Lie Groups in Robotics

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Chapter 1

Introduction

This book is aimed at introducing graduate and undergraduate students in engineering to the applications of Lie groups theory that are relevant to engineering. Instead of focusing on the abstract mathematics initially, it will build intuition by traversing the most fundamental Lie Groups in robotics.

1.1 What is a Lie Group

Definition 1 *A Group is a set G and an associated operator $\cdot$ that is:*

- ***C**losed: if $a \in G$ and $b \in G$, then $a \cdot b \in G$*
- ***A**ssociative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*
- ***I**nverse: $a^{-1} \cdot a = e$*
- ***N**eutral: $a \cdot e = a$*

Definition 2 *A Lie Group is a group that is also a differentiable manifold.*

A differentiable manifold is a topological space resemlbling Euclidean space near each point and locally similar enough to a vector space to apply calculus. Originally Lie Groups were called infinitesmal groups by the creator Sophus Lie (pronounced Lee). The can be thought of as groups of continuous transformations.

Figure 1.1: Sophus Lie

1.2 Applications of Lie Groups

Covered in this Course

- Estimation

  - IEKF : Invariant Extended Kalman Filter
  - Simultaneous Localization and Mapping
- Control of Rigid Bodies
  - Reachable set calculations
  - Geometric control
- Computer Vision
  - Perspective Transforms
  - Homogenous Coordinates

Others Topics not Covered

- Quantum mechanics

Excercises

Questions about Groups

- Is the set of all Integers $\mathbb{Z}$ with the addition operator $+$ a group?
- Is the set of all Integers $\mathbb{Z}$ with the multiplication operator $*$ a group?
- Is the set of all $n \times n$ matrices with the matrix multiplication oerator a group?

Questions about Lie Groups

- Is the set of all Integers $\mathbb{Z}$ with the addition operator $+$ a Lie group?
- Is the set of all Real numbers $\mathbb{R}$ with the addition operator $+$ a Lie group?

Chapter 2

The SO(2) Lie Group

2.1 Group Representation

$SO(2)$ can be represented by any matrix of the from:

$$G(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with the Group operator of matrix multiplication $(\cdot)$, where $\theta \in \mathbb{R}$.

To show that $SO(2)$ is a Lie Group, we much show that it is closed, associative, has inverse, and a neutral element and is a differentiable manifold

Closed

Since $G(\theta_1) \cdot G(\theta_2) = G(\theta_1 + \theta_2)$, $SO(2)$ is **closed** under matrix multiplication.

Associative

$SO(2)$ as a Matrix Lie Group, can inheret associativity from matrix multiplication:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Inverse

$SO(2)$ as a Matrix Lie Group, can inheret the inverse from matrix multiplication, since any element of $SO(2)$ has a non-zero determinant (1) and is invertible.

$$\det G = \cos^2\theta + \sin^2\theta = 1$$

Because the columns of $G(\theta)$ are orthonomal, the inverse is given by the matrix transpose.

$$G^{-1}(\theta) = G^T(\theta)$$

Neutal

$SO(2)$ as a Matrix Lie Group, can inheret the neutral element from matrix multiplication, I.

$$A \cdot I = A$$

Differential Manifold

Is is clear that the group $SO(2)$ is continuous as it inherits this from $\mathbb{R}$. $G(\theta)$, $\theta \in \mathbb{R}$. We will see in the Lie Algebra that the group is locally similar to 1 dimensional Euclidean space and we can perform calculus.

2.2 Lie Algebra

An algebra is a set with associated operators for addition/subtraction, multiplication/division. $\mathcal{R}^3$, the space of 3-dimensions vectors forms an algebra. We are familiar with the cross product and vector addition. Vector addition acts as the addition/subtraction operator and the cross product acts as the multiplication/division operator.

The associated Lie algebra of a Lie group is a linear vector space and already is closed under the addition and subtract operator. The Lie bracket definition 13, which we will study soon, acts as the multiplication/division operator. For matrix Lie groups, we can always find an element of the Lie Algebra, Ω via:

$$\dot{G} = G\Omega$$

$$\Omega = G^{-1}\dot{G}$$

The so2 Lie algebra can be represented by all 2x2 skew symmetric matrices of the form:

$$\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

It is convenient to define a wedge operator such that:

Definition 3 (The Wedge Operator)

$$\mathbb{R} \mapsto so(2)$$

$$\omega^\wedge = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

It is also convient to define a vee operator, the inverse of the wedge operator such that:

Definition 4 (The Vee Operator)

$$so(2) \mapsto \mathbb{R}$$

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}^\vee = \omega$$

2.3 The Exponential Map

Definition 5 (The Lie Group Exponential Map) *The Lie group Exponential map maps from the Lie algebra to the Lie group. For matrix Lie groups, such as $SO(2)$ it is given by the matrix exponential.*

Definition 6 (The Lie Group Logarithm Map) *The Lie group logarithm map maps from the Lie group to the Lie algebra. For matrix Lie groups, such as $SO(2)$ it is given by the matrix logarithm, the inverse of the matrix exponential.*

Definition 7 (Bijective/Invertible Map) *A map is bijective or invertible if it is:*

- *One-one/ injective: For each point in the domain there is one point in the range.*
- *Onto/ surjective: Each point in the range is mapped to by a point in the domain.*

It is often beneficial for us to map vector fields to the Lie algebra to simplify analysis. If we do this, we desire an invertible map, so after we take the logarithm, we can then take the exponential of the Lie algebra to obtain results in the Lie group. We will see later how this can be used for reachable set computation. It is important to note that the matrix exponential is not one to one. A single point in the range, can be reached by multiple points in the domain, since for $so(2)$, the Lie algebra parameter θ and $\theta + 2\pi k$, $k \in \mathbb{Z}$, map to the same point. However, we can restrict the comain of the map to $\{-\pi, \pi\}$ and the map is invertible in this domain.

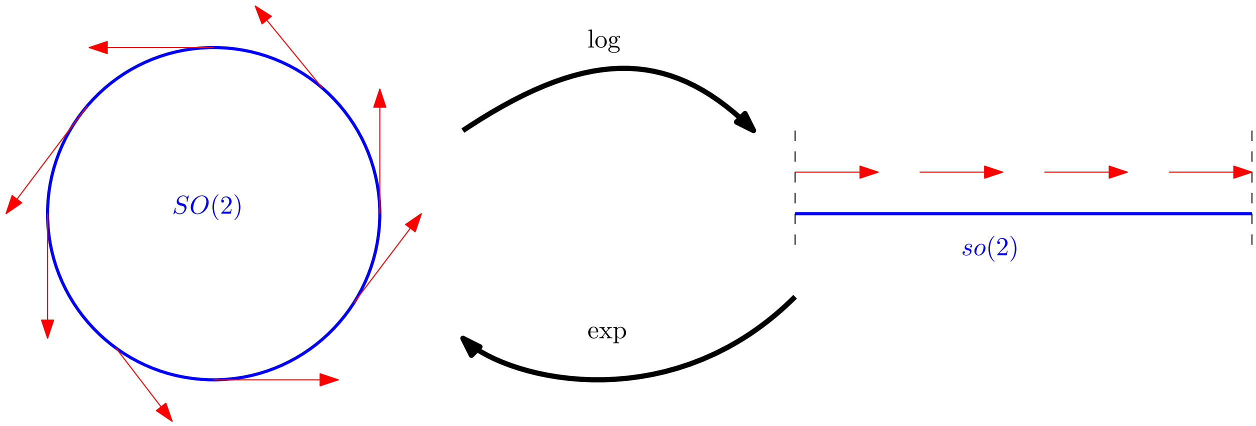

Figure 2.1: Transformation of a Vector Field from the Lie Group to the Lie Algebra

2.4 Vector Fields

On a Lie group, a vector field can be represented by an element of the Lie algebra. This element can then be pushed forward to other group elements, and defining the derivative of the Lie group element at that point.

There are two types of vector fields that may be defined.

Definition 8 (Left Invariant Vector Fields) *A vector field is left invariant, if multiplying on the left by an element of the group does not modify the vector field. Define a group element G_t as the product of a constant group element A, and a time varying group element B_t, $G_t = AB_t$. Here we use the notation that A_t indicates that A is a function of time. Then:*

$$\dot{B}_t = B_t[\omega]^\wedge \implies \dot{G}_t = G_t[\omega]^\wedge$$

$$\dot{G}_t = \frac{d}{dt}(AB_t) = AB_t[\omega]^\wedge = G_t[\omega]^\wedge$$

Definition 9 (Right Invariant Vector Fields) *A vector field is right invariant, if multiplying on the right by an element of the group does not modify the vector field. Define a group element G_t as the product of a time varying group element A_t, and a constant group element B, $G_t = A_tB$.*

$$\dot{A}_t = [\omega]^\wedge A_t \implies \dot{G}_t = [\omega]^\wedge G_t$$

$$\dot{G}_t = \frac{d}{dt}(A_tB) = [\omega]^\wedge A_tB = [\omega]^\wedge G_t$$

Definition 10 (Combing a Vector Field) *A vector field may be combed by a Lie algebra element, if one Lie algebra element can define a vector field on the group without a singularity.*

2.5 The Lie bracket and Adjoint operators, Ad and ad

The Adjoint map maps element from the Lie algebra from one tangent space to another.

$$\dot{G}_t = G_t[\omega_l]^\wedge = [\omega_r]^\wedge G_t$$

$$\dot{G}_tG_t^{-1} = G_t[\omega_l]^\wedge G_t^{-1} = [\omega_r]^\wedge$$

$$[\omega_r]^\wedge = G_t[\omega_l]^\wedge G_t^{-1}$$

Definition 11 (*Ad* operator) *The Adjoint operator maps a Lie algebra element $[\omega]^\wedge$ at the origin(e), to the tangent at G_t is given by:*

$$g \mapsto g$$

where g is an element of the Lie algebra

$$Ad_{G_t}[\omega]^\wedge \equiv G_t[\omega]^\wedge G_t^{-1}$$

Theorem 1 (Linearity of Ad_G operator) *Ad_G is a linear operator on the Lie algebra for any Lie Group (it can be represented as a matrix) when the components of ω are expressed as a vector in $\mathcal{R}^n$, where n is the dimension of the Lie algebra.*

$$\mathcal{R}^n \mapsto \mathcal{R}^n$$

where n is the dimension of the Lie alegbra.

$$Ad_G\omega_1 = \omega_2$$

The use of the linear operator (matrix) form of Ad, as defined in theorem 1 will be implied when multiplying a vector (e.g. $(\omega_2 = Ad_{G_t}\omega_1)$. The use of the operator as defined in definition 11 in will be implied when multiplying by an element of the Lie algebra, (e.g. $Ad_{G_t}([\omega]^\wedge))$

The *ad* operator is derived by taking the derivative of the *Ad* operator at the origin.

Definition 12 (ad_xy operator)

$$ad_{[\omega_1]^\wedge}[\omega_2]^\wedge \equiv [\omega_1]^\wedge[\omega_2]^\wedge - [\omega_2]^\wedge[\omega_1]^\wedge$$

Definition 13 (Lie bracket operator) *The Lie bracket is a binary operator, denoted by $[\cdot,\cdot]$, that acts as the multiplication operator for Lie algebras. It is equivalent to the ad operator.*

$$[[\omega_1]^\wedge,[\omega_2]^\wedge] \equiv ad_{[\omega_1]^\wedge}[\omega_2]^\wedge \equiv [\omega_1]^\wedge[\omega_2]^\wedge - [\omega_2]^\wedge[\omega_1]^\wedge$$

Exercises

1. Prove that $SO2$ is a group.
2. By hand, take the matrix exponential of the Lie algebra of $se(2)$ and show that it maps to the Lie group $SO2$.
3. Which vector fields of $so(2)$ can comb $SO(2)$? Is there a vector-field as a function of θ, defined in $so(2)$ that does not comb $SO(2)$.
4. Write a class in python that represents the group $SO(2)$ and the Lie algebra $so(2)$. It should implement the group product, neutral element, inverse, exponential map, and the logarithm map. Use this class to programmatically find $log(exp^\wedge(\theta_1)exp^\wedge(\theta_2))$. Evaluate this for $\theta_1 = 1$ and $\theta_2 = 2$. Here $G(\theta)$ indicates the group element with parameter θ.