Find the second derivative of the following functions.

Example 1

$$y = \sqrt{\frac{x^2+1}{x^2+4}} = \left(\frac{x^2+1}{x^2+4}\right)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}\left(\frac{x^2+1}{x^2+4}\right)^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left[\frac{x^2+1}{x^2+4}\right]$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2+4}\right)^{-\frac{1}{2}} \cdot \left[\frac{(x^2+4)\cdot\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2+4)}{(x^2+4)^2}\right]$$

$$= \frac{1}{2}\left(\frac{x^2+4}{x^2+1}\right)^{\frac{1}{2}} \cdot \left[\frac{(x^2+4)\cdot 2x - (x^2+1)\cdot 2x}{(x^2+4)^2}\right]$$

$$= \frac{1}{2}\,\frac{(x^2+4)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}} \cdot \left[\frac{\cancel{2x^3} + 8x - \cancel{2x^3} - 2x}{(x^2+4)^2}\right]$$

$$= \frac{(x^2+4)^{\frac{1}{2}}}{2(x^2+1)^{1/2}} \cdot \frac{6x}{(x^2+4)^2}$$

$$= \frac{\overset{3}{\cancel{6}}x\,(x^2+4)^{\frac{1}{2}}}{\cancel{2}(x^2+1)^{1/2}(x^2+4)^2} = \frac{3x}{(x^2+1)^{1/2}(x^2+4)^{2-\frac{1}{2}}}$$

$$= \frac{3x}{(x^2+1)^{1/2}(x^2+4)^{3/2}} = y'$$

$$\Rightarrow y'' = \frac{d}{dx}\left[\frac{3x}{(x^2+1)^{1/2}(x^2+4)^{3/2}}\right]$$

$$= \frac{(x^2+1)^{1/2}(x^2+4)^{3/2}\cdot\frac{d}{dx}(3x) - 3x\cdot\frac{d}{dx}\left[(x^2+1)^{1/2}(x^2+4)^{3/2}\right]}{\left[(x^2+1)^{1/2}(x^2+4)^{3/2}\right]^2}$$

$$= \frac{(x^2+1)^{\frac{1}{2}}(x^2+4)^{3/2}\cdot 3 - 3x\cdot\left[(x^2+1)^{1/2}\cdot\frac{d}{dx}(x^2+4)^{3/2} + \frac{d}{dx}(x^2+1)^{1/2}\cdot(x^2+4)^{3/2}\right]}{(x^2+1)(x^2+4)^3}$$

$$= \frac{3(x^2+1)^{1/2}(x^2+4)^{3/2} - 3x\left[(x^2+1)^{1/2}\,\frac{3}{\cancel{2}}(x^2+4)^{\frac{1}{2}}\cdot\cancel{2}x + \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cdot\cancel{2}x\cdot(x^2+4)^{3/2}\right]}{(x^2+1)(x^2+4)^3}$$

$$= \frac{3(x^2+1)^{1/2}(x^2+4)^{3/2} - 3x\left[3x(x^2+1)^{1/2}(x^2+4)^{1/2} + x(x^2+1)^{-\frac{1}{2}}(x^2+4)^{3/2}\right]}{(x^2+1)(x^2+4)^3}$$

$$= \frac{3(x^2+1)^{1/2}(x^2+4)^{3/2} - 9x^2(x^2+1)^{1/2}(x^2+4)^{1/2} - 3x^2(x^2+1)^{-1/2}(x^2+4)^{3/2}}{(x^2+1)(x^2+4)^3}$$

$$= \frac{3(x^2+1)^{-1/2}(x^2+4)^{1/2}\left[\overbrace{(x^2+1)(x^2+4)}^{\cancel{x^4}+\cancel{4x^2}+x^2+4} \overbrace{-3x^2(x^2+1)}^{-3x^4-3x^2} \overbrace{-x^2(x^2+4)}^{\cancel{-x^4}-\cancel{4x^2}}\right]}{(x^2+1)(x^2+4)^3}$$

note:

$$\frac{\cancel{3}(x^2+1)^{1/2}(x^2+4)^{3/2}}{\cancel{3}(x^2+1)^{-1/2}(x^2+4)^{1/2}} = (x^2+1)^{\frac{1}{2}-(-\frac{1}{2})}(x^2+4)^{3/2-1/2}$$

$$= (x^2+1)^{\frac{1}{2}+\frac{1}{2}}(x^2+4)^{\frac{2}{2}}$$

$$= (x^2+1)^1(x^2+4)^1 = (x^2+1)(x^2+4)$$

$$= \frac{3(x^2+1)^{-1/2}(x^2+4)^{1/2}[-3x^4-2x^2+4]}{(x^2+1)(x^2+4)^3}$$

*note

$$\frac{(x^2+1)^{-1/2}}{(x^2+1)^1} = \frac{1}{(x^2+1)^{1-(-1/2)}} = \frac{1}{(x^2+1)^{1+1/2}} = \frac{1}{(x^2+1)^{3/2}}$$

$$= \frac{3[-3x^4-2x^2+4]}{(x^2+1)^{3/2}(x^2+4)^{5/2}}$$

*

$$\frac{(x^2+4)^{1/2}}{(x^2+4)^3} = \frac{1}{(x^2+4)^{3-1/2}} = \frac{1}{(x^2+4)^{5/2}}$$

OR $$y'' = \boxed{\frac{-9x^4-6x^2+12}{(x^2+1)^{3/2}(x^2+4)^{5/2}}}$$

Example 2

$$f(x) = \frac{x}{\sqrt{x^2+1}} = \frac{x}{(x^2+1)^{1/2}}$$

$$\Rightarrow f'(x) = \frac{(x^2+1)^{1/2} \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}\left[(x^2+1)^{1/2}\right]}{\left[(x^2+1)^{1/2}\right]^2}$$

$$= \frac{(x^2+1)^{1/2} \cdot 1 - x \cdot \cancel{\tfrac{1}{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x}{x^2+1}$$

$$= \frac{(x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}}{x^2+1}$$

$$= \frac{(x^2+1)^{-1/2}\left[(x^2+1) - x^2\right]}{x^2+1}$$

note: $$\frac{(x^2+1)^{1/2}}{(x^2+1)^{-1/2}} = (x^2+1)^{1/2-(-1/2)} = (x^2+1)^{\frac{1}{2}+\frac{1}{2}} = (x^2+1)^{2/2}$$

$$= (x^2+1)^1 = (x^2+1)$$

$$= \frac{(x^2+1)^{-1/2}(\ 1\ )}{x^2+1} = \frac{(x^2+1)^{-1/2}}{(x^2+1)^1}$$

$$= \frac{1}{(x^2+1)^1(x^2+1)^{1/2}} = \frac{1}{(x^2+1)^{1+1/2}} = \frac{1}{(x^2+1)^{3/2}}$$

$$\Rightarrow f'(x) = (x^2+1)^{-3/2}$$

$$\Rightarrow f''(x) = \frac{d}{dx}\left[(x^2+1)^{-3/2}\right]$$

$$= -3/2\,(x^2+1)^{-3/2-1} \cdot \frac{d}{dx}(x^2+1)$$

$$= -\frac{3}{\cancel{2}}(x^2+1)^{-5/2} \cdot \cancel{2}x$$

$$= -3x(x^2+1)^{-5/2}$$

$$= \boxed{\frac{-3x}{(x^2+1)^{5/2}}}$$

3] If a ball is thrown into the air with a velocity of 52 ft/s, its height in feet t seconds later is given by $y = 52t - 16t^2$.

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting for each of the following.

(i) 0.5 seconds

(ii) 0.1 seconds

(iii) 0.05 seconds

(iv) 0.01 seconds

(b) Estimate the instantaneous velocity when $t = 2$

Recall: $D = RT$

$\frac{D}{T} = \frac{RT}{T}$

$\frac{D}{T} = R$

(a)

(i) $[2, 2.5] \Rightarrow V_{avg} = \dfrac{y(2.5) - y(2)}{2.5 - 2}$

$$= \frac{30 - 40}{0.5}$$

$$= \frac{-10}{0.5} = \boxed{-20} \text{ ft/sec}$$

(ii) $[2, 2.1] \Rightarrow V_{avg} = \dfrac{y(2.1) - y(2)}{2.1 - 2}$

$$= \frac{38.64 - 40}{0.1}$$

$$= \frac{-1.36}{0.1} = \boxed{-13.6} \text{ ft/sec}$$

(iii) $[2, 2.05] \Rightarrow V_{avg} = \dfrac{y(2.05) - y(2)}{2.05 - 2}$

$$= \frac{39.36 - 40}{0.05} = \boxed{-12.8} \text{ ft/sec}$$

(iv) $[2, 2.01] \Rightarrow V_{avg} = \dfrac{y(2.01) - y(2)}{2.01 - 2}$

$$= \frac{39.8784 - 40}{0.01}$$

$$= \boxed{-12.16} \text{ ft/sec}$$

(b)

Time Interval	Average Velocity
[2, 2.5]	-20
[2, 2.1]	-13.6
[2, 2.05]	-12.8
[2, 2.01]	-12.16

as $t \to 2$, $V_{avg} \longrightarrow \boxed{-12}$ ft/sec

38] Evaluate the limit, if it exists. (If an answer does not exist, enter DNE.)

$$\lim_{t \to 0} \left(\frac{5}{t} - \frac{5}{t^2+t} \right)$$

$$= \lim_{t \to 0} \left(\frac{5}{t+1} \right)$$

$$= \frac{5}{0+1}$$

$$= \frac{5}{1}$$

$$= \boxed{5}$$

Answer

Note:

$$\frac{5}{t} - \frac{5}{t^2+t}$$

$$= \frac{5}{t} - \frac{5}{t(t+1)}$$

$$= \frac{5\cdot(t+1)}{t\cdot(t+1)} - \frac{5}{t(t+1)}$$

$$= \frac{5t+5}{t(t+1)} - \frac{5}{t(t+1)}$$

$$= \frac{5t+5-5}{t(t+1)}$$

$$= \frac{5\cancel{t}}{\cancel{t}(t+1)}$$

$$= \frac{5}{t+1}$$

note: $\lim_{x\to a} f(x)$

If: ① $f(a) = n \Rightarrow \lim_{x\to a} f(x) = n$

$\boxed{n \neq 0}$

② $f(a) = \frac{n}{0} \Rightarrow \lim_{x\to a} f(x) = DNE$

③ $f(a) = \frac{0}{0} \Rightarrow$ Try * factor $f(x)$ and reduce then substitute a for x again.

* Example:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Try: $x=1 \Rightarrow \dfrac{\sqrt{1}-1}{1-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$

$\Rightarrow$ Try Rationalize numerator.

$$\frac{(\sqrt{x}-1)\cdot(\sqrt{x}+1)}{(x-1)\cdot(\sqrt{x}+1)}$$

$$= \frac{(\sqrt{x})^2 \cancel{+\sqrt{x}} \cancel{-\sqrt{x}} -1}{(x-1)(\sqrt{x}+1)}$$

$$= \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$$

OR Note: $x - 1 = (\sqrt{x})^2 - 1$

$$\Rightarrow \frac{\sqrt{x} - 1}{x - 1} = \frac{\sqrt{x} - 1}{(\sqrt{x})^2 - 1^2}$$

Recall:

$a^2 - b^2 = (a+b)(a-b)$

$$= \frac{\cancel{\sqrt{x} - 1}^{\,1}}{(\sqrt{x} + 1)\cancel{(\sqrt{x} - 1)}}$$

$$= \frac{1}{\sqrt{x} + 1}$$

65. Suppose a sample of a certain substance decayed to 73.2% of its original amount after 300 days. (Round your answers to two decimal places.)

(a)What is the half-life (in days) of this substance?

Recall: $A(t) = A(0)\, e^{Kt}$

Here, $t = 300 \Rightarrow A(300) = 73.2\% \cdot A(0)$

$$\Rightarrow A(0) e^{K(300)} = 0.732 \cdot A(0)$$

$$\Rightarrow e^{300K} = 0.732$$

$$\Rightarrow \ln e^{300K} = \ln 0.732$$

$$\Rightarrow 300K = \ln 0.732$$

$$\Rightarrow K = \frac{\ln 0.732}{300}$$

$$\Rightarrow A(t) = A(0)\, e^{\frac{\ln 0.732}{300} t}$$

note: half life $\Rightarrow A(t) = \frac{1}{2} A(0)$

$$\Rightarrow A(0)\, e^{\frac{\ln 0.732}{300} t} = \frac{1}{2} A(0)$$

$$\Rightarrow e^{\frac{\ln 0.732}{300}t} = \frac{1}{2}$$

$$\Rightarrow \ln e^{\frac{\ln 0.732}{300}t} = \ln \frac{1}{2}$$

$$\Rightarrow \frac{\ln 0.732}{300} t = \ln \frac{1}{2}$$

$$\Rightarrow t = \ln \frac{1}{2} \div \frac{\ln 0.732}{300}$$

$$\Rightarrow t \approx \boxed{666.54} \text{ days}$$

(b) How long would it take the sample to decay to one-third of its original amount?

$$A(t) = \frac{1}{3} A(0)$$

$$\Rightarrow A(0) e^{\frac{\ln 0.732}{300}t} = \frac{1}{3} A(0)$$

$$\Rightarrow e^{\frac{\ln 0.732}{300}t} = \frac{1}{3}$$

$$\Rightarrow \ln e^{\frac{\ln 0.732}{300}t} = \ln \frac{1}{3}$$

$$\Rightarrow \frac{\ln 0.732}{300} t = \ln \frac{1}{3} \Rightarrow t = \ln \frac{1}{3} \div \frac{\ln 0.732}{300}$$

$$\Rightarrow t \approx \boxed{1056.44} \text{ days}$$

The point $P(1, 0)$ lies on the curve $y = \sin\left(\frac{14\pi}{x}\right)$.

(a) If Q is the point $\left(x, \sin\left(\frac{14\pi}{x}\right)\right)$, find the slope of the secant line PQ (correct to four decimal places) for the following values of x.

(i)	2
(ii)	1.5
(iii)	1.4
(iv)	1.3
(v)	1.2
(vi)	1.1
(vii)	0.5
(viii)	0.6
(ix)	0.7
(x)	0.8
(xi)	0.9

Recall: slope of a line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $m_{\overline{PQ}} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Here, we are given $P(1,0) \Rightarrow x_1 = 1,\ y_1 = 0$

(i) $x_2 = 2 \Rightarrow y_2 = \sin\left(\frac{14\pi}{2}\right) = \sin(7\pi) = 0$

$$\Rightarrow m_{\overline{PQ}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-0}{2-1} = \frac{0}{1} = 0 = \boxed{0.0000}$$

(ii) $x_2 = 1.5 \Rightarrow y_2 = \sin\left(\frac{14\pi}{1.5}\right) = \sin(9.\bar{3}\pi) \approx -0.8660254$

$$\Rightarrow m_{\overline{PQ}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-0.8660254 - 0}{1.5 - 1} = \frac{-0.8660254}{0.5} \approx \boxed{-1.7321}$$

(iii) $x_2 = 1.4 \Rightarrow y_2 = \sin\left(\frac{14\pi}{1.4}\right) = 0$

$$\Rightarrow m_{\overline{PQ}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-0}{1.4-1} = \frac{0}{0.4} = \boxed{0.0000}$$

* Similar work for (iv) ~ (ix).

Summary:

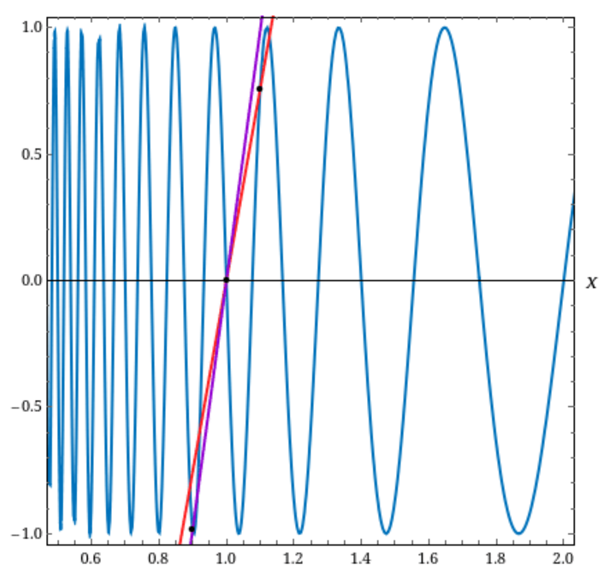
	x	Q	$m_{\overline{PQ}}$
(i)	2	(2, 0.0000)	0.0000
(ii)	1.5	(1.5, -0.8660)	-1.7321
(iii)	1.4	(1.4, 0.0000)	0.0000
(iv)	1.3	(1.3, 0.6631)	2.2104
(v)	1.2	(1.2, -0.8860)	-4.3301
(vi)	1.1	(1.1, 0.7557)	7.5575
(vii)	0.5	(0.5, 0.0000)	0.0000
(viii)	0.6	(0.6, -0.8660)	2.1651
(ix)	0.7	(0.7, 0.0000)	0.0000
(x)	0.8	(0.8, -1.0000)	5.0000
(xi)	0.9	(0.9, -0.9848)	9.8481

Does the slope appear to be approaching a limit?

As x approaches 1, the slopes *do not appear to be approaching any particular value*.

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at *P*.

We see that problems with estimation are caused by the *frequent oscillations* of the graph. The tangent is so steep at *P* that we need to take *x*-values *closer* to 1 in order to get accurate estimates of its slope.



(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P. (Round your answer to two decimal places.)

Choose 2 points sufficiently close to 1.

$\Rightarrow$ use $x_1 = 0.999$ and $x_2 = 1.001$

$\Rightarrow y_1 = \sin\left(\frac{14\pi}{0.999}\right) \approx 0.044012$

$y_2 = \sin\left(\frac{14\pi}{1.001}\right) \approx -0.043924$

$$\Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-0.043924 - 0.044012}{1.001 - 0.999}$$

$$= \frac{-0.087936}{0.002} = -43.968 \approx \boxed{-43.97}$$

Evaluate the limit, if it exists. (If an answer does not exist, enter DNE.)

$$\lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$$

note: $x = -4 \Rightarrow \dfrac{\sqrt{(-4)^2+9} - 5}{-4+4} = \dfrac{\overset{\overset{5}{\shortparallel}}{\sqrt{25}} - 5}{0} = \dfrac{0}{0}$

$\Rightarrow$ need to rationalize radical.

$$\Rightarrow \frac{\sqrt{x^2+9} - 5}{x+4} = \frac{(\sqrt{x^2+9} - 5) \cdot (\sqrt{x^2+9} + 5)}{(x+4) \cdot (\sqrt{x^2+9} + 5)}$$

$$= \frac{(\sqrt{x^2+9})^2 - (5)^2}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \frac{x^2+9 - 25}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \frac{x^2 - 16}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9} + 5)}$$

$$= \frac{x-4}{\sqrt{x^2+9} + 5}$$

$$\Rightarrow \lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$$

$$= \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9} + 5}$$

$$= \frac{-4-4}{\sqrt{(-4)^2+9} + 5} = \frac{-8}{\sqrt{\underbrace{16+9}_{25}} + 5} = \frac{-8}{5+5} = \frac{-8}{10}$$

$$= \boxed{-\frac{4}{5}}$$

Find the limit. (If the limit is infinite, enter '∞' or '-∞', as appropriate. If the limit does not otherwise exist, enter DNE.)

$$\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{7t+t^2}$$

$$= \lim_{t\to\infty} \frac{(\sqrt{t}+t^2)\cdot 1/t^2}{(7t+t^2)\cdot \frac{1}{t^2}}$$

Find largest power of t in denominator and divide all terms in the function by it.
Here: $t^2 \Rightarrow$ divide all terms by t^2
$\Rightarrow$ multiply by $\frac{1}{t^2}$.

$$= \lim_{t\to\infty} \frac{\frac{\sqrt{t}}{t^2} + \frac{t^2}{t^2}}{\frac{7t}{t^2} + \frac{t^2}{t^2}}$$

$$= \lim_{t\to\infty} \frac{\frac{1}{t^{3/2}} + 1}{\frac{7}{t} + 1}$$

Note: $\frac{\sqrt{t}}{t^2} = \frac{t^{1/2}}{t^2} = \frac{1}{t^{2-1/2}} = \frac{1}{t^{\frac{4}{2}-\frac{1}{2}}} = \frac{1}{t^{3/2}}$

$$= \frac{0+1}{0+1}$$

Note: as $t\to\infty$, $\frac{1}{t} \to 0$

$\Rightarrow \frac{1}{t^{3/2}} = \left(\frac{1}{t}\right)^{3/2} \to (0)^{3/2} \to 0$

and $\frac{7}{t} = 7\cdot\frac{1}{t} \to 7\cdot 0 \to 0$

$$= \frac{1}{1}$$

$$= \boxed{1}$$

Find the limit. (If the limit is infinite, enter '∞' or '-∞', as appropriate. If the limit does not otherwise exist, enter DNE.)

$$\lim_{x \to \infty} \frac{\sqrt{1+64x^6}}{3-x^3}$$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{1+64x^6}\right) \cdot 1/x^3}{\left(3-x^3\right) \cdot 1/x^3}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt{1+64x^6}}{x^3}}{\frac{3}{x^3} - \frac{x^3}{x^3}}$$

$$\boxed{\begin{aligned} &\underline{\text{note}}: \text{Need to change } x^3 \text{ into } \sqrt{x^6} \text{ to divide into } \sqrt{} \\ &\Rightarrow x^3 = x^{6/2} = x^{6 \cdot \frac{1}{2}} \\ &= (x^6)^{1/2} = \sqrt{x^6} \end{aligned}}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt{1+64x^6}}{\sqrt{x^6}}}{\frac{3}{x^3} - 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{1+64x^6}{x^6}}}{\frac{3}{x^3} - 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{1}{x^6} + \frac{64\cancel{x^6}}{\cancel{x^6}}}}{\frac{3}{x^3} - 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\cancel{\frac{1}{x^6}}^{\to 0} + 64}}{\cancel{\frac{3}{x^3}}^{\to 0} - 1}$$

$$= \frac{\sqrt{0 + 64}}{0 - 1}$$

$$= \frac{\sqrt{64}}{-1} = \frac{8}{-1} = \boxed{-8}$$

Find an equation of the tangent line to the graph of *f* at the given point.

$$f(x) = \sqrt{x} \quad , \quad (25, 5)$$

Note: we need slope of tangent line first

$\Rightarrow$ need to find $f'(x)$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{2\sqrt{x}}$$

$$\Rightarrow m_{tan} = f'(x) = \frac{1}{2\sqrt{x}}$$

at $(25, 5)$, $x = 25$

$$\Rightarrow m_{tan} = f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{2(5)} = \frac{1}{10}$$

$$\Rightarrow m = \frac{1}{10}, \quad (x_1, y_1) = (25, 5)$$

$$\Rightarrow y - y_1 = m(x - x_1) \text{ becomes}$$

$$y - 5 = \frac{1}{10}(x - 25)$$

$$\Rightarrow y - 5 = \frac{1}{10}x - \frac{\cancel{25}^{5}}{\cancel{10}_{2}}$$

$$\Rightarrow y \cancel{- 5} = \frac{1}{10}x - \frac{5}{2}$$

$$\cancel{+5} \qquad\qquad + 5$$

$$-\frac{5}{2} + \frac{5 \cdot 2}{1 \cdot 2}$$

$$= -\frac{5}{2} + \frac{10}{2} = +\frac{5}{2}$$

$$\Rightarrow y = \frac{1}{10}x + \frac{5}{2}$$

$$\text{OR} \quad y = \boxed{\frac{x}{10} + \frac{5}{2}}$$

35] (a) If $F(x) = 13x/(4 + x^2)$, find $F'(3)$ and use it to find an equation of the tangent line to the curve $y = 13x/(4 + x^2)$ at the point $(3, 3)$.

$$F(x) = \frac{13x}{4+x^2}$$

$$\Rightarrow F'(x) = \frac{(4+x^2)\frac{d}{dx}(13x) - 13x \cdot \frac{d}{dx}(4+x^2)}{(4+x^2)^2}$$

$$= \frac{(4+x^2)\cdot 13 - 13x \cdot 2x}{(4+x^2)^2}$$

$$= \frac{52+13x^2-26x^2}{(4+x^2)^2}$$

$$= \frac{52-13x^2}{(4+x^2)^2}$$

$$\Rightarrow F'(3) = \frac{52-13(3)^2}{(4+3^2)^2} = \frac{52-13(9)}{(4+9)^2}$$

$$= \frac{52-117}{(13)^2} = -\frac{\cancel{65}^{\,5}}{\cancel{169}_{\,13}} = -\frac{5}{13}$$

$$\Rightarrow m_{tan}\Big|_{(3,3)} = -\frac{5}{13}$$

$$y - 3 = -\frac{5}{13}(x-3)$$

$$\Rightarrow \quad y - \cancel{3} = -\frac{5}{13}x + \frac{15}{13} \rightarrow \frac{15}{13}$$

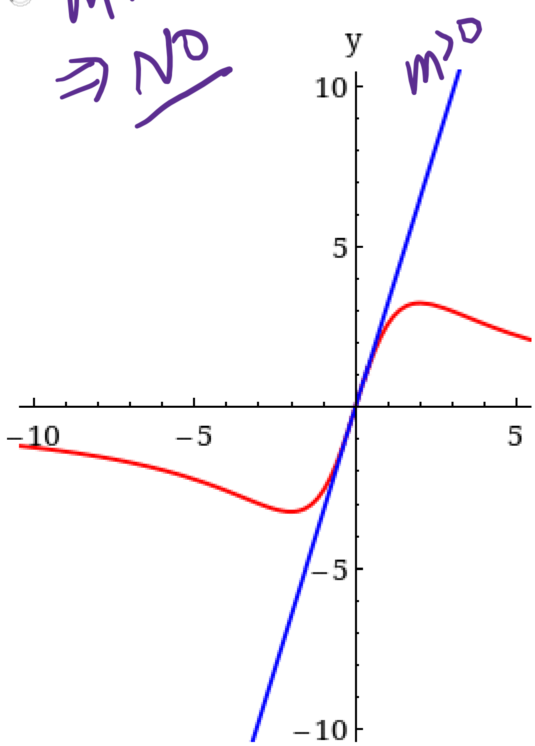
$$\underline{+\cancel{3}} \qquad\qquad \underline{+\frac{3\cdot 13}{1\cdot 13}} \rightarrow \frac{39}{13}$$

$$\underline{\phantom{\frac{39}{13}}}$$

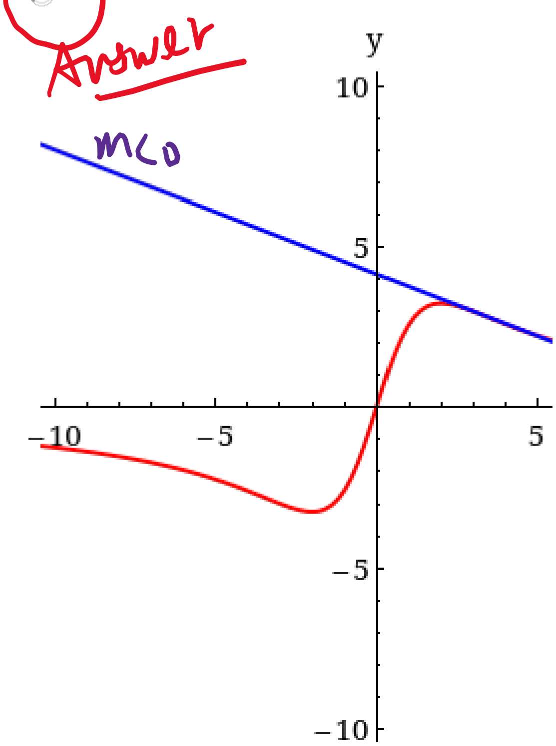
$$\Rightarrow \quad \boxed{y = -\frac{5}{13}x + \frac{54}{13}}$$

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

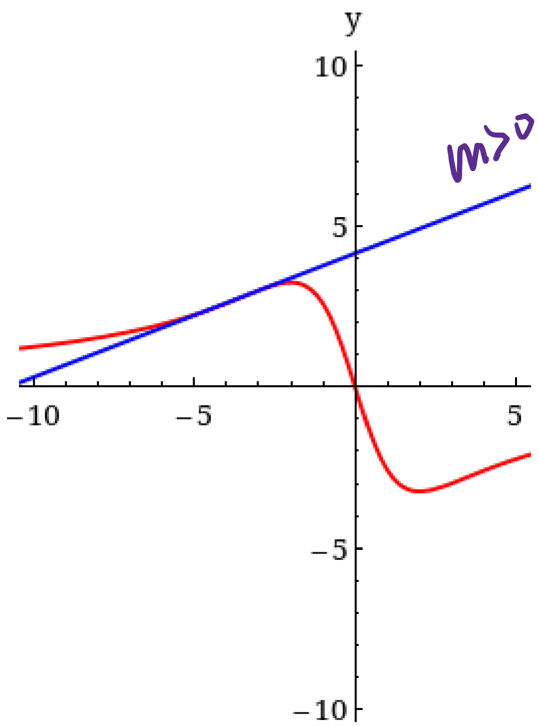
m>0 ⇒ NO



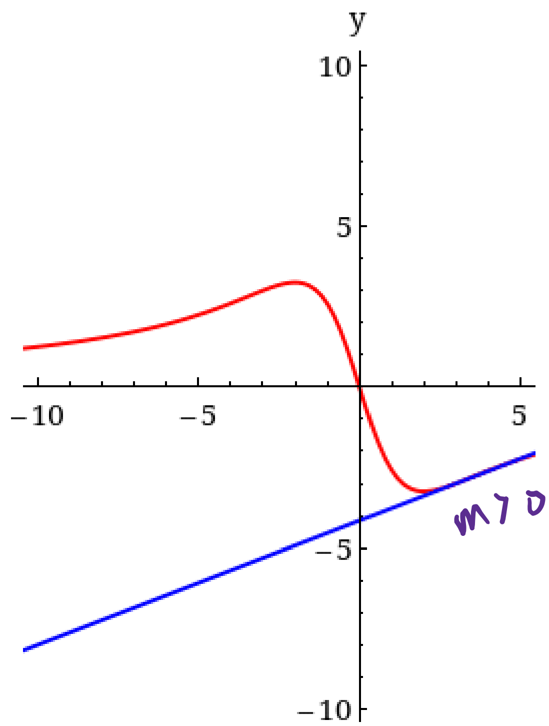
Answer



$m>0 \Rightarrow$ No



$m>0 \Rightarrow$ No



39] Find a parabola with equation $y = ax^2 + bx + c$ that has slope 12 at $x = 1$, slope -16 at $x = -1$, and passes through the point (2, 28).

$$y' = 2ax + b$$

$\boxed{y' = 12 \text{ @ } x = 1}$

$$12 = 2a(1) + b$$

$$12 = 2a + b$$

$\boxed{y' = -16 \text{ @ } x = -1}$

$$-16 = 2a(-1) + b$$

$$-16 = -2a + b$$

$$\cancel{2a} + b = 12$$

$$\cancel{-2a} + b = -16$$

$$\frac{\cancel{2}b}{\cancel{2}} = \frac{-4}{2} \Rightarrow \boxed{b = -2}$$

$\Rightarrow 2a + b = 12$ becomes $2a + (-2) = 12$

$$\Rightarrow 2a \cancel{- 2} = 12$$

$$\quad \cancel{+2} \quad +2$$

$$\Rightarrow \frac{\cancel{2}a}{\cancel{2}} = \frac{14}{2} \Rightarrow \boxed{a = 7}$$

note: parabula passes through $(2, 28)$

$\Rightarrow$ when $x = 2$, $y = 28$

$\Rightarrow y = ax^2 + bx + c$ becomes

$$28 = 7(2)^2 + (-2)(2) + c$$

$$28 = 7(4) + (-4) + c$$

$$28 = 28 - 4 + c$$

$$\underset{-24}{28} = \underset{\cancel{-24}}{\cancel{24}} + c \Rightarrow \boxed{c = 4}$$

$\Rightarrow$ our equation is

$$\boxed{y = 7x^2 - 2x + 4}$$

40] Differentiate.

$$g(t) = \frac{t - \sqrt{t}}{t^{1/6}}$$

$$\Rightarrow g(t) = \frac{t - t^{1/2}}{t^{1/6}} = \frac{t}{t^{1/6}} - \frac{t^{1/2}}{t^{1/6}}$$

$$= t^{1 - 1/6} - t^{1/2 - 1/6}$$

$$= t^{\frac{6}{6} - \frac{1}{6}} - t^{\frac{3}{6} - \frac{1}{6}}$$

$$= t^{\frac{5}{6}} - t^{\frac{2}{6}}$$

$$g(t) = t^{5/6} - t^{1/3}$$

$$\Rightarrow g'(t) = \frac{5}{6} t^{\frac{5}{6} - 1} - \frac{1}{3} t^{\frac{1}{3} - 1}$$

$$= \frac{5}{6} t^{\frac{5}{6} - \frac{6}{6}} - \frac{1}{3} t^{\frac{1}{3} - \frac{3}{3}}$$

$$= \boxed{\frac{5}{6} t^{-1/6} - \frac{1}{3} t^{-2/3}}$$

42. Find the equations of the tangent line and normal line to the given curve at the specified point.

$$y = \frac{\sqrt{x}}{x+8}, \quad \left(1, \frac{1}{9}\right)$$

Tangent line:

$$m = \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sqrt{x}}{x+8}\right]$$

$$= \frac{(x+8)\cdot\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\cdot\frac{d}{dx}(x+8)}{(x+8)^2}$$

$$= \frac{(x+8)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+8)^2}$$

$$= \frac{\frac{x+8}{2\sqrt{x}} - \sqrt{x}}{(x+8)^2}$$

$$\Rightarrow m_{tan}\Big|_{x=1} = \frac{\frac{1+8}{2\sqrt{1}} - \sqrt{1}}{(1+8)^2} = \frac{\frac{9}{2} - 1}{(9)^2} = \frac{\frac{7}{2}}{81} = \frac{7}{162}$$

$$\boxed{\frac{7}{2} \div \frac{81}{1} = \frac{7}{2}\cdot\frac{1}{81} = \frac{7}{162}}$$

$\Rightarrow$ Equation of the tangent line is

$$y - \frac{1}{9} = \frac{7}{162}(x-1)$$

$$y - \frac{1}{9} = \frac{7}{162}x - \frac{7}{162} \Rightarrow \boxed{y = \frac{7}{162x} + \frac{11}{162}}$$

$$+\frac{1}{9} \qquad\qquad + \frac{1}{9} = \frac{18}{162}$$

$$\hookrightarrow -\frac{7}{162} + \frac{18}{162} = \frac{11}{162}$$

note: Normal line is perpendicular to the tangent line and $m_{tan} = \frac{7}{162}$

$\Rightarrow m_N = -\frac{162}{7}$

$\Rightarrow$ Equation of normal line is

$$y - \frac{1}{9} = -\frac{162}{7}(x-1)$$

$$\Rightarrow y - \frac{1}{9} = -\frac{162}{7}x + \frac{162}{7}$$

$$+\frac{1}{9} \qquad\qquad +\frac{1}{9}$$

$$\frac{162}{7} + \frac{1}{9}$$

$$= \frac{162 \cdot 9}{7 \cdot 9} + \frac{1 \cdot 7}{9 \cdot 7}$$

$$= \frac{1458}{63} + \frac{7}{63} = \frac{1465}{63}$$

$$\Rightarrow \boxed{y = -\frac{162}{7}x + \frac{1465}{63}}$$

42] Find the equations of the tangent line and normal line to the given curve at the specified point.

$$y = \frac{\sqrt{x}}{x+4} \ ,\left(1, \frac{1}{5}\right)$$

$$m_{tan} = y' = \frac{d}{dx}\left(\frac{\sqrt{x}}{x+4}\right)$$

note:

$\sqrt{x} = x^{1/2}$

$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2})$

$= \frac{1}{2}x^{1/2 - 1}$

$= \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}}$

$= \frac{1}{2\sqrt{x}}$

$$= \frac{(x+4)\cdot\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\cdot\frac{d}{dx}(x+4)}{(x+4)^2}$$

$$= \frac{(x+4)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot(1)}{(x+4)^2}$$

$$= \frac{\frac{x+4}{2\sqrt{x}} - \sqrt{x}}{(x+4)^2}$$

$$\Rightarrow m_{tan}\Big|_{(1,\frac{1}{5})} = y'(1)$$

$$= \frac{\frac{1+4}{2\sqrt{1}} - \sqrt{1}}{(1+4)^2}$$

$$= \frac{\frac{5}{2} - 1}{5^2} = \frac{\frac{5}{2} - \frac{2}{2}}{25} = \frac{\frac{3}{2}}{25} = \frac{3}{50}$$

$\Rightarrow m_{tan} = \frac{3}{50}$, $(\underset{x_1}{\underset{\uparrow}{1}}, \underset{y_1}{\underset{\uparrow}{\frac{1}{5}}})$

$$y - \frac{1}{5} = \frac{3}{50}(x-1)$$

$$y - \cancel{\frac{1}{5}} = \frac{3}{50}x - \frac{3}{50}$$

$$\underline{+\cancel{\frac{1}{5}}} \qquad\qquad \underline{+\frac{1 \cdot 10}{5 \cdot 10}}$$

$$y = \frac{3}{50}x - \frac{3}{50} + \frac{10}{50}$$

$$\Rightarrow \boxed{y = \frac{3}{50}x + \frac{7}{50}}$$ ← Equation of the tangent line at $(1, \frac{1}{5})$

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note: normal line $\perp$ Tangent line

normal line ↓ $\boxed{m_N = -\frac{50}{3}}$

Tangent line ↓ $m_{tan} = \frac{3}{50}$

$$\Rightarrow \quad y - \frac{1}{5} = -\frac{50}{3}(x-1)$$

$$y - \cancel{\frac{1}{5}} = -\frac{50}{3}x + \frac{50}{3}$$

$$+\cancel{\frac{1}{5}} \qquad\qquad +\frac{1}{5}$$

$$\boxed{y = -\frac{50}{3}x + \frac{253}{15}}$$

Equation of the normal line at $\left(1, \frac{1}{5}\right)$

note:

$$\frac{50 \cdot 5}{3 \cdot 5} + \frac{1 \cdot 3}{5 \cdot 3}$$

$$= \frac{250}{15} + \frac{3}{15}$$

$$= \frac{253}{15}$$

43] Find an equation of the tangent line to the curve at the given point.

$$y = \sec(x) - 6\cos(x), \ P = \left(\frac{\pi}{3}, -1\right)$$

$$y' = \sec(x)\tan(x) - 6\cdot(-\sin(x))$$

$$\Rightarrow y' = \sec(x)\tan(x) + 6\sin(x)$$

$$\Rightarrow m_{tan} = y'\left(\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) + 6\sin\left(\frac{\pi}{3}\right)$$

$$= (2)(\sqrt{3}) + \overset{3}{\cancel{6}}\left(\frac{\sqrt{3}}{\cancel{2}}\right)$$

$$= 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

$\Rightarrow$ Equation of the tangent line is

$$y - (-1) = 5\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$y + 1 = 5\sqrt{3}x - \frac{5\sqrt{3}\pi}{3}$$

$$\boxed{y = 5\sqrt{3}x - \frac{5\sqrt{3}\pi}{3} - 1}$$

If $g(\theta) = \dfrac{\sin(\theta)}{\theta}$, find $g'(\theta)$ and $g''(\theta)$.

$$g'(\theta) = \frac{d}{d\theta}\left(\frac{\sin(\theta)}{\theta}\right)$$ ← use Quotient Rule

$$= \frac{\theta \cdot \frac{d}{d\theta}(\sin(\theta)) - \sin(\theta) \cdot \frac{d}{d\theta}(\theta)}{\theta^2}$$

$$= \frac{\theta \cdot \cos(\theta) - \sin(\theta) \cdot (1)}{\theta^2}$$

$$g'(\theta) = \boxed{\frac{\theta\cos(\theta) - \sin(\theta)}{\theta^2}}$$

$$g''(\theta) = \frac{d}{d\theta}\left(g'(\theta)\right)$$

$$= \frac{d}{d\theta}\left(\frac{\theta\cos(\theta) - \sin(\theta)}{\theta^2}\right)$$

← use Quotient Rule

$$= \frac{\theta^2 \cdot \frac{d}{d\theta}\left(\theta\cos(\theta) - \sin(\theta)\right) - \left(\theta\cos(\theta) - \sin(\theta)\right)\cdot\frac{d}{d\theta}(\theta^2)}{(\theta^2)^2}$$

$$= \frac{\theta^2 \cdot \left[\underbrace{\frac{d}{d\theta}\left(\theta\cos(\theta)\right)}_{\text{use product Rule}} - \frac{d}{d\theta}(\sin(\theta))\right] - \left(\theta\cos(\theta) - \sin(\theta)\right)\cdot 2\theta}{\theta^4}$$

$$= \frac{\theta^2 \cdot \left[\frac{d}{d\theta}(\theta)\cdot\cos(\theta) + \theta\cdot\frac{d}{d\theta}(\cos(\theta)) - \cos(\theta)\right] - 2\theta\left(\theta\cos(\theta) - \sin(\theta)\right)}{\theta^4}$$

$$= \frac{\theta^2 \cdot \left[1\cdot\cos(\theta) + \theta\cdot(-\sin(\theta)) - \cos\theta\right] - 2\theta^2\cos(\theta) + 2\theta\sin(\theta)}{\theta^4}$$

$$= \frac{\theta^2 \cdot \left[\cos(\theta) - \theta\sin(\theta) - \cos(\theta)\right] - 2\theta^2\cos(\theta) + 2\theta\sin(\theta)}{\theta^4}$$

$$= \frac{\cancel{\theta^2\cos(\theta)} - \theta^3\sin(\theta) \cancel{- \theta^2\cos(\theta)} - 2\theta^2\cos(\theta) + 2\theta\sin(\theta)}{\theta^4}$$

$$= \frac{-\theta^3\sin(\theta) - 2\theta^2\cos(\theta) + 2\theta\sin(\theta)}{\theta^4}$$

common factor

$$= \frac{\cancel{\theta}\left(-\theta^2\sin(\theta) - 2\theta\cos(\theta) + 2\sin(\theta)\right)}{\theta^{\cancel{4}\,3}}$$

$$= \boxed{\frac{-\theta^2\sin(\theta) - 2\theta\cos(\theta) + 2\sin(\theta)}{\theta^3}}$$

Find the derivative of the function.

$$f(x) = (2x - 5)^4(x^2 + x + 1)^5$$

use product Rule

$$f'(x) = \frac{d}{dx}\left[(2x-5)^4(x^2+x+1)^5\right]$$

use chain Rule

use chain Rule

$$= \frac{d}{dx}\left[(2x-5)^4\right]\cdot(x^2+x+1)^5 + (2x-5)^4\cdot\underbrace{\frac{d}{dx}\left[(x^2+x+1)^5\right]}_{\text{use chain Rule}}$$

$$= 4(2x-5)^3\cdot\frac{d}{dx}(2x-5)\cdot(x^2+x+1)^5 + (2x-5)^4\cdot 5(x^2+x+1)^4\cdot\frac{d}{dx}(x^2+x+1)$$

$$= 4(2x-5)^3\cdot(2)\cdot(x^2+x+1)^5 + (2x-5)^4\cdot 5(x^2+x+1)^4\cdot(2x+1)$$

$$= 8(2x-5)^3(x^2+x+1)^5 + 5(2x-5)^4(x^2+x+1)^4(2x+1)$$

$$= \overbrace{(2x-5)^3(x^2+x+1)^4}^{\text{GCF}}\left[8(x^2+x+1) + 5(2x-5)(2x+1)\right]$$

$$= (2x-5)^3(x^2+x+1)^4\left[8x^2+8x+8+5(4x^2+2x-10x-5)\right]$$

$$= (2x-5)^3(x^2+x+1)^4\left[8x^2+8x+8+5(4x^2-8x-5)\right]$$

$$= (2x-5)^3(x^2+x+1)^4\left[8x^2+8x+8+20x^2-40x-25\right]$$

$$= \boxed{(2x-5)^3(x^2+x+1)^4(28x^2-32x-17)}$$

Find the derivative of the function.

$$y = \sqrt{\frac{x}{x+5}} = \left(\frac{x}{x+5}\right)^{1/2}$$ ← Differentiate using chain Rule

$$\Rightarrow y' = \frac{1}{2}\left(\frac{x}{x+5}\right)^{1/2-1} \cdot \frac{d}{dx}\left(\frac{x}{x+5}\right)$$ → use Quotient Rule

$$= \frac{1}{2}\left(\frac{x}{x+5}\right)^{-1/2} \cdot \frac{(x+5)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{1}{2}\left(\frac{x}{x+5}\right)^{-1/2} \cdot \frac{(x+5)\cdot 1 - x\cdot 1}{(x+5)^2}$$

$$= \frac{1}{2}\left(\frac{x+5}{x}\right)^{1/2} \cdot \frac{\cancel{x}+5\cancel{-x}}{(x+5)^2}$$

$$= \frac{1}{2}\cdot\frac{(x+5)^{1/2}}{x^{1/2}} \cdot \frac{5}{(x+5)^2}$$

$$= \frac{5(x+5)^{1/2}}{2x^{1/2}(x+5)^2}$$

$$= \frac{5}{2x^{1/2}(x+5)^{2-1/2}} = \boxed{\frac{5}{2x^{1/2}(x+5)^{3/2}}}$$

$$2-\frac{1}{2} = \frac{2\cdot2}{1\cdot2}-\frac{1}{2} = \frac{4}{2}-\frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}$$

49. If $h(x) = \sqrt{7 + 6f(x)}$, where $f(5) = 7$ and $f'(5) = 2$, find $h'(5)$.

note: $h'(x) = \frac{d}{dx}\left(\sqrt{7+6f(x)}\right) = \frac{d}{dx}\left(\left[7+6f(x)\right]^{\frac{1}{2}}\right)$

$$= \frac{1}{2}\left[7+6f(x)\right]^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left[7+6f(x)\right] \leftarrow \text{Chain Rule}$$

$$= \frac{1}{2}\left[7+6f(x)\right]^{-\frac{1}{2}} \cdot \left[0 + 6f'(x)\right]$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\left[7+6f(x)\right]^{1/2}} \cdot \left[\overset{3}{\cancel{6}} f'(x)\right]$$

$$\Rightarrow \quad h'(x) = \frac{3f'(x)}{\sqrt{7+6f(x)}}$$

$$\Rightarrow h'(5) = \frac{3f'(5)}{\sqrt{7+6f(5)}} = \frac{3 \cdot 2}{\sqrt{7+6\cdot 7}} = \frac{6}{\sqrt{7+42}}$$

$$= \frac{6}{\sqrt{49}} = \boxed{\frac{6}{7}} \leftarrow \text{Answer}$$

Find the derivative of the function.

$$h(t) = (t^4 - 1)^4(t^3 + 1)^7$$

$$h'(t) = \frac{d}{dt}\left[(t^4-1)^4(t^3+1)^7\right] \leftarrow \text{use product Rule}$$

use chain Rule

use Chain Rule

$$= \frac{d}{dt}\left[(t^4-1)^4\right]\cdot(t^3+1)^7 + (t^4-1)^4\cdot\frac{d}{dt}\left[(t^3+1)^7\right]$$

$$= 4(t^4-1)^3\cdot\underbrace{\frac{d}{dt}(t^4-1)}\cdot(t^3+1)^7 + (t^4-1)^4\cdot 7(t^3+1)^6\cdot\underbrace{\frac{d}{dt}(t^3+1)}$$

$$= 4(t^4-1)^3\cdot(4t^3)\cdot(t^3+1)^7 + (t^4-1)^4\cdot 7(t^3+1)^6\cdot(3t^2)$$

$$= 16t^3(t^4-1)^3(t^3+1)^7 + 21t^2(t^4-1)^4(t^3+1)^6$$

$$= \underbrace{t^2(t^4-1)^3(t^3+1)^6}_{\text{GCF}}\left[16t(t^3+1) + 21(t^4-1)\right]$$

$$= t^2(t^4-1)^3(t^3+1)^6\left[16t^4 + 16t + 21t^4 - 21\right]$$

$$= \boxed{t^2(t^4-1)^3(t^3+1)^6(37t^4+16t-21)}$$

Find the first and second derivatives of the function.

$$y = xe^{5x}$$

1st Derivative:

$$y' = \frac{d}{dx}\left(xe^{5x}\right) \leftarrow \text{use product Rule}$$

use chain Rule

$$= \underbrace{\frac{d}{dx}(x)} \cdot e^{5x} + x \cdot \frac{d}{dx}\left(e^{5x}\right)$$

$$= 1 \cdot e^{5x} + x \cdot e^{5x} \cdot \underbrace{\frac{d}{dx}(5x)}$$

$$= e^{5x} + xe^{5x} \cdot 5$$

$$= e^{5x} + 5xe^{5x}$$

$$\Rightarrow y' = e^{5x}(1+5x) \text{ or } y' = \boxed{e^{5x}(5x+1)}$$

2^{nd} Derivative:

$$y'' = \frac{d}{dx}(y')$$

$$= \frac{d}{dx}\left[e^{5x}(5x+1)\right] \leftarrow \text{use product Rule}$$

$$= \frac{d}{dx}\left(e^{5x}\right)\cdot(5x+1) + e^{5x}\cdot\underbrace{\frac{d}{dx}(5x+1)}$$

use chain Rule →

$$= e^{5x}\cdot\underbrace{\frac{d}{dx}(5x)}\cdot(5x+1) + e^{5x}\cdot 5$$

$$= e^{5x}\cdot 5\cdot(5x+1) + 5e^{5x}$$

$$= 5e^{5x}\left[(5x+1) + 1\right]$$

$$= 5e^{5x}(5x+\underbrace{1+1}_{+2})$$

$$\Rightarrow y'' = \boxed{5e^{5x}(5x+2)}$$

Find dy/dx by implicit differentiation.

$$8x^3 + x^2y - xy^3 = 4$$

$$\frac{d}{dx}\left(8x^3 + x^2y - xy^3\right) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(8x^3) + \frac{d}{dx}(x^2y) - \frac{d}{dx}(xy^3) = 0$$

$$24x^2 + \left(x^2 \cdot \frac{d}{dx}(y) + \frac{d}{dx}(x^2) \cdot y\right) - \left(x \cdot \frac{d}{dx}(y^3) + \frac{d}{dx}(x) \cdot y^3\right) = 0$$

$$24x^2 + \left(x^2 \cdot \frac{dy}{dx} + 2x \cdot y\right) - \left(x \cdot 3y^2 \cdot \frac{dy}{dx} + 1 \cdot y^3\right) = 0$$

$$\cancel{24x^2} + x^2\frac{dy}{dx} + \cancel{2xy} - 3xy^2\frac{dy}{dx} - \cancel{y^3} = 0$$

$$\cancel{-24x^2} \qquad \cancel{-2xy} \qquad \cancel{+y^3} \qquad \begin{array}{r} -24x^2 \\ -2xy \\ +y^3 \end{array}$$

$$x^2\frac{dy}{dx} - 3xy^2\frac{dy}{dx} = y^3 - 24x^2 - 2xy$$

$$\frac{\frac{dy}{dx}\cancel{\left(x^2 - 3xy^2\right)}}{\cancel{(x^2 - 3xy^2)}} = \frac{y^3 - 24x^2 - 2xy}{x^2 - 3xy^2}$$

$$\frac{dy}{dx} = \boxed{\frac{y^3 - 24x^2 - 2xy}{x^2 - 3xy^2}}$$

Answer

Find the derivative of the function. Simplify if possible.

$$y = \arccos\left(e^{4x}\right)$$

$$y' = \frac{d}{dx}\left(\arccos\left(e^{4x}\right)\right)$$

$$= -\frac{1}{\sqrt{1-\left(e^{4x}\right)^2}} \cdot \frac{d}{dx}\left(e^{4x}\right)$$

$$= -\frac{1}{\sqrt{1-e^{4x\cdot 2}}} \cdot 4e^{4x}$$

$$= \boxed{-\frac{4e^{4x}}{\sqrt{1-e^{8x}}}}$$

Recall:

$$\frac{d}{dx}\left(\cos^{-1}u\right) = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

Here, $u = e^{4x}$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(e^{4x}\right)$$

$$= e^{4x} \cdot \frac{d}{dx}(4x)$$

$$= e^{4x} \cdot 4$$

$$= 4e^{4x}$$

A bacterial culture grows with a constant relative growth rate. After 2 hours there are 600 bacteria, and after 8 hours the count is 75,000.

(a)Find the initial population.

Recall: Exponential Growth function:

$$P(t) = P(0)e^{kt}$$

$P(t)$ = population after time t

$P(0)$ = Initial population

k = Growth constant

t = time

when $t=2$, $P(2) = 600 \Rightarrow P(0)e^{k\cdot 2} = 600 \Rightarrow P(0)e^{2k} = 600$

when $t=8$, $P(8) = 75{,}000 \Rightarrow P(0)e^{k\cdot 8} = 75{,}000 \Rightarrow P(0)e^{8k} = 75{,}000$

$$\Rightarrow \frac{P(8)}{P(2)} = \frac{\cancel{P(0)}e^{8k}}{\cancel{P(0)}e^{2k}} = \frac{75{,}000}{600}$$

$$\Rightarrow e^{8k-2k} = 125$$

$$\Rightarrow \ln\left(e^{6k}\right) = \ln(125)$$

$$\Rightarrow 6k = \ln(5^3)$$

$$\Rightarrow \frac{\cancel{6}k}{\cancel{6}} = \frac{\cancel{3}\ln(5)}{\cancel{6}\,2}$$

$$\Rightarrow k = \frac{\ln(5)}{2} = \frac{1}{2}\ln(5)$$

$$\Rightarrow P(0)\cdot e^{2K} = 600 \quad \text{becomes}$$

$$P(0)\cdot e^{\cancel{2}\cdot\cancel{\frac{1}{2}}\ln(5)} = 600$$

$$\Rightarrow P(0)\cdot e^{\ln(5)} = 600$$

$$\Rightarrow \frac{P(0)\cdot \cancel{5}}{\cancel{5}} = \frac{600}{5} \Rightarrow \boxed{P(0) = 120}$$

(b)Find an expression for the population after *t* hours.

$$P(t) = 120 \cdot e^{\frac{1}{2}\ln(5)\cdot t}$$

$$= 120 \cdot e^{\ln(5)\cdot\frac{1}{2}t} = 120\cdot\left[e^{\ln(5)}\right]^{\frac{1}{2}t}$$

$$= 120 \cdot 5^{\frac{1}{2}t}$$

OR $\boxed{P(t) = 120 \cdot 5^{\frac{t}{2}}}$

(c)Find the number of cells after 7 hours. (Round your answer to the nearest integer.)

$$7 \text{ hours} \Rightarrow t=7 \Rightarrow P(7) = 120\cdot 5^{7/2} \approx \boxed{33,541}$$

(d)Find the rate of growth (in bacteria/hour) after 7 hours. (Round your answer to the nearest integer.)

$$P'(t) = \frac{d}{dt}\left[120\cdot 5^{t/2}\right]$$

$$= 120\cdot\frac{d}{dt}\left[5^{t/2}\right]$$

$$= \overset{60}{\cancel{120}}\cdot 5^{t/2}\cdot\ln(5)\cdot\frac{1}{\cancel{2}}$$

$$= 60\cdot\ln(5)\cdot 5^{t/2} \Rightarrow P'(7) = 60\cdot\ln(5)\cdot 5^{7/2}$$

$$\approx \boxed{26,991}$$

(e) After how many hours will the population reach $\underbrace{200{,}000}_{P(t)}$? (Round your answer to one decimal place.)

Recall from part (b): $P(t) = 120 \cdot 5^{t/2}$

$$\Rightarrow \frac{200{,}000}{120} = \frac{\cancel{120} \cdot 5^{t/2}}{\cancel{120}}$$

$$\Rightarrow \frac{5000}{3} = 5^{t/2}$$

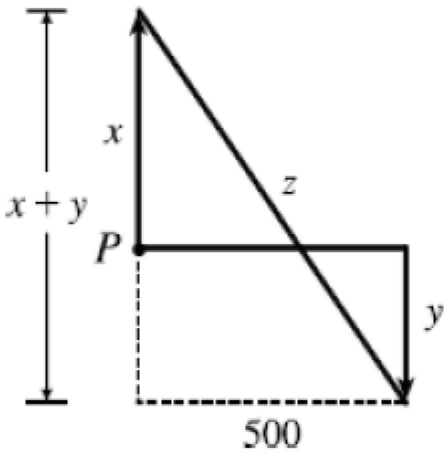
$$\Rightarrow \ln\left(\frac{5000}{3}\right) = \ln\left(5^{t/2}\right)$$

$$\Rightarrow 2\left[\ln\left(\frac{5000}{3}\right)\right] = \cancel{2}\left[\frac{t}{\cancel{2}} \ln(5)\right]$$

$$\Rightarrow \frac{2\ln\left(\frac{5000}{3}\right)}{\ln(5)} = \frac{t\,\cancel{\ln(5)}}{\cancel{\ln(5)}}$$

$$\Rightarrow \boxed{9.2}\,h \approx t$$

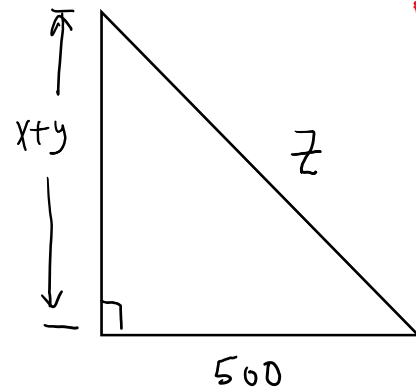
A man starts walking north at 2 ft/s from a point P. Five minutes later a woman starts walking south at 6 ft/s from a point 500 ft due east of P. At what rate (in ft/s) are the people moving apart 15 minutes after the woman starts walking? (Round your answer to two decimal places.)



Let x = Distance the man Walked

y = Distance the woman Walked

$\Rightarrow$



$$\boxed{\text{Find } \frac{dz}{dt}}$$

Given: $\boxed{\begin{array}{l} \frac{dx}{dt} = 2 \text{ ft/sec} \\ \frac{dy}{dt} = 6 \text{ ft/sec} \end{array}}$

$$z^2 = (x+y)^2 + 500^2$$

$$\frac{d}{dt}\left(z^2\right) = \frac{d}{dt}\left[(x+y)^2 + 500^2\right]$$

$$\Rightarrow 2z \cdot \frac{dz}{dt} = \frac{d}{dt}\left[(x+y)^2\right] + \frac{d}{dt}\left[500^2\right]$$

$$\Rightarrow 2z \cdot \frac{dz}{dt} = 2(x+y)\cdot\left(\frac{dx}{dt} + \frac{dy}{dt}\right) + 0$$

$$\boxed{2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)}$$

See Next Page

Recall: $D = R \cdot t$

$$\Rightarrow x = \underbrace{(2 \text{ ft/s})}_{\text{Man's speed}} \cdot \underbrace{(20 \text{ min})}_{\text{Time walked}} \cdot \underbrace{\left(60 \frac{\text{sec}}{\text{min}}\right)}_{\text{Change minutes to seconds}}$$

$$\Rightarrow \boxed{x = 2400 \text{ ft}}.$$

$$y = \underbrace{(6 \text{ ft/sec})}_{\text{Woman's speed}} \underbrace{(15 \text{ min})}_{\text{Time walked}} \underbrace{\left(60 \frac{\text{sec}}{\text{min}}\right)}_{\text{Change minutes to seconds}}.$$

$$\boxed{y = 5400 \text{ ft}}$$

$$z^2 = (x+y)^2 + 500^2$$

$$z^2 = (2400 + 5400)^2 + 250{,}000$$

$$\boxed{z = \sqrt{61{,}090{,}000}}$$

$$2z \cdot \frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

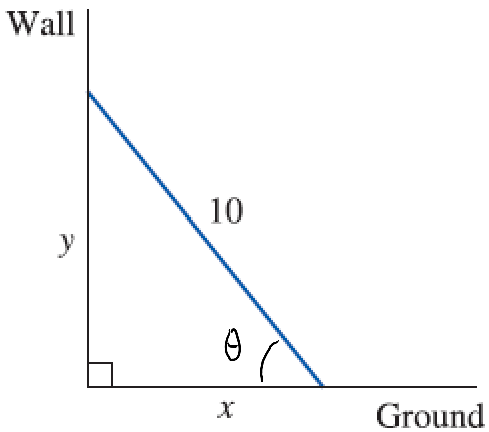
$$\Rightarrow \quad \frac{dz}{dt} = \frac{\cancel{2}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)}{\cancel{2}z}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)}{z}$$

$$= \frac{(2400 + 5400)(2 + 6)}{\sqrt{61{,}090{,}000}}$$

$$\approx \boxed{7.98} \text{ ft/sec}$$

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 0.9 ft/s, how fast (in rad/s) is the angle (in radians) between the ladder and the ground changing when the bottom of the ladder is 8 ft from the wall? (That is, find the angle's rate of change when the bottom of the ladder is 8 ft from the wall.)



Given: $\frac{dx}{dt} = 0.9$

Find: $\frac{d\theta}{dt}$ if $x = 8$

note: We have a right triangle

$\Rightarrow \cos\theta = \frac{x}{10}$

$\Rightarrow 10\cos\theta = x$

$\Rightarrow \frac{d}{dt}(10\cos\theta) = \frac{d}{dt}(x)$

$\Rightarrow 10 \cdot \frac{d}{dt}(\cos\theta) = \frac{dx}{dt}$

$\Rightarrow 10 \cdot (-\sin\theta) \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$

If $x = 8$ and $x^2 + y^2 = 10^2$

$\Rightarrow 8^2 + y^2 = 10^2$

$\Rightarrow 64 + y^2 = 100$

$\Rightarrow y^2 = 36$

$\Rightarrow y = 6$

$\Rightarrow \sin\theta = \frac{y}{10} = \frac{6}{10}$

$\Rightarrow 10 \cdot \left(-\frac{6}{10}\right)\frac{d\theta}{dt} = 0.9$

$\Rightarrow \frac{-6\frac{d\theta}{dt}}{-6} = \frac{0.9}{-6}$

$\Rightarrow \frac{d\theta}{dt} = \frac{9}{10} \div (-6)$

$= \frac{\cancel{9}^{3}}{10} \cdot \left(-\frac{1}{\cancel{6}_{2}}\right) = \boxed{-\frac{3}{20}}$ rad/s

Find the linear approximation $L(x)$ of the function $f(x) = \sqrt{4-x}$ at $a = 0$.

Linear Approximation

$$L(x) \approx f(a) + f'(a)(x-a)$$

$$a=0 \Rightarrow f(a) = f(0) = \sqrt{4-0} = \sqrt{4} = 2$$

$$f'(x) = \frac{d}{dx}\left(\sqrt{4-x}\right) = \frac{d}{dx}\left[(4-x)^{1/2}\right]$$

$$= \frac{1}{2}(4-x)^{1/2-1} \cdot \frac{d}{dx}(4-x)$$

$$= \frac{1}{2}(4-x)^{-1/2} \cdot (0-1)$$

$$= \frac{1}{2} \cdot \frac{1}{(4-x)^{1/2}} \cdot (-1)$$

$$= -\frac{1}{2\sqrt{4-x}}$$

$$\Rightarrow f'(0) = -\frac{1}{2\sqrt{4-0}} = -\frac{1}{2\sqrt{4}} = -\frac{1}{2 \cdot 2} = -\frac{1}{4}$$

$$a=0 \Rightarrow L(x) \approx f(0) + f'(0)(x-0)$$

$$= 2 + \left(-\frac{1}{4}\right)(x) \Rightarrow \boxed{L(x) = 2 - \frac{1}{4}x}$$

Use $L(x)$ to approximate the numbers $\sqrt{3.9}$ and $\sqrt{3.99}$. (Round your answers to four decimal places.)

note:

$$f(x) = L(x)$$

$$\Rightarrow \sqrt{4-x} = 2 - \frac{1}{4}x$$

$$\sqrt{3.9} \Rightarrow 4 - x = 3.9 \Rightarrow -x = 3.9 - 4$$

$$\Rightarrow -x = -0.1$$

$$\Rightarrow x = 0.1$$

$$\Rightarrow \sqrt{3.9} = 2 - \frac{1}{4}(0.1)$$

$$= 2 - 0.25(0.1)$$

$$= 2 - 0.025$$

$$= \boxed{1.975}$$

$$\sqrt{3.99} \Rightarrow 4 - x = 3.99 \Rightarrow -x = 3.99 - 4$$

$$\Rightarrow -x = -0.01$$

$$\Rightarrow x = 0.01$$

$$\Rightarrow \sqrt{3.99} = 2 - \frac{1}{4}(0.01)$$

$$= 2 - 0.25(0.01)$$

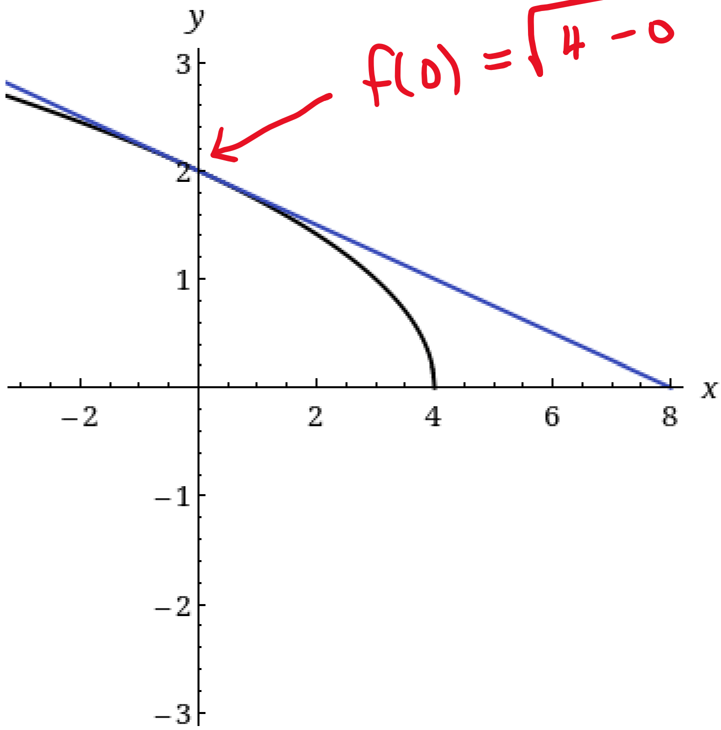
$$= 2 - 0.0025$$

$$= \boxed{1.9975}$$

Graph

$f(0) = \sqrt{4-0} = \sqrt{4} = 2 \Rightarrow$ y-intercept is $(0,2)$

$\Rightarrow$ this is the only graph with y-intercept @ $(0,2)$



Find the differential of each function.

(a)

$$y = \tan(\sqrt{3t})$$

$$\Rightarrow y' = \frac{d}{dt}\left(\tan\sqrt{3t}\right)$$

$$= \sec^2(\sqrt{3t}) \cdot \frac{d}{dt}(\sqrt{3t})$$

$$= \sec^2(\sqrt{3t}) \cdot \frac{1}{2\sqrt{3t}} \cdot \frac{d}{dt}(3t)$$

$$= \sec^2(\sqrt{3t}) \cdot \frac{1}{2\sqrt{3t}} \cdot 3$$

$$\Rightarrow dy = y'dt = \boxed{\frac{3\sec^2(\sqrt{3t})}{2\sqrt{3t}}\,dt}$$

Note:

$$\frac{d}{dt}(\sqrt{3t}) = \frac{d}{dt}(3t)^{1/2}$$

$$= \frac{1}{2}(3t)^{\frac{1}{2}-1} \cdot \frac{d}{dt}(3t)$$

$$= \frac{1}{2}(3t)^{-\frac{1}{2}} \cdot \frac{d}{dt}(3t)$$

$$= \frac{1}{2} \cdot \frac{1}{(3t)^{1/2}} \cdot 3$$

$$= \frac{1}{2\sqrt{3t}} \cdot 3$$

(b)

$$y = \frac{1-v^2}{1+v^2}$$

$$\Rightarrow y' = \underbrace{\frac{(1+v^2)\cdot\frac{d}{dv}(1-v^2) - (1-v^2)\cdot\frac{d}{dv}(1+v^2)}{(1+v^2)^2}}_{\text{Quotient Rule}}$$

$$= \frac{(1+v^2)\cdot(-2v) - (1-v^2)\cdot(2v)}{(1+v^2)^2}$$

$$= \frac{-2v - \cancel{2v^3} - 2v + \cancel{2v^3}}{(1+v^2)^2}$$

$$= \frac{-4v}{(1+v^2)^2} = -\frac{4v}{(1+v^2)^2} \Rightarrow dy = y'dv = \boxed{-\frac{4v}{(1+v^2)^2}\,dv}$$

The radius of a circular disk is given as 26 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error (in cm^2) in the calculated area of the disk. (Round your answer to two decimal places.)

Given: $r = 26$ cm, $dr = 0.2$ cm

Note: Area of a circle $= A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\Rightarrow dA = 2\pi r\,dr$$

$$dA = 2\pi(26)(0.2) \approx \boxed{32.67}\ cm^2$$

(b) What is the relative error? (Round your answer to four decimal places.)
What is the percentage error? (Round your answer to two decimal places.)

Recall:

$$\text{relative error} = \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r\,dr}{\pi r^2}$$

$$\Rightarrow \frac{dA}{A} = \frac{2dr}{r}$$

$$= \frac{2(0.2)}{26}$$

$$= \frac{0.4}{26} = \boxed{0.0154}$$

Since $0.0154 = 1.54\%$, percentage error $= \boxed{1.54\%}$

Use a linear approximation (or differentials) to estimate the given number.

$\sqrt{100.2}$

Let $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

$\Rightarrow dy = f'(x)\,dx = \frac{1}{2\sqrt{x}}\,dx$

Let $x = 100 \Rightarrow dx = \Delta x = 100.2 - 100 = 0.2$

note: $\Delta y \approx dy$

$\Rightarrow f(100.2) - f(100) \approx \frac{1}{2\sqrt{100}}(0.2)$

$\Rightarrow \sqrt{100.2} - \sqrt{100} \approx \frac{1}{2(10)} \cdot \frac{2}{10}$

$\Rightarrow \sqrt{100.2} - 10 \approx \frac{1}{\cancel{20}_{10}} \cdot \frac{\cancel{2}^{1}}{10}$

$\Rightarrow \sqrt{100.2} - 10 \approx \frac{1}{100}$

$\Rightarrow \sqrt{100.2} \approx 10 + \frac{1}{100} = 10 + 0.01 = \boxed{10.01}$

Find the derivative.

$$y = \arctan(6\tanh(x))$$

$$y' = \frac{d}{dx}\left[\arctan\left(6\tanh(x)\right)\right]$$

$$= \frac{1}{1+\left(6\tanh(x)\right)^2} \cdot 6\,\mathrm{sech}^2(x)$$

$$= \boxed{\frac{6\,\mathrm{sech}^2(x)}{1+36\tanh^2(x)}}$$

Recall:

$$\boxed{\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{1}{1+u^2}\cdot\frac{du}{dx}}$$

and

$$\boxed{\frac{d}{dx}\left[\tanh(x)\right] = \mathrm{sech}^2 x}$$

$\downarrow$

Here, $u = 6\tanh(x)$

$$\Rightarrow \frac{du}{dx} = 6\cdot\frac{d}{dx}\left[\tanh(x)\right]$$

$$= 6\cdot\mathrm{sech}^2(x)$$

$$= 6\,\mathrm{sech}^2(x)$$

If sinh(x) = $\frac{12}{35}$, find the values of the other hyperbolic functions at *x*.

Recall: $\cosh^2(x) - \sinh^2(x) = 1$

$$\Rightarrow \cosh^2(x) - \left(\frac{12}{35}\right)^2 = 1$$

$$\Rightarrow \cosh^2(x) - \frac{144}{1225} = 1$$

$$\Rightarrow \cosh^2(x) = 1 + \frac{144}{1225}$$

$$\Rightarrow \cosh^2(x) = \frac{1225}{1225} + \frac{144}{1225}$$

$$\Rightarrow \cosh^2(x) = \frac{1369}{1225}$$

$$\Rightarrow \cosh(x) = \sqrt{\frac{1369}{1225}} = \boxed{\frac{37}{35}}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{12/35}{37/35} = \frac{12}{35} \div \frac{37}{35} = \frac{12}{\cancel{35}_{1}} \cdot \frac{\cancel{35}^{1}}{37} = \boxed{\frac{12}{37}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{1}{12/35} = 1 \div \frac{12}{35} = 1 \cdot \frac{35}{12} = \boxed{\frac{35}{12}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{1}{37/35} = 1 \div \frac{37}{35} = 1 \cdot \frac{35}{37} = \boxed{\frac{35}{37}}$$

$$\coth(x) = \frac{1}{\tanh(x)} = \frac{1}{12/37} = 1 \div \frac{12}{37} = 1 \cdot \frac{37}{12} = \boxed{\frac{37}{12}}$$