}(a-b)^2
\end{aligned}
$$

if a = -0.5 and b = 0.5, then b-a = 1.0, then

$$
\sigma(X)^2 = E[X^2] - [E(X)]^2 = 1/12
$$

We can add 12 times $\sigma^2(X)$ and then extract the square root of the sum to obtain $\sigma(X)$ or we can add 6 times $\sigma^2(X)$, then extract the square root but then, we need to multiply the result by $\sqrt{2}$ to remove the extra 2 in the solution. Proceeding as the latter:

$$
6 * \sigma(X)^2 = E[X^2] - [E(X)]^2 = 1/2
$$

$$
\begin{aligned}
\sigma(X) &= \sqrt{(6 * \sigma(X)^2)} * \sqrt{2} \\
&= \frac{1}{\sqrt{2}}\sqrt{2} \\
&= 1
\end{aligned}
$$

We have demonstrated that we can obtain a normal distribution, $E[X] = 0$ and $\sigma(X) = 1$ by adding 6 times the variance of the uniform distribution and then multiply the rsult by $\sqrt{2}$