(13) $f(x) = x + \frac{4}{x}$, $[0.2, 8]$

$\Rightarrow f(x) = x + 4x^{-1}$

$\Rightarrow f'(x) = 1 - 4x^{-2} = 1 - 4 \cdot \frac{1}{x^2} = 1 - \frac{4}{x^2}$

$\Rightarrow f'(x) = 0$ when $1 - \frac{4}{x^2} = 0$

$\Rightarrow \frac{x^2}{x^2} - \frac{4}{x^2} = 0$

$\Rightarrow \frac{x^2 - 4}{x^2} = 0$

$\Rightarrow x^2 - 4 = 0 \Rightarrow (x+2)(x-2) = 0$

$\Rightarrow x = -2$ or $x = 2$

but $x = -2$ is <u>not</u> in $[0.2, 8]$ so we only need to consider $x = 2$

Since $f(x) = x + \frac{4}{x}$,

$x = 0.2 \Rightarrow f(0.2) = 0.2 + \frac{4}{0.2} = 20.2$ ← largest = Abs. max

$x = 2 \Rightarrow f(2) = 2 + \frac{4}{2} = 2 + 2 = 4$ ← smallest = Abs. min

$x = 8 \Rightarrow f(8) = 8 + \frac{4}{8} = 8 + \frac{1}{2} = 8.5$

$\Rightarrow$ $\boxed{\text{Absolute maximum is } 20.2 \quad \text{Absolute minimum is } 4}$

14. Find the absolute maximum and absolute minimum values of *f* on the given interval.

$$f(x) = \frac{x}{x^2 - x + 4}, \quad [0, 6]$$

$$f'(x) = \frac{(x^2 - x + 4) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(x^2 - x + 4)}{(x^2 - x + 4)^2}$$

$$= \frac{(x^2 - x + 4) \cdot 1 - x \cdot (2x - 1)}{(x^2 - x + 4)^2}$$

$$= \frac{x^2 - x + 4 - 2x^2 + x}{(x^2 - x + 4)^2}$$

$$= \frac{4 - x^2}{(x^2 - x + 4)^2}$$

note: $f'(x) = 0$ if $4 - x^2 = 0 \Rightarrow (2 + x)(2 - x) = 0$

$\Rightarrow 2 + x = 0$ or $2 - x = 0$

$\Rightarrow \cancel{x = -2}$ or $\boxed{x = 2}$

not in $[0, 6]$

only critical number

$$\Rightarrow f(0) = \frac{0}{0^2 - 0 + 4} = \frac{0}{4} = 0 \leftarrow \text{smallest}$$

$$f(2) = \frac{2}{2^2 - 2 + 4} = \frac{2}{6} = \frac{1}{3} \leftarrow \text{largest}$$

$$f(6) = \frac{6}{6^2 - 6 + 4} = \frac{6}{34} = \frac{3}{17}$$

$$\boxed{\begin{array}{l} \text{absolute minimum} = 0 \\ \text{absolute maximum} = \frac{1}{3} \end{array}}$$

15. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(t) = 2\cos(t) + \sin(2t), \quad \left[0, \frac{\pi}{2}\right]$$

$$f'(t) = 2 \cdot \frac{d}{dt}\left[\cos(t)\right] + \frac{d}{dt}\left[\sin(2t)\right]$$

$$= 2\left(-\sin(t)\right) + \cos(2t) \cdot \underbrace{\frac{d}{dt}(2t)}_{2}$$

$$= -2\sin(t) + 2\cos(2t)$$

Critical numbers:

$$-2\sin(t) + 2\cos(2t) = 0$$

$$-2\sin(t) + 2\left(1 - 2\sin^2(t)\right) = 0$$

$$-2\sin(t) + 2 - 4\sin^2(t) = 0$$

$$\Rightarrow -4\sin^2(t) - 2\sin(t) + 2 = 0$$

$$\Rightarrow -2\left[2\sin^2(t) + \sin(t) - 1\right] = 0$$

$$\Rightarrow -2\left[2\sin(t) - 1\right]\left[\sin(t) + 1\right] = 0$$

$$\Rightarrow 2\sin(t) - 1 = 0 \quad \text{OR} \quad \sin(t) + 1 = 0$$

$$\Rightarrow \sin(t) = \frac{1}{2} \quad \text{OR} \quad \sin(t) = -1$$

t in $\left[0, \frac{\pi}{2}\right] \Rightarrow t = \frac{\pi}{6}$

$\boxed{\text{no solution}}$

since $0 \le \sin(t) \le 1$ in $\left[0, \frac{\pi}{2}\right]$

$$t=0 \Rightarrow f(0) = 2\underbrace{\cos(0)} + \sin(2\cdot 0)$$
$$= 2\cdot 1 + \sin(0)$$
$$= 2 + 0 = \boxed{2}$$

$$t=\frac{\pi}{6} \Rightarrow f\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right) + \sin\left(2\cdot\frac{\pi}{6}\right)$$
$$= 2\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3} + \frac{\sqrt{3}}{2}$$
$$= \frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \boxed{\frac{3\sqrt{3}}{2}} \leftarrow \text{largest}$$

$$t=\frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{2}\right) = 2\underbrace{\cos\left(\frac{\pi}{2}\right)} + \sin\left(\cancel{2}^{1}\cdot\frac{\pi}{\cancel{2}}\right)$$
$$= 2(0) + \sin(\pi)$$
$$= 0 + 0 = \boxed{0} \leftarrow \text{smallest}$$

Answers:

Absolute Minimum value $= 0$

Absolute Maximum value $= \frac{3\sqrt{3}}{2}$

17. Find the absolute maximum and absolute minimum values of f on the given interval.

$f(x) = \ln(x^2 + 3x + 14), \quad [-2, 1]$

$$f'(x) = \frac{1}{x^2+3x+14} \cdot \frac{d}{dx}(x^2+3x+14)$$

$$= \frac{1}{x^2+3x+14} \cdot (2x+3) = \frac{2x+3}{x^2+3x+14}$$

$$\Rightarrow f'(x) = 0 \quad \text{if} \quad 2x+3=0 \Rightarrow x = -\frac{3}{2}$$

Critical number in $[2, 1]$

$$\Rightarrow f(-2) = \ln\left[(-2)^2 + 3(-2) + 14\right]$$

$$= \ln[4 - 6 + 14] = \ln(12) \leftarrow \approx 2.48$$

$$f\left(-\frac{3}{2}\right) = \ln\left[\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 14\right]$$

$$= \ln\left[\frac{9}{4} - \frac{9}{2} + 14\right]$$

$$= \ln\left[\frac{9}{4} - \frac{18}{4} + \frac{56}{4}\right]$$

$$= \ln\left[\frac{9-18+56}{4}\right] = \boxed{\ln\left(\frac{47}{4}\right)} \leftarrow \approx 2.46$$

smallest

$$f(1) = \ln\left[(1)^2 + 3(1) + 14\right]$$

$$= \ln[1 + 3 + 14] = \boxed{\ln(18)} \leftarrow \approx 2.89$$

largest

Answers: $\boxed{\text{Absolute minimum} = \ln\left(\frac{47}{4}\right), \text{ Absolute Maximum} = \ln(18)}$

18. After an alcoholic beverage is consumed, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 0.135te^{-2.802t}$$

models the average BAC, measured in g/dL, of a group of eight male subjects, t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC (in g/dL) during the first 5 hours? (Round your answer to three decimal places.)

$0 \le t \le 5 \Rightarrow [0, 5]$

NOTE: To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus, the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Step 1: Critical numbers

$$C(t) = 0.135te^{-2.802t}$$

$$\Rightarrow C'(t) = \frac{d}{dt}(0.135t)\cdot e^{-2.802t} + 0.135t\cdot\frac{d}{dt}\left(e^{-2.802t}\right)$$

$$= 0.135\cdot e^{-2.802t} + 0.135t\cdot e^{-2.802t}\cdot\frac{d}{dt}(-2.802t)$$

$$= 0.135e^{-2.802t} + 0.135te^{-2.802t}\cdot(-2.802)$$

$$= 0.135e^{-2.802t}\left(1 + t\cdot(-2.802)\right)$$

$$= 0.135e^{-2.802t}(1 - 2.802t)$$

$\Rightarrow$ $C'(t) = 0$ when $1 - 2.802t = 0$

$$\Rightarrow 1 = 2.802t$$

$$\Rightarrow \frac{1}{2.802} = t$$

$\Rightarrow$ $t \approx 0.357$ ← only critical number in $[0, 5]$

<u>Step 2</u>: Evaluate function at end points <u>and</u> critical numbers found in step 1.

$$C(0) = 0.135(0)e^{-2.802(0)} = 0$$

$$C(0.357) = 0.135(0.357)e^{-2.802(0.357)} \approx 0.01772\ldots$$

$$C(5) = 0.135(5)e^{-2.802(5)} \approx 0.0000005569\ldots$$

<u>Step 3</u>: Largest from step 2 = absolute maximum
smallest from step 2 = absolute minimum

$\Rightarrow$ Maximum BAC = $\boxed{0.018}$ g/dL

at $t \approx \boxed{0.36}$ hr.

26. Let $f(x) = (x - 3)^{-2}$. Find all values of c in $(2, 5)$ such that $f(5) - f(2) = f'(c)(5 - 2)$.
(Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

$$f'(x) = \frac{d}{dx}\left[(x-3)^{-2}\right] = -2(x-3)^{-3} = -2 \cdot \frac{1}{(x-3)^3} = \frac{-2}{(x-3)^3}$$

$$f(5) = (5-3)^{-2} = (2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$f(2) = (2-3)^{-2} = (-1)^{-2} = \frac{1}{(-1)^2} = \frac{1}{1} = 1$$

$\Rightarrow f(5) - f(2) = f'(c) \cdot (5-2)$ becomes

$$\frac{1}{4} - 1 = \frac{-2}{(c-3)^3}(3)$$

$$\Rightarrow -\frac{3}{4} = \frac{-6}{(c-3)^3}$$

$$\Rightarrow -3(c-3)^3 = 4 \cdot (-6)$$

$$\Rightarrow \frac{\cancel{-3}(c-3)^3}{\cancel{-3}} = \frac{-24}{-3}$$

$$\Rightarrow (c-3)^3 = 8$$

$$\Rightarrow \sqrt[3]{(c-3)^3} = \sqrt[3]{8}$$

$\Rightarrow c - 3 = 2 \Rightarrow c = 5$, which is <u>NOT</u> in $(2,5)$.

$\Rightarrow c = \boxed{\text{DNE}}$

* This does not contradict the Mean Value Theorem since f is not continuous at $x=3$.

28. Suppose that $4 \le f'(x) \le 5$ for all values of x. What are the minimum and maximum possible values of $f(5) - f(1)$?

By the Mean Value Theorem, $f(5) - f(1) = f'(c) \cdot (5-1)$ for some c in $[1, 5]$.

$$\Rightarrow \quad f(5) - f(1) = f'(c) \cdot 4$$

$$\Rightarrow \quad f(5) - f(1) = 4f'(c).$$

<u>Given</u>: $4 \le f'(x) \le 5$

$$\Rightarrow \quad 4 \cdot 4 \le 4 \cdot f'(x) \le 4 \cdot 5$$

$$\Rightarrow \quad 16 \le 4f'(x) \le 20$$

$$\Rightarrow \quad 16 \le f(5) - f(1) \le 20$$

Minimum possible value

Maximum possible value

$y = \tan x = 1$

35. Consider the following. (If an answer does not exist, enter DNE.)

$$f(x) = 7\sin(x) + 7\cos(x), \quad 0 \le x \le 2\pi$$

$$f'(x) = 7 \cdot \cos(x) + 7 \cdot (-\sin(x))$$

$$= 7\cos(x) - 7\sin(x)$$

$\Rightarrow f'(x) = 0$ when $7\cos(x) - \cancel{7\sin(x)} = 0 + 7\sin(x)$

$+ \cancel{7\sin(x)}$

$$\frac{7\cos(x)}{7\cos(x)} = \frac{\cancel{7}\sin(x)}{\cancel{7}\cos(x)}$$

$$1 = \tan(x)$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$f'(x) = 7\cos(x) - 7\sin(x)$:

$x = \frac{\pi}{6} \Rightarrow 7\cos(\frac{\pi}{6}) - 7\sin(\frac{\pi}{6}) = 7 \cdot \frac{\sqrt{3}}{2} - 7 \cdot \frac{1}{2} = \frac{7\sqrt{3}}{2} - \frac{7}{2} = \frac{7\sqrt{3}-7}{2} \approx 2.6$

$= +$

$x = \pi \Rightarrow 7\cos(\pi) - 7\sin(\pi) = 7 \cdot (-1) - 7 \cdot 0 = -7 - 0 = -7 = -$

$x = \frac{3\pi}{2} \Rightarrow 7\cos\left(\frac{3\pi}{2}\right) - 7\sin\left(\frac{3\pi}{2}\right) = 7(0) - 7\cdot(-1)$

$= 0 + 7 = 7 = f$

(a) Find the interval(s) on which f is increasing. (Enter your answer using interval notation.)

$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

(b) Find the interval(s) on which f is decreasing. (Enter your answer using interval notation.)

$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(c) Find the local minimum and maximum values of f.

$f(x) = 7\sin(x) + 7\cos(x)$

max @ $x = \frac{\pi}{4} \Rightarrow f\left(\frac{\pi}{4}\right) = 7\sin\left(\frac{\pi}{4}\right) + 7\cos\left(\frac{\pi}{4}\right)$

$= 7\cdot\frac{\sqrt{2}}{2} + 7\cdot\frac{\sqrt{2}}{2}$

$= \frac{7\sqrt{2}}{2} + \frac{7\sqrt{2}}{2}$

$= \frac{\cancel{14}^{7}\sqrt{2}}{\cancel{2}}$

$= 7\sqrt{2}$ ← Local max. of f

Local min @ $x = \frac{5\pi}{4}$

$\Rightarrow f\left(\frac{5\pi}{4}\right) = 7\sin\left(\frac{5\pi}{4}\right) + 7\cos\left(\frac{5\pi}{4}\right)$

$= 7\cdot\left(-\frac{\sqrt{2}}{2}\right) + 7\left(-\frac{\sqrt{2}}{2}\right)$

$$= -\frac{7\sqrt{2}}{2} - \frac{7\sqrt{2}}{2}$$

$$= -\frac{\cancel{14}^{7}\sqrt{2}}{\cancel{2}}$$

$$= -7\sqrt{2}$$ ← Local min of f

39. Let $f(x) = x^4(x-3)^3$.

(a)Find the critical numbers of the function f. (Enter your answers from smallest to largest.)

$$\Rightarrow f'(x) = \frac{d}{dx}\left[x^4(x-3)^3\right]$$

$$= \frac{d}{dx}(x^4)\cdot(x-3)^3 + x^4\cdot\frac{d}{dx}\left[(x-3)^3\right]$$

$$= 4x^3\cdot(x-3)^3 + x^4\cdot 3(x-3)^2\cdot\underbrace{\frac{d}{dx}(x-3)}_{1}$$

$$= 4x^3(x-3)^3 + 3x^4(x-3)^2$$

$$= x^3(x-3)^2\left[4(x-3) + 3x\right]$$

$$= x^3(x-3)^2(4x-12+3x)$$

$$= x^3(x-3)^2(7x-12)$$

$$\Rightarrow f'(x) = 0 \text{ if } x=0,\ x=3,\ x=\frac{12}{7}$$

$$\Rightarrow x_1 = 0,\ x_2 = 12/7,\ x_3 = 3$$

(b) What does the Second Derivative Test tell you about the behavior of *f* at these critical numbers?

$\rightarrow$ Need f''

$$\Rightarrow f''(x) = \frac{d}{dx}\left[f'(x)\right]$$

$$= \frac{d}{dx}\left[x^3(x-3)^2(7x-12)\right]$$

$$= \frac{d}{dx}(x^3)\cdot(x-3)^2(7x-12) + x^3\cdot\frac{d}{dx}\left[(x-3)^2\right](7x-12) + x^3(x-3)^2\cdot\frac{d}{dx}(7x-12)$$

$$= 3x^2(x-3)^2(7x-12) + x^3\cdot 2(x-3)\cdot 1\cdot(7x-12) + x^3(x-3)^2\cdot 7$$

$$= 3x^2(x-3)^2(7x-12) + 2x^3(x-3)(7x-12) + 7x^3(x-3)^2$$

$$= x^2(x-3)\left[3(x-3)(7x-12) + 2x(7x-12) + 7x(x-3)\right]$$

$$\begin{array}{r} 7x^2-12x \\ -21x+36 \\ \hline (7x^2-33x+36) \end{array}$$

$$= x^2(x-3)\left[21x^2-99x+108+14x^2-24x+7x^2-21x\right]$$

$$= x^2(x-3)(42x^2-144x+108)$$

$$x_1 = 0 \Rightarrow f''(0) = 0^2(0-3)(42\cdot 0^2-144\cdot 0+108) = 0$$

$$x_2 = \frac{12}{7} \Rightarrow f''\left(\frac{12}{7}\right) = \left(\frac{12}{7}\right)^2\left(\frac{12}{7}-3\right)\left(42\left(\frac{12}{7}\right)^2-144\left(\frac{12}{7}\right)+108\right)$$

$$= (+)(-)(-) = +$$

$$x_3 = 3 \Rightarrow f''(3) = 3^2\underbrace{(3-3)}_{0}(42(3)^2 - 144(3) + 108) = 0$$

$f''(0) = 0 \leftarrow$ Inconclusive

$f''(\frac{12}{7}) = + \leftarrow$ Local minimum at $x = 12/7$

$f''(3) = 0 \leftarrow$ Inconclusive

(c)

Recall:

$$f'(x) = x^3(x-3)^2(7x-12)$$

$x = -1 \rightarrow f'(-1) = (-1)^3(-1-3)^2(7\cdot(-1)-12)$
$= (-)(+)(-) = +$

$x = 1 \Rightarrow f'(1) = (1)^3(1-3)^2(7\cdot 1-12)$
$= (+)(+)(-) = -$

$x = 2 \Rightarrow f'(2) = (2)^3(2-3)^2(7\cdot 2-12)$
$= (+)(+)(+) = +$

$$x=4 \Rightarrow f'(4) = 4^3 (4-3)^2 (7\cdot 4 - 12)$$
$$= + \quad + \quad + \quad = +$$

55. Find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin(\theta)}{1+\cos(2\theta)}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\frac{d}{d\theta}(1-\sin(\theta))}{\frac{d}{d\theta}(1+\cos(2\theta))}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{0 - \cos(\theta)}{0 + (-\sin(2\theta)\cdot 2)}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\cancel{-}\cos(\theta)}{\cancel{-}2\sin(2\theta)}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\cos(\theta)}{2\sin(2\theta)}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\frac{d}{d\theta}[\cos(\theta)]}{\frac{d}{d\theta}[2\sin(2\theta)]}$$

$\theta \to \pi/2$

$$\Rightarrow \frac{1-\sin(\frac{\pi}{2})}{1+\cos(\cancel{2}\cdot\frac{\pi}{\cancel{2}})}$$

$$= \frac{1-1}{1+\cos\pi}$$

$$= \frac{1-1}{1+(-1)} = \frac{0}{0}$$

$\Rightarrow$ use L'Hopital's Rule.

$\theta \to \pi/2$

$$\Rightarrow \frac{\cos(\frac{\pi}{2})}{2\sin(\cancel{2}\cdot\frac{\pi}{\cancel{2}})} = \frac{0}{2\cdot 0}$$

$$= \frac{0}{0}$$

$\Rightarrow$ use L'Hopital's Rule again.

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{-\sin(\theta)}{2 \cdot \cos(2\theta) \cdot 2}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{-\sin(\theta)}{4\cos(2\theta)}$$

$$= \frac{-\sin\left(\frac{\pi}{2}\right)}{4\cos\left(\cancel{2} \cdot \frac{\pi}{\cancel{2}}\right)}$$

$$= \frac{-1}{4 \cdot (-1)} = \frac{-1}{-4} = \boxed{\frac{1}{4}}$$

Answer

Consider the equation below. (If an answer does not exist, enter DNE.)

$$f(x) = 5\cos^2(x) - 10\sin(x), \qquad 0 \le x \le 2\pi$$

(a)Find the interval on which f is increasing. (Enter your answer using interval notation.)

$$f'(x) = \frac{d}{dx}\left[5\cos^2(x) - 10\sin(x)\right]$$

$$= 5\cdot\frac{d}{dx}\left[\cos^2(x)\right] - 10\cdot\frac{d}{dx}\left[\sin(x)\right]$$

$$= 5\cdot 2\cos(x)\cdot\frac{d}{dx}\left[\cos(x)\right] - 10\cdot\left[\cos(x)\right]$$

$$= 10\cos(x)\cdot(-\sin(x)) - 10\cos(x)$$

$$= -10\cos(x)\sin(x) - 10\cos(x)$$

$$= -10\cos(x)\left[\sin(x) + 1\right]$$

$$f'(x) = 0 \Rightarrow -10\cos(x)\left[\sin(x)+1\right] = 0$$

$$\Rightarrow \cos(x) = 0 \quad \text{OR} \quad \sin(x) + 1 = 0$$

$$\downarrow \qquad\qquad \sin(x) = -1$$

$$0 \le x \le 2\pi \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad \downarrow$$

$$x = \frac{3\pi}{2}$$

$\Rightarrow$ critical points in $[0, 2\pi]$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$\Rightarrow$ mark the critical points on $[0, 2\pi]$

$\Rightarrow$

Sign of f': $-$ $+$ $-$

Test points:

$x = \frac{\pi}{4} \Rightarrow f'(\frac{\pi}{4}) = \underbrace{-10}_{-}\ \underbrace{\cos(\frac{\pi}{4})}_{+}\ \underbrace{[\sin(\frac{\pi}{4}) + 1]}_{+} = -$ $\Rightarrow$ Decreasing on $(0, \frac{\pi}{2})$

$x = \pi \Rightarrow f'(\pi) = \underbrace{-10}_{-}\ \underbrace{\cos(\pi)}_{-}\ \underbrace{[\sin(\pi) + 1]}_{+} = +$ $\Rightarrow$ Increasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$

$x = \frac{7\pi}{4} \Rightarrow f'(\frac{7\pi}{4}) = \underbrace{-10}_{-}\ \underbrace{\cos(\frac{7\pi}{4})}_{+}\ \underbrace{[\sin(\frac{7\pi}{4}) + 1]}_{+} = -$ $\Rightarrow$ Decreasing on $(\frac{3\pi}{2}, 2\pi)$

$\Rightarrow$ Interval of increase: $(\frac{\pi}{2}, \frac{3\pi}{2})$

Find the interval on which f is decreasing. (Enter your answer using interval notation.)

Interval of decrease: $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

(b) Find the local minimum and maximum values of f.

Local minimum:

$f(\frac{\pi}{2}) = 5\cos^2(\frac{\pi}{2}) - 10\sin(\frac{\pi}{2})$

$= 5(0)^2 - 10(1)$

$= 0 - 10 = \boxed{-10}$

Local Maximum:

$f(\frac{3\pi}{2}) = 5\cos^2(\frac{3\pi}{2}) - 10\sin(\frac{3\pi}{2})$

$= 5(0)^2 - 10(-1)$

$= 0 + 10 = \boxed{10}$

(c) Find the inflection points. (Order your answers from smallest to largest x, then from smallest to largest y.)

$$f''(x) = \frac{d}{dx}\left[-10\cos(x)\left(\sin(x)+1\right)\right]$$ ← use product Rule

$$= \left(\sin(x)+1\right)\cdot\frac{d}{dx}\left[-10\cos(x)\right] + \left(-10\cos(x)\right)\cdot\frac{d}{dx}\left[\sin(x)+1\right]$$

$$= \left(\sin(x)+1\right)\cdot\left[-10\cdot\left(-\sin(x)\right)\right] - 10\cos(x)\cdot\left[\cos(x)\right]$$

$$= \left(\sin(x)+1\right)\left(10\sin(x)\right) - 10\cos^2(x)$$

$$= 10\sin^2(x) + 10\sin(x) - 10\cos^2(x)$$

$$= 10\sin^2(x) + 10\sin(x) - 10\left(1-\sin^2(x)\right)$$

$$= 10\sin^2(x) + 10\sin(x) - 10 + 10\sin^2(x)$$

$$= 20\sin^2(x) + 10\sin(x) - 10$$

$$f''(x) = 0 \Rightarrow 20\sin^2(x) + 10\sin(x) - 10 = 0$$

$$\Rightarrow 10\left(2\sin^2(x) + \sin(x) - 1\right) = 0$$

$$\Rightarrow 10\left(2\sin(x) - 1\right)\left(\sin(x)+1\right) = 0$$

$$\Rightarrow 2\sin(x) - 1 = 0 \quad \text{OR} \quad \sin(x)+1 = 0$$

$$\Rightarrow \sin(x) = \frac{1}{2} \quad \text{OR} \quad \sin(x) = -1$$

$$0 \le x \le 2\pi \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{OR} \quad x = \frac{3\pi}{2}$$

$\cap$ = Concave downward

$\cup$ = Concave upward

Sign of f'': $\cap$ $-$ | $\cup$ $+$ | $\cap$ $-$ | $\cup$ $-$

$0 \quad \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{3\pi}{2} \quad 2\pi$

Test points: $\boxed{\frac{\pi}{12}}$ $\boxed{\frac{\pi}{3}}$ $\boxed{\pi}$ $\boxed{\frac{7\pi}{4}}$

$$f''(x) = 10\left(2\sin(x) - 1\right)\left(\sin(x) + 1\right)$$

$$x = \frac{\pi}{12} \Rightarrow f''\left(\frac{\pi}{12}\right) = \underbrace{10}_{+}\underbrace{\left(2\sin\left(\frac{\pi}{12}\right) - 1\right)}_{-}\underbrace{\left(\sin\left(\frac{\pi}{12}\right) + 1\right)}_{+} = - \Rightarrow \text{C.D. on } \left(0, \frac{\pi}{6}\right)$$

$$x = \frac{\pi}{3} \Rightarrow f''\left(\frac{\pi}{3}\right) = \underbrace{10}_{+}\underbrace{\left(2\sin\left(\frac{\pi}{3}\right) - 1\right)}_{+}\underbrace{\left(\sin\left(\frac{\pi}{3}\right) + 1\right)}_{+} = + \Rightarrow \text{C.U. on } \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$$

$$x = \pi \Rightarrow f''(\pi) = \underbrace{10}_{+}\underbrace{\left(2\sin(\pi) - 1\right)}_{-}\underbrace{\left(\sin(\pi) + 1\right)}_{+} = - \Rightarrow \text{C.D. on } \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$$

$$x = \frac{7\pi}{4} \Rightarrow f''(\pi) = \underbrace{10}_{+}\underbrace{\left(2\sin\left(\frac{7\pi}{4}\right) - 1\right)}_{-}\underbrace{\left(\sin\left(\frac{7\pi}{4}\right) + 1\right)}_{+} = - \Rightarrow \text{C.D. on } \left(\frac{3\pi}{2}, 2\pi\right)$$

$\cap$ | $\cup$ | $\cap$ | $\cap$

$0 \quad \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{3}{2} \quad 2\pi$

Inflection point @ $x = \frac{\pi}{6}$

$\cap \to \cup$

Inflection point @ $x = \frac{5\pi}{6}$

$\cup \to \cap$

Inflection points:

$x = \frac{\pi}{6} \Rightarrow f\left(\frac{\pi}{6}\right) = 5\cos^2\left(\frac{\pi}{6}\right) - 10\sin\left(\frac{\pi}{6}\right)$

$= 5\left(\frac{\sqrt{3}}{2}\right)^2 - 10\left(\frac{1}{2}\right) = 5\left(\frac{3}{4}\right) - 5 = \frac{15}{4} - \frac{20}{4} = -\frac{5}{4}$

$\Rightarrow \boxed{\left(\frac{\pi}{6}, -\frac{5}{4}\right)}$

$x = \frac{5\pi}{6} \Rightarrow f\left(\frac{5\pi}{6}\right) = 5\cos^2\left(\frac{5\pi}{6}\right) - 10\sin\left(\frac{5\pi}{6}\right)$

$= 5\left(-\frac{\sqrt{3}}{2}\right)^2 - 10\left(\frac{1}{2}\right)$

$= 5\left(\frac{3}{4}\right) - 5 = \frac{15}{4} - \frac{20}{4} = -\frac{5}{4}$

$\Rightarrow \boxed{\left(\frac{5\pi}{6}, -\frac{5}{4}\right)}$

Find the interval on which *f* is concave up. (Enter your answer using interval notation.)

$\boxed{\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)}$ ← from work on last page

Find the interval on which *f* is concave down. (Enter your answer using interval notation.)

$\boxed{\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)}$ ← from work on last page

39. Let $f(x) = x^4(x - 1)^3$.

(a)Find the critical numbers of the function f. (Enter your answers from smallest to largest.)
smallest value

Recall: a critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c) = \text{undefined}$.

$\Rightarrow$ we need to find $f'(x)$ first.

Here,
$$f'(x) = \frac{d}{dx}\left[x^4(x-1)^3\right]$$
$$= \frac{d}{dx}(x^4)\cdot(x-1)^3 + x^4\cdot\frac{d}{dx}\left[(x-1)^3\right]$$
$$= 4x^3\cdot(x-1)^3 + x^4\cdot 3(x-1)^2\cdot\underbrace{\frac{d}{dx}(x-1)}_{1}$$
$$= 4x^3(x-1)^3 + 3x^4(x-1)^2$$
$$= x^3(x-1)^2\left[4(x-1) + 3x\right]$$
$$= x^3(x-1)^2(4x - 4 + 3x)$$
$$= x^3(x-1)^2(7x-4)$$

$\Rightarrow f'(x) = 0$ if $x^3(x-1)^2(7x-4) = 0$

$\Rightarrow x^3 = 0$ OR $(x-1)^2 = 0$ OR $7x - 4 = 0$

$\Rightarrow \boxed{x = 0}$ OR $x - 1 = 0$ OR $7x = 4$

$\boxed{x = 1}$ $\boxed{x = 4/7}$

smallest largest

$$\Rightarrow \boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 4/7 \\ x_3 &= 1 \end{aligned}}$$ ← Answers to part (a).

(b)What does the Second Derivative Test tell you about the behavior of *f* at these critical numbers?

Recall: Second Derivative Test

(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c

(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c

(c) If $f''(c)=0$, the test is inconclusive.

$\Rightarrow$ we need to find $f''(x)$ first.

Here, $f''(x) = \frac{d}{dx}\left[x^3(x-1)^2(7x-4)\right]$

$$= \frac{d}{dx}\left[(x-1)^2 \cdot x^3(7x-4)\right]$$

$$= \frac{d}{dx}\left[(x-1)^2(7x^4-4x^3)\right]$$

$$= \frac{d}{dx}\left[(x-1)^2\right](7x^4-4x^3) + (x-1)^2 \cdot \frac{d}{dx}\left[7x^4-4x^3\right]$$

$$= 2(x-1)\cdot 1\,(7x^4-4x^3) + (x-1)^2 \cdot (28x^3-12x^2)$$

$$= 2(x-1)x^3(7x-4) + (x-1)^2 \cdot 4x^2(7x-3)$$

$$= 2x^3(x-1)(7x-4) + 4x^2(x-1)^2(7x-3)$$

$\Rightarrow$ at $x_1=0$, $f''(0) = 2\cdot 0^3(0-1)(7\cdot 0-4) + 4\cdot 0^2(0-1)^2(7\cdot 0-3)$

$$= 0 + 0 = 0$$

At x_1

the second derivative test $\boxed{\text{is inconclusive}}$

at $x_2 = 4/7$, $f''(\frac{4}{7}) = 2(\frac{4}{7})^3(\frac{4}{7}-1)(7\cdot\frac{4}{7}-4) + 4(\frac{4}{7})^2(\frac{4}{7}-1)^2(7\cdot\frac{4}{7}-3)$

$= + \cdot + \cdot - \cdot 0 + + \cdot + \cdot + \cdot +$

$= 0 + + = + \Rightarrow \boxed{f''(\frac{4}{7}) > 0}$

At x_2

the second derivative test $\boxed{\text{indicates a local minimum}}$

at $x_3 = 1$, $f''(1) = 2(1)^3(1-1)(7\cdot 1-4) + 4(1)^2(1-1)^2(7\cdot 1-3)$

$= + \cdot + \cdot 0 \cdot + + + \cdot + \cdot 0 \cdot +$

$= 0 + 0 = \boxed{0}$

At x_3

the second derivative test $\boxed{\text{is inconclusive}}$.

(c)What does the First Derivative Test tell you? Note what the First Derivative Test tells you that Second Derivative Test does not.

Recall: First Derivative Test

Let c = a critical number of a continuous function f.

(a) If f' changes from "+" to "−" from left to right of c, the f has a local maximum at c.

(b) If f' changes from "−" to "+" from left to right of c, then f has a local minimum at c

(c) If no sign change from left side of c to the right side, then neither local max nor min at c.

From part (a), $f'(x) = x^3(x-1)^2(7x-4)$

Test point #1: $\boxed{x=-1} \Rightarrow f'(-1) = (-1)^3(-1-1)^2(7\cdot(-1)-4)$

$= - \cdot + \quad - \quad = +$

Test point #2: $\boxed{x=\frac{1}{4}} \Rightarrow f'(\frac{1}{4}) = (\frac{1}{4})^3(\frac{1}{4}-1)^2(7\cdot\frac{1}{4}-4)$

$= + \cdot + \cdot - = -$

Test point #3: $\boxed{x=\frac{3}{4}} \Rightarrow f'(\frac{3}{4}) = (\frac{3}{4})^3\cdot(\frac{3}{4}-1)^2(7\cdot\frac{3}{4}-4)$

$= + \cdot + \cdot + = +$

Test point #4: $\boxed{x=2} \Rightarrow f'(2) = (2)^3\cdot(2-1)^2(7\cdot 2-4)$

$= + \cdot + \cdot + = +$

Visual aid:

At x_1 $(+ \to -)$

the first derivative test $\boxed{\text{indicates a local maximum}}$

At x_2 $(- \to +)$

the first derivative test $\boxed{\text{indicates a local minimum}}$

At x_3 $(+ \to +)$

the first derivative test $\boxed{\text{indicates neither a minimum nor a maximum}}$

Consider the following. (If an answer does not exist, enter DNE.)

$$C(x) = x^{1/7}(x + 8)$$

Note: $C(x) = x^{1/7}(x+8) = x^{1/7} \cdot x^1 + x^{1/7} \cdot 8$

$= x^{\frac{1}{7}+1} + 8x^{1/7}$

$\Rightarrow C(x) = x^{8/7} + 8x^{1/7}$ ← Easier to differentiate.

$\Rightarrow C'(x) = \frac{8}{7}x^{\frac{1}{7}} + \frac{8}{7}x^{-6/7}$

$\Rightarrow C'(x) = \frac{8}{7}x^{-6/7}(x+1) = \frac{8(x+1)}{7x^{6/7}}$ ← always "+"

$\Rightarrow C'(x) = 0$ when $x+1=0 \Rightarrow x=-1$

Sign of C':

$\Rightarrow$

Test points: $\boxed{-2}$ $\boxed{0}$

$x=-2 \Rightarrow C'(-2) = \frac{8(-2+1)}{+} = \frac{+(-)}{+} = -$ $\Rightarrow$ Decreasing on $(-\infty, -1)$

$x=0 \Rightarrow C'(0) = \frac{8(0+1)}{+} = \frac{+(+)}{+} = +$ $\Rightarrow$ Increasing on $(-1, \infty)$

(a)Find the interval(s) of increase. (Enter your answer using interval notation.)

$\boxed{(-1, \infty)}$

Find the interval(s) of decrease. (Enter your answer using interval notation.)

$\boxed{(-\infty, -1)}$

(b) Find the local minimum value(s). (Enter your answers as a comma-separated list.)

Local minimum at $x=-1$

$$\Rightarrow C(-1) = (-1)^{1/7}(-1+8)$$
$$= -1\,(7) = \boxed{-7}$$

Find the local maximum value(s). (Enter your answers as a comma-separated list.)

Note: only has ↘ to ↗ $\Rightarrow$ NO maximum by 1st Derivative Test.

$$\Rightarrow \boxed{DNE}$$

(c) Find the inflection points.

$$C''(x) = \frac{d}{dx}\left(\frac{8}{7}x^{\frac{1}{7}} + \frac{8}{7}x^{-6/7}\right)$$
$$= \frac{8}{7}\cdot\frac{1}{7}x^{\frac{1}{7}-1} + \frac{8}{7}\cdot\left(-\frac{6}{7}x^{-\frac{6}{7}-1}\right)$$
$$= \frac{8}{49}x^{-6/7} - \frac{48}{49}x^{-\frac{13}{7}}$$
$$= \frac{8}{49}x^{-\frac{13}{7}}(x-6) = \frac{8(x-6)}{49x^{13/7}}$$

Note: You can also use $C'(x) = \dfrac{8(x+1)}{7x^{6/7}}$ and differentiate using the Quotient Rule to get the same $C''(x)$.

$$C''(x) = \frac{8(x-6)}{49x^{13/7}} \Rightarrow \begin{array}{l} C''(x)=0 \text{ when } x-6=0 \Rightarrow x=6 \\ C''(x)=\text{undefined when } x^{13/7}=0 \Rightarrow x=0 \end{array}$$

Sign of C'':

$$x=-1 \Rightarrow C''(-1) = \frac{8(-1-6)}{49(-1)^{13/7}} = \frac{+(-)}{+(-)} = + \Rightarrow \text{C.U. on } (-\infty, 0)$$

$$x=2 \Rightarrow C''(2) = \frac{8(2-6)}{49(2)^{13/7}} = \frac{+(-)}{+(+)} = - \Rightarrow \text{C.D. on } (0, 6)$$

$$x=8 \Rightarrow C''(8) = \frac{8(8-6)}{49(8)^{13/7}} = \frac{+(+)}{+(+)} = + \Rightarrow \text{C.U. on } (6, \infty)$$

$\cup$: Concave up

$\cap$: Concave down

Inflection point at $x=0$

$\cup \to \cap$

Inflection point at $x=6$

$\cap \to \cup$

Smaller x-value: $x=0 \Rightarrow C(0) = 0^{1/7}(0+8) = 0(8) = 0 \Rightarrow \boxed{(0,0)}$

larger x-value: $x=6 \Rightarrow C(6) = 6^{1/7}(6+8) = \sqrt[7]{6}(14) = 14\sqrt[7]{6} \Rightarrow \boxed{(6, 14\sqrt[7]{6})}$

Find the interval(s) where the function is concave up. (Enter your answer using interval notation.)

$\boxed{(-\infty, 0) \cup (6, \infty)}$ ← from last page

Find the interval(s) where the function is concave down. (Enter your answer using interval notation.)

$\boxed{(0, 6)}$ ← from last page

(d) Use the information from parts (a)–(c) to sketch the graph.

49. Consider the following function. (If an answer does not exist, enter DNE.)

$$f(x) = 1 + \frac{5}{x} - \frac{7}{x^2}$$

(a)

Find the vertical asymptote(s). (Enter your answers as a comma-separated list.)

note: domain of f = all real numbers except 0

$\Rightarrow$ dom $f = (-\infty, 0) \cup (0, \infty)$

$\Rightarrow$ $x = 0$ is a possible vertical asymptote. To confirm this, we find the $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(1 + \frac{5}{x} - \frac{7}{x^2}\right) = \lim_{x \to 0^-} \left(\frac{x^2 + 5x - 7}{x^2}\right) = -\infty$$

and $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(1 + \frac{5}{x} - \frac{7}{x^2}\right) = \lim_{x \to 0^+} \left(\frac{x^2 + 5x - 7}{x^2}\right) = -\infty$$

$\Rightarrow$ $\boxed{x = 0}$ is a vertical asymptote.

Find the horizontal asymptote(s). (Enter your answers as a comma-separated list.)

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(1 + \frac{5}{x} - \frac{7}{x^2}\right) = 1 + 0 - 0 = 1$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(1 + \frac{5}{x} - \frac{7}{x^2}\right) = 1 + 0 - 0 = 1$$

$\Rightarrow$ $\boxed{y = 1}$ is the horizontal asymptote.

(b)

$$f(x) = 1 + \frac{5}{x} - \frac{7}{x^2} = 1 + 5x^{-1} - 7x^{-2}$$

$$\Rightarrow f'(x) = 0 - 5x^{-2} + 14x^{-3} = -\frac{5}{x^2} + \frac{14}{x^3} = \frac{-5x}{x^3} + \frac{14}{x^3} = \frac{14-5x}{x^3}$$

$\Rightarrow f'(x) = 0$ if $14 - 5x = 0 \Rightarrow 14 = 5x \Rightarrow \frac{14}{5} = x$

and $f'(x) =$ undefined if $x^3 = 0 \Rightarrow x = 0$

$\Rightarrow$ critical numbers are $x = 0$ and $x = \frac{14}{5}$

$\Rightarrow$

Interval	Test number x	$f'(x) = \frac{14-5x}{x^3}$	Conclusion
$(-\infty, 0)$	-1	$f'(-1) = \frac{14-5(-1)}{(-1)^3} = \frac{14+5}{-1} = \frac{+}{-} = -$	Decreasing
$(0, \frac{14}{5})$	2	$f'(2) = \frac{14-5(2)}{(2)^3} = \frac{14-10}{8} = \frac{+}{+} = +$	Increasing
$(\frac{14}{5}, \infty)$	4	$f'(4) = \frac{14-5(4)}{(4)^3} = \frac{14-20}{64} = \frac{-}{+} = -$	Decreasing

Find the interval(s) of increase. (Enter your answer using interval notation.)

$$\boxed{(0, \frac{14}{5})}$$

Find the interval(s) of decrease. (Enter your answer using interval notation)

$$\boxed{(-\infty, 0), (\frac{14}{5}, \infty)}$$

(c) Find the local maximum and minimum values.

note: From part (b),

min. value $= f(0) = \boxed{\text{DNE}}$

because 0 is <u>not</u> in the domain of f.

max value $= f\left(\frac{14}{5}\right) = 1 + \frac{5}{\frac{14}{5}} - \frac{7}{\left(\frac{14}{5}\right)^2}$

$= 1 + \frac{25}{14} - \frac{7}{\frac{196}{25}}$

$= 1 + \frac{25 \cdot 14}{14 \cdot 14} - \frac{175}{196}$

$= \frac{196}{196} + \frac{350}{196} - \frac{175}{196}$

$= \frac{371}{196} = \boxed{\frac{53}{28}}$ when reduced by 7.

max. value

(d) From part(b) $f'(x) = -5x^{-2} + 14x^{-3}$

$\Rightarrow f''(x) = 10x^{-3} - 42x^{-4} = \frac{10}{x^3} - \frac{42}{x^4} = \frac{10x - 42}{x^4}$

$\Rightarrow f''(x) = 0$ if $10x - 42 = 0 \Rightarrow x = \frac{42}{10} = \frac{21}{5}$

Interval	Test point x	$f''(x) = \frac{10x-42}{x^4}$	conclusion
$(-\infty, 0)$	-1	$f''(-1) = \frac{10(-1)-42}{(-1)^4} = \frac{-10-42}{1} = \frac{-}{+} = -$	Concave downward
$(0, \frac{21}{5})$	2	$f''(2) = \frac{10(2)-42}{(2)^4} = \frac{20-42}{16} = \frac{-}{+} = -$	Concave downward
$(\frac{21}{5}, \infty)$	5	$f''(5) = \frac{10(5)-42}{(5)^4} = \frac{50-42}{625} = \frac{+}{+} = +$	Concave upward

$\Rightarrow$

x-coordinate of inflection point

Find the interval(s) on which *f* is concave up. (Enter your answer using interval notation.)

$$\boxed{\left(\frac{21}{5}, \infty\right)}$$

Find the interval(s) on which *f* is concave down. (Enter your answer using interval notation.)

$$\boxed{(-\infty, 0), \left(0, \frac{21}{5}\right)}$$

Find the inflection point.

$$x = \frac{21}{5} \Rightarrow y = f\left(\frac{21}{5}\right) = 1 + \frac{5}{\frac{21}{5}} - \frac{7}{\left(\frac{21}{5}\right)^2}$$

$$= 1 + \frac{25}{21} - \frac{7}{\frac{441}{25}}$$

$$= 1 + \frac{25 \cdot 21}{21 \cdot 21} - \frac{175}{441}$$

$$= \frac{441}{441} + \frac{525}{441} - \frac{175}{441} = \frac{791}{441} = \frac{113}{63}$$

$\Rightarrow$ Inflection point is $(x, y) = \boxed{\left(\frac{21}{5}, \frac{113}{63}\right)}$

* **Note:** Use exact fraction <u>NOT</u> decimal approximations

(e) Use the information from parts (a)–(d) to sketch the graph of *f*.

Consider the function below. (If an answer does not exist, enter DNE.)

$$f(x) = \frac{x+6}{\sqrt{x^2+6}}$$

(a) Find the exact value of the maximum of *f*.

$$f'(x) = \frac{\sqrt{x^2+6}\cdot\frac{d}{dx}(x+6) - (x+6)\cdot\frac{d}{dx}\left(\sqrt{x^2+6}\right)}{\left(\sqrt{x^2+6}\right)^2}$$

Quotient Rule

$$= \frac{\sqrt{x^2+6}\,(1) - (x+6)\cdot\frac{1}{\cancel{2}\sqrt{x^2+6}}\cdot\cancel{2}x}{x^2+6}$$

$$= \frac{\left(\sqrt{x^2+6} - \frac{x(x+6)}{\sqrt{x^2+6}}\right)\cdot\sqrt{x^2+6}}{(x^2+6)\cdot\sqrt{x^2+6}}$$

$$= \frac{\sqrt{x^2+6}\cdot\sqrt{x^2+6} - \frac{x(x+6)}{\cancel{\sqrt{x^2+6}}}\cdot\cancel{\sqrt{x^2+6}}}{(x^2+6)\cdot(x^2+6)^{1/2}}$$

$$= \frac{\left(\sqrt{x^2+6}\right)^2 - x(x+6)}{(x^2+6)^{1+1/2}}$$

$$= \frac{x^2+6-x^2-6x}{(x^2+6)^{3/2}} = \frac{6-6x}{(x^2+6)^{3/2}} = f'(x)$$

$\Rightarrow \quad f'(x) = 0 \quad$ if $\quad 6 - 6x = 0 \Rightarrow \frac{6}{6} = \frac{6x}{6} \Rightarrow 1 = x$

$\quad\quad\quad

Find the exact value of the minimum of *f*.

Note: No other critical number $\Rightarrow$ no minimum value

$\Rightarrow \boxed{\text{DNE}}$

(b) Find the exact value of *x* at which *f* increases most rapidly.

Note: "increases <u>most</u> rapidly" $\Rightarrow$ maximum rate of change

$\Rightarrow$ maximum of f'.

$\Rightarrow$ Start with $f'(x) = \dfrac{6-6x}{(x^2+6)^{3/2}}$

$$\Rightarrow f''(x) = \frac{12x^2-18x-36}{(x^2+6)^{5/2}}$$ ← from last page

$\Rightarrow$ Critical number for $f'(x) = \dfrac{6-6x}{(x^2+6)^{3/2}}$ is obtained from

$$f''(x) = 0 \Rightarrow \frac{12x^2}{6} - \frac{18x}{6} - \frac{36}{6} = \frac{0}{6}$$

$$\Rightarrow 2x^2 - 3x - 6 = 0$$

not factorable $\Rightarrow$ solve by quadratic formula.

$a = 2,\ b = -3,\ c = -6$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2-4(2)(-6)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+48}}{4} = \frac{3 \pm \sqrt{57}}{4} \Rightarrow x = \frac{3+\sqrt{57}}{4} \approx 2.64$$

OR

$$x = \frac{3-\sqrt{57}}{4} \approx -1.14$$

$$f'''(x) = \frac{d}{dx}\left[\frac{12x^2-18x-36}{(x^2+6)^{5/2}}\right]$$

$$= \frac{(x^2+6)^{5/2}\cdot\frac{d}{dx}(12x^2-18x-36) - (12x^2-18x-36)\cdot\frac{d}{dx}\left((x^2+6)^{5/2}\right)}{\left[(x^2+6)^{5/2}\right]^2}$$

$$= \frac{(x^2+6)^{5/2}\cdot(24x-18) - (12x^2-18x-36)\cdot\frac{5}{\cancel{2}}(x^2+6)^{3/2}(\cancel{2}x)}{(x^2+6)^5}$$

$$= \frac{(x^2+6)^{3/2}\left[(x^2+6)(24x-18) - 5x(12x^2-18x-36)\right]}{(x^2+6)^5}$$

$$= \frac{(x^2+6)^{3/2}\left[24x^3-18x^2+144x-108-60x^3+90x^2+180x\right]}{(x^2+6)^5}$$

$$= \frac{-36x^3+72x^2+324x-108}{(x^2+6)^{5-3/2}} = \frac{-36(x^3-2x^2-9x+3)}{(x^2+6)^{7/2}}$$ always +.

$$f'''\left(\frac{3+\sqrt{57}}{4}\right) \approx f'''(2.64) = \frac{-36(2.64^3-2(2.64)^2-9(2.64)+3)}{+} = \frac{+}{+} = + \Rightarrow \text{Min at } \frac{3+\sqrt{57}}{4}$$

$$f'''\left(\frac{3-\sqrt{57}}{4}\right) \approx f'''(-1.14) = \frac{-36((-1.14)^3-2(-1.14)^2-9(-1.14)+3)}{+} = \frac{-}{+} = - \Rightarrow \text{max at } \frac{3-\sqrt{57}}{4}$$

$\Rightarrow$ Exact value of x at which f increases most rapidly is $x = \boxed{\frac{3-\sqrt{57}}{4}}$

69. Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it.

$$\lim_{x\to 0^+} (\tan(9x))^x$$

note: as $x \to 0^+$

$9x \to 0 \Rightarrow \tan(9x) \to 0$

$\Rightarrow 0^0 \leftarrow$ Indeterminant form.

Solution:

Let $y = (\tan(9x))^x$

$\Rightarrow \ln y = \ln (\tan(9x))^x$

$\Rightarrow \ln y = x \ln(\tan(9x))$

$$\Rightarrow \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln(\tan(9x))$$

$$= \lim_{x\to 0^+} \frac{\ln(\tan(9x))}{\frac{1}{x}}$$

note: as $x \to 0^+$, $9x \to 0 \Rightarrow \tan(9x) \to 0 \Rightarrow \ln(\tan(9x)) \to \infty$

and as $x \to 0^+$, $\frac{1}{x} \to \infty \Rightarrow \frac{\ln(\tan(9x))}{\frac{1}{x}} \to \frac{\infty}{\infty}$

$\Rightarrow$ by L'Hopital's Rule,

$$\lim_{x\to 0^+} \frac{\ln(\tan(9x))}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\left[\ln(\tan(9x))\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

note: $\frac{d}{dx}\left[\ln\left(\tan(9x)\right)\right]$

$$= \frac{1}{\tan(9x)} \cdot \frac{d}{dx}\left[\tan(9x)\right]$$

$$= \frac{1}{\tan(9x)} \cdot \sec^2(9x) \cdot \underbrace{\frac{d}{dx}(9x)}_{9}$$

$$= \frac{1}{\frac{\sin(9x)}{\cos(9x)}} \cdot \frac{1}{\left[\cos(9x)\right]^2} \cdot 9$$

$$= \frac{\cancel{\cos(9x)}}{\sin(9x)} \cdot \frac{1}{\cancel{\left[\cos(9x)\right]}\left[\cos(9x)\right]} \cdot 9$$

$$= \frac{9}{\sin(9x)\cos(9x)}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right]$$

$$= -1\,x^{-2}$$

$$= -\frac{1}{x^2}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\frac{d}{dx}\left[\ln\left(\tan(9x)\right)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]} = \lim_{x\to 0^+} \frac{\frac{9}{\sin(9x)\cos(9x)}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{-9x^2}{\sin(9x)\cos(9x)} \leftarrow *$$

See below for alternate method to find this limit

$$= \lim_{x \to 0^+} \frac{-x \cdot 9x}{\cos(9x) \cdot \sin(9x)}$$

$$= \lim_{x \to 0^+} \frac{-x}{\cos(9x)} \cdot \lim_{x \to 0^+} \frac{9x}{\sin(9x)}$$

$$= \lim_{x \to 0^+} \frac{-x}{\cos(9x)} \cdot \lim_{x \to 0^+} \frac{9x / 9x}{\sin(9x) / 9x}$$

$$= \lim_{x \to 0^+} \frac{-x}{\cos(9x)} \cdot \lim_{x \to 0^+} \frac{1}{\frac{\sin(9x)}{9x}}$$

$$= \frac{-0}{\cos(9 \cdot 0)} \cdot \frac{1}{1}$$

$$= -\frac{0}{1} \cdot \frac{1}{1} = \frac{0}{1} = 0 \leftarrow \lim_{x \to 0^+} \ln y$$

since $y = e^{\ln y}$,

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^{\lim_{x \to 0^+} \ln y} = e^0 = \boxed{1}$$

Answer

* Alternate Method:

$$\lim_{x\to 0^+} \frac{-9x^2}{\sin(9x)\cos(9x)}$$

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\left[-9x^2\right]}{\frac{d}{dx}\left[\sin(9x)\cos(9x)\right]}$$

note: as $x \to 0^+$,

$$\frac{-9x^2}{\sin(9x)\cos(9x)} \to \frac{0}{0}$$

$\Rightarrow$ we can use L'Hopital's Rule again.

$$= \lim_{x\to 0^+} \frac{-18x}{\frac{d}{dx}\left[\sin(9x)\right]\cdot\cos(9x) + \sin(9x)\cdot\frac{d}{dx}\left[\cos(9x)\right]}$$

$$= \lim_{x\to 0^+} \frac{-18x}{\cos(9x)\cdot 9\cdot\cos(9x) + \sin(9x)\cdot\left[-\sin(9x)\right]\cdot 9}$$

$$= \lim_{x\to 0^+} \frac{-18x}{9\cos^2(9x) - 9\sin^2(9x)}$$

$$= \lim_{x\to 0^+} \frac{-18x}{9\left[\cos^2(9x) - \sin^2(9x)\right]}$$

$$= \lim_{x\to 0^+} \frac{-18x}{9\left[\cos 2(9x)\right]}$$

Recall: Double angle formula:

$$\cos^2\alpha - \sin^2\alpha = \cos(2\alpha)$$

$$= \lim_{x \to 0^+} \frac{-\cancel{18}^{\,2}x}{\cancel{9}_{\,1}\cos(18x)}$$

$$= \lim_{x \to 0^+} \frac{-2x}{\cos(18x)}$$

$$= \frac{-2(0)}{\cos(18 \cdot 0)} = \frac{0}{\cos(0)} = \frac{0}{1} = 0$$

$$\Rightarrow \lim_{x \to 0^+} \ln y = 0$$

$$\Rightarrow \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^{\lim_{x \to 0} \ln y} = e^0 = \boxed{1}$$

Same answer as before.