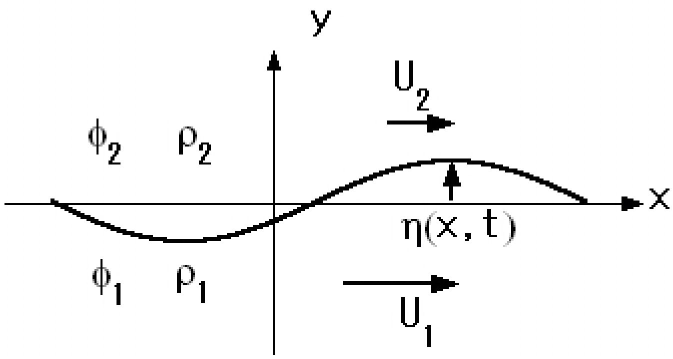
# Kelvin Helmholtz Instability




$$\mathbf{n}\cdot\mathbf{u_{int}} = \mathbf{n}\cdot\mathbf{u_1} = \mathbf{n}\cdot\mathbf{u_2}$$

$$\mathbf{n} = \frac{1}{\sqrt{1+\left(\frac{\partial\eta}{\partial t}\right)}}\left(-\frac{\partial\eta}{\partial t},1\right)$$

Bernoulli Equation:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$q_1 = \nabla\Phi_1 \quad \nabla^2\Phi = 0; \qquad q_2 = \nabla\Phi_2 \quad \nabla^2\Phi_2 = 0$$

$$\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla\Phi_1)^2}{2} + \frac{p_1}{\rho_1} + C_1 \qquad \frac{\partial\Phi_2}{\partial t} + \frac{(\nabla\Phi_2)^2}{2} + \frac{p_1}{\rho_1} + C_2$$

Kinematic Equations at boundary:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi_1}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi_1}{\partial t}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi_2}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi_2}{\partial t}$$

If $\Phi_i = U_i x + \phi_i$ where $\phi_i \ll \Phi_i$, then the kinematic and dynamic condition at the boundary $y = 0$ become:

$$\frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x} = \frac{\partial \phi_i}{\partial y}$$

$$\rho_1 \left( \frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) = \rho_2 \left( \frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right)$$

Now, we can assume that $\eta$ goes as a sinusoidal along the boundary

$$\eta = Ae^{i(kx - \omega t)}$$

$$\phi_i = \bar{\phi}_i e^{\pm ky} e^{i(kx - \omega t)}$$

Substituting the sinusoidal into the kinematic and dynamic condition at the boundary, we get

$$(-i\omega + ikU_i)A = \pm k\bar{\phi}_i$$

$$\rho_1(-i\omega + ikU_1)\phi_1 = \rho_2(-i\omega + ikU_2)\phi_2$$

Writing this system of equations in matrix form, we can find solve the eigenvalue problem to find the dispersion relation:

$$\omega = k\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm ik\frac{\sqrt{\rho_1 \rho_2}\, |U_1 - U_2|}{\rho_1 + \rho_2}$$

**Note: 1 denotes upper boundary and 2 denotes lower boundary**