Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $6x + y = 1$.

We first find slope of the normal line m_N

$\Rightarrow$ use: $6x + y = 1$

$$y = 1 - 6x$$

$$\Rightarrow y' = -6 = m_N$$

$$m_{tan} = -\frac{1}{m_N} = -\frac{1}{-6} = \frac{1}{6}$$

$$\Rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} = \frac{1}{6}$$

$$\Rightarrow 2\sqrt{x} = 6$$

$$\Rightarrow \sqrt{x} = 3$$

$$\Rightarrow x = 9$$

$\Rightarrow y = \sqrt{9} = 3 \Rightarrow$ point $= (9, 3)$

$\Rightarrow$ Equation is $y - 3 = -6(x - 9)$

$$\Rightarrow y - 3 = -6x + 54$$

$$\Rightarrow \boxed{y = -6x + 57}$$

14. At what exact point on the curve $y = 9 + 2e^x - 5x$ is the tangent line parallel to the line $5x - y = 8$?

$$\cancel{5x} - y = 8 \quad -5x$$
$$-\cancel{5x}$$

$$\frac{\cancel{-}y}{\cancel{-}1} = \frac{-5x}{-1} + \frac{8}{-1}$$

$$y = \boxed{5}x - 8$$
$$y = \boxed{m}x + b$$

$$y' = \frac{d}{dx}(9 + 2e^x - 5x)$$
$$= 0 + 2 \cdot e^x - 5$$
$$\Rightarrow y' = 2e^x - 5$$

$$\Rightarrow m_{tan} = 5 \Rightarrow y' = 5$$

$$\Rightarrow 2e^x - \cancel{5} = 5$$
$$+\cancel{5} \qquad +5$$

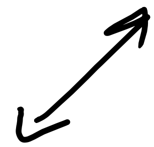
$$\Rightarrow \frac{\cancel{2}e^x}{\cancel{2}} = \frac{10}{2}$$

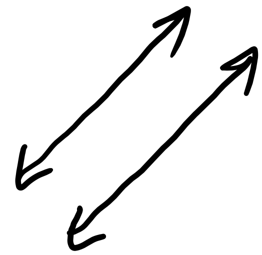
$$\Rightarrow \ln e^x = \ln 5$$

$$\Rightarrow x = \ln(5)$$

$$\Rightarrow y = 9 + 2e^{\ln(5)} - 5\ln(5) \Rightarrow y = 9 + \underbrace{2(5)}_{10} - 5\ln(5) = 19 - 5\ln(5)$$

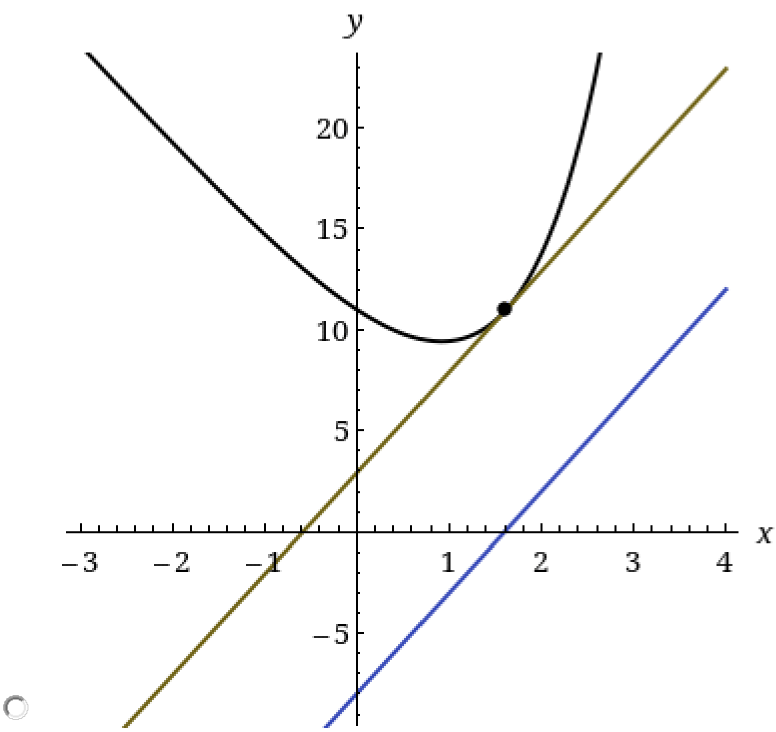
$\Rightarrow$ The point is $\boxed{(\ln(5),\ 19-5\ln(5))}$

Recall: If $m>0$, then 

Here, $m = 5 = m_{tan}$ $\Rightarrow$ 

also, point is approximately $(1.6,\ 11)$

$\Rightarrow$ graph is



$x_1 \quad y_1$

16. Find equations of both lines through the point (2, −3) that are tangent to the parabola

$y = x^2 + x$.

note: $m_{tan} = y'$

$\Rightarrow m_{tan} = 2x + 1$

$\Rightarrow$ If $x = a$, $m_{tan} = 2a + 1$

note: If $x = a$, $y = a^2 + a$

$\Rightarrow (\underset{x_2}{a}, \underbrace{a^2 + a}_{y_2})$ = point where the tangent line touches the parabola

Recall:

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow m = \dfrac{a^2 + a - (-3)}{a - 2}$

$\Rightarrow \boxed{m_{tan}} = m = \dfrac{a^2 + a + 3}{a - 2}$

$\Rightarrow (a-2)(2a+1) = \left(\dfrac{a^2 + a + 3}{\cancel{a - 2}}\right)\cancel{(a-2)}$

$\Rightarrow 2a^2 + \cancel{a} - 4a - 2 = \cancel{a^2} + \cancel{a} + \cancel{3}$

$\quad - a^2 - \cancel{a} \qquad - 3 \qquad \cancel{-a^2} \cancel{-a} \cancel{-3}$

$a^2 - 4a - 5 = 0$

$(a - 5)(a + 1) = 0$

$a - 5 = 0$ OR $a + 1 = 0$

$a = 5$ OR $a = -1$

$(5, 30) \quad | \quad (-1, 0)$

note: $y = x^2 + x$

$a = 5 \Rightarrow y = 5^2 + 5 = 25 + 5 = 30$

$a = -1 \Rightarrow y = (-1)^2 + (-1) = 1 - 1 = 0$

$\boxed{\text{at } (-1, 0)}$ $\quad m_{tan} = 2x + 1 = 2(-1) + 1$

$= -2 + 1 = -1$

$\Rightarrow \quad y - y_1 = m(x - x_1)$ becomes

$y - 0 = -1(x - (-1))$

$y = -1(x + 1) \Rightarrow \boxed{y = -x - 1}$

$\boxed{\text{at } (5, 30)}$ $\quad m_{tan} = 2x + 1 = 2(5) + 1 = 10 + 1 = 11$

$\Rightarrow y - y_1 = m(x - x_1)$ becomes

$y - 30 = 11(x - 5)$

$y - \cancel{30} = 11x - 55$

$\underline{\cancel{+30}} \qquad\qquad \underline{+30}$

$\Rightarrow \quad \boxed{y = 11x - 25}$

Find a second-degree polynomial P such that $P(1) = 6$, $P'(1) = 7$, and $P''(1) = 10$.

2^{nd} degree polynomial $\Rightarrow$ $p(x) = ax^2 + bx + c$, a, b, c = Constants

$p'(x) = 2ax + b$

$p''(x) = 2a$

Given $p(1) = 6$

$\Rightarrow$ $a(1)^2 + b(1) + c = 6$ $\Rightarrow$ $\boxed{a + b + c = 6}$

Given $p'(1) = 7$

$\Rightarrow$ $2a(1) + b = 7$ $\Rightarrow$ $\boxed{2a + b = 7}$

Given $p''(1) = 10$

$\Rightarrow$ $2a = 10$ $\Rightarrow$ $\boxed{a = 5}$

$\Rightarrow$ $\boxed{2a + b = 7}$ $\Rightarrow$ $2(5) + b = 7$

$\Rightarrow$ $10 + b = 7$ $\Rightarrow$ $\boxed{b = -3}$

and $\boxed{a + b + c = 6}$ becomes

$5 + (-3) + c = 6$ $\Rightarrow$ $2 + c = 6$ $\Rightarrow$ $\boxed{c = 4}$

$\Rightarrow$ $\boxed{P(x) = 5x^2 - 3x + 4}$

Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 3)$ has equation $y = 8x - 5$.

$y = mx + b$

Note: ① For $y = ax^2 + bx$, y' = slope of the tangent line

$$\Rightarrow 2ax + b = 8$$

at $(1,3)$, $x = 1 \Rightarrow 2a(1) + b = 8$

$$\Rightarrow \boxed{2a + b = 8} \ ①$$

② point $(1, 3)$ on parabola $\Rightarrow y = ax^2 + bx$

becomes $3 = a(1)^2 + b(1)$

$$\Rightarrow 3 = a + b \quad \text{OR} \quad \boxed{a + b = 3} \ ②$$

$\Rightarrow$ we have the system of equations

$$\begin{cases} ① \ 2a + b = 8 \\ ② \ a + b = 3 \end{cases}$$

$\Rightarrow$ Solve using method of addition (Elimination)

$$\Rightarrow \quad 2a + b = 8$$

$$-1(a + b) = (3)\cdot(-1)$$

$$2a + \cancel{b} = 8$$

$$-a - \cancel{b} = -3$$

$a = 5 \Rightarrow$ Since $a + b = 3$, we get $\cancel{5} + b = 3 - 5$

$\cancel{-5}$

$\Rightarrow b = -2$

$\Rightarrow$ Equation of the parabola $y = ax^2 + bx$

is $y = 5x^2 + (-2)x$

or

$$\boxed{y = 5x^2 - 2x}$$

23. Differentiate.

$$f(t) = \frac{\sqrt[3]{t}}{t-3} = \frac{t^{1/3}}{t-3}$$

$$f'(t) = \frac{(t-3)\cdot \overbrace{\frac{d}{dt}(t^{1/3})}^{1/3\, t^{1/3-1}} - t^{1/3}\cdot \overbrace{\frac{d}{dt}(t-3)}^{1}}{(t-3)^2}$$

$$= \frac{(t-3)\cdot \frac{1}{3}t^{\frac{1}{3}-1} - t^{1/3}\cdot(1)}{(t-3)^2}$$

$$= \frac{(t-3)\cdot \frac{1}{3}t^{-\frac{2}{3}} - t^{\frac{1}{3}}}{(t-3)^2}$$

$$= \frac{\left(\frac{t-3}{3t^{2/3}} - t^{\frac{1}{3}}\right)\cdot 3t^{2/3}}{(t-3)^2\cdot 3t^{2/3}}$$

$$= \frac{\frac{t-3}{\cancel{3}\,\cancel{t^{2/3}}}\cdot \cancel{3}\,\cancel{t^{2/3}} - t^{\frac{1}{3}}\cdot 3t^{\frac{2}{3}}}{3t^{2/3}(t-3)^2}$$

$$= \frac{t-3 - 3t^{\overbrace{\frac{1}{3}+\frac{2}{3}}^{3/3 = 1}}}{3t^{2/3}(t-3)^2}$$

$$= \frac{t-3-3t}{3t^{2/3}(t-3)^2}$$

$$= \boxed{\frac{-2t-3}{3t^{2/3}(t-3)^2}}$$

Answer

25. Find $f'(x)$ and $f''(x)$.

$$f(x) = \frac{x}{x^2 - 8}$$

$$f'(x) = \frac{(x^2-8)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x^2-8)}{(x^2-8)^2}$$ ← Quotient Rule

$$= \frac{(x^2-8)\cdot 1 - x\cdot(2x)}{(x^2-8)^2}$$

$$= \frac{x^2-8-2x^2}{(x^2-8)^2} = \boxed{\frac{-x^2-8}{(x^2-8)^2} = f'(x)}$$

$$f''(x) = \frac{d}{dx}\left[\frac{-x^2-8}{(x^2-8)^2}\right]$$

use Chain Rule

Quotient Rule →

$$= \frac{(x^2-8)^2\cdot\frac{d}{dx}(-x^2-8) - (-x^2-8)\cdot\frac{d}{dx}\left[(x^2-8)^2\right]}{\left[(x^2-8)^2\right]^2}$$

$2x$

$$= \frac{(x^2-8)^2\cdot(-2x) + (x^2+8)\cdot 2(x^2-8)^1\cdot\frac{d}{dx}(x^2-8)}{(x^2-8)^4}$$

$$= \frac{-2x\cancel{(x^2-8)}^2 + 4x\,(x^2+8)\cancel{(x^2-8)}}{(x^2-8)^4}$$

$$= \frac{\cancel{(x^2-8)}\left[-2x(x^2-8) + 4x(x^2+8)\right]}{(x^2-8)^{\cancel{4}\,3}}$$

$$= \frac{-2x^3+16x+4x^3+32x}{(x^2-8)^3}$$

$$= \boxed{\frac{2x^3+48x}{(x^2-8)^3}}$$

31. Differentiate.

$$g(x) = (x + 7\sqrt{x})e^x$$

using product Rule $\Rightarrow g'(x) = \frac{d}{dx}(x+7\sqrt{x}) \cdot e^x + (x+7\sqrt{x}) \cdot \frac{d}{dx}(e^x)$

$$= \left(1 + 7 \cdot \frac{1}{2\sqrt{x}}\right) \cdot e^x + (x+7\sqrt{x}) \cdot e^x$$

$$= e^x \left[\left(1 + \frac{7}{2\sqrt{x}}\right) + (x + 7\sqrt{x})\right]$$

$$= \boxed{e^x \left(1 + \frac{7}{2\sqrt{x}} + x + 7\sqrt{x}\right)}$$

Answer

note:

$$\frac{d}{dx}(\sqrt{x})$$

$$= \frac{d}{dx}(x^{1/2})$$

$$= \frac{1}{2} x^{1/2 - 1}$$

$$= \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

27. If $f(x) = \frac{x^2}{3+x}$, find $f''(3)$.

$$f'(x) = \frac{(3+x)\cdot\frac{d}{dx}(x^2) - x^2\cdot\frac{d}{dx}(3+x)}{(3+x)^2}$$

$$= \frac{(3+x)\cdot(2x) - x^2\cdot(\overbrace{0+1}^{1})}{(3+x)^2}$$

$$= \frac{6x + 2x^2 - x^2}{(3+x)^2}$$

$$\Rightarrow f'(x) = \frac{x^2+6x}{(3+x)^2}$$

$$\Rightarrow f''(x) = \frac{(3+x)^2\cdot\frac{d}{dx}(x^2+6x) - (x^2+6x)\cdot\frac{d}{dx}\left[(3+x)^2\right]}{\left[(3+x)^2\right]^2}$$

$$= \frac{(3+x)^2\cdot(2x+6) - (x^2+6x)\cdot\overbrace{2(3+x)\cdot\overset{\boxed{\frac{d}{dx}(3+x)}}{1}}^{\text{Chain Rule}}}{(3+x)^4}$$

$$= \frac{(3+x)^2(2x+6) - 2(3+x)(x^2+6x)}{(3+x)^4} *$$

* Can simplify further but not necessary here

$$\Rightarrow f''(3) = \frac{(3+3)^2(2\cdot 3+6) - 2(3+3)(3^2+6\cdot 3)}{(3+3)^4}$$

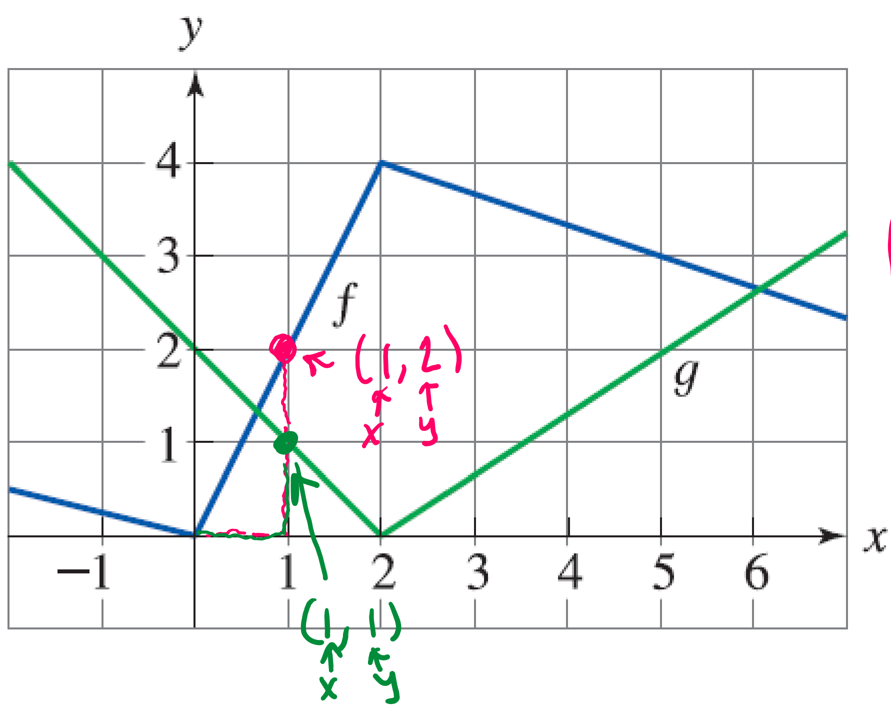
$$= \frac{(6)^2(6+6) - 2(6)(9+18)}{(6)^4}$$

$$= \frac{(36)(12) - 12(27)}{1296}$$

$$= \frac{108}{1296} = \boxed{\frac{1}{12}}$$

Answer

The graphs of the functions *f* and *g* are shown in the figure.



$\boxed{x=1}$

$\boxed{f(1) = 2}$

$f'(1) = m_{tan}\big|_{x=1}$

$= \frac{Rise}{Run} = \frac{+2}{+1} = 2$

$\Rightarrow \boxed{f'(1) = 2}$

$\boxed{g(1) = 1}$, $\boxed{g'(1) = -1}$

$g'(1) = m_{tan}\big|_{x=1}$

$= \frac{Rise}{Run} = \frac{-1}{+1} = -1$

Let $u(x) = f(x)g(x)$ and $v(x) = \dfrac{f(x)}{g(x)}$

x=1

(a) Find $u'(1)$.

note: $u = f(x)\,g(x)$

Product Rule $\Rightarrow$ $u' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$\Rightarrow u'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1)$

$= 2 \cdot 1 + 2 \cdot (-1)$

$= 2 + (-2) = \boxed{0}$

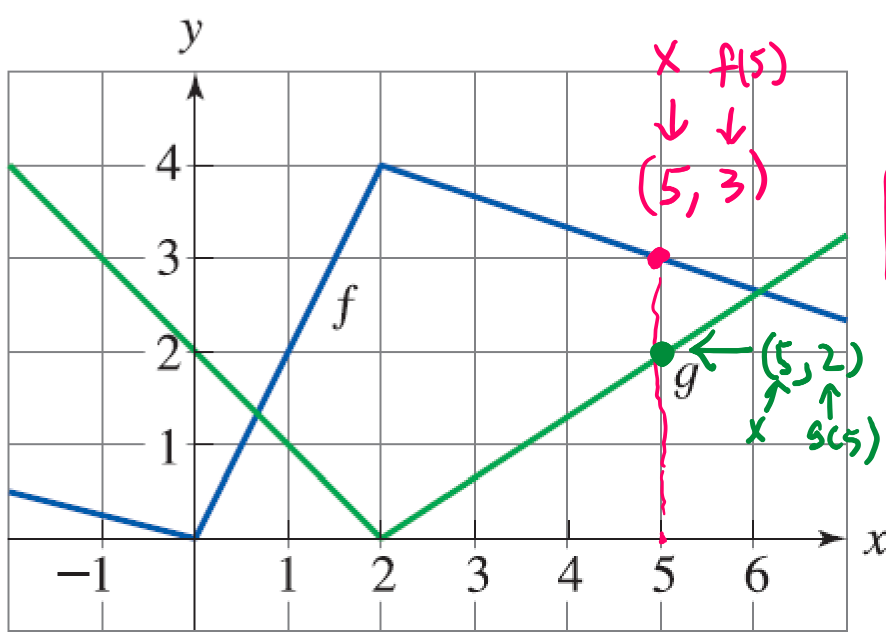
Answer

(b) Find $v'(5)$. $\boxed{x=5}$

note: $v(x) = \dfrac{f(x)}{g(x)}$

$\boxed{\text{Quotient Rule}}$

$$\Rightarrow v'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$



$\boxed{x=5}$

$\boxed{f(5)=3}$

$$f'(5) = \frac{-1 \leftarrow \text{Rise}}{+3 \leftarrow \text{Run}} = -\frac{1}{3}$$

$$\Rightarrow \boxed{f'(5) = -\frac{1}{3}}$$

$\boxed{g(5) = 2}$

$$g'(5) = \frac{+2 \leftarrow \text{Rise}}{+3 \leftarrow \text{Run}} = \frac{2}{3}$$

$$\Rightarrow \boxed{g'(5) = \frac{2}{3}}$$

$$\Rightarrow v'(5) = \frac{g(5)\cdot f'(5) - f(5)\cdot g'(5)}{[g(5)]^2}$$

$$= \frac{2\cdot\left(-\frac{1}{3}\right) - 3\cdot\left(\frac{2}{3}\right)}{(2)^2}$$

$$= \frac{-\frac{2}{3} - \frac{6}{3}}{4} = \frac{-\frac{8}{3}}{\cancel{4}_1} = -\frac{\cancel{8}\,2}{3}\cdot\frac{1}{\cancel{4}_1} = \boxed{-\frac{2}{3}} \leftarrow \text{Answer}$$

Find $f'(x)$ and $f''(x)$.

$$f(x) = \frac{x^2}{9+2x}$$

$$f'(x) = \frac{d}{dx}\left[\frac{x^2}{9+2x}\right]$$

$$= \frac{(9+2x)\cdot\frac{d}{dx}(x^2) - x^2\cdot\frac{d}{dx}(9+2x)}{(9+2x)^2}$$

use Quotient Rule

$$= \frac{(9+2x)\cdot(2x) - x^2\cdot(2)}{(9+2x)^2}$$

$$= \frac{18x+4x^2-2x^2}{(9+2x)^2}$$

$$= \frac{18x+2x^2}{(9+2x)^2}$$

OR $\boxed{\dfrac{2x^2+18x}{(9+2x)^2}}$ ← $f'(x)$

$$f''(x) = \frac{d}{dx}\left[f'(x)\right]$$

$$= \frac{d}{dx}\left[\frac{2x^2+18x}{(9+2x)^2}\right] \leftarrow \text{use Quotient Rule}$$

$$= \frac{(9+2x)^2 \cdot \frac{d}{dx}(2x^2+18x) - (2x^2+18x) \cdot \frac{d}{dx}\left[(9+2x)^2\right]}{\left[(9+2x)^2\right]^2}$$

$$= \frac{(9+2x)^2 \cdot (4x+18) - (2x^2+18x) \cdot \overbrace{\left[2(9+2x) \cdot \frac{d}{dx}(9+2x)\right]}^{\text{Chain Rule}}}{(9+2x)^4}$$

$$= \frac{(9+2x)^2 \cdot 2(2x+9) - (2x^2+18x)\left[2(9+2x) \cdot 2\right]}{(9+2x)^4}$$

$$= \frac{2(9+2x)^3 - (2x^2+18x)\left[4(9+2x)\right]}{(9+2x)^4}$$

$$= \frac{(9+2x)\left[2(9+2x)^2 - (2x^2+18x) \cdot 4\right]}{(9+2x)^4}$$ factor out $(9+2x)$ common factor

$$(9+2x)^2 = (9+2x)(9+2x) = 81 + \underbrace{18x + 18x}_{36x} + 4x^2$$

$$= \frac{\cancel{(9+2x)}\left[2\left(\overbrace{81+36x+4x^2}^{(9+2x)^2}\right) - 4\left(2x^2+18x\right)\right]}{(9+2x)^{\cancel{4}\,3}}$$

$$= \frac{162 + \cancel{72x} + \cancel{8x^2} - \cancel{8x^2} - \cancel{72x}}{(9+2x)^3}$$

$$= \boxed{\frac{162}{(9+2x)^3}} \leftarrow f''(x)$$

34]Suppose that $f(2) = -3$, $g(2) = 5$, $f'(2) = -1$, and $g'(2) = 2$. Find $h'(2)$.

(a) $h(x) = 3f(x) - 4g(x)$

$$\Rightarrow h'(x) = 3f'(x) - 4g'(x) \Rightarrow h'(2) = 3f'(2) - 4g'(2)$$

$$= 3(-1) - 4(2)$$

$$= -3 - 8 = \boxed{-11}$$

(b) $h(x) = f(x)g(x)$

$$\Rightarrow h'(x) = \frac{d}{dx}\left[f(x)g(x)\right]$$

$$= f(x)g'(x) + f'(x)g(x)$$

$$\Rightarrow h'(2) = f(2)g'(2) + f'(2)g(2)$$

$$= (-3)(2) + (-1)(5)$$

$$= -6 + (-5)$$

$$= \boxed{-11}$$

(c) $h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\Rightarrow h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2}$$

$$= \frac{(5)(-1) - (-3)(2)}{[5]^2}$$

$$= \frac{-5 - (-6)}{25} = \frac{-5+6}{25}$$

$$= \boxed{\frac{1}{25}}$$

(d) $h(x) = \frac{g(x)}{1 + f(x)}$

$$\Rightarrow h'(x) = \frac{d}{dx}\left[\frac{g(x)}{1+f(x)}\right]$$

$$= \frac{[1+f(x)]g'(x) - g(x)\cdot f'(x)}{[1+f(x)]^2}$$

$$\Rightarrow h'(2) = \frac{[1+f(2)]g'(2) - g(2)f'(2)}{[1+f(2)]^2}$$

Note: $f(2) = -3$, $g(2) = 5$, $f'(2) = -1$, and $g'(2) = 2$

$$= \frac{[1+(-3)](2) - (5)(-1)}{[1+(-3)]^2}$$

$$= \frac{[-2](2) - (5)(-1)}{[-2]^2}$$

$$= \frac{-4+5}{4} = \boxed{\frac{1}{4}}$$

Differentiate.

$$y = \frac{2x}{3 - \tan(x)}$$

$$y' = \frac{d}{dx}\left[\frac{2x}{3-\tan(x)}\right]$$

Quotient Rule $\Rightarrow$

$$= \frac{[3-\tan(x)]\cdot\frac{d}{dx}(2x) - 2x\cdot\frac{d}{dx}[3-\tan(x)]}{[3-\tan(x)]^2}$$

$$= \frac{[3-\tan(x)]\cdot[2] - 2x\cdot[0-\sec^2(x)]}{[3-\tan(x)]^2}$$

$$= \frac{6 - 2\tan(x) - 2x\cdot[-\sec^2(x)]}{[3-\tan(x)]^2}$$

$$= \boxed{\frac{6-2\tan(x)+2x\sec^2(x)}{[3-\tan(x)]^2}} \quad \text{OR} \quad \boxed{\frac{2[3-\tan(x)+x\sec^2(x)]}{[3-\tan(x)]^2}}$$

4. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$y \sin(8x) = x \cos(2y), \quad (\pi/2, \pi/4)$$

$\boxed{\text{Need } m_{tan} \rightarrow \text{find } \frac{dy}{dx}}$

$$\frac{d}{dx}\left(y \sin(8x)\right) = \frac{d}{dx}\left(x\cos(2y)\right)$$

$\boxed{\text{product Rule}} \rightarrow$

$$\frac{d}{dx}(y)\cdot \sin(8x) + y\cdot\frac{d}{dx}\left[\sin(8x)\right] = \frac{d}{dx}(x)\cdot\cos(2y) + x\cdot\frac{d}{dx}\left[\cos(2y)\right]$$

$$\frac{dy}{dx}\cdot\sin(8x) + y\cdot\underbrace{\cos(8x)\cdot 8}_{\text{chain Rule}} = 1\cdot\cos(2y) + x\cdot\underbrace{\left[-\sin(2y)\cdot 2\cdot\frac{dy}{dx}\right]}_{\text{chain Rule}}$$

$$\frac{dy}{dx}\cdot\sin(8x) + 8y\cos(8x) = \cos(2y) - 2x\sin(2y)\cdot\frac{dy}{dx}$$

$$\Rightarrow \sin(8x)\cdot\frac{dy}{dx} + 2x\sin(2y)\cdot\frac{dy}{dx} = \cos(2y) - 8y\cos(8x)$$

$$\Rightarrow \frac{dy}{dx}\left[\sin(8x) + 2x\sin(2y)\right] = \cos(2y) - 8y\cos(8x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(2y) - 8y\cos(8x)}{\sin(8x) + 2x\sin(2y)}$$

$$\Rightarrow m_{tan}\Big|_{\left(\frac{\pi}{2},\frac{\pi}{4}\right)} = \frac{dy}{dx}\Big|_{\left(\underset{\uparrow x}{\frac{\pi}{2}},\ \underset{\uparrow y}{\frac{\pi}{4}}\right)} = \frac{\cos\left(2\cdot\frac{\pi}{4}\right) - 8\cdot\frac{\pi}{4}\cos\left(8\cdot\frac{\pi}{2}\right)}{\sin\left(8\cdot\frac{\pi}{2}\right) + 2\cdot\frac{\pi}{2}\sin\left(2\cdot\frac{\pi}{4}\right)}$$

$$= \frac{\overbrace{\cos\left(\frac{\pi}{2}\right)}^{0} - 2\pi \overbrace{\cos(4\pi)}^{1}}{\underbrace{\sin(4\pi)}_{0} + \pi \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}}$$

$$= \frac{0 - 2\pi \cdot 1}{0 + \pi \cdot 1}$$

$$= \frac{-2\pi}{\pi} = -2 = m_{tan} \text{ at } \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

$\Rightarrow$ Equation is $y - \frac{\pi}{4} = -2\left(x - \frac{\pi}{2}\right)$

$$\Rightarrow y - \frac{\pi}{4} = -2x + \pi$$

$$+\frac{\pi}{4} \qquad\qquad +\frac{\pi}{4}$$

$$\Rightarrow \boxed{y = -2x + \frac{5\pi}{4}}$$

Find the limit.

$$\lim_{t\to 0} \frac{\tan(8t)}{\sin(4t)}$$

$$= \lim_{t\to 0} \frac{\frac{\sin(8t)}{\cos(8t)}}{\frac{\sin(4t)}{1}}$$

$$= \lim_{t\to 0} \left[\frac{\sin(8t)}{\cos(8t)} \cdot \frac{1}{\sin(4t)} \right]$$

$$= \lim_{t\to 0} \left[\frac{\sin(8t)}{1} \cdot \frac{1}{\cos(8t)} \cdot \frac{1}{\sin(4t)} \right]$$

$$= \lim_{t\to 0} \frac{\sin(8t) \cdot 8t}{1 \cdot 8t} \cdot \lim_{t\to 0} \frac{1}{\cos(8t)} \cdot \lim_{t\to 0} \frac{1 \cdot \frac{1}{4t}}{\sin(4t) \cdot \frac{1}{4t}}$$

$$= \lim_{t\to 0} \left[8t \cdot \frac{\sin(8t)}{8t} \right] \cdot \lim_{t\to 0} \frac{1}{\cos(8t)} \cdot \lim_{t\to 0} \frac{\frac{1}{4t}}{\frac{\sin(4t)}{4t}}$$

$$= \lim_{t\to 0} \frac{8t}{4t} \cdot \lim_{t\to 0} \frac{\sin(8t)}{8t} \cdot \lim_{t\to 0} \frac{1}{\cos(8t)} \cdot \lim_{t\to 0} \frac{1}{\frac{\sin(4t)}{4t}}$$

Note: $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

$$= \underbrace{\lim_{t \to 0} 2}\cdot \underbrace{\lim_{t \to 0} \frac{\sin(8t)}{8t}} \cdot \underbrace{\lim_{t \to 0} \frac{1}{\cos(8t)}} \cdot \frac{1}{\underbrace{\lim_{t \to 0} \frac{\sin(4t)}{4t}}}$$

$$= 2 \cdot 1 \cdot \frac{1}{\cos(0)} \cdot \frac{1}{1}$$

$$= 2 \cdot 1 \cdot \frac{1}{1} \cdot \frac{1}{1}$$

$$= 2 \cdot 1 \cdot 1 \cdot 1$$

$$= \boxed{2}$$

Answer

44. Find the limit.

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\sin(\theta)}$$

$$= \lim_{\theta \to 0} \frac{[\cos(\theta) - 1] \cdot 1/\theta}{[\sin(\theta)] \cdot 1/\theta}$$

$$= \lim_{\theta \to 0} \frac{\frac{\cos(\theta) - 1}{\theta}}{\frac{\sin(\theta)}{\theta}}$$

$$= \frac{\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta}}{\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}} = \frac{0}{1} = \boxed{0}$$

45. Find the limit.

$$\lim_{x \to 0} \frac{\sin(x^8)}{x}$$

$$= \lim_{x \to 0} \frac{x^7 \cdot [\sin(x^8)]}{x^7 \cdot [x]}$$

$$= \lim_{x \to 0} \frac{x^7 \cdot \sin(x^8)}{x^8}$$

$$= \lim_{x \to 0} x^7 \cdot \frac{\sin(x^8)}{x^8}$$

$$= \underbrace{\lim_{x \to 0} x^7}_{} \cdot \underbrace{\lim_{x \to 0} \frac{\sin(x^8)}{x^8}}_{}$$

$$= 0^7 \cdot 1$$

$$= 0 \cdot 1$$

$$= \boxed{0}$$

4b $\displaystyle \lim_{x\to\frac{\pi}{4}} \frac{8-8\tan(x)}{\sin(x)-\cos(x)}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{8-8\frac{\sin(x)}{\cos(x)}}{\sin(x)-\cos(x)} = \lim_{x\to\frac{\pi}{4}} \frac{8\left(1-\frac{\sin(x)}{\cos(x)}\right)}{\sin(x)-\cos(x)}$$

$$= 8\lim_{x\to\frac{\pi}{4}} \frac{\left(1-\frac{\sin(x)}{\cos(x)}\right)\cdot\cos(x)}{(\underbrace{\sin(x)-\cos(x)}_{-(\cos(x)-\sin(x))})\cos(x)}$$

$$= 8\lim_{x\to\frac{\pi}{4}} \frac{\overset{1}{\cancel{\cos(x)-\sin(x)}}}{-\cos(x)(\cancel{\cos(x)-\sin(x)})}$$

$$= -8\lim_{x\to\frac{\pi}{4}} \frac{1}{\cos(x)} = -8\left(\frac{1}{\cos\left(\frac{\pi}{4}\right)}\right)$$

$$= -8\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)$$

$$= \boxed{-8\sqrt{2}}$$

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$$\lim_{x\to 0} \frac{\sin(7x)}{\sin(9x)}$$

$$= \lim_{x\to 0}\left(\sin(7x) \cdot \frac{1}{\sin(9x)}\right)$$

$$= \lim_{x\to 0}\left(7x \cdot \frac{\sin(7x)}{7x} \cdot \frac{1}{\frac{9x \sin(9x)}{9x}}\right)$$

$$= \lim_{x\to 0}\left(7x \cdot \frac{\sin(7x)}{7x} \cdot \frac{1}{9x} \cdot \frac{1}{\frac{\sin(9x)}{9x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{7x}{9x} \cdot \frac{\sin 7x}{7x} \cdot \frac{1}{\frac{\sin 9x}{9x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{7}{9} \cdot \frac{\sin 7x}{7x} \cdot \frac{1}{\frac{\sin 9x}{9x}}\right)$$

$$= \frac{7}{9} \cdot 1 \cdot \frac{1}{1}$$

$$= \boxed{\frac{7}{9}}$$

Find the derivative of the function.

$f(t) = 5t \cdot \sin(\pi t)$

Using product Rule $\Rightarrow$ $f'(t) = \frac{d}{dt}\left[5t \cdot \sin(\pi t)\right]$

$$= \underbrace{\frac{d}{dt}[5t]}_{5} \cdot \sin(\pi t) + 5t \cdot \underbrace{\frac{d}{dt}\left[\sin(\pi t)\right]}_{\text{use chain Rule.}}$$

$$= 5 \cdot \sin(\pi t) + 5t \cdot \cos(\pi t) \cdot \underbrace{\frac{d}{dt}(\pi t)}_{\pi}$$

$$= \boxed{5\sin(\pi t) + 5\pi t \cos(\pi t)}$$

Find the derivative of the function.

$h(t) = (t + 5)^{2/3}(2t^2 - 3)^3$

$$h'(t) = \frac{d}{dt}\left[(t+5)^{2/3}\right]\cdot(2t^2-3)^3 + (t+5)^{2/3}\cdot\frac{d}{dt}\left[(2t^2-3)^3\right]$$

$$= \frac{2}{3}(t+5)^{\frac{2}{3}-1 \leftarrow 3/3}\cdot(1)\cdot(2t^2-3)^3 + (t+5)^{2/3}\cdot 3(2t^2-3)^{3-1}\cdot 4t^{2-1}$$

$$= \frac{2}{3}(t+5)^{-1/3}(2t^2-3)^3 + (t+5)^{\frac{2}{3}}\cdot 12t(2t^2-3)^2$$

$$\frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$= \frac{2(2t^2-3)^3}{3(t+5)^{1/3}} + \frac{12t(2t^2-3)^2(t+5)^{2/3}\cdot 3(t+5)^{1/3}}{1\cdot 3(t+5)^{1/3}}$$

$$= \frac{2(2t^2-3)^3}{3(t+5)^{1/3}} + \frac{36t(2t^2-3)^2(t+5)^1}{3(t+5)^{1/3}}$$

$$= \frac{2(2t^2-3)^3 + 36t(2t^2-3)^2(t+5)}{3(t+5)^{1/3}}$$

$$= \frac{2(2t^2-3)^2\left[(2t^2-3)^1 + 18t(t+5)\right]}{3(t+5)^{1/3}}$$

$$= \frac{2(2t^2-3)^2\left[2t^2-3+18t^2+90t\right]}{3(t+5)^{1/3}}$$

$$= \frac{2(2t^2-3)^2(20t^2+90t-3)}{3(t+5)^{1/3}}$$

Find an equation of the tangent line to the curve at the given point.

$$y = \sin(\sin(x)), \quad (4\pi, 0)$$

Note: $m_{tan} = y' = \frac{d}{dx}\left[\sin(\sin(x))\right]$

$$= \cos(\sin(x)) \cdot \underbrace{\frac{d}{dx}(\sin(x))}$$ Chain Rule

$$= \cos(\sin(x)) \cdot \cos(x)$$

$\Rightarrow$ Slope of the tangent line at $x = 4\pi$

is $y'(4\pi) = \cos(\underbrace{\sin(4\pi)}) \cdot \underbrace{\cos(4\pi)}$

$$= \underbrace{\cos(0)} \cdot 1$$

$$= 1 \cdot 1 = 1$$

Equation: $m = 1, \quad (x_1, y_1) = (4\pi, 0)$

$\Rightarrow$ $y - y_1 = m(x - x_1)$ becomes

$$y - 0 = 1(x - 4\pi)$$

$$\Rightarrow \boxed{y = x - 4\pi}$$ Answer

6. Differentiate the function.

$$P(w) = \frac{8w^2 - 6w + 2}{\sqrt{w}}$$

note: $\boxed{\sqrt{w} = w^{1/2}}$ $\Rightarrow P(w) = \dfrac{8w^2 - 6w + 2}{\sqrt{w}}$

Recall: Exponential properties

① $\dfrac{w^m}{w^n} = w^{m-n}$

② $w^{-n} = \dfrac{1}{w^n}$

$$= \frac{8w^2 - 6w + 2}{w^{1/2}}$$

$$= \frac{8w^2}{w^{1/2}} - \frac{6w}{w^{1/2}} + \frac{2}{w^{1/2}}$$

$$= 8w^{2-1/2} - 6w^{1-1/2} + 2w^{-1/2}$$

$$\Rightarrow P(w) = 8w^{3/2} - 6w^{1/2} + 2w^{-1/2}$$

$$\Rightarrow P'(w) = 8 \cdot \frac{3}{2} w^{3/2-1} - 6 \cdot \frac{1}{2} w^{1/2-1} + 2 \cdot \left(-\frac{1}{2}\right) w^{-\frac{1}{2}-1}$$

$$= \frac{24}{2} w^{1/2} - 3w^{-1/2} - 1w^{-3/2}$$

$$= 12w^{1/2} - \frac{3}{w^{1/2}} - \frac{1}{w^{3/2}}$$

OR $= \boxed{12\sqrt{w} - \dfrac{3}{\sqrt{w}} - \dfrac{1}{w^{3/2}}}$ ← Answer

(60) Finding h'' using the Product Rule

$$h'(x) = x(x^2+4)^{-1/2}$$

$$\Rightarrow h''(x) = \underbrace{\frac{d}{dx}(x)}\cdot(x^2+4)^{-1/2} + x\cdot\underbrace{\frac{d}{dx}\left[(x^2+4)^{-1/2}\right]}$$

$$= 1\cdot(x^2+4)^{-1/2} + x\cdot\left(-\frac{1}{2}\right)(x^2+4)^{-\frac{1}{2}-1=-\frac{1}{2}-\frac{2}{2}}\cdot\underbrace{\frac{d}{dx}(x^2+4)}_{2x}$$

$$= (x^2+4)^{-1/2} - \frac{1}{\cancel{2}}x(x^2+4)^{-\frac{3}{2}}\cdot\cancel{2}x$$

$$= \frac{1}{(x^2+4)^{1/2}} - \frac{x^2}{(x^2+4)^{3/2}}$$

$$= \frac{1\cdot(x^2+4)^1}{(x^2+4)^{1/2}\cdot(x^2+4)^{2/2}} - \frac{x^2}{(x^2+4)^{3/2}}$$

$$= \frac{x^2+4}{(x^2+4)^{3/2}} - \frac{x^2}{(x^2+4)^{3/2}}$$

$$= \frac{\cancel{x^2}+4 - \cancel{x^2}}{(x^2+4)^{3/2}}$$

$$= \boxed{\frac{4}{(x^2+4)^{3/2}}}$$

Answer.

60. Find the first and second derivatives of the function.

$$h(x) = \sqrt{x^2 + 4} = (x^2+4)^{1/2}$$

$$h'(x) = \frac{1}{2}(x^2+4)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x^2+4)$$

Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$$

$$= \frac{1}{\cancel{2}}(x^2+4)^{-\frac{1}{2}} \cdot \overset{1}{\cancel{2}}x$$

$$= x(x^2+4)^{-1/2} \quad \boxed{\text{OR}} \quad \frac{x}{(x^2+4)^{1/2}} = \boxed{\frac{x}{\sqrt{x^2+4}}}$$

$h'(x)$

$$h''(x) = \frac{d}{dx}\left[\frac{x}{(x^2+4)^{1/2}}\right]$$

Quotient Rule

$$= \frac{(x^2+4)^{1/2} \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}\left[(x^2+4)^{1/2}\right]}{\left[(x^2+4)^{1/2}\right]^2}$$

$\frac{1}{\cancel{2}} \cdot \cancel{2} = 1$

$$= \frac{(x^2+4)^{1/2} \cdot 1 - x \cdot \cancel{\tfrac{1}{2}} (x^2+4)^{-1/2} \cdot \cancel{2} x}{(x^2+4)^1}$$

$$= \frac{(x^2+4)^{1/2} - x^2 (x^2+4)^{-1/2}}{(x^2+4)}$$

$$= \frac{\left[(x^2+4)^{1/2} - \dfrac{x^2}{(x^2+4)^{1/2}} \right] \cdot (x^2+4)^{1/2}}{\left[x^2+4 \right] \cdot (x^2+4)^{1/2}}$$

$$= \frac{(x^2+4)^{1/2} \cdot (x^2+4)^{1/2} - \dfrac{x^2}{\cancel{(x^2+4)^{1/2}}} \cdot \cancel{(x^2+4)^{1/2}}^{\,1}}{(x^2+4)^1 \cdot (x^2+4)^{1/2}}$$

$$= \frac{(x^2+4)^{\frac{1}{2}+\frac{1}{2} = \frac{2}{2} = 1} - x^2 \cdot 1}{(x^2+4)^{1+\frac{1}{2} = \frac{2}{2}+\frac{1}{2} = 3/2}}$$

$$= \frac{(x^2+4)^1 - x^2}{(x^2+4)^{3/2}}$$

$$= \frac{\cancel{x^2} + 4 \cancel{- x^2}}{(x^2+4)^{3/2}}$$

$$= \boxed{\frac{4}{(x^2+4)^{3/2}}} \leftarrow h''(x)$$

Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 12)$ and $(2, 6)$.

note: ① slope of tangent line = y'

② horizontal lines have slope = 0

$\Rightarrow$ horizontal tangent lines $\Rightarrow y' = 0$

Here, $y' = 3ax^2 + 2bx + c$

At $(-2, 12)$, $y' = 0 \Rightarrow 3a(-2)^2 + 2b(-2) + c = 0$

$x = -2$

$\Rightarrow 12a - 4b + c = 0$

At $(2, 6)$, $y' = 0 \Rightarrow 3a(2)^2 + 2b(2) + c = 0$

$\Rightarrow 12a + 4b + c = 0$

$x = 2$

$$\Rightarrow \text{Solve} \begin{cases} 12a - 4b + c = 0 \\ 12a + 4b + c = 0 \end{cases}$$

Using addition/elimination method, we get

$$12a - 4b + c = 0$$
$$-1(12a + 4b + c) = (0)\cdot(-1)$$

$$\cancel{12}a - 4b \cancel{+c} = 0$$
$$-\cancel{12}a - 4b - \cancel{c} = 0$$

$$\frac{-8b}{-8} = \frac{0}{-8} \Rightarrow \boxed{b = 0}$$

$\Rightarrow y = ax^3 + bx^2 + cx + d$ becomes $y = ax^3 + cx + d$

For $(-2, 12)$, $x = -2$, $y = 12$, we get

$$12 = a(-2)^3 + c(-2) + d$$
$$\Rightarrow \boxed{12 = -8a - 2c + d}$$

For $(2, 6)$, $x = 2$, $y = 6$, we get

$$6 = a(2)^3 + c(2) + d$$
$$\Rightarrow \boxed{6 = 8a + 2c + d}$$

Solve

$\Rightarrow$ Solve $\begin{cases} -8a - 2c + d = 12 \\ 8a + 2c + d = 6 \end{cases}$

$\Rightarrow$ add:

$$\begin{array}{r} -\cancel{8a} - \cancel{2c} + d = 12 \\ \cancel{8a} + \cancel{2c} + d = 6 \\ \hline \frac{\cancel{2}d}{\cancel{2}} = \frac{18}{2} \end{array} \Rightarrow \boxed{d = 9}$$

$\Rightarrow$ $8a + 2c + d = 6$ becomes

$$8a + 2c + \cancel{9} = 6 \Rightarrow 8a + 2c = -3$$
$$\quad\quad\quad\;\; \underline{-\cancel{9}} \;\; \underline{-9}$$

since $b=0$, $12a - 4b + c = 0$ becomes $12a + c = 0$

$\Rightarrow$ we have

$$\begin{array}{r} 8a + 2c = -3 \\ -2(12a + c) = -2(0) \\ \hline \end{array}$$

$$\begin{array}{r} 8a + \cancel{2c} = -3 \\ -24a - \cancel{2c} = 0 \\ \hline \frac{\cancel{-16}a}{\cancel{-16}} = \frac{-3}{-16} \end{array} \Rightarrow \boxed{a = \frac{3}{16}}$$

$\Rightarrow 8a + 2c = -3$ becomes

$$\cancel{8}^{1}\left(\frac{3}{\cancel{16}_{2}}\right) + 2c = -3$$

$$\Rightarrow 2 \cdot \left(\frac{3}{2} + 2c\right) = (-3) \cdot 2$$

$$\Rightarrow \cancel{3} + 4c = -6$$
$$\underline{-\cancel{3} \qquad\quad -3}$$

$$\Rightarrow \frac{\cancel{4}c}{\cancel{4}} = \frac{-9}{4} \Rightarrow \boxed{c = -\frac{9}{4}}$$

Therefore $y = ax^3 + bx^2 + cx + d$ becomes

$$y = \frac{3}{16}x^3 + 0x^2 + \left(-\frac{9}{4}\right)x + 9$$

which simplifies to

$$\boxed{y = \frac{3}{16}x^3 - \frac{9}{4}x + 9}$$ Answer