

Lie Groups in Robotics

James Goppert

January 22, 2023

Contents

1	Introduction	5
1.1	What is a Lie Group	5
1.2	Applications of Lie Groups	5
2	The $SO(2)$ Lie Group	7
2.1	Group Representation	7
2.2	Lie Algebra	8

Chapter 1

Introduction

This book is aimed at introducing graduate and undergraduate students in engineering to the applications of Lie groups theory that are relevant to engineering. Instead of focusing on the abstract mathematics initially, it will build intuition by traversing the most fundamental Lie Groups in robotics.

1.1 What is a Lie Group

Definition 1 A Group is a set G and an associated operator \cdot that is:

- *Closed*: if $a \in G$ and $b \in G$, then $a \cdot b \in G$
- *Associative*: $(a \cdot b) \cdot c = a(\cdot b \cdot c)$
- *Inverse*: $a^{-1} \cdot a = e$
- *Neutral*: $a \cdot e = a$

Definition 2 A Lie Group is a group that is also a differentiable manifold.

A differentiable manifold is a topological space resembling Euclidean space near each point and locally similar enough to a vector space to apply calculus. Originally Lie Groups were called infinitesimal groups by the creator Sophus Lie (pronounced Lee). They can be thought of as groups of continuous transformations.



Figure 1.1: Sophus Lie

1.2 Applications of Lie Groups

Covered in this Course

- Estimation

- IEKF : Invariant Extended Kalman Filter
 - Simultaneous Localization and Mapping
- Control of Rigid Bodies
 - Reachable set calculations
 - Geometric control
- Computer Vision
 - Perspective Transforms
 - Homogenous Coordinates

Others Topics not Covered

- Quantum mechanics

Excercises

Questions about Groups

- Is the set of all Integers \mathbb{Z} with the addition operator $+$ a group?
- Is the set of all Integers \mathbb{Z} with the multiplication operator $*$ a group?
- Is the set of all $n \times n$ matrices with the matrix multiplication operator a group?

Questions about Lie Groups

- Is the set of all Integers \mathbb{Z} with the addition operator $+$ a Lie group?
- Is the set of all Real numbers \mathbb{R} with the addition operator $+$ a Lie group?

Chapter 2

The $SO(2)$ Lie Group

2.1 Group Representation

$SO(2)$ can be represented by any matrix of the form:

$$G(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

with the Group operator of matrix multiplication (\cdot) , where $\theta \in \mathbb{R}$.

To show that $SO(2)$ is a Lie Group, we must show that it is closed, associative, has inverse, and a neutral element and is a differentiable manifold

Closed

Since $G(\theta_1) \cdot G(\theta_2) = G(\theta_1 + \theta_2)$, $SO(2)$ is **closed** under matrix multiplication.

Associative

$SO(2)$ as a Matrix Lie Group, can inherit associativity from matrix multiplication:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Inverse

$SO(2)$ as a Matrix Lie Group, can inherit the inverse from matrix multiplication, since any element of $SO(2)$ has a non-zero determinant (1) and is invertible.

$$\det G = \cos^2 \theta + \sin^2 \theta = 1$$

Because the columns of $G(\theta)$ are orthonormal, the inverse is given by the matrix transpose.

$$G^{-1}(\theta) = G^T(\theta)$$

Neutral

$SO(2)$ as a Matrix Lie Group, can inherit the neutral element from matrix multiplication, I .

$$A \cdot I = A$$

Differential Manifold

It is clear that the group $SO(2)$ is continuous as it inherits this from \mathbb{R} . $G(\theta)$, $\theta \in \mathbb{R}$. We will see in the Lie Algebra that the group is locally similar to 1 dimensional Euclidean space and we can perform calculus.

2.2 Lie Algebra

For matrix Lie groups, we can always find an element of the Lie Algebra, Ω via:

$$\dot{G} = G\Omega$$

$$\Omega = G^{-1}\dot{G}$$

The $so(2)$ Lie algebra can be represented by all 2x2 skew symmetric matrices of the form:

$$\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

It is convenient to define a wedge operator such that:

Definition 3 (The Wedge Operator)

$$\begin{aligned} \mathbb{R} &\mapsto so(2) \\ \omega^\wedge &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \end{aligned}$$

It is also convenient to define a vee operator, the inverse of the wedge operator such that:

Definition 4 (The Vee Operator)

$$\begin{aligned} so(2) &\mapsto \mathbb{R} \\ \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}^\vee &= \omega \end{aligned}$$

Definition 5 (The Lie Group Exponential Map) *The Lie group Exponential map maps from the Lie algebra to the Lie group. For matrix Lie groups, such as $SO(2)$ it is given by the matrix exponential.*

Definition 6 (The Lie Group Logarithm Map) *The Lie group logarithm map maps from the Lie group to the Lie algebra. For matrix Lie groups, such as $SO(2)$ it is given by the matrix logarithm, the inverse of the matrix exponential.*

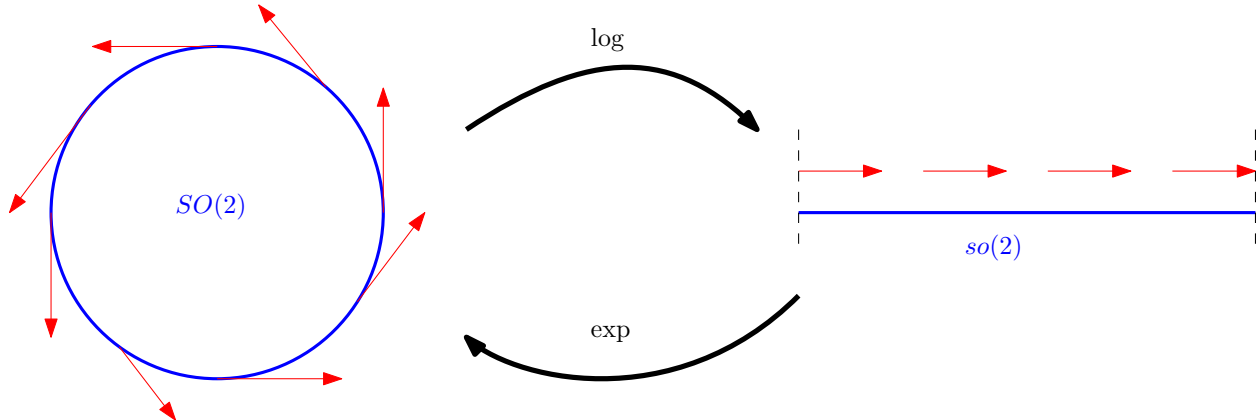


Figure 2.1: Transformation of a Vector Field from the Lie Group to the Lie Algebra

Definition 7 (Bijective/Invertible Map) *A map is bijective or invertible if it is:*

- *One-one/ injective: For each point in the domain there is one point in the range.*
- *Onto/ surjective: Each point in the range is mapped to by a point in the domain.*

It is often beneficial for us to map vector fields to the Lie algebra to simplify analysis. If we do this, we desire an invertible map, so after we take the logarithm, we can then take the exponential of the Lie algebra to obtain results in the Lie group. We will see later how this can be used for reachable set computation. It is important to note that the matrix exponential is not one to one. A single point in the range, can be reached by multiple points in the domain, since for $so(2)$, the Lie algebra parameter θ and $\theta + 2\pi k$, $k \in \mathbb{Z}$, map to the same point. However, we can restrict the domain of the map to $\{-\pi, \pi\}$ and the map is invertible in this domain.