Lie Groups in Robotics

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Chapter 1

Introduction

This book is aimed at introducing graduate and undergraduate students in engineering to the applications of Lie groups theory that are relevant to engineering. Instead of focusing on the abstract mathematics initially, it will build intuition by traversing the most fundamental Lie Groups in robotics.

1.1 What is a Lie Group

Definition 1 *A Group is a set G and an associated operator $\cdot$ that is:*

- ***C**losed: if $a \in G$ and $b \in G$, then $a \cdot b \in G$*
- ***A**ssociative: $(a \cdot b) \cdot c = a(\cdot b \cdot c)$*
- ***I**nverse: $a^1 \cdot a = e$*
- ***N**eutral: $a \cdot e = e$*

Definition 2 *A Lie Group is a group that is also a differentiable manifold.*

A differentiable manifold is a topological space resemlbling Euclidean space near each point and locally similar enough to a vector space to apply calculus. Originally Lie Groups were called infinitesmal groups by the creator Sophus Lie (pronounced Lee). The can be thought of as groups of continuous transformations.

Figure 1.1: Sophus Lie

1.2 Applications of Lie Groups

Covered in this Course

- Estimation

  - IEKF : Invariant Extended Kalman Filter
  - Simultaneous Localization and Mapping

- Control of Rigid Bodies
  - Reachable set calculations
  - Geometric control

- Computer Vision
  - Perspective Transforms
  - Homogenous Coordinates

Others Topics not Covered

- Quantum mechanics

Excercises

Questions about Groups

- Is the set of all Integers $\mathbb{Z}$ with the addition operator $+$ a group?
- Is the set of all Integers $\mathbb{Z}$ with the multiplication operator $*$ a group?
- Is the set of all $n \times n$ matrices with the matrix multiplication oerator a group?

Questions about Lie Groups

- Is the set of all Integers $\mathbb{Z}$ with the addition operator $+$ a Lie group?
- Is the set of all Real numbers $\mathbb{R}$ with the addition operator $+$ a Lie group?

Chapter 2

The $SO(2)$ Lie Group

2.1 Group Representation

$SO(2)$ can be represented by any matrix of the from:

$$G(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with the Group operator of matrix multiplication $(\cdot)$, where $\theta \in \mathbb{R}$.

To show that $SO(2)$ is a Lie Group, we much show that it is closed, associative, has inverse, and a neutral element and is a differentiable manifold

Closed

Since $G(\theta_1) \cdot G(\theta_2) = G(\theta_1 + \theta_2)$, $SO(2)$ is **closed** under matrix multiplication.

Associative

$SO(2)$ as a Matrix Lie Group, can inheret associativity from matrix multiplication:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Inverse

$SO(2)$ as a Matrix Lie Group, can inheret the inverse from matrix multiplication, since any element of $SO(2)$ has a non-zero determinant (1) and is invertible.

$$\det G = \cos^2\theta + \sin^2\theta = 1$$

Because the columns of $G(\theta)$ are orthonomal, the inverse is given by the matrix transpose.

$$G^{-1}(\theta) = G^T(\theta)$$

Neutal

$SO(2)$ as a Matrix Lie Group, can inheret the neutral element from matrix multiplication, I.

$$A \cdot I = A$$

Differential Manifold

Is is clear that the group $SO(2)$ is continuous as it inherits this from $\mathbb{R}$. $G(\theta)$, $\theta \in \mathbb{R}$. We will see in the Lie Algebra that the group is locally similar to 1 dimensional Euclidean space and we can perform calculus.

2.2 Lie Algebra

For matrix Lie groups, we can always find an element of the Lie Algebra, Ω via:

$$\dot{G} = G\Omega$$

$$\Omega = G^{-1}\dot{G}$$

The so2 Lie algebra can be represented by all 2x2 skew symmetric matrices of the form:

$$\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

It is convenient to define a wedge operator such that:

Definition 3 (The Wedge Operator)

$$\mathbb{R} \mapsto so(2)$$

$$\omega^{\wedge} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

It is also convient to define a vee operator, the inverse of the wedge operator such that:

Definition 4 (The Vee Operator)

$$so(2) \mapsto \mathbb{R}$$

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}^{\vee} = \omega$$