1. (May 2000, #5)
An insurance company has agreed to make payments to a worker age x who was injured at work.

   (a) The payments are 150,000 per year, paid annually, starting immediately and continuing for the remainder of the worker's life.
   (b) After the first 500,000 is paid by the insurance company, the remainder will be paid by a reinsurance company.
   (c)
$$ {}_tp_x = \begin{cases} (0.7)^t & \text{for } 0 \leq t \leq 5.5, \\ 0 & \text{otherwise.} \end{cases} $$
   (d) $i = 0.05$

Calculate the actuarial present value of the payments to be made by the reinsurer.

Solution:
If the worker survives for three years, the reinsurance will pay 100,000 at time $t = 3$ and everything after that. So the actuarial present value of the reinsurer's portion of the claim is

$$100,000 \times \left(\frac{0.7}{1.05}\right)^3 + 150,000 \times \left[\left(\frac{0.7}{1.05}\right)^4 + \left(\frac{0.7}{1.05}\right)^5\right] = 79,012.$$

2. (May 2000, #10)
Taxicabs leave a hotel with a group of passengers at a Poisson rate $\lambda = 10$ per hour. The number of people in each group taking a cab is independent and has the following probabilities:

Number of People	Probability
1	0.60
2	0.30
3	0.10

Using the normal approximation, calculate the probability that at least 1050 people leave the hotel in a cab during a 72-hour period.

3. (May 2000, #11)
A company provides insurance to a concert hall for losses due to power failure. You are given:

   (a) The number of power failures in a year has a Poisson distribution with mean 1.
   (b) The distribution of ground up losses due to a single power failure is:

x	$\Pr\{\text{loss} = x\}$
10	0.3
20	0.3
50	0.4

   (c) The number of power failures and the amounts of losses are independent.
   (d) There is an annual deductible of 30.

Calculate the expected amount of claims paid by the insurer in one year.

4. (May 2000, #2)
Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:

   (a) 60% of the coins are worth 1;

(b) 20% of the coins are worth 5; and

(c) 20% of the coins are worth 10.

Calculate the conditional expected value of the coins Tom found during his one-hour walk today, given that among the coins he found, exactly ten were worth 5 each.

<u>**Solution:**</u>
0.5 coins per minute is 30 coins per hour.

$$\lambda_1 = .6 * 30 = 18, \qquad \lambda_5 = .2 * 30 = 6, \qquad \lambda_{10} = .2 * 30 = 6$$

In "thinning" a Poisson process, the thinned processes are independent of each other so the fact that he found 10 nickels does not affect the number of pennies and dimes he found.

$$\mathrm{E}[X] = 1 \cdot \mathrm{E}[N_1] + 5 \cdot 10 + 10 \cdot \mathrm{E}[N_10] = 18 + 50 + 60 = 128.$$

<u>Ans: (C)</u>
This is a thinning problem. The total number N of coins he finds in the hour has a Poisson(30) distribution. Let N_1, N_5, and N_{10} be the number of coins worth 1, 5, and 10, resp. Then each of these has a Poisson distribution, and $\lambda_1 = 0.6 \cdot 30 = 18$, $\lambda_5 = 0.2 \cdot 30 = 6$, and $\lambda_{10} = 0.2 \cdot 30 = 6$.
N_1, N_5 and N_{10} are all independent. The expected number of pennies he finds is 18, we know he found ten nickels, and the expected number of dimes he finds is 6, so the expected total is $18 \cdot 1 + 10 \cdot 5 + 6 \cdot 10 = 18 + 50 + 60 = 128$.

5. (May 2000, #37)
Given:

(a) p_k denotes the probability that the number of claims equals k for $k = 0, 1, 2, \ldots$.

(b) $\dfrac{p_n}{p_m} = \dfrac{m!}{n!}$ for $n \geq 0$ and $m \geq 0$.

Using the corresponding zero-modified claim count distribution with $p_0^M = 0.1$, calculate p_1^M.

<u>**Solution:**</u>
Special case of $\dfrac{p_n}{p_m} = \dfrac{m!}{n!}$ is when $m = n-1$ in which case we get $\dfrac{p_n}{p_{n-1}} = \dfrac{1}{n}$, so that

$$p_n = \frac{1}{n} p_{n-1}.$$

This means that the number of claims follows an (a,b,0) distribution with $a = 0$ and $b = 1$. So the number of claims follows the Poisson(1) distribution.

For the corresponding zero-modified claim count distribution, the PMF values p_k^M are given by

$$\begin{aligned}
p_k^M &= \left(\frac{1 - p_0^M}{1 - p_0}\right) p_k \\
&= \left(\frac{1 - 0.1}{1 - e^{-1}}\right) \cdot \frac{e^{-1}}{k!} \\
p_1^M &= \left(\frac{1 - 0.1}{1 - e^{-1}}\right) \cdot e^{-1} = \boxed{0.52378.}
\end{aligned}$$

6. (May 2000, #4)
You are given:

(a) The claim count N has a Poisson distribution with mean Λ.

(b) Λ has a gamma distribution with mean 1 and variance 2.

Calculate the probability that $N = 1$.

7. (May 2000, #40)
Rain is modeled as a Markov process with two states:

   (a) If it rains today, the probability that it rains tomorrow is 0.50.

   (b) If it does not rain today, the probability that it rains tomorrow is 0.30.

   Calculate the limiting probability that it rains on two consecutive days.

   Solution:
   Here's the idea. First, set up the "stochastic matrix" (or "probability transition matrix") P for this problem. The find the state-state or long term probabilities, by finding the eigenvector corresponding to $\lambda = 1$. From this extract the probability that it will be raining on a given day in the distant future. Then you can answer the question very simply.

   Details:
   Denote by $\pi_{i,n}$ the probability that the system is in state i on day n.
   The given info means:

$$\begin{aligned} \pi_{0,n+1} &= \Pr\{\text{No rain tomorrow}\} \\ &= \Pr\{\text{No rain tomorrow} \mid \text{No rain today}\} \times \Pr\{\text{No rain today}\} \\ &\quad + \Pr\{\text{No rain tomorrow} \mid \text{Rain today}\} \times \Pr\{\text{Rain today}\} \\ \pi_{0,n+1} &= 0.7\pi_{0,n} + 0.5\pi_{1,n} \\ \text{and similarly } \pi_{1,n+1} &= 0.3\pi_{0,n} + 0.5\pi_{1,n} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\pi}_{n+1} &= \boldsymbol{\pi}_n \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \\ \boldsymbol{\pi} &= \boldsymbol{\pi} \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \\ \mathbf{0} &= \boldsymbol{\pi} \begin{bmatrix} -0.3 & 0.3 \\ 0.5 & -0.5 \end{bmatrix} \\ \boldsymbol{\pi} &= \left[\frac{5}{8}, \frac{3}{8}\right] \end{aligned}$$

   so in the distant future the chance of a rain on a given day is $\frac{3}{8}$. Finally (here's the "very simply" part), the probability that it will also rain in the next day, given that it's raining on the one day, 1/2, so the probability of two consecutive days of rain is $\frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16} = \boxed{0.1875}$.

8. (May 2000, #8)
For a two-year term insurance on a randomly chosen member of a population:

   (a) 1/3 of the population are smokers and 2/3 are nonsmokers.

   (b) The future lifetimes follow a Weibull distribution with:

$\tau = 2$	$\theta = 1.5$	for smokers,
$\tau = 2$	$\theta = 2.0$	for nonsmokers.

(c) The death benefit is 100,000 payable at the end of the year of death.

(d) $i = 0.05$.

Calculate the actuarial present value of this insurance.

Solution:
For each of the two classes of (smokers or nonsmokers) we can use "first principles" to compute the APV of a death benefit payable *immediately*, and then the fact that

$$\delta \bar{A}_x = i A_x.$$

Then we can condition on class.

Here are the details. If $f(t)$ denotes the PDF of the future lifetime random variable, we have

$$\bar{A}_x = \int_0^\infty e^{-\delta t} f(t)\, dt.$$

From the tables provided with the exam, we have the PDF :

9. (Nov 2000, #23)
Worker's compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. Two percent of the claims exceed 30,000. Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.

Solution:
Whether a claim exceeds 30,000 or not can be represented by a Bernoulli(p) distribution with $p = 0.02$. The number of claims exceeding 30,000 in n months is a compound Poisson with frequency $\lambda 100n$ and Bernoulli(p) severity. We have

$$\begin{aligned}
M_X(t) &= 1 - p + pe^t, \\
M_S(t) &= e^{\lambda(M_X(t)-1)} = e^{\lambda p(e^t-1)} \\
&= \text{The MGF for a Poisson}(\lambda p) \text{ distribution} \\
\Pr\{S = k\} &= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \\
\Pr\{S = k\} &= \frac{(2n)^k e^{-2n}}{k!} \\
\Pr\{S > 2\} &= 1 - (\Pr\{S = 0\} + \Pr\{S = 1\} + \Pr\{S = 2\}) \\
\ldots &
\end{aligned}$$

10. (Nov 2000, #23)
Workers' compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. Two percent of the claims exceed 30,000. Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.

Solution:
Whether a claim exceeds 30,000 or not can be represented by a Bernoulli(p) distribution

with $p = 0.02$. The number of claims exceeding 30,000 in n months is a compound Poisson with frequency $\lambda 100n$ and Bernoulli(p) severity. We have

$$\begin{aligned}
M_X(t) &= 1 - p + pe^t, \\
M_S(t) &= e^{\lambda(M_X(t)-1)} = e^{\lambda p(e^t-1)} \\
&= \text{The MGF for a Poisson}(\lambda p) \text{ distribution} \\
\Pr\{S = k\} &= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \\
\Pr\{S = k\} &= \frac{(2n)^k e^{-2n}}{k!} \\
\Pr\{S > 2\} &= 1 - (\Pr\{S = 0\} + \Pr\{S = 1\} + \Pr\{S = 2\}) \\
\ldots
\end{aligned}$$

11. (Nov 2000, #29)
Job offers for a college graduate arrive according to a Poisson process with mean 2 per month. A job offer is acceptable if the wages are at least 28,000. Wages offered are mutually independent and follow a lognormal distribution with $\mu = 10.12$ and $\sigma = 0.12$. Calculate the probability that it will take a college graduate more than 3 months to receive an acceptable job offer.

12. (Nov 2000, #34)
The Town Council purchases weather insurance for their annual July 14 picnic.

    (a) For each of the next three years, the insurance pays 1000 on July 4 if it rains on that day.

    (b) Weather is modeled by a Markov chain, with two states, as follows:

        i. The probability that it rains on a day is 0.50 if it rained on the prior day.

        ii. The probability that it rains on a day is 0.20 if it did not rain on the prior day.

    (c) $i = 0.10$

    Calculate the single benefit premium for this insurance purchased one year before the first scheduled picnic.

    Solution:
    Chance of rain on a random day in the distant future, p, $p = 2/7$.

    $$\text{Premium} = \frac{2}{7} \cdot 1000 \cdot \left(\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.13}\right)$$

    $$\text{Premium} = \boxed{710.53}$$

13. (Nov 2000, #8)
The number of claims, N, made on an insurance portfolio follows the following distribution:

n	$\Pr\{N = n\}$
0	0.7
2	0.2
3	0.1

If a claim occurs, the benefit is 0 or 10 with probability 0.8 and 0.2, respectively. The number of claims and the benefit for each claim are independent.

Calculate the probability that aggregate benefits will exceed expected benefits by more than 2 standard deviations.

Solution:

$$
\begin{aligned}
\mathrm{E}[N] &= 0.7 \\
\mathrm{Var}[N] &= 2^2 \cdot 0.2 + 3^2 \cdot 0.1 - (0.7)^2 = 1.21 \\
\mathrm{E}[X] &= 2 \\
\mathrm{Var}[X] &= 10^2 \cdot 0.2 - (2)^2 = 16 \\
\mathrm{Var}[S] &= \mathrm{E}[N]\,\mathrm{Var}[X] + \mathrm{E}[X]^2\,\mathrm{Var}[N] = 0.7 \cdot 16 + 4 \cdot 1.21 = 16.04 \\
\mathrm{Std}[S] &= 4.005 \\
\mathrm{E}[S] + 2\mathrm{Std}[S] &= 1.4 + 2 \cdot 4.005 = 9.4 \\
\Pr\{S > 9.04\} &= 1 - \Pr\{S = 0\} = 1 - \left(0.7 + 0.2 \cdot 8^2 - 0.1 \cdot 8^3\right) = 0.12.
\end{aligned}
$$

14. (May 2001, #25)
For a discrete probability distribution, you are given the recursion relation

$$p_k = \frac{2}{k} p_{k-1} \quad \text{for} k = 1, 2, 3, \ldots.$$

Determine p_4.

Solution:
This is an (a,b,0) recursion formula with $a = 0$ and $b = 2$ to the distribution is Poisson(2). So $p_k = \dfrac{e^{-2}2^4}{4!} = 0.09022$.

15. (Course 3, Nov 2003, #4)
Computer maintenance costs for a department are modeled as follows:

(a) The distribution of the number of maintenance calls each machine will need in a year is Poisson with mean 3.
(b) The cost for a maintenance call has mean 80 and standard deviation 200.
(c) The number of maintenance calls and the costs of the maintenance calls are all mutually independent.

The department must buy a maintenance contract to cover repairs if there is at least a 10% probability that aggregate maintenance costs in a given year will exceed 120% of the expected costs. Using the normal approximation for the distribution of the aggregate maintenance costs. calculate the minimum number of coniputers needed to avoid purchasing a maintenance contract.

Solution:

Aggregate repair cost is a compound random variable, call it S. If there are, say, n computers, then $N \sim$ Poisson($3n$). So,

$$
\begin{aligned}
\mathrm{E}[S] &= \mathrm{E}[N]\,\mathrm{E}[X] = 3n \times 80 = 240n \\
\mathrm{Var}[S] &= \mathrm{E}[N]\,\mathrm{Var}[X] + \mathrm{E}[X]^2\,\mathrm{Var}[N] = 3n(200^2 + 80^2) = (373.1\sqrt{n})^2.
\end{aligned}
$$

$$
\begin{aligned}
& \Pr\{S > 1.2\mathrm{E}[S]\} = 0.1 \quad \Longrightarrow \quad \Pr\{S \le 288n\} = 0.9 \\
\Longrightarrow \quad & \Pr\left\{\frac{S - 240n}{373.1\sqrt{n}} \le \frac{288n - 240n}{373.1\sqrt{(n)}}\right\} = 0.9 \\
\Longrightarrow \quad & \Phi\left(\frac{48\sqrt{n}}{373.1}\right) = 0.9 \quad \Longrightarrow \quad \frac{48\sqrt{n}}{373.1} = 1.28155 \quad \Longrightarrow \quad n = 99.229
\end{aligned}
$$

<u>Ans: C</u>

16. (Nov 2004, #15)

Two types of insurance claims are made to an insurance company. For each type. the number of clainis follows a Poisson distribution and the amount of each claim is uniformly distributed as follows:

Type of Claim	Poisson Paramter λ for # claims	Range of each Claim Amt
I	12	(0,1)
II	4	(0,5)

The numbers of claims of the two types are independent and the claim amounts and claim numbers are independent. Calculate the normal approximation to the probability that the total of claim amounts exceeds 18.

Solution:

Let $N_1,\ \ N_2$ denote the random variable for # of claims for Type I and II in 2 years

$X_1,\ \ X_2$ denote the claim amount for Type I and II

$S_1 =$ total claim amount for type I in 2 years

$S_2 =$ total claim amount for Type II at time in 2 years

$S = S_1 + S_2 =$ total claim amount in 2 years

$\{S_1\} \rightarrow$ compound poisson $\lambda_1 = 2\times 6 = 12 \quad X_1 \sim U(0,1)$

$\{S_2\} \rightarrow$ compound poisson $\lambda_2 = 2\times 2 = 4 \quad X_2 \sim U(0,5)$

$$E(N_1) = Var(N_1) = 2\times 6 = 12$$

$$E(S_1) = E(N_1)E(X_1) = (12)(0.5) = 6$$

$$\begin{aligned} Var(S_1) &= E(N_1)Var(X_1) + Var(N_1)\left(E(X_1)\right)^2 \\ &= (12)\frac{(1-0)}{12} + (12)(0.5)^2 \\ &= 4 \end{aligned}$$

$$E(N_2) = Var(N_2) = 2\times 2 = 4$$

With formulas corresponding to those for S_1,

$$E(S_2) = 4\times\frac{5}{2} = 10$$

$$Var(S_2) = 4\times\frac{(5-0)^2}{12} + 4\left(\frac{5}{2}\right)^2 = 33.\overline{3}$$

$$E(S) = E(S_1) + E(S_2) = 6 + 10 = 16$$

Since S_1 and S_2 are independent,

$$Var(S) = Var(S_1) + Var(S_2) = 4 + 33.\overline{3} = 37.\overline{3}$$

$$\Pr(S > 18) = \Pr\left(\frac{S-16}{\sqrt{39.\overline{3}}} > \frac{2}{\sqrt{37.\overline{3}}} = 0.327\right)$$

Using normal approximation

$$\begin{aligned} \Pr(S > 18) &= 1 - \Phi(0.327) \\ &= 0.37 \end{aligned}$$

17. (Nov 2004, #26)

Customers arrive at a store at a Poisson rate that increases linearly from 6 per hour at 1:00 p.m. to 9 per hour at 2:00 p.m.

Calculate tile probability that exactly 2 customers arrive between 1:00 p.m. and 2:00 p.m.

18. (May 2005 #17)
For a collective risk model the number of losses has a Poisson distribution with $\lambda = 20$. The common distribution of the individual losses has the following characteristics:

(a) $\mathrm{E}[X] = 70$.

(b) $\mathrm{E}[X \wedge 30] = 25$.

(c) $\Pr\{X > 30\} = 0.75$

(d) $\mathrm{E}\left[X^2 | X > 30\right] = 9000$.

An insurance covers aggregate losses subject to an ordinary deductible of 30 per loss.

Calculate the variance of the aggregate payments of the insurance.

19. (May 2005 #18)
For a collective risk model:

(a) The number of losses has a Poisson distribution with $\lambda = 2$.

(b) The common distribution of the individual losses is:

x	$f_X(x)$
1	0.6
2	0.4

An insurance covers aggregate losses subject to a deductible of 3. Calculate the expected aggregate payments of the insurance.

20. (May 2005 #19)
A discrete probability distribution has the following properties:

(a) $p_k = c\left(1 + \frac{1}{k}\right)p_{k-1}$.

(b) $p_0 = 0.5$

Calculate c.

Solution:
We immediately see that this distribution is an $(a, b, 0)$ distribution with $a = b = c$. Inspection of the possibilities in the Tables provided shows that the distribution is the Negative Binomial with $r = 2$ and
$$p_0 = 0.5 = \frac{\beta}{1+\beta},$$
so that $\beta = \sqrt{2} - 1$ and $c = \dfrac{\sqrt{2}-1}{\sqrt{2}} = 0.2929$.

21. (May 2005 #25)
Beginning with the first full moon in October deer are hit by cars at a Poisson rate of 20 per day. The time between when a deer is hit and when it is discovered by highway maintenance has an exponential distribution with a mean of 7 days. The number hit and the times until they are discovered are independent.

Calculate the expected number of deer that will be discovered in the first 10 days following the first full moon in October.

Solution:
The Poisson process of deer-hits can be decomposed into two processes: Deer hit but not found, and deer hit and found.
If a single deer is hit before time $t = 10$, then what is the probability that it is found by time $t = 10$? With T the wait-time until the dead deer is found, we have:

$$\begin{aligned}
\Pr\{\text{fnd by } t = 10\} &= \int \Pr\{\text{fnd by } t = 10 \mid \text{hit at time } s\} \Pr\{\text{hit at time } s\} \\
&= \int_0^{10} F_T(10 - s)\frac{1}{10}\, ds \\
&= \frac{1}{10}\int_0^{10} 1 - e^{(s-10)/7}\, ds = \frac{3}{10} + \frac{7}{10}e^{-10/7} \approx 0.4678.
\end{aligned}$$

So, hit-and-found deer happen at a Poisson rate of $0.4678 \times 20 \times 10 = 93.56$
Ans: E

22. (May 2005 #30)
The repair costs for boats in a marina have the following characteristics:

Boat type	Number of boats	Probability that repair is needed	Mean of repair cost given a repair	Variance of repair cost given a repair
Power boats	100	0.3	300	10,000
Sailboats	300	0.1	1000	400,000
Luxury Yachts	50	0.6	4000	2,000,000

At most one repair is required per boat each year.

The marina budgets an amount, Y, equal to the aggregate mean repair costs plus the standard deviation of the aggregate repair costs.

Calculate Y.

23. (May 2005 #31)
The repair costs for boats in a marina have the following characteristics:

Boat type	Number of boats	Probability that repair is needed	Mean of repair cost given a repair	Variance of repair cost given a repair
Power boats	100	0.3	300	10,000
Sailboats	300	0.1	1000	400,000
Luxury Yachts	50	0.6	5000	2,000,000

At most one repair is required per boat each year.

The marina budgets an amount, Y, equal to the aggregate mean repair costs plus the standard deviation of the aggregate repair costs.

Calculate Y.

Solution:
For the i^{th} individual boat, let X_i denote the repair costs, and let B_i be the indicator R.V. (Bernoulli) for whether that boat needed any repairs. We have:

$$\begin{aligned}
\mathrm{E}[X_i] &= \mathrm{E}[\mathrm{E}[X_i|B_i]] = p\mu \\
\mathrm{Var}[X_i] &= \mathrm{E}[\mathrm{Var}[X_i|B_i]] + \mathrm{Var}[\mathrm{E}[X_i|B_i]] = p\sigma^2 + \left[p\mu^2 - p^2\mu^2\right] \\
&= p\left(\sigma^2 + \mu^2(1-p)\right)
\end{aligned}$$

	p	μ	σ	$\mathrm{E}[X_i]$	$\mathrm{Var}[X_i]$
Powerboats	0.3	300	10,000	90	21,900
Sailboats	0.1	1,000	400,000	100	130,000
Luxury Yachts	0.6	5,000	2,000,000	3,000	7,200,000

So the mean and standard deviation of the aggregate repair costs are:

$$\begin{aligned} 100 \times 90 + 300 \times 100 + 50 \times 3,000 &= 189,000 \\ \sqrt{100 \times 21,900 + 300 \times 130,000 + 50 \times 7,200,000} &= 20,030 \end{aligned}$$

so the repair budget should be $189,000 + 20,030 = 209,030$

Ans: B

24. (May 2005 #32)
For an insurance:

(a) Losses can be 100, 200 or 300 with respective probabilities 0.2, 0.2, and 0.6.

(b) The insurance has an ordinary deductible of 150 per loss.

(c) Y^P is the claim payment per payment random variable.

Calculate Var(Y P).

25. (May 2005 #34)
The distribution of a loss, X , is a two-point mixture:

(a) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

(b) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

Calculate $\Pr\{X \leq 200\}$.

Solution:

$$\begin{aligned} \Pr\{X \leq 200\} &= \Pr\{X \leq 200|pt1\}\Pr\{pt1\} + \Pr\{X \leq 200|pt2\}\Pr\{pt2\} \\ &= 0.8 \times F_1(200) + 0.2 \times F_2(200) \\ &= 0.8 \times \left[1 - \left(\frac{100}{100+200}\right)^2\right] + 0.2 \times \left[1 - \left(\frac{3000}{3000+200}\right)^4\right] \\ &= 0.7566 \end{aligned}$$

26. (May 2005 #40)
For aggregate losses, S:

(a) The number of losses has a negative binomial distribution with mean 3 and variance 3.6.

(b) The common distribution of the independent individual loss amounts is uniform from 0 to 20.

Calculate the 95^{th} percentile of the distribution of S as approximated by the normal distribution.

Solution:

$$
\begin{aligned}
\mu &= \mathrm{E}[N]\,\mathrm{E}[X] = 3 \times 10 = 30, \\
\sigma^2 &= \mathrm{Var}[N]\,\mathrm{E}[X]^2 + \mathrm{E}[N]\,\mathrm{Var}[X] \\
&= 3.6 \times 100 + 3 \times 100/12 = 460 = (21.45)^2 \\
\Pr\{S \le \alpha\} &= 0.95 \\
\Pr\{30 + 21.45Z \le \alpha\} &= 0.95 \\
\Pr\{Z \le (\alpha - 30)/21.45\} &= 0.95 \\
(\alpha - 30)/21.45 &= 1.645 \\
\alpha = 65.28 &
\end{aligned}
$$

Ans: C

27. (May 2005 #5)
Kings of Fredonia drink glasses of wine at a Poisson rate of 2 glasses per day.

Assassins attempt to poison the king's wine glasses. There is a 0.01 probability that any given glass is poisoned. Drinking poisoned wine is always fatal instantly and is the only cause of death.

The occurrences of poison in the glasses and the number of glasses drunk are independent events.

Calculate the probability that the current king survives at least 30 days.

Solution:
The poisoned glasses occur at a Poisson rate of $\lambda = 2 \times 0.01 = 1/50$ glasses per day.

Denote by T the wait-time until the first poisoned glass.
We know that $T \sim \text{Exponential}(\lambda)$. We want $\Pr\{T > 30\}$ days.

$$\Pr\{T > 30\} = 1 - \Pr\{T \le 30\} = 1 - F(30) = 1 - \left[1 - e^{-30\lambda}\right] = e^{-0.6} \approx 0.549$$

Ans: (D)

28. (May 2005 #6)
Insurance losses are a compound Poisson process where:

(a) The approvals of insurance applications arise in accordance with a Poisson process at a rate of 1000 per day.
(b) Each approved application has a 20% chance of being from a smoker and an 80% chance of being from a non-smoker.
(c) The insurances are priced so that the expected loss on each approval is -100.
(d) The variance of the loss amount is 5000 for a smoker and is 8000 for a non-smoker.

Calculate the variance for the total losses on one day's approvals.

Solution:
Assuming that the losses from each application are independent, we can decompose the total loss S into the losses S_1 due to smokers and the losses S_2 due to non-smokers, and

$\text{Var}[S] = \text{Var}[S_1] + \text{Var}[S_2]$. Each of S_1 and S_2 are compound random variables with Poisson frequency. We have:

$$\begin{aligned}
\text{Var}[S_1] &= \text{Var}[N_1]\,\text{E}[X_1]^2 + \text{E}[N_1]\,\text{Var}[X_1] \\
&= 200 \times 100^2 + 200 \times 5000 = 3,000,000 \\
\text{Var}[S_2] &= \text{Var}[N_2]\,\text{E}[X_2]^2 + \text{E}[N_2]\,\text{Var}[X_2] \\
&= 800 \times 100^2 + 800 \times 8000 = 14,400,000 \\
\text{Var}[S] &= 3,000,000 + 14,400,000 = 17,400,000
\end{aligned}$$

Ans: E

29. (May 2005 #9)
A loss X follows a 2-parameter Pareto to distribution with $\alpha = 2$ and unspecified parameter θ. You are given:

$$\text{E}[X - 100|X > 100] = \frac{5}{3}\text{E}[X - 50|X > 50].$$

Calculate $\text{E}[X - 150|X > 150]$.

30. (Course M, Nov 2006, #9)
A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts are 1, 2, 3, ..., without limt. The probability that any given Payout is equal to i is $1/2^i$. Payouts are independent.

Calculate the probability that there are no payouts of 1, 2, or 3 in a given 20 minute period.

Solution:
The game makes payouts of amount i at a Poisson rate of λ_i per hour, where $\lambda_i = 5/2^i$. The probability we're looking for is

$$\left(e^{-\lambda_1/3}\right) \cdot \left(e^{-\lambda_2/3}\right) \cdot \left(e^{-\lambda_3/3}\right) = e^{-(\lambda_1+\lambda_2+\lambda_3)/3} = e^{-35/24} \approx 0.2326$$

Ans D

31. (Course M, Nov 2006, #10)
You arrive at a subway station at 6:15. Until 7:00, trains arrive at a Poisson rate of 1 train per 30 minutes. Starting at 7:00, they arrive at a Poisson rate of 2 trains per 30 minutes.

Calculate your expected waiting time until a train arrives.

Solution:
Let T be the waiting time for the first train. We have

$$S_T(t) = \Pr\{T > t\} = \begin{cases} e^{-t/30} & \text{for } t \leq 45 \\ e^{-45/30} \times e^{-(t-45)/15} & \text{otherwise,} \end{cases}$$

and so

$$\begin{aligned}
\text{E}[T] &= \int_0^\infty S_T(t)\,dt \\
&= \int_0^{45} e^{-t/30}\,dt + e^{-45/30}\int_{45}^\infty e^{-(t-45)/15}\,dt = 30 - 15e^{-1.5} \approx 26.653.
\end{aligned}$$

Ans: D

32. (Course M, Nov 2006, #22)
The annual number of doctor visits for each individual in a family of 4 has a geometric distribution with mean 1.5. The annual numbers of visits for the family members are mutually independent. An insurance pays 100 per doctor visit beginning with the 4^{th} visit per family.

Calculate the expected payments per year for this family.

Solution:
Let N be the total number of annual doctor visits for the whole family. This is a sum of four independent Geometric random variables, so it is a Negative Binomial with $r = 4$.
The payments per year for this family is $100 \max(0, N-3)$, so we need $\mathrm{E}[\max(0, N-3)]$.
Denote $p_n = \Pr\{N = n\}$. We have:

$$\begin{aligned}
\mathrm{E}[\max(0, N-3)] &= \sum_{n=0}^{\infty} \max(0, n-3) \cdot p_n = \sum_{n=4}^{\infty} (n-3) \cdot p_n \\
&= \sum_{n=4}^{\infty} n p_n - 3 \sum_{n=4}^{\infty} p_n \\
&= \Big(\mathrm{E}[N] - (p_1 + 2p_2 + 3p_3)\Big) - 3\Big(1 - (p_0 + p_1 + p_2 + p_3)\Big) \\
&= \mathrm{E}[N] - 3 + 3p_0 + 2p_1 + p_2 \\
&= 4 \cdot \frac{3}{2} - 3 + 3 \cdot \frac{16}{625} + 2 \cdot \frac{192}{3125} + \frac{288}{3125} \\
&= 3.29184
\end{aligned}$$

Ans: D

33. (Course M, Nov 2006, #30)
You are the producer for the television show Actuarial Idol. Each year, 1000 actuarial clubs audition for the show. The probability of a club being accepted is 0.20.

The number of members of an accepted club has a distribution with mean 20 and variance 20. Club acceptances and the numbers of club members are mutually independent.

Your annual budget for appearing on the show equals 10 times the expected number of persons plus 10 times the standard deviation of the number of persons.

Calculate your annual budget for persons appearing on the show.

Solution:

The number S of people appearing on the show is a compound random variable where N is the number of clubs accepted and X is the number of members in an accepted club.

$$\begin{aligned}
N &\sim \text{Binomial}(1000, 0.2) \\
\mathrm{E}[N] &= 1000 \times 0.2 = 200 \\
\mathrm{Var}[N] &= 1000 \times 0.2 \times (1 - 0.2) = 160 \\
\mathrm{E}[S] &= \mathrm{E}[N]\,\mathrm{E}[X] = 200 \times 20 = 4000 \\
\mathrm{Var}[S] &= \mathrm{Var}[N]\,\mathrm{E}[X]^2 + \mathrm{Var}[X]\,\mathrm{E}[N] \\
&= 160 \times 20^2 + 20^2 \times 200 = 68,000 = (260.768)^2 \\
\text{Budget} &= 10 \times 4000 + 10 \times 260.77 = 42,607.68
\end{aligned}$$

Ans: A

34. (Course M, Nov 2006, #32)
For an aggregate loss distribution S:

(a) The number of claims has a negative binomial distribution with $r = 16$ and $\beta = 6$.
(b) The claim amounts are uniformly distributed on the interval $(0, 8)$.
(c) The number of claims and claim amounts are mutually independent.

Using the normal approximation for aggregate losses, calculate the premium such that the probability that aggregate losses will exceed the premium is 5%

Solution:

$$\begin{aligned}
\mathrm{E}[N] &= r\beta = 96 \\
\mathrm{Var}[N] &= r\beta(1+\beta) = 672 \\
\mathrm{E}[X] &= (b+a)/2 = 4 \\
\mathrm{Var}[X] &= (b-a)^2/12 = 16/3 \\
\mu = \mathrm{E}[S] &= \mathrm{E}[N]\,\mathrm{E}[X] = 96 \times 4 = 384 \\
\sigma^2 = \mathrm{Var}[S] &= \mathrm{Var}[N]\,\mathrm{E}[X]^2 + \mathrm{Var}[X]\,\mathrm{E}[N] \\
&= 672 \times 4^2 + (16/3) \times 96 = 11,264 = (106.132)^2 \\
\Pr\{S \geq \pi\} &= 0.05 \\
\Pr\left\{\frac{S-\mu}{\sigma} \leq \frac{\pi-\mu}{\sigma}\right\} &= 0.95 \\
\pi &= \mu + \sigma\Phi^{-1}(0.95) = 384 + 106.132 \times 1.64485 = 558.57
\end{aligned}$$

Ans: D

35. (Course M, Nov 2006, #39)
The random variable N has a mixed distribution:

(a) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.
(b) With probability $1-p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

Which of the following is a correct expression for $\Pr\{(\} N = 2)$?

Solution:

$$\begin{aligned}
\Pr\{N=2\} &= \Pr\{N=2|m=2\}\Pr\{m=2\} \\
&\quad + \Pr\{N=2|m=4\}\Pr\{m=4\} \\
&= p\begin{pmatrix}2\\2\end{pmatrix}(0.5)^2(0.5)^{2-2} + (1-p)\begin{pmatrix}4\\2\end{pmatrix}(0.5)^2(0.5)^{4-2} \\
&= \frac{1}{4}\cdot p + \frac{3}{8}(1-p) = \frac{3}{8} - \frac{1}{8}p
\end{aligned}$$

Ans: E

36. (Course M, Nov 2006, #40)
A compound Poisson distribution has $\lambda = 5$ and claim amount distribution as follows:

x	$p(x)$
100	0.80
500	0.16
1000	0.04

Calculate the probability that aggregate claims will be exactly 600.

Solution:

$$\begin{aligned}
\Pr\{S = 600\} &= \sum_{n=0}^{\infty} \Pr\{S = 600 | N = n\} \Pr\{N = n\} \\
&= \Pr\{S = 600 | N = 2\} \Pr\{N = 2\} \\
&\quad + \Pr\{S = 600 | N = 6\} \Pr\{N = 6\} \\
&= 2 \times 0.8 \times 0.16 \times \left(\frac{5^2 e^{-5}}{2!}\right) + 0.80^6 \times \left(\frac{5^6 e^{-5}}{6!}\right) \\
&= 0.05989
\end{aligned}$$

(The factor of 2 in the first term reflects the fact that there could be a 100 then a 500, or a 500 then a 100.)

<u>Ans: D</u>

37. (May 2000, #7)
In a triple decrement table, lives are subject to decrements of death (d), disability (i), and withdrawal (w).

You are given:

(i) The total decrement is uniformly distributed over each year of age.

(ii) $l_x^{(\tau)} = 25,000$.

(iii) $l_{x+1}^{(\tau)} = 23,000$.

(iv) $m_x^{(d)} = 0.02$

(v) $m_x^{(w)} = 0.05$

Calculate q_x^i, the probability of decrement by disability at age x.

38. (May 2000, #1)
Given:

(a) $\mathring{e}_0 = 25$,

(b) $l_x = \omega - x$ for $0 \leq x \leq \omega$,

(c) $T(x)$ is the future lifetime random variable.

Calculate $\mathrm{Var}[T(10)]$.

Solution:
Since the survival function is linear, we know that $T(x) \sim \text{Uniform}(0, \omega - x)$, so that

$$\begin{aligned}
\mathrm{E}[T(x)] &= \frac{b+a}{2} = \frac{\omega - x}{2}, \\
\mathrm{Var}[T(x)] &= \frac{(b-a)^2}{12} = \frac{(\omega - x)^2}{12}.
\end{aligned}$$

Since $\mathrm{E}[T(0)] = \mathring{e}_0 = 25$, we know that $\Omega = 50$, so $\mathrm{Var}[T(10)] = (50-10)^2/(12) = 133.\bar{3}$.

39. (May 2000, #12)
For a certain mortality table, you are given:

(i) $\mu(80.5) = 0.0202$

(ii) $\mu(81.5) = 0.0408$

(iii) $\mu(82.5) = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

Solution:
We use

- the fact that under UDD, for x integers and $s \in [0,1]$ we have

$$ {}_sq_x = s \cdot q_x \tag{1} $$

and

$$ \mu_{x+s} \frac{q_x}{1 - s \cdot q_x}, \tag{2} $$

and

- the principle

$$ {}_{s+t}q_x = {}_sq_x + ({}_sp_x({}_tq_{x+s}) \tag{3} $$

which arises from the fact that $\{T_x \leq s+t\}$ is the union of two disjoint events, $\{T \leq s\}$ and $\{T_x \in (s, s+t]\}$.

From (2) and the given problem data we obtain

$$ q_{80} = 0.02, \qquad q_{81} = 0.04, \quad \text{and} \quad q_{82} = 0.06. $$

Then

$$ \begin{aligned} {}_2q_{80.5} &= {}_{0.5}q_{80.5} + ({}_{0.5}p_{80.5}) \cdot ({}_{1.5}q_{81}) \quad \text{from (3)} \\ {}_2q_{80.5} &= {}_{0.5}q_{80.5} + ({}_{0.5}p_{80.5}) \cdot \left[q_{81} + (p_{81})({}_{0.5}q_{82})\right] \quad \text{from (3) again.} \end{aligned} $$

Now ${}_{0.5}p_{80.5} = (p_{80})/({}_{0.5}p_{80}) = (p_{80})/(1 - {}_{0.5}q_{80}) = (p_{80})/(1 - 0.5 \cdot q_{80}) = 0.98/0.99$, so

$$ \begin{aligned} {}_2q_{80.5} &= \frac{0.01}{0.99} + \frac{0.98}{0.99} \cdot [0.04 + (0.96)(0.5 \cdot 0.06)] \\ &= \boxed{0.07821} \end{aligned} $$

40. (May 2000, #13)
An investment fund is established to provide benefits on 400 independent lives age x.

(i) On January 1, 2001, each life is issued a 10-year deferred whole life insurance of 1000, payable at the moment of death.

(ii) Each life is subject to a constant force of mortality of 0.05.

(iii) The force of interest is 0.07.

Calculate the amount needed in the investment fund on January 1, 2001, so that the probability, as determined by the normal approximation, is 0.95 that the fund will be sufficient to provide these benefits.

Solution:
For one such policy the mean $1000 \times {}_{10|}\bar{A}_x$ is $1000 \times \mu e^{-10(\mu+\delta)}/(\mu+\delta)$ so the mean for the portfolio, call it μ, is

$$ \mu = 400 \times 1000 \times {}_{10|}\bar{A}_x = 50,199.035. $$

The variance for one policy is $1000^2 \times ({}_{10|}{}^2\bar{A}_x - {}_{10|}\bar{A}_x{}^2)$ so

$$\begin{aligned}
\sigma &= \sqrt{400 \times 1000^2 \times ({}_{10|}{}^2\bar{A}_x - {}_{10|}\bar{A}_x{}^2)} \\
\sigma &= 20,000 \times \sqrt{{}_{10|}{}^2\bar{A}_x - {}_{10|}\bar{A}_x{}^2} \\
&= 3073.143
\end{aligned}$$

Now, approximating the distribution of the payout as a normal random variable, we have:

$$\begin{aligned}
\Pr\{X \le \xi\} &= 0.95 \\
\Pr\left\{\frac{X-\mu}{\sigma} \le \frac{\xi-\mu}{\sigma}\right\} &= 0.95 \\
\Pr\left\{Z \le \frac{\xi-\mu}{\sigma}\right\} &= 0.95 \\
\frac{\xi-\mu}{\sigma} &= 1.6449 \\
\xi &= \mu + 1.6449 \times \sigma = 55254.05.
\end{aligned}$$

41. (May 2000, #14)
The Rejection Method is used to generate a random variable with density function $f(x)$ by using the density function $g(x)$ and constant c as the basis, where:

$$\begin{aligned}
f(x) &= 12(x - 2x^2 + x^3), \qquad 0 < x < 1, \\
g(x) &= 1, \qquad 0 < x < 1.
\end{aligned}$$

The constant c has been chosen so as to minimize n, the expected number of iterations needed to generate a random variable from $f(x)$.

Calculate n.

(A) 1.52
(B) 1.78
(C) 1.86
(D) 2.05
(E) 2.11

42. (May 2000, #15)
In a double decrement table:

(a) $l_{30}^{(\tau)} = 1000$
(b) $q_{30}^{\prime(1)} = 0.100$
(c) $q_{30}^{\prime(2)} = 0.300$
(d) ${}_{1|}q_{30}^{(1)} = 0.075$
(e) $l_{32}^{(\tau)} = 472$

Calculate $q_{31}^{(2)}$

Solution:
First, isolate $q_{31}^{(2)}$ in "${}_2q_{30}^{(\tau)} = q_{30}^{(\tau)} + {}_{1|}q_{30}^{(\tau)}$",

$$\begin{aligned}
{}_2q_{30}^{(\tau)} &= q_{30}^{(\tau)} + {}_{1|}q_{30}^{(\tau)} \\
&= q_{30}^{(\tau)} + p_{30}^{(\tau)} \cdot q_{31}^{(\tau)} \\
&= q_{30}^{(\tau)} + p_{30}^{(\tau)} \cdot \left(q_{31}^{(1)} + q_{31}^{(2)}\right) \\
{}_2q_{30}^{(\tau)} &= q_{30}^{(\tau)} + {}_{1|}q_{30}^{(1)} + p_{30}^{(\tau)} \cdot \underline{q_{31}^{(2)}} \\
&\implies \underline{q_{31}^{(2)} = \left({}_2q_{30}^{(\tau)} - q_{30}^{(\tau)} - {}_{1|}q_{30}^{(1)}\right)/p_{30}^{(\tau)}}
\end{aligned}$$

Now get the three numbers on the right side of the last equation:

$$\begin{aligned}
{}_2q_{30}^{(\tau)} &= 1 - p_{30}^{(\tau)} = 1 - \frac{l_{32}^{(\tau)}}{l_{30}^{(\tau)}} = 1 - \frac{472}{1000} = 0.528 \\
{}_{1|}q_{30}^{(\tau)} &= 0.075 \quad \text{(given)}, \\
p_{30}^{(\tau)} &= \left(p_{30}^{'(1)}\right)\left(p_{30}^{'(2)}\right) = \left(1 - q_{30}^{'(1)}\right)\left(1 - q_{30}^{'(2)}\right) = (1-0.100)(1-0.300) = 0.630 \\
q_{30}^{(\tau)} &= 1 - p_{30}^{(\tau)} = 0.37,
\end{aligned}$$

So:

$$\underline{q_{31}^{(2)} = \frac{0.528 - 0.37 - 0.075}{0.630} = 0.1317.}$$

43. (May 2000, #20)
For a last-survivor insurance of 10,000 on independent lives (70) and (80), you are given:

(a) The benefit, payable at the end of the year of death, is paid only if the second death occurs during year 5.

(b) Mortality follows the Illustrative Life Table.

(c) $i = 0.03$

Calculate the actuarial present value of this insurance.

Solution:
Let F be the event that the second death occurs in year 5. We have:

$$\begin{aligned}
F &= \left\{\left[(T_x \le 5) \text{ and } (T_y \in (4,5])\right] \text{ OR } \left[(T_y \le 5) \text{ and } (T_x \in (4,5])\right]\right\} \\
\Pr\{F\} &= ({}_5q_x)({}_{4|}q_y) + ({}_5q_y)({}_{4|}q_x) - ({}_{4|}q_x)({}_{4|}q_y) \\
&= \left(\frac{S(70)-S(75)}{S(70)}\right)\left(\frac{S(84)-S(85)}{S(80)}\right) \\
&\quad + \left(\frac{S(80)-S(85)}{S(80)}\right)\left(\frac{S(74)-S(75)}{S(70)}\right) \\
&\quad - \left(\frac{S(74)-S(75)}{S(70)}\right)\left(\frac{S(84)-S(85)}{S(80)}\right) \\
\Pr\{F\} &= 0.027220
\end{aligned}$$

Now $A = \Pr\{F\}\, v^5 = 0.027220/1.03^5 = \underline{0.0234819}$

44. (Nov 2000, #23)
Workers' compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. Two percent of the claims exceed 30,000. Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.

Solution:
Whether a claim exceeds 30,000 or not can be represented by a Bernoulli(p) distribution with $p = 0.02$. The number of claims exceeding 30,000 in n months is a compound Poisson with frequency $\lambda 100n$ and Bernoulli(p) severity. We have

$$\begin{aligned}
M_X(t) &= 1 - p + pe^t, \\
M_S(t) &= e^{\lambda(M_X(t)-1)} = e^{\lambda p(e^t-1)} \\
&= \text{The MGF for a Poisson}(\lambda p) \text{ distribution} \\
\Pr\{S = k\} &= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \\
\Pr\{S = k\} &= \frac{(2n)^k e^{-2n}}{k!} \\
\Pr\{S > 2\} &= 1 - (\Pr\{S = 0\} + \Pr\{S = 1\} + \Pr\{S = 2\}) \\
\cdots
\end{aligned}$$

45. (May 2000, #29)
For a whole life annuity-due of 1 on (x), payable annually:

(a) $q_x = 0.01$
(b) $q_{x+1} = 0.05$
(c) $i = 0.05$
(d) $\ddot{a}_{x+1} = 6.951$

Calculate the change in the actuarial present value of this annuity-due if p_{x+1} is increased by 0.03.

Solution:
Note that

$$\ddot{a}_x = 1 + vp_x + {}_2E_x\, \ddot{a}_{x+2} = 1 + vp_x + v^2 p_x p_{x+1} \cdot \ddot{a}_{x+2}.$$

If p_{x+1} is replaced by $p_{x+1} + 0.03$ then the new APV, call it $\ddot{a}'_x$, is

$$\begin{aligned}
\ddot{a}'_x &= 1 + vp_x + v^2 p_x (p_{x+1} + 0.03) \cdot \ddot{a}_{x+2} \\
&= \ddot{a}_x + 0.03 \times v^2 p_x \cdot \ddot{a}_{x+2},
\end{aligned}$$

so the change in the APV, call it Δ-APV, is

$$\Delta\text{−APV} = 0.03 \times v^2 p_x \cdot \ddot{a}_{x+2}.$$

Now, $\ddot{a}_{x+1} = 1 + vp_{x+1}\ddot{a}_{x+2}$. Solving this for $\ddot{a}_{x+2}$ and using it (and the other given data) in Δ-APV, we have:

$$\begin{aligned}
\Delta\text{−APV} &= 0.03 \times v^2 p_x \cdot \ddot{a}_{x+2} \\
&= 0.03 \times vp_x(\ddot{a}_{x+1} - 1)/p_{x+1} = \underline{0.177188}
\end{aligned}$$

46. (May 2000, #3)
An insurance on (x) pays 400 at the end of the year of death if (x) dies within the next two years.

(a) The annual premium amount P for this policy is the net level annual premium, which is \$74.33.

(b) The effective annual interest rate is 10%.

(c) The policy value ${}_1V$ after one year is \$16.58

Calculate the variance of the loss at issue.

Solution:
We have:

$$
\begin{aligned}
{}_0L = 400Z^{1}_{x:\overline{2}|} - P\ddot{Y}_{x:\overline{2}|} &= \begin{cases} 400v - P & \text{w/ prob. } q_x \\ 400v^2 - P(1+v) & \text{w/ prob. } {}_{1|}q_x = (1-q_x)q_{x+1} \\ -P(1+v) & \text{w/ prob. } {}_2p_x = (1-q_x)(1-q_{x+1}) \end{cases} \\
&= \begin{cases} 289.30 & \text{w/ prob. } q_x \\ 188.67 & \text{w/ prob. } {}_{1|}q_x \\ -141.90 & \text{w/ prob. } {}_2p_x \end{cases}
\end{aligned}
$$

So we need the probability values q_x and q_{x+1}.
<u>To find q_{x+1}</u>, use the given value of ${}_1V$:

$$
\begin{aligned}
{}_1V = 16.58 &= 400A^{\,1}_{x+1:\overline{1}|} - P\ddot{a}_{x+1:\overline{1}|} \\
&= 400vq_{x+1} - P = \frac{400}{1.1}q_{x+1} - 74.33 \qquad \Longrightarrow \qquad q_{x+1} = 0.2500025
\end{aligned}
$$

<u>To find q_x</u>, use the fact that ${}_0V = 0$ (because P is a *net* premium), and a recursive formula to relate ${}_0V$ to ${}_1V$:

$$
\begin{aligned}
0 = {}_0V &= 400vq_x - P + E_x \cdot ({}_1V) \\
0 &= \frac{400}{1.1}q_x - 74.33 + \frac{1-q_x}{400} \cdot 16.58 \qquad \Longrightarrow \qquad q_x = 0.1700041730
\end{aligned}
$$

So we have

$$
\begin{aligned}
q_x &= 0.1700041730 \\
{}_{1|}q_x &= (1-q_x)q_{x+1} = 0.207501 \\
{}_2p_x &= 1 - q_x - {}_{1|}q_x = 0.622494827
\end{aligned}
$$

So

$$
{}_0L = 400Z^{1}_{x:\overline{2}|} - P\ddot{Y}_{x:\overline{2}|} = \begin{cases} 289.30 & \text{w/ prob. } 0.1700041730 \\ 188.67 & \text{w/ prob. } 0.207501 \\ -141.90 & \text{w/ prob. } 0.622494827 \end{cases}
$$

and so $\mathrm{Var}\left[{}_0L\right] = 34,149$.

47. (May 2000, #3)
For a fully discrete two-year term insurance of 400 on (x):

(a) $i = 0.1$.

(b) $400P^{1}_{x:\overline{2}|} = 74.33$.

(c) $400\,{}_1V^{\,1}_{x:\overline{2}|} = 16.58$

(d) The contract premium equals the benefit premium.

Calculate the variance of the loss at issue.

Solution:
The loss at issue is the benefit present value random variable minus the premium stream present value random variable.
The reserve ${}_tV$ is the present value t years after issue of the benefit to be paid, minus the present value of the premiums yet to be collected. Since at $t = 1$ there is only one more year to go in the contract, and since premia are colllected at the beginning of the year, the present value of the remaining premiums to be collected is exactly one annual premium, that is the amount given in 1. We have

$$16.58 = 400\,{}_1V^{\;1}_{x:\overline{2}|} = 400vq_{x+1} - 400P^{1}_{x:\overline{2}|} = 400q_{x+1}/1.1 - 74.33.$$

Solving for q_{x+1} we have

$$q_{x+1} = \frac{(16.58 + 74.33)1.1}{400} = 0.2500025$$

48. (May 2000, #39)
For a continuous whole life annuity of 1 on (x):

(a) $T(x)$, the future lifetime of (x), follows a constant force of mortality 0.06.
(b) The force of interest is 0.04.

Calculate $\Pr\left\{\bar{a}_{\overline{T(x)}|} > \bar{a}_x\right\}$.

Solution:

$$\begin{aligned}
\bar{a}_x &= \frac{1 - e^{-\delta x}}{\delta} = 10, \\
\bar{a}_{\overline{T(x)}|} &= \frac{1 - e^{-\delta T(x)}}{\delta} \\
\Pr\left\{\bar{a}_{\overline{T(x)}|} > \bar{a}_x\right\} &= \Pr\left\{\frac{1 - e^{-\delta T(x)}}{\delta} > 10\right\} \\
&= \Pr\left\{T(x) > -\log(1 - 10\delta)/\delta\right\} = \Pr\left\{T(x) > 12.77\right\} \\
&= e^{-12.77 \cdot 0.06} = \underline{0.464758}
\end{aligned}$$

49. (May 2000, #6)
A special purpose insurance company is set up to insure one single life. The risk consists of a single possible claim.

(a) The claim amount distribution is:

Amount	Probability
100	0.60
200	0.40

(b) The probability that the claim does not occur by time t is $1/(1 + t)$.
(c) The insurer's surplus at time t is $U(t) = 60 + 20t - S(t)$, where $S(t)$ is the aggregate claim amount paid by time t.
(d) The claim is payable immediately.

Calculate the probability of ruin.

Solution:
Denote by T the time-of-claim. From the problem statement we have

$$\Pr\{T < t\} = 1 - \frac{1}{1+t} = \frac{t}{1+t}.$$

Clearly $\Pr\{T < \infty\} = 1$, that is, there is eventually a claim.
Ruin occurs if for any $t > 0$ we have $U(t) \leq 0$. If this happens, it happens at time T. So we need to compute $\Pr\{U(t) \leq 0\}$. Now,

$$\begin{aligned}
\Pr\{U(T) <\leq 0\} &= \Pr\{60 + 20T + S(T) \leq 0\} = \Pr\left\{T \leq \frac{S(T) - 60}{20}\right\} \\
&= \Pr\left\{\left[T \leq \frac{S(T) - 60}{20}\right] \mid \text{small claim}\right\} \Pr\{\text{small claim}\} \\
&\quad + \Pr\left\{\left[T \leq \frac{S(T) - 60}{20}\right] \mid \text{large claim}\right\} \Pr\{\text{large claim}\} \\
&= 0.6\Pr\{T \leq 2\} + 0.4\Pr\{T \leq 7\} = 0.75.
\end{aligned}$$

ANSWER: D

50. (May 2000 #6)
A special purpose insurance company is set up to insure one single life. The risk consists of a single possible claim.

(a) The claim amount distribution is:

Amount	Probability
100	0.60
200	0.40

(b) The probability that the claim does not occur by time t is $1/(1+t)$.

(c) The insurer's surplus at time t is $U(t) = 60 + 20t - S(t)$, where $S(t)$ is the aggregate claim amount paid by time t.

(d) The claim is payable immediately.

Calculate the probability of ruin.

(A) 4/7
(B) 3/5
(C) 2/3
(D) 3/4
(E) 7/8

Solution:
Denote by T the time-of-claim. From the problem statement we have

$$\Pr\{T < t\} = 1 - \frac{1}{1+t} = \frac{t}{1+t}.$$

Clearly $\Pr\{T < \infty\} = 1$, that is, there is eventually a claim.

Ruin occurs if for any $t > 0$ we have $U(t) \le 0$. If this happens, it happens at time T. So we need to compute $\Pr\{U(t) \le 0\}$. Now,

$$\begin{aligned}
\Pr\{U(T) <\le 0\} &= \Pr\{60 + 20T + S(T) \le 0\} = \Pr\left\{T \le \frac{S(T)-60}{20}\right\} \\
&= \Pr\left\{\left[T \le \frac{S(T)-60}{20}\right] \mid \text{small claim}\right\}\Pr\{\text{small claim}\} \\
&\quad + \Pr\left\{\left[T \le \frac{S(T)-60}{20}\right] \mid \text{large claim}\right\}\Pr\{\text{large claim}\} \\
&= 0.6\Pr\{T \le 2\} + 0.4\Pr\{T \le 7\} = 0.75.
\end{aligned}$$

ANSWER: D

51. (May 2000, #9)
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:

(i) Mortality follows De Moivre's law with $\omega = 85$.

(ii) $i = 0$.

(iii) Level benefit premiums are payable continuously for 10 years.

Calculate the net premium policy value at the end of five years.

Solution:
Since $i = 0$ we have $v = 1$ and $\bar{a}_{\overline{T_x}|} = T_x$ (this follows from a quick reflection on the meaning of $\bar{a}_{\overline{t}|}$, or from $\bar{a}_{\overline{T_x}|} = (1 - e^{\delta T_x})/\delta$ and L'Hopital's Rule), so $\bar{a}_x = \mathring{e}_x$. Since also, survival follows the deMoivre's law, we have:

$$\begin{aligned}
{}_nE_x &= {}_tp_x = \frac{85 - x - n}{85 - x}, \\
\text{and } \bar{a}_x &= \mathring{e}_x = \frac{85 - x}{2}, \\
\text{so that } {}_{n|}\bar{a}_x &= {}_tE_x\, \bar{a}_{x+n} = \frac{(85 - x - n)^2}{2(85 - x)}, \\
\text{and } \bar{a}_{x:\overline{n}|} &= \bar{a}_x - {}_{n|}\bar{a}_x = \frac{(85 - x)^2 - (85 - x - n)^2}{2(85 - x)}
\end{aligned}$$

Now the level benefit premium — let's call it π — is determined from the equivalence principle,

$$\begin{aligned}
0 &= {}_{10|}\bar{a}_{35} - \pi \bar{a}_{35:\overline{10}|} \\
\Longrightarrow \pi &= \frac{{}_{10|}\bar{a}_{35}}{\bar{a}_{35:\overline{10}|}} = \frac{(85 - 35 - 10)^2}{(85 - 35)^2 - (85 - 35 - 10)^2} = \frac{16}{9}
\end{aligned}$$

Now, prospectively,

$$\begin{aligned}
{}_5V &= {}_{5|}\bar{a}_{40} - \pi \bar{a}_{40:\overline{5}|} \\
&= \frac{(85 - 40 - 5)^2}{2(85 - 40)} - \left(\frac{16}{9}\right)\frac{(85 - 40)^2 - (85 - 40 - 5)^2}{2(85 - 40)} = \frac{760}{81} \approx \boxed{9.3827}
\end{aligned}$$

Or, retrospectively,

$$
\begin{aligned}
{}_5V &= \frac{-1}{{}_5E_{35}}\left(-\pi\bar{a}_{35:\overline{5}|}\right) \\
&\qquad \bar{a}_{35:\overline{5}|} = \frac{(85-35)^2-(85-35-5)^2}{2(85-35)} = \frac{19}{4} \\
&\qquad {}_5E_{35} = \frac{85-35-5}{85-35} = \frac{9}{10} \\
&= \frac{16}{9}\cdot\frac{19}{4}\bigg/\frac{9}{10} = \frac{760}{81} \approx \boxed{9.3827}
\end{aligned}
$$

52. (Nov 2000, #1)
For independent lives (x) and (y):

(a) $q_x = 0.05$
(b) $q_y = 0.10$
(c) Deaths are uniformly distributed over each year of age.

Calculate ${}_{0.75}q_{xy}$

53. (Nov 2000, #7 (book 6^{th} ed #13.16))
For a multiple decrement table, you are given:

(a) Decrement (1) is death, decrement (2) is disability, and decrement (3) is withdrawal.
(b) $q_{60}^{\prime(1)} = 0.010$
(c) $q_{60}^{\prime(2)} = 0.050$
(d) $q_{60}^{\prime(3)} = 0.100$
(e) Withdrawals occur only at the end of the year.
(f) Mortality and disability are uniformly distributed over each year of age in the associated single decrement tables.

Calculate $q_{60}^{(3)}$

Solution:
Ans: (C). Decrement (3) can only happen at the end of the year if neither of the other two decrements happened during the year, so we have

$$
\begin{aligned}
\Pr\{\text{going out by decrement (3)}\} &= \Pr\{\text{surviving to year's end}\} \\
&\quad \times \Pr\{\text{not suviving (3) at year's end)}\} \\
&= 0.10 \times \Pr\{\text{surviving to year's end}\}.
\end{aligned}
$$

Since decrement (3) cannot happen during the year, we have

$$
\begin{aligned}
\Pr\{\text{surviving to year's end}\} &= \Pr\{(\text{surviving (1)}) \text{ AND } (\text{surviving (2)})\} \\
&= \Pr\{\text{surviving (1)}\}\cdot\Pr\{\text{surviving (2)}\} \\
&= (1-0.01)\cdot(1-0.05) \\
&= 0.9405, \\
\text{so, } \Pr\{\text{going out by decrement (3)}\} &= 0.10\times 0.9405 = 0.09405.
\end{aligned}
$$

54. (Nov 2000, #9)
For a special whole life insurance of 100,000 on (x), you are given:

(a) $\delta = 0.06$

(b) The death benefit is payable at the moment of death.

(c) If death occurs by accident during the first 30 years, the death benefit is doubled.

(d) $\mu_x^{(\tau)}(t) = 0.008$ for $t \geq 0$.

(e) $\mu_x^{(1)}(t) = 0.001$ for $t \geq 0$, where $\mu_x^{(1)}$ is the force of decrement due to death by accident.

Calculate the single benefit premium for this insurance.

Solution:
Model this insurance as (whole life insurance) + (30-year accident term insurance).

In general, the APV for an insurance covering times t from 0 to n we have

$$A = \mathrm{E}\left[e^{-\delta T}\right] = \int_0^n e^{-\delta t} f_T(t)\, dt.$$

For the whole-life insurance we have $n = \infty$ and $f_T(t) = \mu^{(\tau)} e^{-\mu^{(\tau)}}$, and
for the term accident insurance we have $n = 30$ and $f_T(t) = \mu^{(1)} e^{-\mu^{(\tau)}}$. So,

$$\begin{aligned}
A &= \left(10^5 \int_0^\infty \left(e^{-\delta t}\right)\left(\mu^{(\tau)} e^{-\mu^{(\tau)}}\right) dt\right) + \left(10^5 \int_0^{30} \left(e^{-\delta t}\right)\left(\mu^{(1)} e^{-\mu^{(\tau)}}\right) dt\right) \\
&= 10^5 \cdot \left(\frac{\mu^{(\tau)}}{\mu^{(\tau)} + \delta} + \frac{\mu^{(1)}(1 - e^{-30(\mu^{(\tau)} + \delta)})}{\mu^{(\tau)} + \delta}\right) \\
&= 10^5 \cdot \left(\frac{0.008}{0.068} + \frac{0.001 \cdot (1 - e^{-(30)\cdot(0.068)})}{0.068}\right) \\
&= 13044.08
\end{aligned}$$

55. (Nov 2000, #12)
For a 20-year term insurance on (70):

(a) The death benefit, payable at the end of the year of death, is the net premium policy value plus 1000.

(b) The one year mortality rate q_{x+t} is equal to 0.03, for all $t \geq 0$.

(c) $i = 0.07$

Calculate the single benefit premium for this insurance.

Solution:

$$\text{(Reserve at } n) \;+\; \text{(Premium at } n) \;= vp_x \cdot \text{(Reserve at } n+1) + vq_x \cdot \text{(Death Ben. at } n+1)$$

$$
\begin{aligned}
{}_nV + \pi_n &= vp_x({}_{n+1}V) + vq_{x+n}\big[1000 + ({}_{n+1}V)\big] \\
({}_nV) - v({}_{n+1}V) + \pi_n &= 1000vq_{x+n} \quad \text{for each}
\end{aligned}
$$

$$
\begin{aligned}
(n=0) \qquad -\underbrace{v({}_1V)} + \pi_0 &= 1000vq_x \\
(n=1) \qquad {}_1V - v({}_2V) &= 1000vq_{x+1} \\
\underbrace{v({}_1V)} - v^2({}_2V) &= 1000v^2q_{x+1} \\
(\text{add eq'ns w/} \underbrace{\quad}) \qquad \pi_0 - v^2({}_2V) &= 1000(vq_x + v^2q_{x+1}) \\
(n=2) \qquad {}_2V - v({}_3V) &= 1000vq_{x+2} \\
v^2\,{}_2V - v^3({}_3V) &= 1000v^3q_{x+2} \\
(\text{add eq'ns}) \qquad \pi_0 - v^3({}_3V) &= 1000(vq_x + v^2q_{x+1} + v^3q_{x+2})
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
\pi_0 - v^{21}({}_{20}V) &= 1000\sum_{k=0}^{19} v^{k+1}q_{x+k} \\
\pi_0 - 0 &= 1000\sum_{k=0}^{19} v^{k+1}q_{x+k} \\
\pi_0 &= 1000 \cdot (0.03)a_{\overline{20|}} = 317.82
\end{aligned}
$$

56. (Nov 2000, #14 (book 6^{th} ed, #13.12))
For a double decrement table with $l_{40}^{(\tau)} = 2000$:

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{'(1)}$	$q_x^{'(2)}$
40	0.24	0.10	0.25	y
41	—	—	0.20	2y

Calculate $l_{42}^{(\tau)}$

Solution:
ANS: (A)

$$
\begin{aligned}
q_{40}^{(\tau)} &= q_{40}^{(1)} + q_{40}^{(2)} = 0.24 + 0.10 = 0.34, \\
\text{and also } q_x^{(\tau)} &= 1 - p_x^{(\tau)} = 1 - (p_{40}^{'(1)})(p_{40}^{'(2)}) = 1 - (1-0.25)(1-y) \\
\text{so, } 0.34 &= 1 - (1-0.25)(1-y) \\
&\implies \underline{y = 0.12} \\
p_{40}^{(\tau)} &= 1 - q_{40}^{(\tau)} = 1 - 0.34 = 0.66 \\
p_{41}^{(\tau)} &= (1 - q_{41}^{'(1)})(1 - q_{40}^{'(2)}) = (1-0.20)(1-2y) = (0.8)(0.76) = 0.608 \\
{}_2p_{40}^{(\tau)} &= (p_{40}^{(\tau)})(p_{41}^{(\tau)}) = (0.66)(0.608) = 0.40128 \\
l_{42} &= ({}_2p_{40}^{(\tau)}) \cdot l_{40} = 2000 \cdot (0.40128) = 802.56.
\end{aligned}
$$

57. (Nov 2000, #20)
Y is the present-value random variable for a special 3-year temporary life annuity-due on (x). You are given:

(a) ${}_tp_x = 0.9^t$, $t \geq 0$.

(b) K is the curtate-future-lifetime random variable for (x).

(c) $Y = \begin{cases} 1.00 & \text{for } K = 0, \\ 1.87 & \text{for } K = 1, \\ 2.72 & \text{for } K = 2, 3, \ldots. \end{cases}$

Calculate $\mathrm{Var}[Y]$.

Solution:

$$
\begin{aligned}
{}_{k|}q_x &= {}_tp_x - {}_{t+1}p_x = 0.9^t(0.1), \\
Y &= \begin{cases} 1.00 & \text{w/ prob. } 0.1, \\ 1.87 & \text{w/ prob. } 0.09, \\ 2.72 & \text{w/ prob. } 0.81 \end{cases} \\
\mathrm{E}[Y] &= 1 \cdot 0.1 + 1.87 \cdot 0.09 + 2.72 \cdot 0.81 = 2.4715, \\
\mathrm{E}\left[Y^2\right] &= 1^2 \cdot 0.1 + 1.87^2 \cdot 0.09 + 2.72^2 \cdot 0.81 = 6.407425, \\
\mathrm{Var}[Y] &= 6.407425 - 2.4715^2 = \boxed{0.29911275}
\end{aligned}
$$

58. (Nov 2000, #24 (book 6^{th} ed., #13.1)
For students entering a three-year law school, you are given:

(a) The following double decrement table:

	For a student at the beginning of that academic year, Probability of:		
Academic Year	Academic Failure	Withdrawal for All Other Reasons	Survival Through Academic Year
1	0.40	0.20	—
2	—	0.30	—
3	—	—	0.60

(b) Ten times as many students survive year 2 as fail during year 3.

(c) The number of students who fail during year 2 is 40% of the number of students who survive year 2.

Calculate the probability that a student entering the school will withdraw for reasons other than academic failure before graduation.

Solution:
Let's call "academic failure" decrement (1), and "withdrawal for other reasons", decrement (2), let l_0 be the number of students entering law school, and let l_x the number still in school after x years.

Consider the given coniditon (b). The number of students surviving year 2 is the same as the number that make it to year 3, which is l_3. And the number of student who fail in year 3 is the number that make it to year 3 times the percentage failing that year, which is $l_3 q_2^{(1)}$. So given (b) is saying:

$$
l_3 = 10(l_3 \cdot q_2^{(1)}) \qquad \Longrightarrow \underline{(i): \ \ q_2^{(1)} = 0.10}
$$

Further:

$$\begin{aligned}
p_0^{(\tau)} &= 1 - q_0^{(\tau)} = 1 - (q_0^{(1)} + q_0^{(2)}) = \underline{0.40 = p_0^{(\tau)} \ (ii)} \\
l_1 &= (l_0) p_0^{(\tau} = 0.40\, l_0. \\
\text{From (c): } l_1 q_1^{(1)} &= 0.40 \cdot (l_1 p_1^{(\tau)}) \\
q_1^{(1)} &= 0.40 \cdot (1 - q_1^{(1)} - q_1^{(2)}) = 0.40 \cdot (1 - q_1^{(1)} - 0.30) \\
&\quad \Longrightarrow \underline{(iii): \ \ q_1^{(1)} = 0.20} \\
1 &= q_1^{(1)} + q_1^{(2)} + p_1^{(\tau)} \qquad \Longrightarrow \underline{(iv): \ \ p_1^{(\tau)} = 0.50} \\
1 &= q_2^{(1)} + q_2^{(2)} + p_2^{(\tau)} = 0.1 + q_2^{(2)} + 0.6 \qquad \Longrightarrow \underline{(v): \ \ q_2^{(2)} = 0.30}
\end{aligned}$$

So we've got our table filled in:

Academic Year	Academic Failure	Withdrawal for All Other Reasons	Survival Through Academic Year
1	0.40	0.20	(ii) 0.40
2	(iii) 0.20	0.30	(iv) 0.50
3	(i) 0.10	(v) 0.30	0.60

Now

$$\begin{aligned}
{}_2q_0^{(\tau)} &= q_0^{(2)} + (p_0^{(\tau)})(q_1^{(1)}) + ({}_2p_0^{(\tau)})q_2^{(2)} \\
&= 0.20 + (0.40) \cdot (0.30) + (0.4) \cdot (0.5) \cdot (0.30) = 0.3800.
\end{aligned}$$

59. (Nov 2000, #28)
A decreasing term life insurance on (80) pays $(20 - k)$ at the end of the year of death if (80) dies in year $k + 1$, for $k = 0, 1, 2, \ldots, 19$.

You are given:

(a) $i = 0.06$
(b) For a certain mortality table with $q_{80} = 0.2$, the single benefit premium for this insurance is 13.
(c) For this same mortality table, except that $q_{80} = 0.1$, the single benefit premium for this insurance is P.

Calculate P.

Solution:

$$\begin{aligned}
13 &= \sum_{k=0}^{19} (20 - k) v^{k+1} \, {}_{k|}q_{80} \\
&= \frac{20}{1.06} \times (0.2) + \frac{v}{1.06} \times (1 - 0.2) \sum_{j=0}^{18} (19 - j) \, {}_{j|}p_{81} \\
13 &= \frac{20}{1.06} \times (0.2) + \frac{1}{1.06} \times (1 - 0.2) \sum_{j=0}^{18} (19 - j) \, {}_{j|}p_{81} \\
13 &= \frac{(0.2)20}{1.06} + \frac{(0.8)}{1.06} \cdot X
\end{aligned}$$

where $X = \sum_{j=0}^{18} (19-j)\ {}_{j|}p_{81}$,

$$X = (13 \cdot 1.06 - 4)/0.8 = 12.225$$

Similarly,

$$P = \frac{(0.1)20}{1.06} + \frac{(0.9)}{1.06} \cdot X$$

$$\boxed{P = 12.2665}$$

60. (Nov 2000, #30)
For independent lives (50) and (60), $\mu(x) = \dfrac{1}{100-x}$.
Calculate $\mathring{e}_{\overline{50:60}}$

61. (Nov 2000, #36)
Given:

(a) $\mu(x) = F + e^{2x}$, $x \geq 0$.

(b) ${}_{0.4}p_0 = 0.5$.

Calculate F.

Solution:

$$\underline{\frac{1}{2} = {}_{0.4}p_0} = S_0(2/5) = \exp\left(-\int_{z=0}^{2/5} \mu(z)\,dz\right)$$

$$= \exp\left(-\int_{z=0}^{2/5} F + e^{2z}\,dz\right)$$

$$\left[-\int_{z=0}^{4/5} F + e^{2z}\,dz = \frac{-2}{5}F + \frac{1}{2}\left(1 - e^{4/5}\right)\right]$$

$$\frac{1}{2} = \exp\left(\frac{-2}{5}F + \frac{1}{2}\left(1 - e^{4/5}\right)\right)$$

Now solve for F: $\boxed{F = 0.200942}$

62. (Nov 2000, #4)
Amber, age 25, has a future lifetime random variable survival funtion given by $S_{25}(t) = 1 - t/75$. If she takes up skydiving for the next year, her assumed mortality will be such that for the coming year only, she will have a constant force of mortality of 0.1.

Calculate the decrease in the 11-year temporary complete life expectancy for Amber if she takes up skydiving.

Solution:
The n-year temporary complete life expectancy is

$$\mathring{e}_{x:\overline{n}|} = \int_{t=0}^{n} {}_tp_x\ dt.$$

In this problem $x = 25$ and $n = 11$, and “de Moivre” means $S(x) = (\omega - x)/\omega$ which means ${}_tp_x = 1 - t/(\omega - x)$, so that

$$\mathring{e}_{25:\overline{11|}} = \int_{t=0}^{11} 1 - t/75\,dt = 10.1933.$$

Under the modification, for $t > 1$ we have ${}_tp_{[x]} = p_{[x]} \cdot ({}_{(t-1)}p_{[x]+1}) = p_{[x]} \cdot ({}_{(t-1)}p_{x+1})$, and so we have

$$\begin{aligned}
\mathring{e}_{x:\overline{n|}} = \int_{t=0}^{n} {}_tp_{[x]}\ dt &= \int_{t=0}^{1} {}_tp_{[x]}\ dt + \int_{t=1}^{n} {}_tp_{[x]}\ dt \\
&= \int_{t=0}^{1} {}_tp_{[x]}\ dt + \int_{t=1}^{n} p_{[x]} \cdot ({}_{(t-1)}p_{x+1})\,dt \\
&= \int_{t=0}^{1} {}_tp_x\ dt + p_{[x]} \int_{t=0}^{n-1} {}_tp_{x+1}\ dt \\
&= \int_{t=0}^{1} e^{-0.1t}\,dt + e^{-0.1} \int_{t=0}^{n-1} 1 - t/74\,dt \\
&= 9.3886.
\end{aligned}$$

Difference is 0.8047

63. (Nov 2000, #7)
For a multiple decrement table, you are given:

(a) Decrement (1) is death, decrement (2) is disability, and decrement (3) is withdrawal.
(b) $q_{60}^{\prime(1)} = 0.010$
(c) $q_{60}^{\prime(2)} = 0.050$
(d) $q_{60}^{\prime(3)} = 0.100$
(e) Withdrawals occur only at the end of the year.
(f) Mortality and disability are uniformly distributed over each year of age in the associated single decrement tables.

Calculate q_{60}^{3}

64. (May 2001, #6)
For a multiple decrement model on (60):

(a) $\mu_{60}^{(1)}(t)$ follows the Illustrative Life Table.
(b) $\mu_{60}^{(\tau)}(t) = 2\mu_{60}^{(1)}(t)$ for $t \geq 0$.

Calculate ${}_{11|}q_{60}^{(\tau)}$, the probability that decrement occurs during the 11th year.

Solution:
If $\mu_x(t)$ is a given force-of-mortality function, we know that

$${}_tp_x = e^{-\int_0^t \mu_x(s)\,ds}$$

and so if we double μ then we square the survival function:

$$e^{-\int_0^t 2\mu_x(s)\,ds} = e^{-2\int_0^t \mu_x(s)\,ds} = \left(e^{-\int_0^t \mu_x(s)\,ds}\right)^2 = ({}_tp_x)^2.$$

So

$$\begin{aligned}
{}_{11|}q_{60}^{(\tau)} &= {}_{10}q_{60}^{(\tau)}\,q_{70}^{(\tau)} = {}_{10}q_{60}^{(\tau)}\,(1 - p_{70}^{(\tau)}) \\
&= \left(\frac{S(70)}{S(60)}\right)^2 \left(1 - \left[\frac{S(71)}{S(72)}\right]^2\right) = 0.042612
\end{aligned}$$

65. (May 2001, #9)
(x) and (y) are two lives with identical expected mortality. You are given:

(a) $P_x = P_y = 0.1$
(b) $P_{\overline{xy}} = 0.06$
(c) $d = 0.06$

Calculate the premium P_{xy}

66. (May 2001, #10 (book 6^{th} ed #13.6))
For students entering a college, you are given the following from a multiple decrement model:

(a) 1000 students enter the college at $t = 0$.
(b) Students leave the college for failure (1) or all other reasons (2).
(c) $\mu^{(1)}(t) = \mu$ and $\mu^{(2)}(t) = 0.04$, both for $t \in [0, 4]$.
(d) 48 students are expected to leave the college during their first year due to all causes.

Calculate the expected number of students who will leave because of failure during their fourth year.

Solution:

We want $l_3 q_3^{(1)}$.
First, let's get $\mu^{(\tau)}$. Since $\mu^{(\tau)} = \mu^{(1)} = \mu^{(2)}$ and (from given (d)) $(1000 - 48) = l_0\, q_0^{(\tau)} = 1000e^{-\mu^{(\tau)}}$, we find $\mu^{(\tau)} = 0.04919$ so that $\mu^{(1)} = 0.00919$.
Now,

$$\begin{aligned} l_3 q_3^{(1)} &= 1000 \cdot {}_3p_0^{(\tau)} \cdot q_3^{(1)} \\ &= 1000 \cdot e^{-3\mu^{(\tau)}} \int_0^1 \mu^{(1)} e^{-t\mu^{(\tau)}}\, dt \\ &= 1000 \cdot e^{-3\times 0.04919} \left(\frac{0.00919}{0.04919}\right)\left(1 - e^{-0.04919}\right) = \underline{7.7373} \end{aligned}$$

67. (May 2001, #13)
Mr. Ucci has only 3 hairs left on his head and he won't be growing any more.

(a) The future mortality of each hair follows ${}_kq_x = 0.1(k+1)$ for $k = 0, 1, 2, 3,$, and x is Mr. Ucci's age
(b) Hair loss follows the hyperbolic assumption at fractional ages.
(c) The future lifetimes of the 3 hairs are independent.

Calculate the probability that Mr. Ucci is bald (has no hair left) at age $x + 2.5$.

68. (May 2001, #18)
For a special fully discrete 20-year endowment insurance on (55):

(a) Death benefits in year k are given by $b_k = (21 - k),\ k = 1, 2, \ldots, 20$
(b) The maturity benefit is 1.
(c) Annual benefit premiums are level.
(d) ${}_kV$ denotes the benefit reserve at the end of year k for $k = 1, 2, \ldots, 20$.
(e) ${}_{10}V = 5.0$
(f) ${}_{19}V = 0.6$
(g) $q_{65} = 0.10$

(h) i= 0.08

Calculate ${}_{11}V$.

Solution:
Prospectively, ${}_{t}V$ =APV future benefits - APV future premiums:

$$ {}_{19}V = v - \pi $$

so $\pi = 1 - {}_{19}V = 0.3259$.

$$
\begin{aligned}
{}_{10}V + \pi &= vp_{10}({}_{11}V) + vq_{10}b_{11} \\
5.0 + 0.3259 &= \frac{0.9}{1.08}({}_{11}V) + \frac{0.1}{1.08} \cdot 10 \\
{}_{11}V &= 5.28
\end{aligned}
$$

69. (May 2001, #23)
A continuous two-life annuity pays:

- 100 while both (30) and (40) are alive;
- 70 while (30) is alive but (40) is dead; and
- 50 while (40) is alive but (30) is dead.

The actuarial present value of this annuity is 1180. Continuous single life annuities paying 100 per year are available for (30) and (40) with actuarial present values of 1200 and 1000, respectively.

Calculate the actuarial present value of a two-life continuous annuity that pays 100 while at least one of them is alive.

70. (May 2001, #31)
For a special fully discrete 3-year term insurance on (x):

(a) Level benefit premiums are paid at the beginning of each year.

(b)

k	b_{k+1}	q_{x+k}
0	200,000	0.03
1	150,000	0.06
2	100,000	0.09

(c) $i = 0.06$

Calculate the initial benefit reserve for year 2.

Solution:
The initial benefit reserve for year 2 is ${}_{1}V + \pi$.

We have to compute π, and since the benefit changes each year, we have to compute it from first principles. We have

k	q_{x+k}	p_{x+k}	${}_{k\|}q_x$
0	0.03	0.97	0.03
0	0.06	0.94	$(0.97) \cdot (0.06) = 0.0582$
0	0.09		$(0.97) \cdot (0.94) \cdot (0.09) = 0.082062$

$$
\begin{aligned}
A &= 100,000\Big(2 \cdot (0.03)v + 1.5 \cdot (0.0582)v^2 + 1 \cdot (0.082062)v^3 = 20,320.13 \\
\ddot{a} &= \sum_{k=0}^{3-2} {}_kE_x = 1 + (0.97)v + (0.97)(0.94)v^2 = 2.72659 \\
\pi &= A/\ddot{a} = 7452.57
\end{aligned}
$$

Retrospectively,

$$
\begin{aligned}
{}_1V &= \frac{1}{E_x}\left(\pi - vq_x \cdot 200,000\right) \\
&= \frac{1.06}{0.97}\left(7452.57 - \frac{0.03 \times 200,000}{1.06}\right) = 1958.48 \\
\text{ANS} = {}_1V + \pi &= 1958.48 + 7452.57 = \underline{9411.01}
\end{aligned}
$$

71. (May 2001, #35)
You have calculated the actuarial present value of a last-survivor whole life insurance of 1 on (x) and (y). You assumed:

(a) The death benefit is payable at the moment of death.
(b) The future lifetimes of (x) and (y) are independent, and each life has a constant force of mortality with $\mu = 0.06$.
(c) $d = 0.05$

Your supervisor points out that these are not independent future lifetimes. Each mortality assumption is correct, but each includes a common shock component with constant force 0.02.

Calculate the increase in the actuarial present value over what you originally calculated.

Solution:
Using the formula pattern "$\overline{A} = \dfrac{\mu}{\mu + \delta}$" (which is valid whenever μ and δ are constant).
Without the common shock information:

$$
\begin{aligned}
\overline{A}_x &= \overline{A}_y = \frac{0.06}{0.6 + 0.05} = \frac{6}{11} \\
\overline{A}_{xy} &= \frac{0.06 + 0.06}{0.6 + 0.06 + 0.05} = \frac{12}{17} \\
\overline{A}_{\overline{xy}} &= \overline{A}_x + \overline{A}_y - \overline{A}_{xy} = \frac{72}{187}
\end{aligned}
$$

With the common shock information, since μ_x includes the common shock, then the force of mortality μ_x^* to which (x) is subject but (y) is not satisfies $\mu_x = \mu_x^* + \lambda$ or $0.06 = \mu_x^* + 0.02$ or $\mu_x^* = 0.04$. Similarly $\mu_y^* = 0.04$. So,

$$\mu_{xy} = \mu_x^* + \mu_x^* + \lambda = 0.10.$$

Now repeat the calculations above:

$$
\begin{aligned}
\overline{A}_x &= \overline{A}_y = \frac{0.06}{0.6 + 0.05} = \frac{6}{11} \\
\overline{A}_{xy} &= \frac{0.1}{0.1 + 0.05} = \frac{2}{3} \\
\overline{A}_{\overline{xy}} &= \overline{A}_x + \overline{A}_y - \overline{A}_{xy} = \frac{14}{33}
\end{aligned}
$$

Difference is $\dfrac{14}{33} - \dfrac{72}{187} \approx \boxed{0.039216}$

72. (May 2001, #38)
Lottery Life issues a special fully discrete whole life insurance on (25):

(a) At the end of the year of death there is a random drawing. With probability 0.2, the death benefit is 1000. With probability 0.8, the death benefit is 0.
(b) At the start of each year, including the first, while (25) is alive, there is a random drawing. With probability 0.8, the level premium π is paid. With probability 0.2, no premium is paid.
(c) The random drawings are independent.
(d) Mortality follows the Illustrative Life Table.
(e) $i = 0.06$
(f) π is determined using the equivalence principle.

Calculate the benefit reserve at the end of year 10.

73. (May 2001, #5)
For an annual 20-payment whole life insurance of 1000 on (x), you are given:

(a) $i = 0.06$
(b) $q_{x+19} = 0.01254$
(c) The level (for 20 years) annual benefit premium is 13.72.
(d) The net premium policy value at the end of year 19 is 342.03.

Calculate the level annual benefit premium for an annual whole life insurance of 1000 on (x+20).

Solution:
We're being asked to compute (a thousand times) P_{x+20}, which we could do if we knew either of $\ddot{a}_{x+20}$ or A_{x+20} (because $A_{x+20} = 1 - d\ddot{a}_{x+20}$ and we know i). The trick here is recognize that, after 20 years, there will be no more premiums, so the benefit reserve $1000({}_{20}V)$ at that time must equal the APV A_{x+20} of the benefit itself at that time, and we're given enough information to compute ${}_{20}V$.

Here are the details.

$$\begin{aligned}
1000({}_{19}V) + P &= 1000vp_{x+19}({}_{20}V) + 1000vq_{x+19} \\
342.03 + 13.72 &= \frac{1000(1-0.01254)}{1.06}({}_{20}V) + \frac{1000 \cdot 0.01254}{1.06} \\
1000({}_{20}V) &= 369.184 \\
A_{x+20} &= {}_{20}V = 0.369184 \\
P_{x+20} &= \frac{dA_{x+20}}{1-A_{x+20}} = 0.03313 \\
1000P_{x+20} &= 33.13
\end{aligned}$$

74. (Nov 2001, #12)
A fund is established by collecting an amount P from each of 100 independent lives age 70. The fund will pay the following benefits:

- 10, payable at the end of the year of death, for those who die before age 72, or
- P, payable at age 72, to those who survive.

You are given:

(a) Mortality follows the Illustrative Life Table.
(b) $i = 0.08$

Calculate P, using the equivalence principle.

Solution:
The EPV of the benefit for an individual in this group is

$$10A^{1}_{70:\overline{2}|} + P\,{}_2E_{70}$$

and since each of the 100 lives-age-70 is independent, the EPV for the whole group is just 100 time this amount. So the equivalence principle gives

$$\begin{aligned} 0 &= 1000A^{1}_{70:\overline{2}|} + 100P\,{}_2E_{70} - 100P \\ 0 &= 10A^{1}_{70:\overline{2}|} - P\,(1 - {}_2E_{70}) \\ P &= \frac{10A^{1}_{70:\overline{2}|}}{1 - {}_2E_{70}} \end{aligned}$$

75. (Nov 2001, #20)
Don, age 50, is an actuarial science professor. His career is subject to two decrements:

(a) Decrement 1 is mortality. The associated single decrement table follows De Moivre's law with $\omega = 100$.

(b) Decrement 2 is leaving academic employment, with $\mu^{(2)}_{50} = 0.05$ for $t \geq 0$.

Calculate the probability that Don remains an actuarial science professor for at least five but less than ten years.

Solution:

The quantity we seek is ${}_{5|5}q^{(\tau)}_{50}$ which we can write cleanly in terms of the survival function, so let's get the survival function, ${}_tp^{(\tau)}_{50}$.

The phrase "asssociated single decrement table follows deMoivre's law" indicates that ${}_tp'^{(1)}_{50} = 1 - \frac{t}{50}$, and the fact that decrement (2) has a constant hazard rate indicates that ${}_tp'^{(2)}_{50} = e^{-t/20}$. So

$$\begin{aligned} {}_tp^{(\tau)}_{50} &= \left({}_tp'^{(1)}_{50}\right)\left({}_tp'^{(2)}_{50}\right) = \frac{1}{50}(50 - t)e^{-t/20}, \\ {}_{5|5}q^{(\tau)}_{50} &= {}_5p^{(\tau)}_{50} - {}_{10}p^{(\tau)}_{50} \\ &= \frac{1}{50}\left(45e^{-5/20} - 40e^{-10/20}\right) = \underline{0.215696} \end{aligned}$$

76. (Nov 2001, #21)
For a double decrement model:

(a) In the single decrement table associated with cause (1), $q'^{(1)}_{40} = 1/10$ and these decrements are uniformly distributed over the year.

(b) In the single decrement table associated with cause (2), $q'^{(1)}_{40} = 1/8$ and all decrements occur at time 0.7.

Calculate $q^{(2)}_{40}$

Solution:

If decrement occurs due to cause (2), it can only occur at time $t = 0.7$. So the $q_{40}^{(2)}$ is the probability that (40) fails at time $t = 0.7$. This is turn is the probability that (40) survives for all $t \in [0, 0.7)$ and then fails at that time due to the second cause.

So we have to compute the probability that (40) survives for all $t \in [0, 0.7)$.

For times $t \in [0.0, 0.7)$, there is only decrement (1) is active. Since decrement (1) is uniformly distributed over the year and $q_{40}^{'(1)} = 1/10$, we have ${}_tq_{40}^{'(1)} = t/10$ and ${}_tp_{40}^{'(1)} = 1 - t/10$ for all $t \in [0, 1]$. Hence the probability of survival up to time $t = 0.7$ is $1 - 0.7/10 = 0.93$.

So $\underline{q_{40}^{(2)} = 0.93/8 = 0.11625}$

77. (Nov 2001, #24)
For a special 2-payment whole life insurance on (80):

(a) Premiums of π are paid at the beginning of years 1 and 3.

(b) The death benefit is paid at the end of the year of death.

(c) There is a partial refund of premium feature:
If (80) dies in either year 1 or year 3, the death benefit is $1000 + \pi/2$
Otherwise, the death benefit is 1000.

(d) Mortality follows the Illustrative Life Table.

(e) $i = 0.06$

Calculate π, using the equivalence principle.

Solution:

$$
\begin{aligned}
\mathrm{E}[\text{Loss}] = 0 &= \left(1000A_{80} + \frac{\pi}{2}vq_{80} + \frac{\pi}{2}v^3({}_{2|}q_{80})\right) - \pi\left(1 + v^2{}_2p_{80}\right) \\
&= 1000A_{80} - \pi\left(1 + v^2{}_2p_{80} - \frac{1}{2}vq_{80} - \frac{1}{2}v^3{}_{2|}q_{80}\right) \\
\pi &= \frac{1000A_{80}}{1 + v^2{}_2p_{80} - \frac{1}{2}vq_{80} - \frac{1}{2}v^3{}_{2|}q_{80}}
\end{aligned}
$$

Now run the numbers.

78. (Nov 2001, #25)
For a special fully continuous whole life insurance on (65):

(a) The death benefit at time t is $b_t = 1000e^{0.04t},\ t \geq 0$

(b) Level benefit premiums are payable for life.

(c) $\mu_{65}(t) = 0.02$ for $t \geq 0$

(d) $\delta = 0.04$

Calculate ${}_2V$, the benefit reserve at the end of year 2.

Solution:
We need the premium:

$$\begin{aligned}
0 &= A - \pi \bar{a}_x \\
&= \left(\int_0^\infty v^t b_t f_{T_{65}}(t)\, dt \right) - \pi \bar{a}_x \\
&= \left(\int_0^\infty (e^{-0.04t})(1000e^{0.04t})(0.02e^{-0.02t})\, dt \right) - \pi \bar{a}_x \\
&= 1000 \left(\int_0^\infty 0.02e^{-0.02t}\, dt \right) - \frac{\pi}{0.02+0.04} \\
\pi &= 60
\end{aligned}$$

Now

$$\begin{aligned}
{}_2V &= \left(\int_0^\infty (e^{-0.04t})(1000e^{0.04t+2})(0.02e^{-0.02t})\, dt \right) - \frac{60}{0.06} \\
&= 1000(e^{0.08} - 1) = 83.2871
\end{aligned}$$

79. (Nov 2001, #26)
You are given:

(a) $A_x = 0.28$

(b) $A_{x+20} = 0.40$

(c) $A_{x:\overset{1}{\overline{20|}}} = 0.25$

(d) $i = 0.05$

Calculate $a_{x:\overline{20|}}$

Solution:
"Endowment insurance is term plus pure endowment":

$$\begin{aligned}
A_{x:\overline{n|}} &= A^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}} = \left(A_x - A_{x:\overset{1}{\overline{n|}}}\, A_{x+n} \right) + A_{x:\overset{1}{\overline{n|}}} \\
&= A_x + A_{x:\overset{1}{\overline{n|}}} \left(1 - A_{x+n}\right) \\
&= 0.28 + 0.25(1 - 0.4) = 0.43, \\
\ddot{a}_{x:\overline{n|}} &= \frac{1 - A_{x:\overline{n|}}}{d} = 11.97, \\
a_{x:\overline{n|}} &= \ddot{a}_{x:\overline{n|}} - 1 + A_{x:\overset{1}{\overline{n|}}} = 11.22
\end{aligned}$$

80. (Nov 2001, #38)
For a triple decrement model:

(a)

x	$q_x^{'(1)}$	$q_x^{'(2)}$	$q_x^{'(3)}$
63	0.02000	0.03000	0.25000
64	0.02500	0.03500	0.20000
65	0.03000	0.04000	0.15000

(b) $q_{65}^{(1)} = 0.02716$

(c) Each decrement has a constant force over each year of age.

Calculate ${}_2q_{64}^{(1)}$

Solution:

Following Example 13.2(c) in the text we note that

$$ {}_2q_{64}^{(1)} = q_{64}^{(1)} + p_{64}^{(\tau)} \cdot q_{65}^{(1)}. \tag{4}$$

In general, $q_x^{(i)} = \int_0^1 {}_tp_x^{(\tau)}\ \mu_{x+t}^{(i)}\,dt$, and in the special case of constant HRF for $t =\in [0,1]$ this simplifies to

$$q_x^{(i)} = \frac{\log(p_x^{'(i)})}{\log(p_x^{'(\tau)})} q_x^{(\tau)}$$

(see "Case B1" in the big table on page 352).
So (4) becomes

$$ {}_2q_{64}^{(1)} = \frac{\log(p_{64}^{'(i)})}{\log(p_{64}^{'(\tau)})} q_{64}^{(\tau)} + p_{64}^{(\tau)} \cdot \frac{\log(p_{65}^{'(i)})}{\log(p_{65}^{'(\tau)})} q_{65}^{(\tau)}$$

We have all the $p_x^{'(i)}$'s, and we also need:

$$\begin{aligned}
p_{64}^{(\tau)} &= = \left(p_{64}^{'(1)}\right)\left(p_{64}^{'(2)}\right)\left(p_{64}^{'(3)}\right)\left(1-q_{64}^{'(1)}\right)\left(1-q_{64}^{'(2)}\right)\left(1-q_{64}^{'(3)}\right) = 0.7527,\\
q_{64}^{(\tau)} &= = 1 - 0.7527 = 0.2473,\\
p_{65}^{(\tau)} &= = \left(p_{65}^{'(1)}\right)\left(p_{65}^{'(2)}\right)\left(p_{65}^{'(3)}\right)\left(1-q_{65}^{'(1)}\right)\left(1-q_{65}^{'(2)}\right)\left(1-q_{65}^{'(3)}\right) = 0.79152,\\
q_{65}^{(\tau)} &= = 1 - 0.79152 = 0.20848
\end{aligned}$$

So . . .

81. (Nov 2001, #39)
For a special 3-year deferred whole life annuity-due on (x):

(a) $i = 0.04$

(b) The first annual payment is 1000.

(c) Payments in the following years increase by 4% per year.

(d) There is no death benefit during the three year deferral period.

(e) Level benefit premiums are payable at the beginning of each of the first three years.

(f) $e_x = 11.05$ is the curtate expectation of life for (x).

(g)

k	1	2	3
${}_kp_x$	0.99	0.98	0.97

Calculate the annual benefit premium.

Solution:

82. (Nov 2001, #4)
There are two different insurance whole life policies on (x) with a death benefit of 5000, policy (I) and policy (II). Policy (I) pays pays at the end of the year of death, and policy (II) pays at at the end of the year of death <u>if</u> (x) survives at least one year.

For each policy, the contract premium is the annual net level premium.

Additionally you are given:

(a) $q_x = 0.05$

(b) $v = 0.90$

(c) $\ddot{a}_x = 5.00$

(d) The net premium policy value ${}_{10}V^{(I)}$ for policy (I) is 0.20000.

Calculate the net premium policy value ${}_{10}V^{(II)}$ for policy (II).

Solution:
Prospectively, for $t > 1$,

$$ {}_{10}V \;= 5000 \cdot A_{x+10} - \pi \ddot{a}_{x+10}. $$

The net annual premium for this policy, call it π, is

$$ \pi = \frac{5000 \cdot v \cdot p_x \cdot A_{x+1}}{\ddot{a}_x}. $$

There is enough information given to allow us to compute the following:

A_x

π

Also, for a fully discrete whole life policy on (x) with sum-insured of 1, the premium P_x is satisfies $0 = A_x + P_x \ddot{a}_x$. There's enough information given to allow us to compute P_x.

The benefit reserve for that policy after 10 years, ${}_{10}V_x$, is

$$ {}_{10}V_x = A_{x+10} - P_x \ddot{a}_{x+10}, $$

and so there enough information to compute A_{x+10} and $\ddot{a}_{x+10}$.

$$ {}_{10}V = 5000 \cdot A_{x+10} - \pi \ddot{a}_{x+10} = 5000 \cdot 0.6 - 455 \cdot 4 = \boxed{1180} $$

83. (Nov 2001, #40)
For a special fully discrete 10-payment whole life insurance on (30) with level annual benefit premium π:

(a) The death benefit is equal to 1000 plus the refund, without interest, of the benefit premiums paid.

(b) $A_{30} = 0.102$

(c) ${}_{10|}A_{30} = 0.088$

(d) $(IA)^{1}_{30:\overline{10|}} = 0.078$.

(e) $\ddot{a}_{30:\overline{10|}} = 7.747$.

Calculate π.

Solution:
Think about the meaning of, "...the refund, without interest, of the benefit premiums paid", if $K_{30} = k$ for various values of k, and compare the the values of $(IA)^{1}_{30:\overline{10|}}$:

If $K_x = k$	0	1	2	...	9	10	11	12	...
Amt refunded	π	2π	3π	...	10π	10π	10π	10π	...
$(IA)^{1}_{30:\overline{10|}}$:	1	2	3	...	10	0	0	0	...

reflecting the fact that the policy is a "10-payment" policy.

So the APV of the premium refund portion of the benefit can be written as

$$ \pi\Big((IA)^{1}_{30:\overline{10|}} + 10 \cdot {}_{10|}A_{30} \Big) $$

So the Equivalence Principle in this problem becomes

$$\begin{aligned}
0 &= 1000A_{30} + \pi\Big((IA)^{1}_{30:\overline{10|}} + 10 \cdot {}_{10|}A_{30}\Big) - \pi\ddot{a}_{30:\overline{10|}} \\
0 &= 1000A_{30} + \pi\Big[(IA)^{1}_{30:\overline{10|}} + 10 \cdot {}_{10|}A_{30} - \ddot{a}_{30:\overline{10|}}\Big] \\
0 &= 1000 \cdot 0.102 + \pi\Big[0.078 + 10 \cdot 0.088 - 7.747\Big] \\
& \boxed{\pi = 15.024}
\end{aligned}$$

84. (Nov 2001, #8)
Each of 100 independent lives purchase a single premium 5-year deferred whole life insurance of 10 payable at the moment of death. You are given:

(a) $\mu = 0.04$

(b) $\delta = 0.06$

(c) F is the aggregate amount the insurer receives from the 100 lives.

Using the normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

Solution:
For a single policy, present value random variable Z for the insurance benefit is given by

$$Z = 10 \begin{cases} 0 & \text{if } T_x \le 5 \\ e^{-\delta T_x} & \text{if } T_x > 5 \end{cases}$$

and

$$\begin{aligned}
\mathrm{E}[Z] &= \int_{t=5}^{\infty} 10e^{-\delta t}{}_tp_x\mu_{x+t}\,dt \\
&= 10\int_{s=0}^{\infty} e^{-\delta(s+5)}{}_{s+5}p_x\mu_{x+5+s}\,dt \\
&= 10e^{-5\delta}{}_5p_x\int_{s=0}^{\infty} e^{-\delta s}{}_sp_{x+1}\mu_{x+5+s}\,dt \\
&= 10\left(e^{-0.06\cdot 5}e^{-0.04\cdot 5}\right)\int_0^{\infty} e^{-0.06t}\cdot 0.04\cdot e^{-0.04t}\,dt = \underline{2.42612} \\
\mathrm{E}\left[Z^2\right] &= 100\int_{t=5}^{\infty} e^{-2\delta t}{}_tp_x\mu_{x+t}\,dt \\
&= 100\left(e^{-0.12\cdot 5}e^{-0.04\cdot 5}\right)\int_0^{\infty} e^{-0.12t}\cdot 0.04\cdot e^{-0.04t}\,dt = \underline{11.2332}
\end{aligned}$$

so that $\mathrm{Std}[Z] = \sqrt{11.2332 - (2.42612)^2} = \sqrt{5.347142} = 2.31239$.
Now if L denotes the aggegate loss, then

$$\begin{aligned}
L &= -F + \sum_{i=1}^{100} Z_i \\
\mathrm{E}[L] &= 100\cdot\mathrm{E}[Z_i] - F = 100\cdot 2.42612 - F = 242.612 - F \\
\mathrm{Std}[L] &= \sqrt{\mathrm{Var}\left[\sum_{i=1}^{100} Z_i\right]} = 10\cdot 2.21239 = 22.1239
\end{aligned}$$

Now approximate L as normal random variable and set $\Pr\{L \le 0\} = 0.95$, and solve for F. With Φ denoting the standard normnal CDF, we have

$$\begin{aligned}
0.95 &= \Pr\{L \le 0\} = \Pr\left\{\frac{L - \mathrm{E}[L]}{\mathrm{Std}[L]} \le \frac{-\mathrm{E}[L]}{\mathrm{Std}[L]}\right\} \approx \Pr\left\{\mathcal{Z} \le \frac{-\mathrm{E}[L]}{\mathrm{Std}[L]}\right\} = \Phi\left(\frac{-\mathrm{E}[L]}{\mathrm{Std}[L]}\right) \\
\Phi^{-1}(0.95) &= \frac{-\mathrm{E}[L]}{\mathrm{Std}[L]} \\
1.64485 &= \frac{F - 242.612}{22.1239} \quad \Longrightarrow \boxed{F = 279}
\end{aligned}$$

85. (Nov 2002, #2)
For a triple decrement table, you are given:

(a) $\mu_x^{(1)}(t) = 0.3$ for $t > 0$,
(b) $\mu_x^{(2)}(t) = 0.5$ for $t > 0$,
(c) $\mu_x^{(3)}(t) = 0.7$ for $t > 0$,

Calculate $q_x^{(2)}$.

Solution:

We have $\mu_x^{(\tau)}(t) = 1.5$, and:

$$q_x^{(2)} = \int_0^1 {}_sp_x^{(\tau)}\mu_x^{(2)}(s)\,ds = 0.5\int_0^1 e^{-1.5s}\,ds = \frac{0.5}{1.5}(1 - e^{-1.5}) = 0.25896$$

86. (Nov 2002, #13)
For a double-decrement table where cause 1 is death and cause 2 is withdrawal, you are given:

(a) Deaths are uniformly distributed over each year of age in the single-decrement table.
(b) Withdrawals occur only at the end of each year of age.
(c) $l_x^{(\tau)} = 1000$
(d) $q_x^{(2)} = 0.40$
(e) $d_x^{(1)} = 0.45 d_x^{(2)}$

Calculate $p_x^{'(2)}$.

Solution:

For $t \in [0, 1)$, only decrement (1) is in effect and it is uniformly distributed so

$${}_tp_x^{\tau} = {}_tp_x^{(1)} = {}_tp_x^{'(1)} = 1 - tq_x^{'(1)} \quad \text{for } t \in [0, 1),$$

and $p_x^{(1)} = p_x^{'(1)}$

$$\begin{aligned}
d_x^{(2)} &= l_x^{(\tau)} \cdot q_x^{(2)} = 400 \\
d_x^{(1)} &= 0.45 \times 400 = 180 \\
q_x^{(1)} &= 0.180 \quad \text{because decrement (2) can't happen midyear} \\
d_x^{(\tau)} &= 580, \\
q_x^{(\tau)} &= 0.580 \\
p_x^{(tau)} &= 0.420 \\
q_x^{(2)} &= 0.400 \\
q_x^{(2)} &= p_x^{(\tau)} q_x^{'(2)}
\end{aligned}$$

87. (Nov 2002, #3)
You are given:

(a) the following select-and-ultimate mortality table with 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	x +3
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(b) $i = 0.03$

Calculate the actuarial present value of a 2- year deferred 2-year term insurance on 60 .

Solution:

$$\begin{aligned}
{}_{2|}A^{\,1}_{[60]:\overline{2}|} &= v^3{}_{2|}q_{[60]} + v^4{}_{3|}q_{[60]} \\
&= v^3\left(p_{[60]}\right)\left(p_{[60]+1}\right)\left(q_{[60]+2}\right) + v^4\left(p_{[60]}\right)\left(p_{[60]+1}\right)\left(p_{[60]+2}\right)\left(q_{[60]+3}\right) \\
&= v^3\left(p_{[60]}\right)\left(p_{[60]+1}\right)\left(\left(q_{[60]+2}\right) + v^2\left(p_{[60]+2}\right)\left(q_{60+3}\right)\right) \\
&= \frac{1}{1.03^3}(1-0.09)(1-0.11)\left((0.13) + \frac{1}{1.03}(1-0.13)(0.15)\right) \\
&\boxed{= 0.190258}
\end{aligned}$$

88. (Nov 2002, #32)
For a continuously increasing whole life insurance on (x), you are given:

(a) The force of mortaility is constant.
(b) $\delta = 0.06$
(c) ${}^2\bar{A}_x = 0.25$

Calculate $(\overline{IA})_x$

Solution:
Since $\int_0^\infty \mu e^{-(\delta+\mu)t}\,dt = \frac{\mu}{\delta+\mu}$, we have

$$ {}^2A_x = 0.25 = \frac{\mu}{2\cdot 0.06 + \mu} \implies \mu = 0.04$$

Then

$$(\overline{IA})_x = \int_{t=0}^\infty \mu t e^{-(\delta+\mu)t}\,dt = \frac{\mu}{(\mu+\delta)^2} = \frac{0.4}{0.1^1} = 4.$$

89. (Nov 2002, #38)
A large machine in the ABC Paper Mill is 25 years old when ABC purchases a 5-year term insurance paying a benefit in the event the machine breaks down. Given:

(a) Annual benefit premiums of 6643 are payable at the beginning of the year.
(b) A benefit of 500,000 is payable at the moment of breakdown.
(c) Once a benefit is paid, the insurance contract is terminated.
(d) Machine breakdowns follow De Moivre's law with $l_x = 100 - x$.

(e) $i = 0.06$

Calculate the benefit reserve for this insurance at the end of the third year.

90. (Nov 2002, #39)
For a whole life insurance of 1 on (x), you are given:

(a) The force of mortaility is $\mu_x(t)$.
(b) The benefits are payable at the moment of death.
(c) $\delta = 0.06$
(d) $\bar{A}_x = 0.60$

Calculate the revised actuarial present value of this insurance assuming $\mu_x(t)$ is increased by 0.03 for all t, and δ is decreased by 0.03.

Solution:

$$\begin{aligned}
0.60 &= \bar{A}_x = 1 - \delta \bar{a}_x = 1 - 0.06\bar{a}_x \quad \Longrightarrow \quad \bar{a}_x = 20/3 \\
20/3 &= \bar{a}_x = \int_0^\infty e^{-0.06t} {}_tp_x^{old}\, dt \qquad (\text{“current payments” form}) \\
&= \bar{a}_x = \int_0^\infty e^{-\delta_{old} t} {}_tp_x^{old}\, dt \\
&= \bar{a}_x = \int_0^\infty e^{-\delta_{new} t} {}_tp_x^{new}\, dt \\
\bar{A}_x &= 1 - 0.03 \cdot 20/3 = 0.80
\end{aligned}$$

91. (Simular to: C3, Nov 2003, #7)
A whole life policy provides that upon accidental death as a passenger on an airplane a benefit of 1,000,000 will be paid. If death occurs from other accidental causes, a death benefit of 500,000 will be paid. If death occurs from a cause other than an accident, a death benefit of 250,000 will be paid.

You are given:

(a) Death benefits are payable at the moment of death.
(b) $\mu^{(1)} = 1/2,000,000$ where (1) indicates accidental death as a passenger on an airplane.
(c) $\mu^{(2)} = 1/250,000$ where (2) indicates death from other accidental causes.
(d) $\mu^{(3)} = 1/10,000$ where (3) indicates non-accidental death.
(e) $\delta = 0.06$

Calculate the single benefit premium for this insurance.

Solution:

decrement # j	force of decrement μ_j	Benefit if caused by decr. j
1	$\mu^{(1)} = 1/2,000,000$	$B_1 = \$1,000,000$
2	$\mu^{(2)} = 1/250,000$	$B_2 = \$500,000$
3	$\mu^{(3)} = 1/10,000$	$B_3 = \$250,000$
	$\mu^{(\tau)} = \mu^{(1)} + \mu^{(2)} + \mu^{(3)} = 209/2{,}000{,}000$	

Survival function ${}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)\, ds}$.

Density function f_{T_x} breaks up as follows:

$$\begin{aligned}
f_{T_x}(t) &= \mu_x^{(\tau)}\ {}_tp_x^{(\tau)} \\
&= \left(\mu_x^{(1)}(t)+\mu_x^{(1)}(t)+\mu_x^{(1)}(t)\right)\ {}_tp_x^{(\tau)} \\
&= f_{T_x}^{(1)}(t)+f_{T_x}^{(2)}(t)+f_{T_x}^{(3)}(t)
\end{aligned}$$

Since the μ's are constant, the $f_{T_x}^{(j)}$'s take the following simple form:

$$f_{T_x}^{(j)}(t)=\mu^{(j)}e^{-\mu^{(\tau)}t}.$$

The single benefit premium is the APV of the benefit, which is:

$$\begin{aligned}
\text{APV Ben.} &= \sum_{j=1}^{3}\int_0^\infty e^{-\delta t}B_j f_{T_x}^{(j)}(t) \\
&= \sum_{j=1}^{3}\int_0^\infty B_j\mu^{(j)}e^{-(\delta+\mu^{(\tau)})t} \\
&= \sum_{j=1}^{3}\int_0^\infty \frac{B_j\mu^{(j)}}{\delta+\mu^{(\tau)}} = \frac{\frac{1{,}000{,}000}{2{,}000{,}000}+\frac{500{,}000}{250{,}000}+\frac{250{,}000}{10{,}000}}{\frac{6}{100}+\frac{209}{2{,}000{,}000}} = \boxed{457.536}
\end{aligned}$$

92. (Nov 2003, #11)
For a fully discrete whole life insurance of 1000 on (60), the annual benefit premium was calculated using the following:

(a) $i=0.06$
(b) $q_{60}=0.01376$
(c) $1000A_{60}=369.33$
(d) $1000A_{61}=383.00.$

A particular insured is expected to experience a first-year mortality rate ten times the rate used to calculate the annual benefit premium. The expected mortality rates for all other years are the ones originally used.

Calculate the expected loss at issue for this insured, based on the original benefit premium.

Solution:
The original premium is $1000P_{60}$,

$$P_{60}=\frac{A_{60}}{\ddot{a}_{60}}=\frac{dA_{60}}{1-A_{60}}=\frac{(0.06/1.06)\cdot 0.36933}{1-0.36933}=0.033148.$$

The whole life benefit APV A_{60} can be written as a one-year term benefit APV plus a one-year deferred whole life benefit APV:

$$\begin{aligned}
A_{60} &= A^{\ 1}_{60:\overline{1}|}+E_{60}A_{61} \\
&= vq_{60}+v(1-q_{60})A_{61}.
\end{aligned}$$

The first-year mortality rate q_{60} is magnified by 10.

$$\begin{aligned}
A_{60}^{NEW} &= 10vq_{60}+v(1-10q_{60})A_{61} \\
&= \big(0.1376+(1-0.1376)\cdot 0.383\big)/1.06=0.44141 \\
\mathrm{E}\left[L\right] &= \left(1000+\frac{P}{d}\right)A_{60}^{NEW}-\frac{P}{d} \\
&= (1000+585.62)\cdot 0.44141-585.62 \\
&= 114.29
\end{aligned}$$

93. (Nov 2003, #11)
For a fully discrete whole life insurance of 1000 on (60), the annual benefit premium was calculated using the following:

(a) $i = 0.06$
(b) $q_{60} = 0.01376$
(c) $1000A_{60} = 369.33$
(d) $1000A_{61} = 383.00$.

A particular insured is expected to experience a first-year mortality rate ten times the rate used to calculate the annual benefit premium. The expected mortality rates for all other years are the ones originally used.

Calculate the expected loss at issue for this insured, based on the original benefit premium.

(A) 72
(B) 86
(C) 100
(D) 114
(E) 128

Solution:
The original premium is $1000P_{60}$,

$$P_{60} = \frac{A_{60}}{\ddot{a}_{60}} = \frac{dA_{60}}{1-A_{60}} = \frac{(0.06/1.06)\cdot 0.36933}{1-0.36933} = 0.033148.$$

The whole life benefit APV A_{60} can be written as a one-year term benefit APV plus a one-year deferred whole life benefit APV:

$$\begin{aligned} A_{60} &= A^{1}_{60:\overline{1}|} + E_{60}A_{61} \\ &= vq_{60} + v(1-q_{60})A_{61}. \end{aligned}$$

The first-year mortality rate q_{60} is magnified by 10.

$$\begin{aligned} A^{NEW}_{60} &= 10vq_{60} + v(1-10q_{60})A_{61} \\ &= \big(0.1376 + (1-0.1376)\cdot 0.383\big)/1.06 = 0.44141 \\ \mathrm{E}[L] &= \Big(1000 + \frac{P}{d}\Big)A^{NEW}_{60} - \frac{P}{d} \\ &= (1000 + 585.62)\cdot 0.44141 - 585.62 \\ &= 114.29 \end{aligned}$$

ANS: D

94. (Nov 2003, #21)
For a special fully discrete whole life insurance on (40):

(a) The death benefit is 1000 for the first 20 years; 5000 for the next 5 years; 1000 thereafter.
(b) The annual benefit premium is $1000\,P_{40}$ for the first 20 years; $5000\,P_{40}$ for the next 5 years; π thereafter.
(c) Mortality follows the Illustrative Life Table.
(d) $i = 0.06$

Calculate ${}_{21}V$, the benefit reserve at the end of year 21 for this insurance.

95. (C3, Nov 2003, #22)
For a whole life insurance of 1 on (41) with death benefit payable at the end of year of death, you are given:

(a) $i = 0.05$
(b) $p_{40} = 0.9972$
(c) $A_{41} - A_{40} = 0.00822$
(d) ${}^2A_{41} - {}^2A_{40} = 0.00433$
(e) Z is the present-value random variable for this insurance.

Calculate Var(Z).

Solution:
Use $A_{41} - A_{40} = 0.00822$ plus recursion formula to get $A_{41} = 0.21650$, and similar for ${}^2A_{41}$.
Here are the details:

$$\begin{aligned}
A_{41} - 0.00822 = A_{40} &= vq_{40} + vp_{40}A_{41} \\
A_{41} - 0.00822 &= (1/1.05)(1-0.9972) + (1/1.05)0.9972A_{41} \\
&\Longrightarrow \underline{A_{41} = 0.216496} \\
{}^2A_{41} - 0.00433 = {}^2A_{40} &= v^2q_{40} + v^2p_{40}{}^2A_{41} \\
{}^2A_{41} - 0.00433 &= (1/1.05)^2(1-0.9972) + (1/1.05)^2 0.9972\left({}^2A_{41}\right) \\
&\Longrightarrow \underline{{}^2A_{41} = 0.0719262} \\
\text{Var}[Z] &= 0.0719262 - (0.216496)^2 = \boxed{0.025056.}
\end{aligned}$$

96. (Nov 2003, #38)
For a fully discrete 10-payment whole life insurance of 100,000 on (x), you are given:

(a) $i = 0.05$
(b) $q_{x+9} = 0.011$
(c) $q_{x+10} = 0.012$
(d) $q_{x+11} = 0.014$
(e) The level annual benefit premium is 2078.
(f) The benefit reserve at the end of year 9 is 32,535.

Calculate $100,000\,A_{x+11}$.

97. (Nov 2004, #3)
For a fully continuous whole life insurance of 1 on (x), you are given:

(a) The forces of mortality and interest are constant,
(b) ${}^2\overline{A}_x = 0.20$,
(c) The net level annual premium for a unit whole life insurance paying immediately upon death is 0.03.
(d) ${}_0L$ is the loss-at-issue random variable based on the benefit premium

Calculate Var $\left[{}_0L\right]$.

Solution:
Since δ and μ are constant we have $\overline{A}_x = \mu/(\mu+\delta)$ and $\overline{a}_x = 1/(\mu+\delta)$ so that the equivalence principle becomes

$$0 = \frac{\mu}{\mu+\delta} - \frac{P}{\mu+\delta},$$

so $\underline{\mu = 0.03}$ according to the given data.

Then ${}^2\overline{A}_x = 0.20$ and ${}^2\overline{A}_x = 0.03/(0.03+2\delta)$ mean that $0.03/(0.03+2\delta) = 0.20$ so that $\underline{\delta = 0.06}$.

We'll also need: $\overline{A}_x = \mu/(\mu+\delta) = 0.03/0.09 = 1/3$.

Now,

$$\begin{aligned}
{}_0L &= \overline{Z}_x + P\overline{Y}_x = \left(1+\frac{P}{\delta}\right)\overline{Z}_x - \frac{P}{\delta} \\
\mathrm{Var}\left[{}_0L\right] &= \left(1+\frac{P}{\delta}\right)^2 \mathrm{Var}\left[\overline{Z}_x\right] = \left(1+\frac{P}{\delta}\right)^2 \left[\ {}^2\overline{A}_x - \overline{A}_x^2\right] \\
&= \left(1+\frac{0.03}{0.06}\right)^2 \times (0.20 - (1/3)^2) = 0.20.
\end{aligned}$$

98. (Nov 2004, #1)
For a special whole life insurance on (x), payable at the moment of death:

(a) $\mu_x(t) = 0.05,\ t \geq 0$
(b) $\delta = 0.08$
(c) The death benefit at time t is $b_t = e^{0.06t},\ t \geq 0$.
(d) Z is the present value random variable for this insurance at issue.

Calculate $\mathrm{Var}[Z]$.

Solution:

$$\begin{aligned}
\mathrm{E}[Z] &= \int_0^\infty e^{0.06t} e^{-\delta t} \mu e^{-\mu t}\, dt \\
&= 0.05 \int_0^\infty e^{0.06-0.08-0.05}\, dt \\
&= \frac{0.05}{0.07} = \\
\mathrm{E}\left[Z^2\right] &= \int_0^\infty \left(e^{0.06t} e^{-\delta t}\right)^2 \mu e^{-\mu t}\, dt \\
&= \frac{0.05}{0.09} = \\
\mathrm{Var}[Z] &= \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = 0.045351474
\end{aligned}$$

99. (Nov 2004, #2)
For a group of individuals all age x, you are given:

(a) 25% are smokers (s); 75% are nonsmokers (ns).
(b)

k	q^s_{x+k}	q^{ns}_{x+k}
0	0.10	0.05
1	0.20	0.10
2	0.30	0.15

(c) $i = 0.02$.

Calculate $1000\, A^{\,1}_{x:\overline{2}|}$ for an individual chosen at random from this group.

Solution:

$$A^{\,1}_{x:\overline{2}|} = v\,{}_{0|}q_x + v^2\,{}_{1|}q_x = \frac{q_x}{1.02} + \frac{(1-q_x)q_{x+1}}{1.02^2}$$

$$\text{Smokers:} \quad A^{\,1}_{x:\overline{2}|} = \frac{0.10}{1.02} + \frac{(0.90)0.20}{1.02^2} = 0.27105,$$

$$\text{Non-smokers:} \quad A^{\,1}_{x:\overline{2}|} = \frac{0.05}{1.02} + \frac{(0.95)0.10}{1.02^2} = 0.14033$$

$$\text{25-75 weighting:} \quad 1000\, A^{\,1}_{x:\overline{2}|} = (0.25)\cdot 0.27105 + (0.75)\cdot 0.14033 = \boxed{0.17301}$$

100. (Nov 2004, #31)
You are given:

(a) The future lifetimes of (50) and (50) are independent.

(b) Mortality follows the Illustrative Life Table.

(c) Deaths are uniformly distributed over each year of age.

Calculate the force of failure at duration 10.5 for the last survivor status of (50) and (50).

101. (Nov 2004, #37)
Z is the present value random variable for a 15-year pure endowment of 1 on (x):

(a) The force of mortality is constant over the 15-year period.

(b) $v = 0.9$.

(c) $\mathrm{Var}[Z] = 0.065\mathrm{E}[Z]$.

Calculate q_x.

Solution:

$$
\begin{aligned}
Z &= \begin{cases} v^{15} & \text{if (x) survives 15 years,} \\ 0 & \text{otherwise.} \end{cases} \\
\Pr\{\text{(x) survives 15 yrs }\} &= {}_{15}p_x = e^{-15\mu} \qquad \text{since “constant force”} \\
Z &= \begin{cases} v^{15} & \text{w/ prob. } e^{-15\mu} \\ 0 & \text{w/ prob. } 1-e^{-15\mu}. \end{cases} \\
\mathrm{E}[Z] &= v^{15}e^{-15\mu} \\
\mathrm{E}\left[Z^2\right] &= (v^{15})^2 e^{-15\mu} \\
\mathrm{Var}[Z] &= \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = v^{30}e^{-15\mu}\left(1-e^{-15\mu}\right) \\
\mathrm{Var}[Z] = 0.065\mathrm{E}[Z] \implies \frac{65}{1000} &= \frac{v^{30}e^{-15\mu}\left(1-e^{-15\mu}\right)}{v^{15}e^{-15\mu}} = v^{15}(1-e^{-15\mu}) \\
1 - \frac{65}{1000\cdot 0.9^{15}} &= e^{-15\mu} \\
& \underline{\mu = 0.025290671} \\
q_x &= 1 - e^{-\mu} \\
& \boxed{q_x = 0.02497}
\end{aligned}
$$

102. (Nov 2004, #9)
For a special fully continuous last survivor insurance of 1 on (x) and (y), you are given:

(a) T_x and T_y are independent.
(b) $\mu_{x+t} = 0.08$ for $t > 0$,
(c) $\mu_{y+t} = 0.04$ for $t > 0$,
(d) $\delta = 0.06$
(e) π is the annual benefit premium payable until the first of (x) and (y) dies.

Calculate π.

Solution:
Because π is the benefit premium, it is computed from the equivalance principle:

$$
0 = \overline{A}_{\overline{xy}} - \pi\overline{a}_{xy} \quad \implies \quad \pi = \frac{\overline{A}_{\overline{xy}}}{\overline{a}_{xy}}
$$

Use:

$$
\begin{aligned}
\overline{A}_{\overline{xy}} + \overline{A}_{xy} &= \overline{A}_x + \overline{A}_x \\
\overline{A}_{xy} &= \frac{\mu}{\mu+\delta} \quad \text{where } \mu = \mu_x + \mu_y, \\
\overline{a}_{xy} &= \frac{1-\overline{A}_{xy}}{\delta}
\end{aligned}
$$

<u>ANS:</u> 0.054857

103. (May 2005 #1)
A 3 year insurance on (x) pays 1000 at the end of the year of death or at the end of three years, whichever occurs first. You are given:

(a) ${}_{k}L$ is the present value of future net loss random variable at time k.

(b) The effective annual interest rate is 10%.

(c) $\ddot{a}_{x:\overline{3}|} = 2.70182$.

(d) Premiums are determined by the equivalence principle. There are no policy expenses.

Calculate ${}_{1}L$, given that (x) dies in the second year from issue.

<u>**Solution:**</u>
Denote the premium by P. Then

$$P = 1000 \cdot \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{1 - d\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = 1000 \cdot \frac{1 - (0.1/1.1) \cdot (2.70182)}{2.70182} = 279.11$$

Now if the insured dies during the second year of the policy, then the present value of loss random variable reflects the fact the at time $t = 2$ we have to pay 1000 which has a present value at time 1 of $1000v$ or 909.09, but we collected a premium payment of 279.21 at the beginning of the year, so the loss is $\boxed{629.88}$.

104. (May 2005 #10)
The scores on the final exam in Ms. B's Latin class have a normal distribution with mean θ and standard deviation equal to 8. θ a random variable with a normal distribution with mean equal to 75 and standard deviation equal to 6.

Each year, Ms. B chooses a student at random and pays the student 1 times the student's score. However, if the student udent fails the exam (score ≤ 65), then there is no payment.

Calculate the conditional probability that the payment is less than 90, given that there is a payment.

(A) 0.77
(B) 0.85
(C) 0.88
(D) 0.92
(E) 1.00

<u>**Solution:**</u>
Let X be the exam score. We have

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[\mathrm{E}[X|\Theta]] = \mathrm{E}[\Theta] = 75, \\ \mathrm{Var}[X] &= \mathrm{Var}[\mathrm{E}[X|\Theta]] + \mathrm{E}[\mathrm{Var}[X|\Theta]] = \mathrm{Var}[\Theta] + \mathrm{E}\left[8^2\right] = 6^2 + 8^2 = 10^2. \end{aligned}$$

$$\begin{aligned} \Pr\{X \leq 90 | X > 65\} &= \frac{\Pr\{65 < X \text{ and } X \leq 90\}}{\Pr\{65 < X\}} \\ &= \frac{\Pr\{(65-75)/10 < Z \text{ and } Z \leq (90-75)/10\}}{\Pr\{(65-75)/10 < Z\}} \\ &= \frac{\Phi(1.5) - \Phi(-1.0)}{1 - \Phi(-1.0)} \\ &= \frac{\Phi(1.5) - 1 + \Phi(1.0)}{\Phi(1.0)} = \frac{0.93319 - 1 + 0.84134}{0.84134} = 0.9206 \end{aligned}$$

<u>Ans: D</u>

105. (May 2005 #13)
For a fully discrete whole life insurance of b on (x), you are given:

(a) $q_{x+9} = 0.02904$.
(b) $i = 0.03$
(c) The initial benefit reserve for policy year 10 is 343.
(d) The net amount at risk for policy year 10 is 872.
(e) $\ddot{a}_x = 14.65976$.

Calculate the terminal benefit reserve for policy year 9.

106. (May 2005 #14)
For a special fully discrete 2-year endowment insurance of 1000 on (x), you are given:

(a) The first year benefit premium is 668.
(b) The second year benefit premium is 258.
(c) $d = 0.06$

Calculate the level annual premium using the equivalence principle.

Solution:
Since "benefit premium" means premiums computed so that $\mathrm{E}[L] = 0$, we have:

$$\mathrm{E}[L] = 0 \quad = \quad 1000A_{x:\overline{2}|} - 668 - 258vp_x \qquad (v = 1 - d).$$

We need to find p_x, using this equation and information about $A_{x:\overline{2}|}$:

$$\begin{aligned} A_{x:\overline{2}|} &= A^{1}_{x:\overline{2}|} + {}_2E_x \\ &= \left[vq_x + v^2\ {}_{1|}q_x\right] + v^2\ {}_2p_x \\ &= v(1 - p_x) + v^2\left[{}_{1|}q_x + {}_2p_x\right] \\ &= v(1 - p_x) + v^2 p_x \\ &= v + p_x(v^2 - v) = v - vdp_x \end{aligned}$$

So we have

$$\begin{aligned} 0 &= 1000\,[v - vdp_x] - 668 - 258vp_x \\ 668 - 1000v &= p_x\,(1000vd - 258v) \\ -272.00 &= -298.920p_x \qquad \Longrightarrow \underline{p_x = 0.90994} \end{aligned}$$

So,

$$\begin{aligned} A_{x:\overline{2}|} &= v - vdp_x = 0.88868, \\ \text{and} \quad P_{x:\overline{2}|} &= \frac{dA_{x:\overline{2}|}}{1 - A_{x:\overline{2}|}} = 0.47898 \end{aligned}$$

so the level annual premium is 478.98.

107. (May 2005 #15)
For an increasing 10-year term insurance, you are given:

(a) $b_{k+1} = 100,000(1 + k)$.
(b) Benefits are payable at the end of the year of death.
(c) Mortality follows the Illustrative Life Table.
(d) $i = 0.06$
(e) The single benefit premium for this insurance on (41) is 16,736.

Calculate the single benefit premium for this insurance on (40).

Solution:
We are being asked to compute $100,000 \times (\ IA)_{40:10}$. The quantity b_{k+1} represents the amount of the benefit paid at the end of the year of failure if failure occurs after the insured survived for exactly k years. We have:

$$
\begin{aligned}
(IA)^{\ 1}_{40:\overline{10|}} &= \sum_{k=0}^{9} (b_{k+1}/100,000)\, v^{k+1}\ {}_{k|}q_{40} \\
&= \sum_{k=0}^{9} (1+k)\, v^{k+1}\ {}_{k|}q_{40} \\
&= \sum_{k=0}^{9} v^{k+1}\ {}_{k|}q_{40} + \sum_{k=0}^{9} k v^{k+1}\ {}_{k|}q_{40} \\
&= A^{\ 1}_{40:\overline{10|}} + vp_{40} \sum_{k=1}^{9} k v^{k}\ {}_{(k-1)|}q_{41} \\
&= A^{\ 1}_{40:\overline{10|}} + vp_{40} \sum_{j=0}^{8} (j+1) v^{j+1}\ {}_{j|}q_{41} \\
&= A^{\ 1}_{40:\overline{10|}} + vp_{40} \left[(IA)^{\ 1}_{40:\overline{10|}} - 10v^{10}\ {}_{10|}q_{41} \right] \\
&= A^{\ 1}_{40:\overline{10|}} + vp_{40} \left[(\$16,736/100,000) - 10v^{10}\ {}_{9|}q_{41} \right].
\end{aligned}
$$

We have:

$$
\begin{aligned}
A^{\ 1}_{40:\overline{10|}} &= A_{40} - {}_{10}E_{40}\, A_{50} \\
&= 0.161\,3242 - (9611\,0194) \cdot (1.06^{-10} \cdot 0.249\,0475 \\
&= 0.0276\,6678 \\
{}_{10|}q_{41} &= \frac{S(50) - S(51)}{S(41)} = 0.00\,5705\,448
\end{aligned}
$$

so that

$$
\begin{aligned}
(IA)^{\ 1}_{40:\overline{10|}} &= A^{\ 1}_{40:\overline{10|}} + vp_{40} \left[(\$16,736/100,000) - 10v^{10}\ {}_{9|}q_{41} \right] \\
&= 0.02766678 - vp_{40} \left[0.16736 - 10v^{10} \times 0.00\,5705\,448 \right] \\
&= 0.1551425.
\end{aligned}
$$

<u>Ans: 15,514.25</u>

108. (May 2005 #2)
For a double-decrement model:

(a) ${}_tp_x^{\prime(1)} = 1 - t/60$ for $0 \le t \le 60$,

(b) ${}_tp_x^{\prime(2)} = 1 - t/40$ for $0 \le t \le 40$.

Calculate $\mu_{40}^{(\tau)}(20)$.

Solution:

One approach:

$$\begin{aligned}
{}_tp_x^{\prime(\tau)} &= \left({}_tp_x^{\prime(1)}\right)\left({}_tp_x^{\prime(2)}\right) = \frac{t^2-100t+2400}{2400} \\
\frac{-d}{dt}\,{}_tp_x^{\prime(\tau)} &= \frac{t-50}{1200} \\
\mu_{40}^{(\tau)}(t) &= \frac{-d/dt\ {}_tp_x^{\prime(\tau)}}{{}_tp_x^{\prime(\tau)}} = \frac{2(t-50)}{(60-t)(40-t)} \\
\mu_{40}^{(\tau)}(20) &= \frac{3}{40} = 0.075
\end{aligned}$$

Another approach: For $j = 1$ and $j = 2$ we have $\mu_{40}^{(j)}(t) = \left(-d/dt\ {}_tp_x^{\prime(\tau)}\right) / \left({}_tp_x^{\prime(\tau)}\right)$ so

$$\begin{aligned}
\mu_{40}^{(1)}(t) &= \frac{1}{60-t} \quad \text{so} \quad \mu_{40}^{(1)}(20) = \frac{1}{40} \\
\mu_{40}^{(2)}(t) &= \frac{1}{40-t} \quad \text{so} \quad \mu_{40}^{(2)}(20) = \frac{1}{20} \\
\mu_{40}^{(\tau)}(t) &= \mu_{40}^{(1)}(t) + \mu_{40}^{(2)}(t) \quad \text{so} \quad \mu_{40}^{(\tau)}(20) = \frac{3}{40} = 0.075.
\end{aligned}$$

109. (May 2005 #20)
A fully discrete 3-year term insurance of 10,000 on (40) is based on a double-decrement model, death and withdrawal:

(a) Decrement 1 is death.
(b) $\mu_{40}^{(1)} = 0.02$.
(c) Decrement 2 is withdrawal, which occurs at the end of the year.
(d) $q_{40+k}^{\prime(2)} = 0.04$
(e) $v = 0.95$

Calculate the actuarial present value of the death benefits for this insurance.

110. (May 2005 #21)
You are given:

(a) $\mathring{e}_{30:\overline{40}|} = 27.692$
(b) $S_0(x) = 1 - \frac{x}{\omega}$ for $0 \le x \le \omega$
(c) $T(x)$ is the future lifetime random variable for (x).

Calculate $\mathrm{Var}\left[T(30)\right]$.

Solution:

$$\begin{aligned}
{}_tp_x &= \frac{\omega - x - t}{\omega - x} \\
\mathring{e}_{30:\overline{40}|} = 27.692 &= \int_{t=0}^{40} \frac{\omega - 30 - t}{\omega - 30}\,dt = \int_{t=0}^{40} 1\,dt - \frac{1}{\omega-30}\int_{t=0}^{40} t\,dt \\
27.692 &= 40 - \frac{800}{\omega - 30} \\
\omega &= 95 \\
T(30) &\sim \mathrm{Uniform}(0, 65) \\
\mathrm{Var}\left[T(30)\right] &= \frac{65^2}{12} = \boxed{352.083}
\end{aligned}$$

111. (May 2005 #22)
For a fully discrete 5-payment 10-year decreasing term insurance on (60), you are given:

(a) $b_{k+1} = 1000(10-k)$
(b) Level benefit premiums are payable for five years and equal 218.15 each.
(c) $q_{60+k} = 0.02 + 0.001k$ for $k = 0, 1, 2, \ldots, 9$.
(d) $i = 0.06$.

Calculate ${}_2V$, the benefit reserve at the end of year 2.

Solution:
Retrospective approach:

$$\begin{aligned}
{}_2V &= \underline{\text{Actuarial Accumulated Value}} \text{ of: } \Big\{\text{Prem's to time 2}\Big\} - \Big\{\text{Ins Cvg to time 2}\Big\} \\
&= \frac{1}{{}_2E_{60}} \times \underline{\text{APV}} \text{ of: } \Big\{\text{Prem's to time 2}\Big\} - \Big\{\text{Ins Cvg to time 2}\Big\} \\
&= \frac{1}{{}_2E_{60}} \Big[\{\pi(1+v\,p_{60})\} - 1000\Big\{10v({}_{0|}q_{60}) + 9v^2({}_{1|}q_{60})\Big\}\Big] \\
&= \frac{1}{v^2 p_{60}p_{61}} \Big[\{\pi(1+v\,p_{60})\} - 1000\big\{10v\,q_{60} + 9v^2\,p_{60}q_{61}\big\}\Big] \\
&= \frac{1.06^2}{(0.98)(0.9790)} \Big[\{218.15(1+0.98/1.06)\} - 10^3\big\{0.2/1.06 + 9(0.98)(0.021)/1.06^2\big\}\Big] \\
&= 77.6590
\end{aligned}$$

112. (May 2005 #26)
You are given:

(a) $\mu_x(t) = 0.03$ for $t > 0$.
(b) $\delta = 0.05$.
(c) $T(x)$ is the future lifetime random variable.
(d) g is the standard deviation of $\bar{a}_{\overline{T(x)}|}$.

Calculate $\Pr\Big\{\bar{a}_{\overline{T(X)}|} > \bar{a}_x - g\Big\}$.

Solution:
We need to compute the variance:

$$g^2 = \mathrm{Var}\left[\bar{a}_{\overline{T(x)}|}\right] = \mathrm{Var}\left[\frac{1-e^{-\delta T(x)}}{\delta}\right] = \frac{\mathrm{Var}\left[e^{-\delta T(x)}\right]}{\delta^2} = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}$$

We have:

$$\begin{aligned}
\bar{A}_x &= \frac{\mu}{\mu+\delta} = 0.375 \\
{}^2\bar{A}_x &= \frac{\mu}{\mu+2\delta} = 0.23076923
\end{aligned}$$

so that

$$g = \sqrt{\frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}} = 6.00481.$$

and

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = 12.5$$

so that $\bar{a}_x - g = 11.1573$

...

<u>Ans: $e^{-0.03\times 7.85374} \approx 0.79$</u>

113. (May 2005 #27)
(50) is an employee of XYZ Corporation. Future employment with XYZ follows a double decrement model:

(a) Decrement 1 is retirement, with

$$\mu_{50}^{(1)}(t) = \begin{cases} 0.00 & \text{for } 0 \le t < 5, \\ 0.02 & \text{for } 5 \le t. \end{cases}$$

(b) Decrement 2 is leaving employment with XYZ for all other causes, with

$$\mu_{50}^{(2)}(t) = \begin{cases} 0.05 & \text{for } 0 \le t < 5, \\ 0.03 & \text{for } 5 \le t. \end{cases}$$

(c) If (50) leaves employment with XYZ, he will never rejoin XYZ.

Calculate the probability that (50) will retire from XYZ before age 60.

114. (May 2005 #28)
For a life table with a one-year select period, you are given:

(a)

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$\mathring{e}_{[x]}$
80	1000	90		8.5
81	920	90		

(b) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{[81]}$.

<u>Solution:</u>
Complete the table:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$p_{[x]}$	$q_{[x]}$
80	1000	90	910	0.910	0.0900
81	920	90	830	0.9022	0.0978

We have

$$\begin{aligned} \mathring{e}_{[x]} &= \int_{t=0}^{\infty} {}_tp_{[x]}\, dt = \int_{t=0}^{1} {}_tp_{[x]}\, dt + \int_{t=1}^{\infty} {}_tp_{[x]}\, dt = \mathring{e}_{[x]:\overline{1}|} + \int_{t=0}^{\infty} {}_{t+1}p_{[x]}\, dt \\ &= \mathring{e}_{[x]:\overline{1}|} + \int_{t=0}^{\infty} p_{[x]}\, {}_tp_{[x]+1}\, dt = \mathring{e}_{[x]:\overline{1}|} + p_{[x]}\mathring{e}_{[x]+1}. \end{aligned}$$

We also have

$$\mathring{e}_{[x]:\overline{1}|} = \frac{1+p_{[x]}}{2} = \frac{q_{[x]}}{2} + p_{[x]},$$

so that

$$\mathring{e}_{[x]} = \frac{q_{[x]}}{2} + p_{[x]}\left(1 + \mathring{e}_{[x]+1}\right).$$

Using this and the fact that the select period is only one year, we have

$$\mathring{e}_{[x]} = \frac{1}{2}q_{[x]} + p_{[x]}\left(1 + \mathring{e}_{x+1}\right) \tag{5}$$

$$\text{and} \quad \mathring{e}_{x+1} = \frac{\mathring{e}_{[x]} - (1 + p_{[x]})/2}{p_{[x]}}. \tag{6}$$

Using these we have the fact that $p_{81} = \ell_{82}/\ell_{81} = 830/910 = .91209$

$$\begin{aligned}
\mathring{e}_{81} &= \frac{\mathring{e}_{[80]} - (1 + p_{[80]}/2)}{p_{[80]}} = 8.2912, \\
\mathring{e}_{82} &= \frac{\mathring{e}_{81} - (1 + p_{81}/2)}{p_{81}} = 8.0430 \\
\mathring{e}_{[81]} &= \frac{q_{[81]}}{2} + p_{[81]}\left(1 + \mathring{e}_{82}\right) = 8.206
\end{aligned}$$

115. (May 2005 #29)
For a fully discrete 3-year endowment insurance of 1000 on (x):

(a) $i = 0.05$

(b) $p_x = p_{x+1} = 0.7$

Calculate the second year terminal benefit reserve.

Solution:
Looking forward (prospectively) from the end of the second year, we see that we'll collect exactly one more premium (at the beginning of the third year), and then at the end of the third year we'll be paying 1000, no matter what.
So ${}_2V + 1000P_{x:\overline{3}|} = 1000v$. So we need $P_{x:\overline{3}|}$..

$$\begin{aligned}
A_{x:\overline{3}|} &= vq_x + v^2\,{}_{1|}q_x + v^3\,{}_2q_x + v^3 \\
&= 0.3v + (0.7)(0.3)v^2 + (0.7)^2v^3 = 0.899471 \\
\ddot{a}_{x:\overline{3}|} &= \frac{1 - A_{x:\overline{3}|}}{d} = 2.11111111 \\
P_{x:\overline{3}|} &= \frac{A_{x:\overline{3}|}}{a_{x:\overline{3}|}} = 0.426065
\end{aligned}$$

So ${}_2V = 1000(v - P_{x:\overline{3}|}) = \underline{526.3158}$

116. (May 2005 #3)

117. (May 2005 #35)
For a special fully discrete 5-year deferred whole life insurance of 100,000 on (40), you are given:

(a) The death benefit during the 5-year deferral period is return of benefit premiums paid without interest.

(b) Annual benefit premiums are payable only during the deferral period.

(c) Mortality follows the Illustrative Life Table.

(d) $i = 0.06$

(e) $(IA)^{1}_{40:\overline{5}|} = 0.04042$

Calculate the annual benefit premiums.

Solution:
APV of Benefits paid: $10^5\ {}_{5|}A_{40} + \pi(IA)^{1}_{40:\overline{5|}}$
APV of Premiums received: $\pi\ddot{a}_{40:\overline{5|}}$

$$\begin{aligned}
\mathrm{E}[L] &= 0 = 10^5\ {}_{5|}A_{40} + \pi(IA)^{1}_{40:\overline{5|}} - \pi\ddot{a}_{40:\overline{5|}} \\
&= 10^5\ {}_{5|}A_{40} - \pi\left[\pi\ddot{a}_{40:\overline{5|}} - (IA)^{1}_{40:\overline{5|}}\right] \\
\pi &= 10^5\ {}_{5|}A_{40} / \left[\pi\ddot{a}_{40:\overline{5|}} - (IA)^{1}_{40:\overline{5|}}\right] \\
&= (0.20120)\cdot(0.73529)/\left[4.4401 - 0.04042\right] = 3363
\end{aligned}$$

118. (May 2005 #37)
Company ABC sets the contract premium for a continuous life annuity of 1 per year on (x) equal to the single benefit premium calculated using:

(a) $\delta = 0.03$
(b) $\mu_x(t) = 0.02$ for $t \geq 0$

However, a revised mortality assumption reflects future mortality improvement and is given by

$$\mu_x(t) = \begin{cases} 0.02 & \text{for } t \leq 10 \\ 0.01 & \text{for } t > 10 \end{cases}$$

Calculate the expected loss at issue for ABC (using the revised mortality assumption) as a percentage of the contract premium.

Solution:
The "single benefit premium" for a life annuity is the EPV of that annuity, which is $\bar{a}_x$:

$$\bar{a}_x = \int_0^\infty e^{-\delta t}\,{}_tp_x\,dt = \int_0^\infty e^{-\delta t}e^{-\mu t}\,dt = \frac{1}{\mu+\delta} = \frac{1}{0.05} = 20.$$

Under the revised mortality assumptions, the expected present value — call it a — of the life annuity on (x) can be viewed as the sum of a 10-year temporary life annuity on (x), plus a deferred whole life annuity on (x+10). So

$$a = \bar{a}_{x:\overline{10|}} + {}_{10}E_x\,\bar{a}_{x+10} = \int_0^{10} e^{-0.05t}\,dt + e^{-0.05\cdot 10}\int_0^\infty e^{-0.04t}\,dt = 23.0327$$

So we lose 3.0327 in this policy. As a percentage of the contract premium, this is $(100 \cdot 3.0327/20)\%$ or $\boxed{15.16\%}$.

119. (May 2005 #38)
A group of 1000 lives each age 30 sets up a fund to pay 1000 at the end of the first year for each member who dies in the first year, and 500 at the end of the second year for each member who dies in the second year. Each member pays into the fund an amount equal to the single benefit premium for a special 2-year term insurance, with:

(a) Benefits:

k	b_{k+1}
0	1000
1	500

(b) Mortality follows the ILT
(c) $i = 0.06$

The actual experience of the fund is as follows:

k	Interest Rate Earned	Number of Deaths
0	0.070	1
1	0.069	1

Calculate the difference, at the end of the second year, between the expected size of the fund as projected at time 0 and the actual fund.

Solution:
From the tables we have $q_{30} = 0.00152894$ and $q_{31} = 0.00160886$, so that ${}_{0|}q_{30} = 0.00152894$, and ${}_{1|}q_{30} = (1 - 0.00152894) \cdot 0.00160886 = 0.00160640$. So the EPV of th insurance is

$$\frac{1000}{1.06} \cdot 0.00152894 + \frac{500}{1.06^2} \cdot 0.00160640 = 2.15724,$$

so the total amount collected is \$2157.24. The expected amount of this remaining after 2 years is zero.
In the first year this amount grows to $\$2157.24 \cdot 1.07 = 2308.25$ and then \$1,000 is removed leaving \$1308.25.
In the second year this amount grows to $\$1308.25 \cdot 1.069 = 1398.52$ and then \$500 is removed leaving \$898.52.
So the difference, at the end of the second year, between the expected size of the fund as projected at time 0 and the actual fund is \$898.52.

120. (May 2005 #7)
Z is the present-value random variable for a whole life insurance of b payable at the moment of death of (x).

You are given:

(a) $\delta = 0.04$
(b) $\mu_x(t) = 0.02$ for $t \geq 0$.
(c) The single benefit premium for this insurance is equal to $\mathrm{Var}[Z]$.

Calculate b.

Solution:
The random variable Z is equal to bZ_x, and the single benefit premium is $\mathrm{E}[Z]$. We need to equate $\mathrm{E}[Z]$ with $\mathrm{Var}[Z]$ and solve for b.

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{Var}[Z] \\ bA_x &= b^2 \left({}^2A_x - A_x^2 \right) \\ b &= \frac{A_x}{{}^2A_x - A_x^2} \end{aligned}$$

Since the force of interest and force of mortality are both constant, we have

$$\begin{aligned} A_x &= \frac{\mu}{\mu + \delta} = \frac{1}{3} \\ {}^2A_x &= \frac{\mu}{\mu + 2\delta} = \frac{1}{5}, \end{aligned}$$

so that

$$b = \frac{1/3}{1/5 - 1/9} = 15/4 = \underline{3.75}$$

121. (Nov 2005, #1)
For a special whole life insurance on (x), you are given:

(a) Z is the present value random variable for this insurance.
(b) Death benefits are paid at the moment of death.
(c) $\mu_x(t) = 0.02$
(d) $\delta = 0.08$
(e) $b_t = e^{0.03t}$

Calculate $\mathrm{Var}[Z]$.

Solution:

$$\begin{aligned}
Z &= (\text{Amt Paid at time } T_x) \times (\text{Discount factor at time } T_x) \\
&= e^{0.03t} \cdot e^{-0.08t}, \\
Z &= e^{-0.05t} \\
\mathrm{E}[Z] &= \int_0^\infty Z \cdot \mathrm{PDF}_{T_x}(t)\, dt \\
&= \int_0^\infty e^{-0.05t} \cdot (0.02e^{-0.02t})\, dt = \frac{1}{50}\int_0^\infty e^{-7t/100}\, dt = \frac{100}{7 \cdot 50} \\
&= \frac{2}{7} \\
\mathrm{E}\left[Z^2\right] &= \int_0^\infty (e^{-0.05t})^2 \cdot (0.02e^{-0.02t})\, dt = \frac{1}{50}\int_0^\infty e^{-12t/100}\, dt = \frac{100}{12 \cdot 50} \\
&= \frac{1}{6} \\
\mathrm{Var}[Z] &= \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = \frac{1}{6} - \frac{4}{49} = 0.085034014
\end{aligned}$$

$$\boxed{\mathrm{Var}[Z] = 0.08503}$$

122. (C3 Nov 2005, #13)
The actuarial department for the SharpPoint Corporation models the lifetime of pencil sharpeners from purchase using a generalized DeMoivre model with $S_0(x) = (1 - x/\omega)^\alpha$, for $\alpha > 0$ and $0 < x < \omega$.

A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:

(a) The new complete expectation of life at purchase is half what it was previously.
(b) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.
(c) ω remains the same.

Calculate the original value of α.

123. (C3, Nov 2005, #14)
You are given:

(a) T is the future lifetime random variable.
(b) $\mu(t) = \mu, \ \ t \geq 0$.

(c) $\mathrm{Var}[T] = 100$.

Calculate $\mathring{e}_{25:\overline{10}|}$

124. (C3, Nov 2005, #25)
For a special 3-year term insurance on (x) , you are given:

(a) Z is the present-value random variable for this insurance.

(b) $q_{x+k} = 0.02(k+1)$ for $k = 0, 1, 2$.

(c) The following benefits are payable at the end of the year of death:

k	b_{k+1}
0	300
1	350
2	400

(d) $i = 0.06$

Calculate $\mathrm{Var}[Z]$

Solution:

${}_{0\mid}q_x = \Pr\{K_x = 0\}$	${}_{1\mid}q_x = \Pr\{K_x = 1\}$	${}_{2\mid}q_x = \Pr\{K_x = 2\}$
$\underline{0.02}$	$0.98 \cdot 0.04 = \underline{0.0392}$	$0.98 \cdot 0.96 \cdot 0.06 = \underline{0.056448}$

$$Z = \begin{cases} 300 \cdot 1.06^{-1} & \text{with probability } 0.02, \\ 350 \cdot 1.06^{-2} & \text{with probability } 0.0392, \\ 400 \cdot 1.06^{-3} & \text{with probability } 0.056448, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathrm{E}[Z] &= 0.02 \cdot 300 \cdot 1.06^{-1} + 0.0392 \cdot 350 \cdot 1.06^{-2} + 0.056448 \cdot 400 \cdot 1.06^{-3} \\ &= 36.829 \\ \mathrm{E}\left[Z^2\right] &= 0.02 \cdot (300 \cdot 1.06^{-1})^2 + 0.0392 \cdot (350 \cdot 1.06^{-2})^2 + 0.056448(\cdot 400 \cdot 1.06^{-3})^2 \\ &= 11772.6 \end{aligned}$$

$$\mathrm{Var}[Z] = \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = 11772.6 - (36.829)^2 = \boxed{10416.2}$$

125. (C3, Nov 2005, #25b)
An insurance on (x) pays $300,000 + 50,000K_x$ if $K_x \leq 2$ and zero otherwise. Given that $q_{x+k} = 0.02(k+1)$ for $k = 0, 1, 2$ and the $i = 0.06$, compute the mean and variance of the present value random variable for this insurance.

Solution:

${}_{0\mid}q_x = \Pr\{K_x = 0\}$	${}_{1\mid}q_x = \Pr\{K_x = 1\}$	${}_{2\mid}q_x = \Pr\{K_x = 2\}$
$\underline{0.02}$	$0.98 \cdot 0.04 = \underline{0.0392}$	$0.98 \cdot 0.96 \cdot 0.06 = \underline{0.056448}$

$$
Z = \begin{cases} 300 \cdot 1.06^{-1} & \text{with probability } 0.02, \\ 350 \cdot 1.06^{-2} & \text{with probability } 0.0392, \\ 400 \cdot 1.06^{-3} & \text{with probability } 0.056448, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
\mathrm{E}[Z] &= 0.02 \cdot 300 \cdot 1.06^{-1} + 0.0392 \cdot 350 \cdot 1.06^{-2} + 0.056448 \cdot 400 \cdot 1.06^{-3} \\
&= 36.829 \\
\mathrm{E}\left[Z^2\right] &= 0.02 \cdot (300 \cdot 1.06^{-1})^2 + 0.0392 \cdot (350 \cdot 1.06^{-2})^2 + 0.056448(\cdot 400 \cdot 1.06^{-3})^2 \\
&= 11772.6 \\
\\
\mathrm{Var}[Z] &= \mathrm{E}\left[Z^2\right] - \mathrm{E}[Z]^2 = 11772.6 - (36.829)^2 = \boxed{10416.2}
\end{aligned}
$$

126. (C3 Nov 2005, #31)
The graph of a piecewise linear survival function, $S_0(x)$, consists of 3 line segments with endpoints $(0,1)$, $(25, 0.50)$, $(75, 0.40)$, and $(100, 0)$.

Calculate $\dfrac{_{20|55}q_{15}}{_{55}q_{35}}$

Solution:

$$
\frac{_{20|55}q_{15}}{_{55}q_{35}} = \frac{_{20}p_{15} \cdot {_{55}q_{35}}}{_{55}q_{35}} = {_{20}p_{15}} = \frac{S_0(35)}{S_0(15)}
$$

Since 35 is 1/5 of the way from 25 to 75, $S_0(35)$ is 1/5 of the way from 0.50 to 0.40 which is 0.48, so $S_0(35) = 0.48$.
Since 15 is 3/5 of the way from 0 to 25, $S_0(15)$ is 3/5 of the way from 1.00 to 0.50 which is 0.70, so $S_0(15) = 0.70$.
So $\boxed{\dfrac{_{20|55}q_{15}}{_{55}q_{35}} = \dfrac{0.48}{0.70} = 0.68571}$.

127. (C3 Nov 2005, #32)
For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:

(a) For non-smokers, $\mu^n(x) = 0.08$ for $x \geq 30$,
(b) For smokers, $\mu^s(x) = 0.16$ for $x \geq 30$.

Calculate q_{80} for a life randomly selected from those surviving to age 80.

Hint: Initially the ratio of smokers/nonsmokers is known (50/50). How does this ratio change over time? Initially (at age 30), there are equal numbers of non-smokers and smokers. What's the ratio by age 80?

Solution:

$$
\begin{aligned}
\text{chance of non-smoker reaching age 80} &= {_{50}p^n}_{30} &= e^{-0.08 \cdot 50} \\
\text{chance of smoker reaching age 80} &= {_{50}p^s}_{30} &= e^{-0.16 \cdot 50}
\end{aligned}
$$

$$
\frac{\text{chance of non-smoker reaching age 80}}{\text{chance of smoker reaching age 80}} = e^{0.08 \cdot 50} = e^4.
$$

So the ratio of non-smokers to smokers, amoung those surviving to age 80, is $e^4 : 1$. So

$$\Pr\{\text{Selecting a non-smoker}\} = \frac{e^4}{e^4+1},$$
$$\Pr\{\text{Selecting a smoker}\} = \frac{1}{e^4+1}$$

Now just take a weighted average:

$$q_{80} \text{ for a randomly selected (80)} \quad = \quad \frac{e^4 \cdot (1-e^{-0.08}) + 1 \cdot (1-e^{-0.16})}{e^4+1} = \boxed{0.07816}$$

128. (C3, Nov 2006, #12)
For a whole life insurance of 1 on (x) with benefits payable at the moment of death, you are given:

(a) $\delta_t = \begin{cases} 0.02 & \text{for } t < 12, \\ 0.03 & \text{for } t \geq 12 \end{cases}$

(b) $\mu_x(t) = \begin{cases} 0.04 & \text{for } t < 5, \\ 0.05 & \text{for } t \geq 5 \end{cases}$

Calculate the actuarial present value of this insurance.

Solution:

After simplifying,

- Discount factor is $\exp\left(-\int_0^t \delta_z \, dz\right) = \begin{cases} e^{-0.02t} & \text{for } t < 12 \\ e^{0.12} \cdot e^{-0.03t} & \text{for } t \geq 12 \end{cases}$
- PDF for T_x is $\frac{-d}{dt} \exp\left(-\int_0^t \delta_z \, dz\right) = \begin{cases} 0.04 \cdot e^{-0.04t} & \text{for } t < 5 \\ e^{0.05} 0.05 \cdot e^{-0.05t} & \text{for } t \geq 5 \end{cases}$

$$\begin{aligned}
\bar{A}_x &= \int_{t=0}^{\infty} (\text{Discount factor}) \cdot (\text{PDF}) \, dt \\
&= 0.04 \int_{t=0}^{5} e^{-0.06t} \, dt + 0.05 \cdot e^{0.12} \int_{5}^{12} e^{0.07t} \, dt + 0.05 e^{0.17} \int_{12}^{\infty} e^{-0.08t} \, dt \\
&= \frac{0.04}{0.06} \cdot (1-e^{-0.3}) + \frac{0.05 \cdot e^{0.05}}{0.07} \cdot (e^{-0.35} - e^{-0.84}) + \frac{0.05 e^{0.17}}{0.08} e^{-0.96} \\
& \boxed{\bar{A}_x = 0.661422}
\end{aligned}$$

129. (Nov 2006, #25)
You are given:

(a) The future lifetimes of (40) and (50) are independent.
(b) The survival function for (40) is based on a constant force of mortality, $\mu = 0.05$.
(c) The survival function for (50) follows DeMoivre,s law with $\omega = 110$.

Calculate the probability that (50) dies within 10 years and dies before (40).

Solution:
With (x) being (50) and (y) being (40), they're asking for ${}_n q^{1}_{xy}$, for which we have:

$${}_n q^{1}_{xy} \; = \int_{t=0}^{n} {}_t p_{xy} \; f_{T_x}(t) \, dt.$$

From the given data, we have:

$$\begin{aligned}
{}_tp_{40} &= e^{-0.05t} \\
{}_tp_{50} &= \frac{60-t}{60} \\
\mu_{50}(t) &= \frac{1}{60-t}.
\end{aligned}$$

So,

$${}_nq^1_{xy} = \int_{t=0}^{10} \frac{e^{-0.05t}}{60}\, dt = 0.13115.$$

130. (C3 Nov 2006, # 33)
You are given:

(i) Y is the present value random variable for a continuous whole life annuity of 1 per year on (40).
(ii) Mortality follows DeMoivre's Law with $\omega = 120$.
(iii) $\delta = 0.05$

Calculate the 75^{th} percentile of the distribution of Y.

<u>**Solution:**</u>
The present value of a continuous annuity paying 1 per years for t years is

$\bar{a}_{\overline{t}|} = \int_{z=0}^{t} e^{-\delta z}\, dz = \frac{1-e^{-\delta t}}{\delta}$, so $Y = \bar{a}_{\overline{T_{40}}|} = \frac{1-e^{-\delta T_{40}}}{\delta}$. The 75^{th} percentile of Y is that value (call it y_*) for which $\Pr\{Y \le y_*\} = 0.75$. So,

$$\begin{aligned}
0.75 = \Pr\{Y \le y_*\} &= \Pr\left\{\frac{1-e^{-\delta T_{40}}}{\delta} \le y_*\right\} \\
&= \Pr\{T_{40} \le -\ln(1-\delta y_*)/\delta\} \\
0.75 &= \Pr\{T_{40} \le t_*\} \qquad \text{where } t_* = -\ln(1-\delta y_*)/\delta.
\end{aligned}$$

Since T_{40} is DeMoivre (uniform) on $[0, 80]$,

$$0.75 = \Pr\{T_{40} \le t_*\} \quad \Longrightarrow \quad 0.75 = t_*/80 \quad \Longrightarrow \quad t_* = 60.$$

131. (C3 Nov 2006, # 33b)
Calculate the mean, the standard deviation, and the median of the random variable $\bar{a}_{\overline{T_{40}}|}$, assuming that $\omega = 120$, that ${}_tp_{40} = 1 - t/80$, and that $\delta = 0.05$.

<u>**Solution:**</u>
The present value of a continuous annuity paying 1 per years for t years is

$$\bar{a}_{\overline{t}|} = \int_{z=0}^{t} e^{-\delta z}\, dz = \frac{1-e^{-\delta t}}{\delta} = 20\left(1-e^{-t/20}\right), \quad \text{so} \quad \underline{\bar{a}_{\overline{T_{40}}|} = 20\left(1-e^{T_{40}/20}\right).}$$

<u>Mean</u>

$$\mathrm{E}\left[\bar{a}_{\overline{T_{40}}|}\right] = \bar{a}_{40} = \int_0^{80} \left(20\left(1-e^{-t/20}\right)\right) \frac{1}{80}\, dt = \boxed{15.091578}.$$

<u>Standard Deviation</u>

$$\begin{aligned}
\mathrm{E}\left[\left(\bar{a}_{\overline{T_{40}}|}\right)^2\right] &= \int_0^{80} \left(20\left(1-e^{-t/20}\right)\right)^2 \frac{1}{80}\, dt = 25.890628, \\
\mathrm{Std}\left[\bar{a}_{\overline{T_{40}}|}\right] &= \mathrm{E}\left[\left(\bar{a}_{\overline{T_{40}}|}\right)^2\right] - \left(\mathrm{E}\left[\bar{a}_{\overline{T_{40}}|}\right]\right)^2 = \boxed{5.088283}.
\end{aligned}$$

Median The median of Y is that value (call it y_*) for which $\Pr\{Y \le y_*\} = 0.5$. So,

$$\begin{aligned} 0.5 = \Pr\{Y \le y_*\} &= \Pr\left\{\frac{1-e^{-\delta T_{40}}}{\delta} \le y_*\right\} \\ &= \Pr\{T_{40} \le -\ln(1-\delta y_*)/\delta\} \\ 0.5 &= \Pr\{T_{40} \le t_*\} \quad \text{where } t_* = -\ln(1-\delta y_*)/\delta. \\ 0.5 &= 1-(1-t_*/80) = t_*/80 \\ 40 &= t_* \quad \boxed{y_* = 20\left(1-e^{-40/20}\right) = 17.2933} \end{aligned}$$

132. (C3, Nov 2006, #35)
For a whole life insurance of 1000 on (80), with death benefits payable at the end of the year of death, you are given:

(a) Mortality follows a select and ultimate mortality table with a one-year select period.

(b) $q_{[80]} = 0.5 \cdot q_{80}$

(c) $i = 0.06$

(d) $1000A_{80} = 679.80$

(e) $1000A_{81} = 689.52$

Calculate $A_{[80]}$

Solution:

$$\begin{aligned} A_{80} &= vq_{80} + vp_{80}A_{81} \\ 0.67980 &= 1.06^{-1}\big(q_{80} + 0.68952(1-q_{80})\big) \\ &\underline{q_{80} = 0.100064} \end{aligned}$$

$$\begin{aligned} A_{[80]} &= vq_{[80]} + vp_{[80]}A_{[80]+1} \\ A_{[80]} &= vq_{[80]} + vp_{[80]}A_{81} \qquad \text{because select period is one year} \\ A_{[80]} &= 1.06^{-1}\big(0.5 \cdot 0.10064 + (1-0.5 \cdot 0.100064) \cdot 0.68952\big) \\ &\boxed{1000A_{[80]} = 665.417} \end{aligned}$$

133. (May 2007 #1)
You are given:

(a) ${}_3p_{70} = 0.95$

(b) ${}_2p_{71} = 0.96$

(c) $\int_{71}^{75} \mu_x \, dx = 0.107$.

Calculate ${}_5p_{70}$.

Solution:
Strategy: Write ${}_5p_{70} = (p_{70}) \cdot ({}_4p_{71})$.
Use (a) and (b) to get p_{70},
and use (c) to get ${}_4p_{71}$.

Use (a) and (b) to get p_{70}.

$$\frac{{}_3p_{70}}{{}_2p_{71}} = \frac{S(73)/S(70)}{S(73)/S(71)} = \frac{S(71)}{S(70)} = p_{70}$$

$$\underline{p_{70} = 0.95/0.96}$$

Use (c) to get ${}_4p_{71}$:

$$
\begin{aligned}
{}_tp_x = \exp\left(-\int_{z=0}^{t} \mu(x+z)\,dz\right) &= \exp\left(-\int_{x}^{x+t} \mu(z)\,dz\right) \\
&\exp\left(-\int_{71}^{75} \mu_x(z)\,dz\right) = e^{-0.107}
\end{aligned}
$$

$$\underline{{}_4p_{71} = e^{-0.107}}.$$

Now,

$${}_5p_{70} = p_{70}\,{}_4p_{71} = \frac{95e^{-0.107}}{96} \approx \underline{0.8892 = {}_5p_{70}}.$$

134. (May 2007 #02)
You are given:

(a) $\mu_x(t) = c$ for $t > 0$
(b) $\delta = 0.08$
(c) $\bar{A}_x = 0.3443$
(d) $T(x)$ is the future lifetime random variable for (x).

Calculate Var $\left[\bar{a}_{\overline{T(x)}|}\right]$

<u>Solution:</u>

135. (C3 May 2007, 04)
For a fully discrete whole life insurance of 150,000 on (x), you are given:

(a) ${}^2A_x = 0.0143$
(b) $A_x = 0.0653$
(c) The annual premium is determined using the equivalence principle.
(d) L is the loss-at-issue random variable.

Calculate the standard deviation of L.

<u>Solution:</u>

$$
\begin{aligned}
L &= 150,000 \times Z_x - \pi \times \ddot{Y}_x \\
&= 150,000 \times Z_x - \frac{\pi}{d} \times (1 - Z_x) \\
\mathrm{E}[L] = 0 \quad \Longrightarrow \quad & \frac{\pi}{d} = 150,000 \times \frac{A_x}{1 - A_x} = 10,479.30 \\
L &= 150,000 \times Z_x - 10,479.30 \times \ddot{Y}_x \\
&= (150,000 + 10,479.30) \times Z_x + 10,479.30 \\
\mathrm{Var}[L] &= (150,000 + 10,479.30)^2 \left({}^2A_x - A_x^2\right) = 2.58460869 \times 10^8 \\
\mathrm{Std}[L] &= 16076.72
\end{aligned}
$$

136. (C3 May 2007, 07)

Solution:

137. (C3 May 2007, 10)
For whole life insurances of 1000 on (65) and (66):

(a) Death benefits are payable at the end of the year of death.
(b) The interest rate is 10% for 2008 and 6% for 2009 and thereafter.
(c) $q_{65} = 0.010$ and $q_{66} = 0.012$.
(d) The actuarial present value on December 31^{st}, 2007, of the insurance on (66) is 300.

Caclulate the actuarial present value on December 31^{st}, 2007, of the insurance on (65).

Solution:
Because of the differing interest rate, we have to go to first-principles. Denote by $1000A^*_{65}$ and $1000A^*_{66}$ the APV's of the two insurances at the end of 2007, and let A_x have it's usual meaning, computed at an interest rate of 6%. We have:

$$\begin{aligned}
\frac{300}{1000} = A^*_{66} &= vq_{66} + vp_{66}A_{67} \\
\frac{3}{10} = &= \frac{0.010}{1.1} + \frac{1-0.012}{1.1}A_{67} \\
\implies A_{67} &= 0.32186 \\
A_{66} &= vq_{66} + p_{66}A_{67} \\
&= \frac{0.012}{1.06} + \frac{1-0.012}{1.06} \cdot 0.32186 = 0.3113 \\
A^*_{65} &= vq_{65} + p_{65}A_{66} \\
&= \frac{0.010}{1.1} + \frac{1-0.010}{1.06} \cdot 0.3113 = 0.289
\end{aligned}$$

138. (C3 May 2007, #13)
A 3-year term insurance pays 1000 at the end of the year of death of (x) if that death occurs within three years, and level premium are payable annually. You are given:

(a) $i = 0.10$
(b) The mortality rates and net premium policy values are given by:

k	q_{x+k}	${}_{k+1}V$
0	0.3	95.833
1	0.4	120.833
2	0.5	0

(c) ${}_1L$ is the present value net loss random variable at time 1, based on the benefit premium.
(d) K_x is the full number of years in the future lifetime of (x).

Calculate $\mathrm{Var}\left[{}_1L \mid K_x \geq 1\right]$.

Solution:
After one year, given that $K_x \geq 1$, the remaining contract is a two-year term insurance, and

Since the policy is a three year term insurance, then the benefit reserve after two years is ${}_2V = 1000vq_{x+2} - P$ from which we can find P:

$$\begin{aligned} {}_2V &= 1000vq_{x+2} - P \\ P &= 1000vq_{x+2} - {}_2V = \frac{1000}{1.1}0.5 - 120.833 = 333.71. \end{aligned}$$

Now, one year into this 3-year term insurance, there are three possibilities given that the insured has survived the first year (e.g, that $K_x \geq 1$) : The insured could die between $t = 1$ and $t = 2$, which is when $K_x = 1$, or the insured could die between $t = 2$ and $t = 3$, which is when $K_x = 2$, or the insured could survive the term if the insurance, which is $K_x \geq 3$. We have

$$\begin{aligned} \Pr\{K_x = 1\} &= q_{x+1} = 0.4 \\ \Pr\{K_x = 2\} &= q_{1|}x + 1 = (1 - q_{x+1})q_{x+2} = (1 - 0.4) \cdot 0.5 = 0.3, \\ \Pr\{K_x \geq 3\} &= {}_2p_{x+1} = 1 - (0.3 - 0.4) = 0.3. \end{aligned}$$

So:

$$\begin{aligned} {}_1L &= \left\{ \begin{array}{cc} 1000v - P & \text{w/ prob. } 0.4, \\ 1000v^2 - P(1+v) & \text{w/ prob. } 0.3, \\ -P(1+v) & \text{w/ prob. } 0.3 \end{array} \right\} = \left\{ \begin{array}{rl} 575.38 & \text{w/ prob. } 0.4, \\ 189.36 & \text{w/ prob. } 0.3, \\ -637.08 & \text{w/ prob. } 0.3 \end{array} \right. \\ \mathrm{E}\left[{}_1V\right] &= 575.38 \cdot 0.4 + 189.36 \cdot 0.3 + (-637.08) \cdot 0.3 = 95.836, \\ \mathrm{E}\left[\left({}_1V\right)^2\right] &= (575.38)^2 \cdot 0.4 + (189.36)^2 \cdot 0.3 + (-637.08)^2 \cdot 0.3 = 264,943, \\ \mathrm{Var}\left[{}_1V\right] &= 264,943 - 95.836^2 = \boxed{255,758} \end{aligned}$$

139. (C3 May 2007, 19)
For whole life insurance of 1 on (40), you are given:

(a) $i = 0.06$

(b) $p_{50} = p_{51} = p_{52}$

(c) The net premium policy values at durations 10 and 13 are equal.

(d) $\ddot{a}_{50} = 10.0$

Calculate p_{50}.

Solution:
Assuming a level net premium of π, we have

$${}_tV = A_{40+t} - \pi\ddot{a}_{40+t} = (1 - d\ddot{a}_{40+t}) - \pi\ddot{a}_{40+t} = 1 + (d+\pi)\ddot{a}_{40+t}.$$

Because the net premium policy values at durations 10 and 13 are equal, we have ${}_{10}V = {}_{13}V$ means $1 + (d+\pi)\ddot{a}_{50} = 1 + (d+\pi)\ddot{a}_{53}$ which means $\underline{\ddot{a}_{50} = \ddot{a}_{53}}$.

Since $p_{50} = p_{51} = p_{52}$ we have ${}_2p_{50} = (p_{50})^2$ and ${}_3p_{50} = (p_{50})^3$, so that

$$\begin{aligned} \ddot{a}_{50} &= 1 + vp_{50}\ddot{a}_{51} \\ &= 1 + vp_{50}\left(1 + vp_{51}\ddot{a}_{52}\right) \\ &= 1 + vp_{50}\left(1 + vp_{51}\left[1 + vp_{52}\ddot{a}_{53}\right]\right) \\ \ddot{a}_{50} &= 1 + vp_{50} + \left(vp_{50}\right)^2 + \left(vp_{50}\right)^3\ddot{a}_{53}. \end{aligned}$$

Since $\ddot{a}_{50} = 10 = \ddot{a}_{53}$, the above equation means

$$\begin{aligned} 10 &= 1 + vp_{50} + (vp_{50})^2 + 10\,(vp_{50})^3 \\ 10(1-z^3) &= 1 + z + z^2, \qquad \text{where z} = vp_{50} \\ 10(1-z^3) &= \frac{1-z^3}{1-z} + 10z^3 \\ 10(1-z^3) &= \frac{1-z^3}{1-z} \\ 10 &= \frac{1}{1-z} \qquad \Longrightarrow \qquad z = \frac{9}{10} \qquad \Longrightarrow \qquad p_{50} = \frac{1.06 \cdot 9}{10} = \boxed{0.954.} \end{aligned}$$

140. (May 2007 #21)
You are given the following information about a new model for buildings with limiting age ω.

(a) The expected number of buildings surviving at age x will be $l_x = (\omega - x)^a$ for $x < \omega$.
(b) The new model predicts a 1/3 higher complete life expectency (over the previous deMoivre model with the same ω) for buildings aged 30.
(c) The complete life expectency for buildings aged 60 under the new model is 20 years.

Calculate the complete life expectency under the previous deMoivre model for buildings aged 70.

Solution:
If $l_x = (\omega - x)^a$ for $x < \omega$ then

$$\begin{aligned} \mathring{e}_x &= \int_{t=0}^{\omega - x} {}_tp_x \, dt = \int_{t=0}^{\omega - x} \frac{l_{x+t}}{l_x} \, dt = \frac{1}{(\omega - x)^a} \int_{t=0}^{\omega - x} (\omega - x - t)^a \, dt \\ &= \frac{\omega - x}{a+1} \end{aligned}$$

From the problem statement we have

$$\begin{aligned} \frac{4}{3} &= \frac{\left[\frac{\omega - 30}{a+1}\right]}{\left[\frac{\omega - 30}{2}\right]} \qquad \Longrightarrow \qquad \underline{a + 1 = 3/2} \\ \mathring{e}_{60} &= 20 = \frac{\omega - 60}{a+1} = \frac{2}{3} \cdot (\omega - 60) \qquad \Longrightarrow \qquad \underline{\omega = 90} \\ \mathring{e}_{70} &= (90 - 70)/2 = 10. \end{aligned}$$

141. (May 2007 #21)
The future lifetime random variable for a certain model of building follows a DeMoivre(ω) (what we in Math 4260 have called the "gen-DeMoivre($\omega, 1$)" distribution).

A new model of building is proposed, with a future lifetime random variable following a gen-DeMoivre(ω, a) distribution, with the same ω.

You are given the following additional information:

(a) The survival function for one of the new model of buildings is $S_0(x) = \left(\frac{\omega - x}{\omega}\right)^a$ for $x < \omega$.
(b) The new model predicts a 1/3 higher complete life expectency (over the previous old model with the same ω) for buildings aged 30.
(c) The complete life expectency for buildings aged 60 under the new model is 20 years.

Calculate the complete life expectency of the old model, for buildings aged 70.

Solution:
If $S_0(x) = [(\omega - x)/\omega]^a$ for $x < \omega$ then

$$\begin{aligned} \mathring{e}_x &= \int_{t=0}^{\omega-x} {}_tp_x \, dt = \int_{t=0}^{\omega-x} \frac{S_0(x+t)}{S_0(x)} \, dt = \frac{1}{(\omega-x)^a} \int_{t=0}^{\omega-x} (\omega - x - t)^a \, dt \\ &= \frac{\omega - x}{a+1} \end{aligned}$$

With $\mathring{e}_x^*$ denoting the complete future expection of life of the new model of buuldings, and from the problem statement, we have:

$$\begin{aligned} \frac{4}{3} &= \frac{\left[\frac{\omega-30}{a+1}\right]}{\left[\frac{\omega-30}{2}\right]} \quad \Longrightarrow \quad \underline{a+1 = 3/2} \\ \mathring{e}_{60}^* &= 20 = \frac{\omega - 60}{a+1} = \frac{2}{3} \cdot (\omega - 60) \quad \Longrightarrow \quad \underline{\omega = 90} \\ \mathring{e}_{70} &= (90 - 70)/2 = 10. \end{aligned}$$

<u>ANS: 10 years</u>

142. (May 2007 #22)
For a special whole-life insurance on (40), you are given:

(a) The death benefit is 1000 for the first 10 years and 2500 thereafter.
(b) Death benefits are payable at the moment of death.
(c) Z is the present value random variable.
(d) Mortality follows deMoivre's law with $\omega = 100$.
(e) $\delta = 0.10$.

Calculate $\Pr\{Z > 700\}$

Solution:
Denote by T the future lifetime random variable. It takes values in $(0, 60]$. We have:

$$Z = \begin{cases} 1000e^{-T/10} & \text{if } 0 < T \le 10 \\ 2500e^{-T/10} & \text{if } 10 < T \le 60. \end{cases}$$

The variable Z depends piecewise continuously on T, and is decreasing everywhere except at $T = 10$ where it jumps up. We have

$$\begin{aligned} Z &= 700 \quad \text{when } T = 3.567, \\ \text{and } Z &= 700 \quad \text{when } T = 12.73 \end{aligned}$$

So

$$\begin{aligned} \Pr\{Z > 700\} &= \Pr\{T \le 3.567 \quad \underline{\text{or}} \quad 10 < T \le 12.73\} \\ &= \frac{3.567}{60} + \frac{(12.73 - 10)}{60} = 0.1049 \end{aligned}$$

143. (C3 May 2007, #24)
For a three-year temporary life annuity due of 100 on (75), you are given:

- (i) $\displaystyle\int_0^x \mu_z \, dz = 0.01x^{1.2}$

- (ii) $i = 0.11$

Calculate the actuarial present value of this annuity.

Solution:
Use the "current payments" formula

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$$

which in this case comes to

$$\ddot{a}_{75:\overline{3}|} = 1 + vp_{75} + v^2{}_2p_{75}.$$

Since ${}_kp_{75} = \dfrac{S_0(75+k)}{S_0(75)}$ and $S_0(x) = \exp(-\int_{z=0}^{x} \mu_z\,dz) = \exp(-x^{1.2}/100)$, we have

$${}_kp_{75} = \exp\left(-\left[(75+k)^{1.2} + 75^{1.2}\right]/100\right),$$

and so

$$\begin{aligned} p_{75} = {}_1p_{75} &= = \exp\left(-\left[76^{1.2} - 75^{1.2}\right]/100\right) = 0.971907 \\ {}_2p_{75} &= \exp\left(-\left[77^{1.2} - 75^{1.2}\right]/100\right) = 0.944532 \\ 100\ddot{a}_{75:\overline{3}|} &= 100 \cdot \left(1 + \frac{0.971907}{1.11} + \frac{0.944532}{1.11^2}\right) = \boxed{264.22} \end{aligned}$$

144. (C3 May 2007, #24b)

You are given that $\int_0^x \mu_z\,dz = 0.01x^{1.2}$ and that $i = 0.08$, calculate the expected present value of a life annuity on (75) paying 100 at the end of each year for four years.

Solution:
The expected present value of a life annuity on (75) paying 100 at the end of each year for four years is denoted "$100a_{75:\overline{40}|}$." Use the "current payments" formula

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$$

which in this case comes to

$$\ddot{a}_{75:\overline{4}|} = 1 + vp_{75} + v^2{}_2p_{75} + v^3{}_3p_{75}.$$

Since ${}_kp_{75} = \dfrac{S_0(75+k)}{S_0(75)}$ and $S_0(x) = \exp\left(-\int_{z=0}^{x} \mu_z\,dz\right) = \exp\left(-x^{1.2}/100\right)$, we have

$${}_kp_{75} = \frac{\exp\left(-x^{1.2}/100\right)}{\exp\left(-75^{1.2}/100\right)} = \exp\left(-\left[(75+k)^{1.2} - 75^{1.2}\right]/100\right),$$

and so

$$\begin{aligned} p_{75} &= = \exp\left(-\left[76^{1.2} - 75^{1.2}\right]/100\right) = 0.971907, \\ {}_2p_{75} &= \exp\left(-\left[77^{1.2} - 75^{1.2}\right]/100\right) = 0.944532, \\ {}_3p_{75} &= \exp\left(-\left[78^{1.2} - 75^{1.2}\right]/100\right) = 0.917861, \\ {}_4p_{75} &= \exp\left(-\left[79^{1.2} - 75^{1.2}\right]/100\right) = 0.891876, \\ 100\ddot{a}_{75:\overline{3}|} &= 100 \cdot \left(\frac{0.971907}{1.08} + \frac{0.944532}{1.08^2} + \frac{0.917861}{1.08^2} + \frac{0.891876}{1.08^2}\right) = \boxed{309.40} \end{aligned}$$

145. (May 2007 #27)

For a special whole-life insurance, you are given:

(a) $b_t = e^{-t}$ for $t > 0$,

(b) μ is a constant,

(c) $\delta = 0.06$,

(d) $Z = e^{-T}v^T$, where T is the future lifetime random variable, and

(e) $\mathrm{E}[Z] = 0.03636$.

Calculate $\mathrm{Var}[Z]$.

Solution:

Note that $Z = e^{-T}v^T = e^{-T}e^{-\delta T} = e^{-(1+\delta)T} = e^{-1.06T}$. We have:

$$\begin{aligned}
0.03636 = \mathrm{E}[Z] &= \int_{t=0}^{\infty} Zf(t)\,dt \\
&= \int_{t=0}^{\infty} e^{-1.06t}\cdot \mu e^{-\mu t}\,dt \\
&= \frac{\mu}{1.06+\mu}\int_{t=0}^{\infty}(1.06+\mu)e^{-(1.06+\mu)t}\,dt \\
&= \frac{\mu}{1.06+\mu} \\
&\Longrightarrow \underline{\mu = 0.04}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[Z^2\right] &= \int_{t=0}^{\infty} Z^2 f(t)\,dt \\
&= 0.04\int_{t=0}^{\infty} e^{-2.16t}\,dt = \frac{0.04}{2.16} = 0.01852, \\
\mathrm{Var}[Z] &= 0.01852 - (0.03636)^2 = 0.017198
\end{aligned}$$

146. (C3 May 2007, 29)

For a special fully discrete, 30-year deferred, annual life annuity-due of 200 on (30), you are given:

(a) The single benefit premium is refunded without interest at the end of the year of death if death occurs during the deferral period.

(b) Mortality follows the Illustrative Life Table.

(c) $i = 0.06$

Calculate the single benefit premium for this annuity.

Solution:

The single benefit premium is the expected value of the benefits.

$$\begin{aligned}
P &= 200\,{}_{30|}\ddot{a}_{30} + PA^{1}_{30:\overline{30|}} \\
&= 200\left({}_{30}E_{30}\right)\ddot{a}_{60} + P\left(A_{30} - {}_{30}E_{30}A_{60}\right) \\
&= 200\left({}_{30}E_{30}\right)\left(\frac{1-A_{60}}{d}\right) + P\left(A_{30} - {}_{30}E_{30}A_{60}\right)
\end{aligned}$$

Now ${}_{30}E_{30} = \dfrac{\ell_{60}}{\ell_{30}\cdot 1.06^{30}} = \dfrac{8188073}{9501380\cdot 1.06^{30}} = 0.15004415$, and $A_{60} = 0.36913104$, so

$$\begin{aligned}
P &= 200\cdot 0.15004415\left(\frac{1-0.36913104}{0.05660377}\right) + P\left(0.10248356 - 0.15004415\cdot 0.36913104\right) \\
P &= 334.45898 + P\cdot 0.04709761 \quad \Longrightarrow \boxed{P = 350.99}
\end{aligned}$$

147. (Nov 2008 #12)

You are given the force of mortality, $\mu(x) = \dfrac{1}{3(100-x)}$. Calculate the probability that a life aged 70 will die between ages 75 and 80

Solution:

$$\begin{aligned}
-\int_{z=x}^{x+t} \mu(z)\,dz &= \frac{-1}{3}\int_{z=x}^{x+t} \frac{1}{100-z}\,dz \\
&= \frac{-1}{3}\int_{u=100-x-t}^{100-x} \frac{1}{u}\,dz = \log\left(\frac{100-x-t}{100-x}\right)^{1/3} \\
{}_tp_x &= \exp\left(-\int_x^{x+t} \mu(z)\,dz\right) = \left(\frac{100-x-t}{100-x}\right)^{1/3} \\
{}_{5|5}q_{70} &= {}_{75}p_{70} - {}_{80}p_{70} = \underline{0.067455}
\end{aligned}$$

148. (Nov 2008 #13)
You are given the curtate expectation of life at age x and x+1:

$$e_x = 5.000 \qquad e_{x+1} = 4.530.$$

Calculate q_x.

Solution:

$$\begin{aligned}
e_x &= p_x(1+e_{x+1}) \\
q_x &= 1 - p_x - 1 = \frac{e_x}{1+e_{x+1}} = 1 - \frac{5.000}{4.530} = \underline{0.095841}
\end{aligned}$$

149. (Nov 2008 #14)

You are given this table:

Age(x)	l_x	d_x
0		100
1		
2		
3		
4	125	75
5		50

You are also given:

$$\begin{aligned}
{}_{3|2}q_1 &= 0.329, \\
{}_3p_0 &= 0.427
\end{aligned}$$

Caclulate the number of lives in the cohort at age 3.

Solution:

$$\begin{aligned}
l_5 &= l_4 - d_4 = 125 - 75 = 50 \\
l_6 &= l_5 - d_5 = 50 - 50 = 0 \quad \underline{\text{so } \omega = 6},
\end{aligned}$$

$$\begin{aligned}
0.329 &= {}_{3|2}q_1 = \frac{\ell_4 - \ell_6}{\ell_1} = \frac{\ell_4}{\ell_1} = \frac{\ell_4}{\ell_0 - d_0} = \frac{125}{l_0 - 100} \qquad \Longrightarrow \underline{\ell_0 = 480} \\
0.427 &= {}_3p_0 = \frac{l_3}{l_0} \quad \Longrightarrow \underline{l_3 = 205} \qquad \boxed{\text{205 alive at age 3}}
\end{aligned}$$

150. (MLC May 2012, #1)
For a 2-year select and ultimate mortality model, you are given:

(a) $q_{[x]+1} = 0.95\, q_{x+1}$
(b) $\ell_{76} = 98,153$
(c) $\ell_{77} = 96,124$

Calculate $\ell_{[75]+1}$.

Solution:
From given items b. and c., we have $q_{76} = 1 - \ell_{77}/\ell_{76} = 0.020672$, so that (from a.) $q_{[75]+1} = 0.95 q_{76} = 0.01963822$.
Now,

$$\begin{aligned} q_{[75]+1} &= 1 - \frac{\ell_{[75]+2}}{\ell_{[75]+1}} = 1 - \frac{\ell_{77}}{\ell_{[75]+1}} \\ 0.01963822 &= = 1 - \frac{96,124}{l_{[75]+1}} \qquad \Longrightarrow \boxed{\ell_{[75]+1} = 98,049.5} \end{aligned}$$

151. (C3 May 2012, #3)
For a fully discrete 3-year term insurance of 10,000 on (40), you are given:

(a) μ_{40+t}, $t \geq 0$, is a force of mortality consistent with the Illustrative Life Table
(b) $\mu_{40+t} + 0.02$, $t = 0, 1,$ and 2,is the force of mortality for this insured.
(c) $i = 0.06$

Calculate the annual benefit premium for this insurance.

Solution:

$\ldots p_x = e^{-0.02} p_x^{ILT}$ so

$$p_{40}^{ILT} = 0.997219, \quad p_{41}^{ILT} = 0.997018, \quad p_{41}^{ILT} = 0.996798,$$
$$p_{40} = 0.977473, \quad p_{41} = 0.977276, \quad p_{42} = 0.977060$$

$${}_2p_{40} = 0.977473 \times 0.977276 = 0.955260 \qquad {}_3p_{40} = 0.95526 \times 0.97706 = 0.933347$$

$$\begin{aligned} \ddot{a}_{40:\overline{3}|} &= 1 + vp_{40} + v^2\, {}_2p_{40} = 1 + \frac{0.977473}{1.06} + \frac{0.955260}{1.06^2} = 2.772322 \\ A_{40:\overline{3}|} &= 1 - d\ddot{a}_{40:\overline{3}|} = 1 + \frac{0.06}{1.06} \cdot 2.772322 = 0.843076 \\ A^{\,1}_{40:\overline{3}|} &= A_{40:\overline{3}|} - v^3\, {}_3p_{40} = 0.843076 - \frac{0.933347}{1.06^3} = 0.059420 \\ \pi &= 10,000 \frac{A^{\,1}_{40:\overline{3}|}}{\ddot{a}_{40:\overline{3}|}} \quad \text{from the Eqiv. Princ.} \\ & \boxed{\pi = 214.33} \end{aligned}$$

152. (C3, May 2012, #24)
For a three-year term insurance of 10,000 on (65), payable at the end of the year of death, you are given:

(a)

x	q_x
65	0.00355
66	0.00397
67	0.0044

(b) Forward interest rates at the date of issue of the contract, expressed as annual rates, are as follows:

Start time	End Time	Annual Forward Rate
0	3	0.050
1	3	0.070
2	3	0.091

Calculate the expected present value of this insurance.

Solution:
Let A be the expected value of this insurance. We have:

$$\begin{aligned}
\frac{A}{10,000} &= \frac{q_{65}}{1+i_0(0,1)} + \frac{{}_{1|}q_{65}}{\big(1+i_0(0,2)\big)^2} + \frac{{}_{2|}q_{65}}{\big(1+i_0(0,3)\big)^2} \\
&= \frac{0.00355}{1+i_0(0,1)} + \frac{(1-0.00355)\cdot 0.00397}{\big(1+i_0(0,2)\big)^2} + \frac{(1-0.00355)\cdot(1-0.00397)\cdot 0.0044}{\big(1+i_0(0,3)\big)^2} \\
&= \frac{0.00355}{1+i_0(0,1)} + \frac{0.0039559}{\big(1+i_0(0,2)\big)^2} + \frac{0.0043670\cdot 0.0044}{\big(1+i_0(0,3)\big)^2}
\end{aligned}$$

- Given $i_0(0,3) = 0.050$, so $\underline{\big(1+i_0(0,3)\big)^3 = 1.157625}$
- Find $\big(1+i_0(0,2)\big)^2$:

$$\begin{aligned}
\big(1+i_0(0,2)\big)^2\big(1+i_0(2,3)\big) &= \big(1+i_0(0,3)\big)^3 \\
\big(1+i_0(0,2)\big)^2 1.091 &= 1.050^3 \qquad \Longrightarrow \qquad \underline{\big(1+i_0(0,2)\big)^2 = 1.06107}
\end{aligned}$$

- Find $1+i_0(0,1)$:

$$\begin{aligned}
\big(1+i_0(0,1)\big)\big(1+i_0(1,3)\big)^2 &= \big(1+i_0(0,3)\big)^3 \\
\big(1+i_0(0,1)\big)\cdot 0.070^2 &= 1.050^3 \qquad \Longrightarrow \qquad \underline{\big(1+i_0(0,1)\big) = 1.01111}
\end{aligned}$$

$$A = 10,000\cdot\left(\frac{0.00355}{1.01111} + \frac{0.0039559}{1.06107} + \frac{0.0043670}{1.157625}\right) = \boxed{110.12}$$

153. (Nov 2012, #1)
For two lives, (80) and (90), with independent future lifetimes, you are given:

k	p_{80+k}	p_{90+k}
0	0.9	0.6
1	0.8	0.5
2	0.7	0.4

Calculate the probability that the last survivor will die in the third year.

Solution:

$$\begin{aligned}
{}_{2|}q_{\overline{xy}} &= {}_{2|}q_x + {}_{2|}q_y - {}_{2|}q_{xy} \\
&= {}_2p_x\, q_{x+2} + {}_2p_y\, q_{y+2} - {}_2p_{xy}\, q_{x+2:y+2} \\
&= {}_2p_x\,(1 - p_{x+2}) + {}_2p_x\,(1 - p_{x+2}) - {}_2p_{xy}\,(1 - p_{x+2:y+2}) \\
&= {}_2p_x\,(1 - p_{x+2}) + {}_2p_y\,(1 - p_{y+2}) - {}_2p_x\,{}_2p_y\,(1 - (p_{x+2})(p_{y+2}))
\end{aligned}$$

$$\begin{aligned}
{}_2p_x &= p_x p_{x+1} = 0.72 \\
{}_2p_y &= p_y p_{y+1} = 0.30
\end{aligned}$$

So

$${}_{2|}q_{\overline{xy}} = (0.72)(1 - 0.7) + (0.3)(1 - 0.4) - (0.72)(0.3)\,(1 - (0.7)(0.4)) = 0.24048$$

154. (MLC Nov 2012, #2)
You are given:

(a) An excerpt from a select and ultimate life table with a select period of 3 years:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
60	80,000	79,000	77,000	74,000	63
61	78,000	76,000	73,000	70,000	64
62	75,000	72,000	69,000	67,000	65
63	71,000	68,000	66,000	65,000	66

(b) Deaths follow a constant force of mortality over each year of age.

Calculate 1000 ${}_{2|3}q_{[60]+0.75}$.

Solution:
I recall that

$${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x = \frac{l_{x+u} - l_{x+u+t}}{l_x}$$

and so

$${}_{u|t}q_{[x]+s} = \frac{l_{[x]+u+s} - l_{[x]+u+t+s}}{l_{[x]+s}} \quad \text{for } s \in [0,1],$$

$$\boxed{{}_{2|3}q_{[60]+0.75} = \frac{l_{[60]+2.75} - l_{[60]+5.75}}{l_{[60]+0.75}}}.$$

So we need these three l-values.

We also have to account for the fractional years. Under the *"constant force"* assumption, "${}_sp_x = (p_x)^s$ for $s \in [0,1]$", we quickly obtain

$$\boxed{l_s = (l_x)^{1-s} \cdot (l_{x+1})^s, \quad \text{for } s \in [0,1].}$$

Now we're ready to compute:

$$\begin{aligned}
l_{[60]+0.75} &= (l_{[60]})^{0.25} \cdot (l_{[60]+1})^{0.75} = (80,000)^{0.25} \cdot (79,000)^{0.75} = 79,248.82, \\
l_{[60]+2.75} &= (l_{[60]+2})^{0.25} \cdot (l_{[60]+3})^{0.75} = (77,000)^{0.25} \cdot (74,000)^{0.75} = 74,738.86 \\
l_{[60]+5.75} &= (l_{[60]+5})^{0.25} \cdot (l_{[60]+6})^{0.75} = (67,000)^{0.25} \cdot (65,000)^{0.75} = 65494.33,
\end{aligned}$$

so that

$$1000 \cdot \frac{l_{[60]+2.75} - l_{[60]+5.75}}{l_{[60]+0.75}} = \boxed{116.65 = 1000\ {}_{2|3}q_{[60]+0.75}}$$

155. (C3 Nov 2012 #03)
You are given:

(a) $S_0(t) = \left(1 - \frac{t}{\omega}\right)^{1/4}$ for $0 \le t \le \omega$.

(b) $\mu_{65} = \frac{1}{180}$

Calculate the curtate expectation of life at age 106.

Solution:
We need to find ω. Using the relation $\mu_x = -S'(x+t)/S(x)$,
we find that $\mu_x = \dfrac{1}{4(\omega - x)}$, so that $\mu_{65} = 1/180$ implies $4(\omega - 65) = 180$ or $\underline{\omega = 110.}$

Now with $\omega = 110$ we can proceed with the backward recursive relationship, $e_x = p_x(1 + e_{x+1})$, where

$$p_x = \left(\frac{110 - x - 1}{110 - x}\right)^{1/4}.$$

$$\begin{aligned}
& & e_{110} = e_{109} &= 0 \\
p_{108} = \left(\frac{110-109}{108}\right)^{1/4} &= 0.84090 & e_{108} &= 0.84090 \\
p_{107} = \left(\frac{110-108}{107}\right)^{1/4} &= 0.90360 & e_{107} = 0.90360 \cdot (1 + 0.84090) &= 1.66344 \\
p_{106} = \left(\frac{110-107}{106}\right)^{1/4} &= 0.93061 & e_{106} = 0.9036 \cdot (1 + 01.66344) &= \boxed{2.47861}
\end{aligned}$$

156. (C3 Nov 2012 #03-b)
You are given:

(a) $S_0(t) = \left(1 - \frac{t}{\omega}\right)^{1/3}$ for $0 \le t \le \omega$.

(b) $\mu_{35} = \frac{1}{210}$

Find ω.

Solution:
We need to find ω. Using the relation $\mu_x = -S'(x)/S(x)$,
we find that $\mu_x = \dfrac{1}{3(\omega - x)}$, so that $\mu_{65} = 1/210$ implies $3(\omega - 35) = 210$ or $\underline{\omega = 105.}$

157. (Nov 2012 #04)
A certain whole life policy on (40) pays \$50,000 at the end of year of death if (40) dies in the first 20 years of the policy, and it pays \$100,000 at the end of the year of death otherwise. The level annual premiums are are payable for 20 years.

Assuming that mortality follows the Illustrative Life Table and that $i = 0.06$, Calculate the net premium policy value at the end of year 10 for this insurance.

Solution:
First we need to premiums. From the equivalence principle, we have

$$0 = 50,000\,(A_{40} + {}_{20}E_{40}A_{60}) - P\ddot{a}_{40:\overline{20}|} \quad \Longrightarrow \quad P = \frac{50,000\,(A_{40} + {}_{20}E_{40}A_{60})}{\ddot{a}_{40:\overline{20}|}}$$

We compute the following values:

$$A_{40} = 0.1613242, \qquad A_{60} = 0.369131, \qquad {}_{20}E_{40} = 0.2741367, \qquad \ddot{a}_{40}20 = 11.761256$$

so that $P = \dfrac{50,000\,(0.1613242 + 0.2741367 \cdot 0.369131)}{11.761256} = 1,116.02.$
Now the policy value at $t = 10$ is

$$\begin{aligned}
{}_{10}V &= 50,000 \cdot (A_{50} + {}_{10}E_{50} A_{60}) - P\ddot{a}_{50:\overline{10}|} \\
&\quad A_{50} = 0.2490\,4748\,5 \\
&\quad {}_{10}E_{50} = 0.5108\,0642\,1 \\
&\quad \ddot{a}_{50:\overline{10}|} = 7.5737106 \\
{}_{10}V &= 50,000 \cdot (0.2490475 + 0.51080642 \cdot 0.369131) - 1116.02 \cdot 7.5737106 = \boxed{13,427.69}
\end{aligned}$$

158. (Nov 2012 #05)
A modified annual two-year endowment insurance on (x) for 2,000 pays an *additional* death benefit equal to the net premium policy value at the end of the first year.

For year 2, calculate the net premium policy value just *before* the maturity benefit is paid.

You are given:

(a) $i = 0.10$

(b) $q_x = 0.150$ and $q_{x+1} = 0.165$.

Calculate the level annual benefit premium.

Solution:
This is a situation where comment #5 after Def'n 2 is applicable. It says,

> "If we are calculating a policy value at an integer duration, end of a year, there may be premiums and/or expenses and/or benefits payable at precisely that time and we need to be careful about which cash flows are included in our future loss random variable. It is the usual practice to regard a premium and any premium-related expenses due at that time as future payments and any insurance benefits (i.e. death or maturity claims) and related expenses as past payments."

We have ${}_0V = 0$ and ${}_2V = 2000$. The first of these is because we are using "benefit" premium which means Equivalence Principle. The second of these is because just before the maturity benefit is paid, we have to have that amount on hand in reserve.

Denote by π the level annual premium. If (x) fails in the first year, then the payout is $2000 + {}_1V$, and since ${}_0V = 0$ we have

$$\begin{aligned}
\left({}_0V + \pi\right)(i+1) = 1.1\pi &= q_x(2000 + {}_1V) + p_x\,{}_1V \\
1.1\pi &= {}_1V + 300 \\
{}_1V &= 1.1\pi - 300
\end{aligned}$$

If (x) fails in the second year, then the payout is $2000 + {}_2V$. Also, ${}_2V = 2000$. So,

$$\begin{aligned}
\left({}_1V + \pi\right)(i+1) &= q_{x+1}(2000 + {}_2V) + p_{x+1} \cdot ({}_2V) \\
\left({}_1V + \pi\right) &= (0.165 \times 4000 + 0.835 \times 2000)/1.1 = 2118.18 \\
(1.1\pi - 300 + \pi) &= 2118.18 \quad \Longrightarrow \underline{\pi = 1151.52}
\end{aligned}$$

159. (Nov 2012 #06)
For a special fully continuous 10-year increasing term insurance, you are given:

(i) The death benefit is payable at the moment of death and increases linearly from 10,000 to 110,000.

(ii) $\mu = 0.01$

(iii) $\delta = 0.05$

(iv) The annual premium rate is 450.

(v) Premium-related expenses equal 2% of premium, incurred continuously.

(vi) Claims-related expenses equal 200 at the moment of death.

(vii) ${}_tV$ denotes the gross premium reserve at time t for this insurance.

(viii) You estimate ${}_{9.6}V$ using Euler's method with step size 0.2 and the derivative of ${}_tV$ at time 9.6.

(ix) Your estimate of ${}_{9.8}V$ is 126.68.

Calculate the estimate of ${}_{9.6}V$.

Solution:

Thiele's differential equation and the backward Euler step:

$$\begin{aligned}
\frac{d}{dt}\,{}_tV &= (\delta_t + \mu_{x+t})\,{}_tV - (S_t + E_t)\mu_{x+t} + P_t - e_t \\
{}_tV &\approx \frac{{}_{t+h}V + h\big((S_t + E_t)\mu_{x+t} - P_t + e_t\big)}{1 + h\big(\delta_t + \mu_{x+t}\big)}
\end{aligned}$$

In this problem the Euler step becomes, at $t = 9.6$ and ${}_{9.8}V = 126.68$,

$$\begin{aligned}
{}_tV &\approx \frac{\overbrace{126.68}^{{}_{9.8}V} + 0.2\left[\left(\overbrace{10,000\cdot(9.6+1)}^{S_{9.6}} + 200\right)\cdot 0.01 - 450 + 0.02\cdot 450\right]}{1 + 0.2\,(0.05 + 0.01)} \\
&= \boxed{247.905}
\end{aligned}$$

160. (Nov 2012, #10)
For a special 3-year term life insurance policy on (x) and (y) with dependent future lifetimes, you are given:

(a) A death benefit of 100,000 is paid at the end of the year of death if both (x) and (y) die within the same year. No death benefits are payable otherwise.

(b) $p_{x+k} = 0.84366$ for $k = 0, 1, 2$.

(c) $p_{y+k} = 0.86936$ for $k = 0, 1, 2$.

(d) $p_{x+k:y+k} = 0.77105$ for $k = 0, 1, 2$.

(e)

Maturity (in years)	Annual Effective Spot Rate
1	3%
2	8%
3	10%

Calculate the expected present value of the death benefit.

Solution:
In order for both (x) and (y) to fail in the same year, say year k, both (x) and (y) must survive $k-1$ complete years *and then* the lives aged (x+k-1) and (y+k-1) must both fail within one year. Let's analyze this sequence of events:

- The probability that both (x) and (y) must survive $k-1$ complete years is the probability that the joint-life status (xy) survives $k-1$ years, which is ${}_{(k-1)}p_{xy}$. In terms of the given information,

$$ {}_{(k-1)}p_{xy} = (0.77105)^{k-1}. $$

- The probability that the lives aged (x+k-1) and (y+k-1) both fail within one year is the probability that the last-survivor status $(\overline{\mathrm{x+k-1:y+k-1}})$ fails within one years, which is $q_{\overline{x+k-1:y+k-1}}$. And, in terms of the given information,

$$
\begin{aligned}
q_{\overline{x+k-1:y+k-1}} &= q_{x+k-1} + q_{y+k-1} - q_{x+k-1:y+k-1} \\
&= 1 - \Big(p_{x+k-1} + p_{y+k-1} - p_{x+k-1:y+k-1} \Big) \\
&= 1 - (0.84366 + 0.86936 - 0.77105) \\
&= 0.05803.
\end{aligned}
$$

So the probability that they both fail in year k is

$$ \Pr\{\text{both fail in year } k\} = (0.77105)^{k-1} \cdot 0.05803. $$

So, the EPV of the death benefit is

$$ 100,000 \cdot \left(\frac{0.05803}{1.03} + \frac{0.77105 \cdot 0.05803}{1.08^2} + \frac{0.77105^2 \cdot 0.05803}{1.1^3} \right) = \boxed{12062.09} $$

161. (Nov 2012 #11)
For a whole life insurance of 1000 on (70), you are given:

(a) Death benefits are payable at the end of the year of death.
(b) Mortality follows the Illustrative Life Table.
(c)

Maturity (in years)	Annual effective spot rate
1	1.6%
2	2.6%

(d) For the year starting at time $k-1$ and ending at time k, $k = 3, 4, 5, \ldots$, the one-year forward rate is 6%.

Calculate the expected present value of the death benefits.

Solution:
Let A denote the EPV of the death benefits, and let $v(k)$ denote the present-value discount factor for a payment k years from now. From the problem data, we obtain

$$
\begin{aligned}
v(1) &= 1.016^{-1} \\
v(2) &= 1.026^{-2} \\
v(j) &= \frac{1}{1.026^2 \cdot 1.06^{j-2}} = v(2)v^{j-2} \quad \text{for } j \geq 3, \text{ where } v = \frac{1}{1.06}
\end{aligned}
\tag{7}
$$

So,

$$
\begin{aligned}
\frac{A}{1000} &= \sum_{k=0}^{\infty} v(k+1)\, {}_{k|}q_{70} \\
&= v(1)q_{70} + v(2)\, {}_{1|}q_{70} + \sum_{k=2}^{\infty} v(k+1)\, {}_{k|}q_{70} \\
&\qquad \text{(change of indices: } j = k-2) \\
&= v(1)q_{70} + v(2)\, {}_{1|}q_{70} + {}_2p_{70}\, v(2) \sum_{j=0}^{\infty} v^{j+1}\, {}_{j|}q_{72} \\
&= v(1)q_{70} + v(2)\, {}_{1|}q_{70} + {}_2p_{70}\, v(2) A_{70}
\end{aligned}
$$

where we've used (7) and the fact that $_{(j+2)|}q_{70} = {}_2p_{70}\ {}_{j|}q_{72}$, and where A_{72} is as in the Illustrative Life Table.

From the Illustrative Life Table we find:

$$q_{70} = 0.03318, \quad {}_{1|}q_{70} = 0.03506, \quad {}_2q_{70} = 0.93176, \quad \text{and } A_{72} = 0.54560,$$

and so

$$\boxed{A = 548.889}$$

162. (C3 Nov 2012 #12)
A party of scientists arrives at a remote island. Unknown to them, a hungry tyrannosaur lives on the island. You model the future lifetimes of the scientists as a three-state model, where:

State 0: no scientists have been eaten.

State 1: exactly one scientist has been eaten.

State 2: at least two scientists have been eaten.

You are given:

(i) Until a scientist is eaten, they suspect nothing, so

$$\mu_t^{01} = 0.01 + 0.02 \cdot 2^t, \; t > 0.$$

(ii) Until a scientist is eaten, they suspect nothing, so the tyrannosaur may come across two together and eat both, with

$$\mu_t^{02} = 0.5\mu_t^{01}, \; t > 0.$$

(iii) After the first death, scientists become much more careful, so

$$\mu_t^{12} = 0.01\mu_t^{01}, \; t > 0.$$

Calculate the probability that no scientists are eaten in the first year.

Solution:
The probability that no scientist has been eaten by time 1 is the probability that the system remains in state 0 up to time 1, which is p_1^{00} which is given by

$$p_1^{00} = \exp\left(-\int_0^1 \mu_t^{01} + \mu_t^{02}\, dt\right) = 0.943385$$

163. (C3 Nov 2012, #13)
You are given:

(i) The following excerpt from a triple decrement table:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
50	100,000	490	8,045	1,100
51	90,365		8,200	
52	80,000			

(ii) All decrements are uniformly distributed over each year of age in the triple decrement table.

(iii) $q_x^{*(3)}$ is the same for all ages.

Calculate $q_{51}^{*(1)}$

Solution:
What can we tell immediately from the table? Since $\ell_{x+1}^{(\tau)} = \ell_x^{(\tau)} - \left(d_x^{(1)} + d_x^{(2)} + d_x^{(3)}\right)$, we know that $d_{51}^{(1)} + d_{51}^{(3)} = \ell_{51}^{(\tau)} - \ell_{52}^{(\tau)} - d_{51}^{(2)}$ so that

$$d_{51}^{(1)} + d_{51}^{(3)} = 2165. \tag{8}$$

Now, because all decrements are uniformly distributed over each year of age, the key relationship for this problem is equation (8.30) AMLCR(ed.2) which is

$$p_x^{*(j)} = \left(p_x^{00}\right)^{\left(p_x^{0j}/p_x^{0\bullet}\right)}, \tag{9}$$

Since $p_x^{00} = \ell_{x+1}^{(\tau)}/\ell_x^{(\tau)}$, and $p_x^{0j}/p_x^{0\bullet} = d_x^{(j)}/\left(d_x^{(1)} + d_x^{(2)} + d_x^{(3)}\right)$, and because $q_x^{*(j)} = 1 - p_x^{*(j)}$, this becomes

$$q_x^{*(j)} = 1 - \left(\ell_{x+1}^{(\tau)}/\ell_x^{(\tau)}\right)^{\left(d_x^{(j)}/\left(d_x^{(1)}+d_x^{(2)}+d_x^{(3)}\right)\right)} \tag{10}$$

Since $q_x^{*(3)}$ is the same for all ages, let's call it $q^{*(3)}$, from equation (10) we have

$$q^{*(3)} = 1 - \left(\ell_{51}^{(\tau)}/\ell_{50}^{(\tau)}\right)^{\left(d_{50}^{(3)}/\left(d_{50}^{(1)}+d_{50}^{(2)}+d_{50}^{(3)}\right)\right)} = 0.0115\,000 \tag{11}$$

From this, the given table, and (8) we get

$$0.0115 \quad = \quad 1 - (80,000/90,365)^{\left(d_{51}^{(3)}/(8200+2165)\right)} \quad \to \quad \underline{d_{51}^{(3)} = 984}$$

and so, from (8), $\underline{d_{51}^{(1)} = 1181.}$
Finally, using the now-known values of $d_{51}^{(j)}$ in (10), we obtain

$$q_{51}^{*(1)} = 1 - \left(\ell_{52}^{(\tau)}/\ell_{51}^{(\tau)}\right)^{\left(d_{51}^{(1)}/\left(d_x^{(1)}+d_x^{(2)}+d_x^{(3)}\right)\right)} = 1 - (80000/90365)^{1181/(8200+2165)} = \boxed{0.0137856}.$$

164. (C3 Nov 2012 #16)
You are evaluating the financial strength of companies based on the following multiple state model:

Three-state model diagram with labels
"State0 Solvent", "State1 Bankrupt", and "State3 Liquidated,"
and transition $0 \to 1$ $1 \to 0$, and $1 \to 2$ possible

For each company, you assume the following constant transition intensities:

(i) $\mu^{01} = 0.02$
(ii) $\mu^{10} = 0.06$
(iii) $\mu^{12} = 0.1$

Using Kolmogorov's forward equations with step $h = 1/2$, calculate the probability that a company currently Bankrupt will be Solvent at the end of one year.

Solution:

We want ${}_1p^{01}$ based on a Euler stepsize of 1/2.

$$
\begin{aligned}
\frac{d}{dt}{}_t\mathbf{P} &= {}_t\mathbf{P}\ \mathbf{U} = \begin{bmatrix} {}_tp^{00} & {}_tp^{01} & {}_tp^{02} \\ {}_tp^{10} & {}_tp^{11} & {}_tp^{12} \\ {}_tp^{20} & {}_tp^{21} & {}_tp^{22} \end{bmatrix} \begin{bmatrix} -0.02 & 0.02 & 0 \\ 0.06 & -0.16 & 0.1 \\ 0 & 0 & 0 \end{bmatrix} \\
I + h\mathbf{U} &= \begin{bmatrix} 0.99 & 0.01 & 0 \\ 0.03 & 0.92 & 0.05 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}_0\mathbf{P} &= I \\
{}_h\mathbf{P} &= I + h\mathbf{U} \\
{}_{2h}\mathbf{P} &= (I + h\mathbf{U})^2 \\
[{}_{2h}\mathbf{P}\ ]_{12} &= \boxed{{}_{2h}p^{10} = 0.05730}
\end{aligned}
$$

165. (Nov 2012 #17)

Solution:

166. (Nov 2012 #18)

For a special whole life insurance on (40), you are given:

(a) The death benefit is payable at the end of the year of death, and is 1000 during the first 11 years and 5000 thereafter.

(b) Expenses, payable at the beginning of the year, are 100 in year 1 and 10 in years 2 and later.

(c) π is the level annual premium, determined using the equivalence principle.

(d) $G = 1.02\pi$ is the level annual gross premium.

(e) Mortality follows the Illustrative Life Table.

(f) $i = 0.06$

Calculate the gross premium policy value at the end of year 1 for this insurance.

Solution:

The Equivalence Principle implies

$$
\begin{aligned}
0 &= 1000 \cdot A_{40} + 4000{}_{11}E_{40}A_{51} + 90 + 10\ddot{a}_{40} - \pi\ddot{a}_{40} \\
&= 683.9648 + 90 + 10 \cdot 14.816605 - 14.816605 \cdot \pi \quad \Longrightarrow \quad \underline{\pi = 62.2363.}
\end{aligned}
$$

Now the required policy value is

$$
\begin{aligned}
{}_1V &= 1000 \cdot A_{41} + 4000{}_{10}E_{41}A_{51} + 10\ddot{a}_{41} - 1.02 \cdot 62.2363\ddot{a}_{41} \\
&= 168.6916 + 555.5441 + 146.8645 - 932.3108 = \boxed{-61.21}
\end{aligned}
$$

167. (Nov 2012 #19)

For whole life annuities-due of 15 per month on each of 200 lives age 62 with independent future lifetimes, you are given:

(a) $i = 0.06$

(b) $A_{62}^{(12)} = 0.4075$ and $^{2}A_{6}^{(12)}2 = 0.2105$

(c) π is the single premium to be paid by each of the 200 lives.

(d) S is the present value random variable at time 0 of total payments made to the 200 lives.

Using the normal approximation, calculate π such that $\Pr\{200\pi > S\} = 0.90$

Solution:

168. (Nov 2012, #21)
For a fully continuous whole life insurance issued on (x) and (y), you are given:

(a) The death benefit of 100 is payable at the second death.

(b) Premiums are payable until the first death.

(c) The future lifetimes of (x) and (y) are dependent.

(d) ${}_tp_{xy} = e^{-0.01t}/4 + 3e^{-0.03t}/4$

(e) ${}_tp_x = e^{-0.01t}$

(f) ${}_tp_y = e^{-0.02t}$

(g) $\delta = 0.05$

Calculate the annual benefit premium rate for this insurance.

Solution:

$$
\begin{aligned}
\bar{a}_{xy} &= \int_0^\infty e^{-\delta t}\, {}_tp_{xy}\, dt = 13.54167, \\
\overline{A}_{xy} &= 0.3229167 \\
\overline{A}_x &= 1/6 \\
\overline{A}_y &= 2/7 \\
\overline{A}_{\overline{xy}} &= 0.1294643 \\
\pi &= 0.956044
\end{aligned}
$$

169. (C3 Nov 2012 #24)
An insurance company is designing a special 2-year term insurance. Transitions are modeled as a four-state homogeneous Markov model with states:

(H) Healthy

(Z) infected with virus "Zebra"

(L) infected with virus "Lion"

(D) Death

The annual transition probability matrix is given by:

$$
\begin{array}{c}
\text{H} \\ \text{Z} \\ \text{L} \\ \text{D}
\end{array}
\begin{bmatrix}
0.90 & 0.05 & 0.04 & 0.01 \\
0.10 & 0.20 & 0 & 0.70 \\
0.20 & 0 & 0.20 & 0.60 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

You are given:

(i) Only one transitions can occur during any given year.

(ii) 250 is payable at the end of the year in which you become infected with either virus.

(iii) For lives infected with either virus, 1000 is payable at the end of the year of death.

(iv) The policy is issued only on Healthy lives.

(v) $i = 0.05$

Calculate the actuarial present value of the benefits at policy issue.

Solution:

Possible transitions	probability	discounted benefits	APV
H $\to$Z	0.05	$250v$	11.904762
H $\to$L	0.04	$250v$	9.523810
H $\to$Z $\to$ D	$0.05(0.7) = 0.035$	$1000v^2$	31.746032
H $\to$L$\to$ D	$0.04(0.6) = 0.024$	$1000v^2$	21.768707
H $\to$H $\to$ Z	$0.90(0.05) = 0.045$	$250v^2$	10.204082
H $\to$H $\to$ L	$0.90(0.04) = 0.036$	$250v^2$	8.163265

Sum of these gives total APV =93.31066

170. (Nov 2012 #25)

For a fully discrete 10-year term life insurance policy on (x), you are given:

(i) Death benefits are 100,000 plus the return of all gross premiums paid without interest.

(ii) Expenses are 50% of the first year's gross premium, 5% of renewal gross premiums and 200 per policy expenses each year.

(iii) Expenses are payable at the beginning of the year.

(iv) $A^{1}_{x:\overline{10}|} = 0.17094$

(v) $(IA)^{1}_{x:\overline{10}|} = 0.96728$

(vi) $\ddot{a}_{x:\overline{10}|} = 6.8865$

Calculate the gross premium using the equivalence principle.

Solution:

171. (C3 May 2013 # 02)
P&C Insurance Company is pricing a special fully discrete 3-year term insurance policy on (70). The policy will pay a benefit if and only if the insured dies as a result of an automobile accident. You are given:

(i)

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	Benefit
70	1000	80	10	40	5,000
71	870	94	15	60	7,500
72	701	108	18	82	10,000

where $d_x^{(1)}$ represents deaths from cancer, $d_x^{(2)}$ represents deaths from automobile accidents, and $d_x^{(3)}$ represents deaths from all other causes.

(ii) $i = 0.06$

(iii) Level premiums are determined using the equivalence principle.

Calculate the annual premium.

Solution:
Let A denote the EPV of the benefit, let P represent the annual premiums, and let a denote the EPV of a 3-year annuity on (70), so that $P = A/a$. We have

$$\begin{aligned} A &= \frac{5000}{1.06}\cdot\frac{10}{1000}+\frac{7500}{1.06^2}\cdot\frac{15}{1000}+\frac{10000}{1.06^3}\cdot\frac{18}{1000} = 298.4259 \\ a &= 1+\frac{870}{1000\cdot 1.06}+\frac{701}{1000\cdot 1.06^2} = 2.4446422 \\ P &= \frac{298.4259}{2.4446422} = \boxed{122.07.} \end{aligned}$$

172. (C3 May 2013, #3)
For a special fully discrete 20-year endowment insurance on (40), you are given:

(a) The only death benefit is the return of annual net premiums accumulated with interest at 5% to the end of the year of death.

(b) The endowment benefit is 100,000.

(c) Mortality follows the Standard Ultimate Life Table.

(d) $i = 0.05$.

Calculate the annual net premium.

Solution:
Let π denote the annual net premium.

Condition on whether (40) survives the 20 year endowment period or not.

$$0 = \mathrm{E}[L] = \mathrm{E}[L|K_x < 20]\Pr\{K_x < 20\} + \mathrm{E}[L|K_x \geq 20]\Pr\{K_x \geq 20\}$$

$\mathrm{E}[L|K_x < 20]$ In this case there is no endowment payment, and the premiums are returned with interest, with the interest rate being the same as what we used to discount the present value. So $\mathrm{E}[L|K_x < 20] = 0$.

$\mathrm{E}[L|K_x \geq 20]$ In this case there is no premium refund, only the 20-year endowment. Since the endowment is paid 20 years from now, it's present value is $v^{20}\cdot 10^5$, and since the premiums are paid for a period of 20 years certain, the EPV of the premium payments is $\pi\ddot{a}_{\overline{20|}}$. So $\mathrm{E}[L|K_x \geq 20] = v^{20}\cdot 10^5 - \pi\ddot{a}_{\overline{20|}}$. So the Equivalence Principle gives

$$0 = \left(v^{20}\cdot 10^5 - \pi\ddot{a}_{\overline{20|}}\right) {}_{20}p_x$$

or $\pi = \dfrac{v^{20}\cdot 10^5}{\ddot{a}_{\overline{20|}}}$, or $\boxed{\pi = 2880.25}$

173. (May 2013, #4)
You are given the following information for two independent lives aged 45 and 65.

(a) Mortality for the life age 45 follows the Illustrative Life Table.
(b) Mortality for the life age 65 follows: $f(y) = 0.2e^{-0.2y}$.

Calculate ${}_{10}p_{\overline{xy}}$

174. (May 2013, #4)
Employment for Joe is modeled according to a two-state homogeneous Markov modeled with states:

Actuary (Ac)
Professional Hockey Player (H)

You are given:

(i) Transitions occur December 31 of each year. The one-year transition probabilities are:

$$\begin{array}{c} \text{Ac} \\ \text{H} \end{array} \begin{bmatrix} 0.4 & 06 \\ 0.8 & 0.2 \end{bmatrix}$$

(ii) Mortality for Joe depends on his employment:

$$\begin{aligned} q^{Ac}_{35+k} &= 0.10 + 0.05k \qquad \text{for } k = 0, 1, 2, \\ q^{H}_{35+k} &= 0.25 + 0.05k \qquad \text{for } k = 0, 1, 2. \end{aligned}$$

(iii) $i = 0.08$

On January 1, 2013, Joe turned 35 years old and was employed as an actuary. On that date, he purchased a 3-year pure endowment of 100,000. Calculate the expected present value at issue of the pure endowment.

Solution:
There are four possible career paths for Joe, shown

Path	Prob{Path}	Prob{Survival|Path}	product
$A \to A \to A$	0.4×0.4	$0.9 \cdot (0.9 - 0.05) \cdot (0.9 - 0.1)$	0.09792
$A \to A \to H$	0.4×0.6	$0.9 \cdot (0.9 - 0.05) \cdot (0.75 - 0.1)$	0.11934
$A \to H \to A$	0.6×0.8	$0.9 \cdot (0.75 - 0.05) \cdot (0.9 - 0.1)$	0.24192
$A \to H \to H$	0.6×0.2	$0.9 \cdot (0.75 - 0.05) \cdot (0.75 - 0.1)$	0.04914

So Joe's probality of surviving three years is $0.09792 + 0.11934 + 0.24192 + 0.04914 = 0.50832$, so the EPV of the endowment is $100,000 \cdot 0.50832 \cdot 1.08^{-3} = \boxed{40352.08}$

175. (May 2013, #5)
You are given the following information:

Age	l_x	l_y
25	920	
26	915	
27	905	880
28		850
29		810

- Assume the future lifetimes of an individual (x) age 25 and an individual (y) age 27 are indpendent.

Calculate the two-year temporary curtate expectation of the last-survivor status, $e_{\overline{25:27}:\overline{2}|}$

Solution:

$$\begin{aligned}
e_{\overline{25:27}:\,\overline{2}|} &= p_{\overline{25:27}} + {}_2p_{\overline{25:27}} \\
&= 1 - q_{\overline{25:27}} + 1 - {}_2q_{\overline{25:27}} \\
&= 2 - (q_{25})(q_{27}) - ({}_2q_{25})({}_2q_{27}) \\
&= 2 - (1-p_{25})(1-p_{27}) - (1-{}_2p_{25})(1-{}_2p_{27}) \\
&= 2 - \left(1 - \frac{915}{920}\right)\left(1 - \frac{850}{880}\right) - \left(1 - \frac{905}{920}\right)\left(1 - \frac{810}{880}\right) \\
&= 1.99852
\end{aligned}$$

176. (C3, May 2013, #6)
You are given the following double-decrement table:

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
48			0.075			260
49	0.020	0.060		3,700		
50					132	

Calculate ${}_3q_{48}^{(1)}$.

Solution:

$$\begin{aligned}
{}_3q_{48}^{(1)} &= q_{48}^{(1)} + p_{48}^{(\tau)}\, q_{49}^{(1)} + {}_2p_{48}^{(\tau)}\, q_{50}^{(1)} \\
&= q_{48}^{(1)} + (1 - q_{48}^{(\tau)})q_{49}^{(1)} + (1 - q_{48}^{(\tau)})(1 - q_{49}^{(\tau)})q_{50}^{(1)}
\end{aligned}$$

Fill in what's needed in the table.

(a) $q_{49}^{(1)} + q_{49}^{(2)} = \underline{q_{49}^{(\tau)} = 0.080.}$

(b) $l_{49}^{(\tau)} = l_{48}p_{48}^{(\tau)} = l_{48}(1 - q_{48}^{(\tau)})$
implies $3700 = (0.920) \cdot l_{48}$, which implies $\underline{l_{48}^{(\tau)} = 4000.}$

(c) $l_{48}^{(\tau)} \cdot q_{48}^{(\tau)} = d_{48}^{(\tau)} = d_{48}^{(1)} + d_{48}^{(2)}$
implies $4021.74 \times 0.075 = d_{48}^{(1)} + 260$, which implies $\underline{d_{48}^{(1)} = 40}$

(d) $(d_{48}^{(1)})/(l_{48}^{(\tau)}) = \underline{q_{48}^{(1)} = 0.010}$

(e) $l_{49}^{(\tau)} \cdot (1 - q_{49}^{(\tau)}) = \underline{l_{50}^{(\tau)} = 3404}$

(f) $(d_{50}^{(1)})/(l_{50}^{(\tau)}) = \underline{q_{50}^{(1)} = 0.038778}$

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
48	(d) 0.01		0.075	(b) 4000	(c) 40	260
49	0.020	0.060	(a) 0.08	3,700		
50	(f) 0.03778			(e) 3404	132	

$$\begin{aligned}
{}_3q_{48}^{(1)} &= q_{48}^{(1)} + (1 - q_{48}^{(\tau)})q_{49}^{(1)} + (1 - q_{48}^{(\tau)})(1 - q_{49}^{(\tau)})q_{50}^{(1)} \\
&= 0.01 + (1 - 0.075) \times 0.02 + (1 - 0.075) \times (1 - 0.08) \times 0.03778 \\
&= \underline{0.0615}
\end{aligned}$$

177. (C3, May2013, #07)
You are given:

(a) $q_{60} = 0.01$

(b) Using $i = 0.05$, $A_{60:\overline{3}|} = 0.86545$

Using $i = 0.045$, calculate $A_{60:\overline{3}|}$.

Solution:

$$\begin{aligned}
A_{60:\overline{3}|i} &= \frac{q_{60}}{(1+i)} + \frac{{}_{1|}q_{60}}{(1+i)^2} + \frac{{}_2p_{60}}{(1+i)^3} \\
&= \frac{0.01}{(1+i)} + \frac{0.99q_{61}}{(1+i)^2} + \frac{1-\big(0.01+(0.99q_{61}\big)}{(1+i)^3} \\
A_{60:\overline{3}|i} &= \left(\frac{0.01}{1+i} + \frac{0.99}{(1+i)^3}\right) + \left(\frac{0.99}{(1+i)^2} - \frac{0.99}{(1+i)^3}\right) q_{61}. \qquad (12)
\end{aligned}$$

From the problem data we then find

$$\begin{aligned}
A_{60:\overline{3}|0.05} &= \left(\frac{0.01}{1.05} + \frac{0.99}{1.05^3}\right) + \left(\frac{0.99}{1.05^2} - \frac{0.99}{1.05^3}\right) q_{61} \\
0.86545 &= 1.798056365 + 0.042759961 \cdot q_{61} \\
&\Longrightarrow \quad \underline{q_{61} = 0.017}
\end{aligned}$$

Using this value and $i = 0.045$ in (12),

$$\begin{aligned}
A_{60:\overline{3}|0.045} &= \left(\frac{0.01}{1.045} + \frac{0.99}{1.045^3}\right) + \left(\frac{0.99}{1.045^2} - \frac{0.99}{1.045^3}\right) \cdot 0.017 \\
&= \boxed{0.877766724}
\end{aligned}$$

178. (C3 May 2013 #11)

For a one-year term insurance on (45), whose mortality follows a double decrement model, you are given:

(i) The death benefit for cause (1) is 1000 and for cause (2) is F.

(ii) Death benefits are payable at the end of the year of death.

(iii) $q_{45}^{(1)} = 0.04$ and $q_{45}^{(2)} = 0.20$

(iv) $i = 0.06$

(v) Z is the present value random variable for this insurance.

Calculate the value of F that minimizes $\mathrm{Var}[Z]$

Solution:

A value of F minimizes $\mathrm{Var}[Z]$ if and only if it minimizes $\mathrm{Var}\left[\tilde{Z}\right]$ where $\tilde{Z} = 1.06 \cdot Z$, and since Z is the present value random variable for this insurance,

$$\begin{aligned}
\tilde{Z} &= \begin{cases} 1000 & \text{w/ probability } 0.04 \\ F & \text{w/ probability } 0.2 \\ 0 & \text{w/ probability } 0.76. \end{cases} \\
\mathrm{E}\left[\tilde{Z}\right] &= 1000 \cdot 0.04 + 0.2 \cdot F = 40 + \frac{F}{5}, \\
\mathrm{E}\left[\tilde{Z}^2\right] &= 1000^2 \cdot 0.04 + 0.2 \cdot F^2 = 40,000 + \frac{F^2}{5}, \\
\mathrm{Var}\left[\tilde{Z}\right] &= \left(40,000 + \frac{F^2}{5}\right) - \left(40 + \frac{F}{5}\right)^2 = \frac{4}{25}\left(F^2 - 100F + 240000\right)
\end{aligned}$$

so $\mathrm{Var}[Z]$ is $\boxed{\text{minimized when } F = 50}$.

179. (C3 May 2013 #13)
An automobile insurance company classifies its insured drivers into three risk categories. The risk categories and expected annual claim costs are as follows:

Risk Category	Expected Annual Claim Cost
Low	100
Medium	300
High	600

The pricing model assumes:

- At the end of each year, 75% of insured drivers in each risk category will renew their insurance.
- $i = 0.06$.
- All claim costs are incurred mid-year.

For those renewing, 70% of Low Risk drivers remain Low Risk, and 30% become Medium Risk, 40% of Medium Risk drivers remain Medium Risk, 20% become Low Risk, and 40% become High Risk. All High Risk drivers remain High Risk.

Today the Company requires that all new insured drivers be Low Risk. The present value of expected claim costs for the first three years for a Low Risk driver is 317. Next year the company will allow 10% of new insured drivers to be Medium Risk.

Calculate the percentage increase in the present value of expected claim costs for the first three years per new insured driver due to the change.

Solution:

$$\begin{array}{c} \text{L} \\ \text{M} \\ \text{H} \end{array} \qquad Q = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.4 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q^2 = \begin{bmatrix} 0.55 & 0.33 & 0.12 \\ 0.22 & 0.22 & 0.56 \\ 0 & 0 & 1 \end{bmatrix}$$

Here are the calculations for what we'll call C_M, the EPV of claims costs for someone who starts out in the Medium Risk class.

Path	Prob{Path}	EPV{claims|path}	product
$\text{M} \to \text{L} \to L$	$0.2 \cdot 0.7$	$\frac{300}{1.06^{1/2}} + \frac{100 \cdot 0.75}{1.06^{3/2}} + \frac{100 \cdot 0.75^2}{1.06^{5/2}}$	57.2227
$\text{M} \to \text{L} \to M$	$0.2 \cdot 0.3$	$\frac{300}{1.06^{1/2}} + \frac{100 \cdot 0.75}{1.06^{3/2}} + \frac{300 \cdot 0.75^2}{1.06^{5/2}}$	30.3590
$\text{M} \to \text{M} \to L$	$0.4 \cdot 0.2$	$\frac{300}{1.06^{1/2}} + \frac{300 \cdot 0.75}{1.06^{3/2}} + \frac{100 \cdot 0.75^2}{1.06^{5/2}}$	43.6944
$\text{M} \to \text{M} \to M$	$0.4 \cdot 0.4$	$\frac{300}{1.06^{1/2}} + \frac{300 \cdot 0.75}{1.06^{3/2}} + \frac{300 \cdot 0.75^2}{1.06^{5/2}}$	102.949
$\text{M} \to \text{M} \to H$	$0.4 \cdot 0.4$	$\frac{300}{1.06^{1/2}} + \frac{300 \cdot 0.75}{1.06^{3/2}} + \frac{600 \cdot 0.75^2}{1.06^{5/2}}$	126.289
$\text{M} \to \text{H} \to H$	$0.4 \cdot 1.0$	$\frac{300}{1.06^{1/2}} + \frac{600 \cdot 0.75}{1.06^{3/2}} + \frac{600 \cdot 0.75^2}{1.06^{5/2}}$	398.189
		Sum:	758.703

So if 90% of the portfolio starts out at Low risk and 10% starts out at Medum risk, then the expected claims will be $0.9 \times 316.97 + 0.1 \times 758.703 = 361.1433$.

And since $361.1433/316.97 = 1.139$, the increase is $\boxed{13.9\%}$

180. (C3 May 2013, #15)

For fully discrete whole life insurance policies of 10,000 issued on 600 lives with independent future lifetimes, each age 62, you are given:

(i) Mortality follows the Illustrative Life Table.

(ii) $i = 0.06$

(iii) Expenses of 5% of the first year gross premium are incurred at issue.

(iv) Expenses of 5 per policy are incurred at the beginning of each policy year.

(v) The gross premium is 102% of the benefit premium.

(vi) ${}_0L$ is the aggregate present value of future loss at issue random variable.

Calculate $\Pr({}_0L < 60{,}000)$, using the normal approximation.

Solution:

181. (MLC May 2013, #19)
You are given:

(a) The following extract from a mortality table with a one-year select period:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	x+1
65	1000	40		66
66	995	45		67

(b) Deaths are uniformly distributed over each year of age.

(c) $\mathring{e}_{[65]} = 15.0$

Calculate $\mathring{e}_{[66]}$

182. (May 2013, #20)
Scientists are searching for a vaccine for a disease. You are given:

- 100,000 lives age x are exposed to the disease.
- Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1.
- The probability that the vaccine will be available is 0.2.
- For each life during year 1, $q_x = 0.02$.
- For each life during year 2, $q_{x+1} = 0.01$ if the vaccine has been given, and $q_{x+1} = 0.02$ if it has not been given.

Calculate the standard deviation of the number of survivors at the end of year 2.

Solution:
We're going to use (A.7) of Appendix A.4 in the text, which is

$$\mathrm{Var}[N] = \mathrm{E}[\mathrm{Var}[N|P]] + \mathrm{Var}[\mathrm{E}[N|P]]$$

for random variable N and P which we will define, where P is the probability that (x) survives

- If the vaccine is not available:
  then for an individual (x), $q_x = q_{x+1} = 0.02$, so that $p_x = p_{x+1} = 0.98$, so that

$$_2p_x = p_x p_{x+1} = 0.98^2 = 0.9604$$

- If the vaccine is available, then similarly

$$_2p_x = 0.98 \cdot 0.99 = 0.9702.$$

So for a given individual, the probability of that individual's surviving two years is itself a random variable (depending on whether the vaccine is developed or not), call it P:

$$\begin{aligned}
P &\stackrel{\text{def}}{=} \Pr\{\text{ (x) survives two years }\} = \begin{cases} 0.9604 & \text{w/ prob. } 0.8 \\ 0.9702 & \text{w/ prob. } 0.2 \end{cases} \\
\mathrm{E}[P] &= 0.2 \cdot 0.9702 + 0.8 \cdot 0.9604 = 0.96236 \\
\mathrm{E}\left[P^2\right] &= 0.2 \cdot 0.9702^2 + 0.8 \cdot 0.9604^2 = 0.926152136 \\
\mathrm{Var}[P] &= \mathrm{E}\left[P^2\right] - \mathrm{E}[P]^2 = 0.926152136 - 0.96236^2 = 0.0000153664.
\end{aligned}$$

Now let N be the number of people of the original 100,000 that survive two years, so that

$$N|P \sim \mathrm{Binomial}(100000, P),$$

so that

$$\begin{aligned}
\mathrm{E}[N|P] &= 100000\,P, \\
\mathrm{Var}[N|P] &= 100000\,P(1-P).
\end{aligned}$$

Now

$$\begin{aligned}
\mathrm{Var}[\mathrm{E}[N|P]] &= \mathrm{Var}[100000\,P] = 100000^2\mathrm{Var}[P] = 100000^2 \cdot 0.0000153664 = 153664 \\
\mathrm{E}[\mathrm{Var}[N|P]] &= \mathrm{E}[100000\,P(1-P)] = 100000 \cdot (0.96236 - 0.926152136) = 3620.79
\end{aligned}$$

so that $\mathrm{Var}[N] = 153664 + 3620.79 = 157284.79$, so

$$\boxed{\mathrm{Std}[N] = 396.59.}$$

183. (C3, May 2013 #21)
You are given:

(a) $\delta_t = 0.06,\ t \geq 0$

(b) $\mu_x(t) = 0.01,\ t \geq 0$

(c) Y is the present value random variable for a continuous annuity of 1 per year, payable for the lifetime of (x) with 10 years certain.

Calculate $\Pr\{Y > \mathrm{E}[Y]\}$

Solution:

$$\begin{aligned}
Y = \bar{a}_{\overline{10}|} + {}_{10}I_x a_{\overline{T_{x+10}}|} &= \frac{1-e^{-0.06\cdot 10}}{0.06} + \left(e^{-(\mu+\delta)\cdot 10}\right)\frac{1-e^{-0.06\cdot T_{x+10}}}{0.06} \\
\mathrm{E}[Y] &= 7.51981 + 8.27642\left(1 - \frac{\mu}{\mu+\delta}\right) = 14.6139 \\
\Pr\{Y > \mathrm{E}[Y]\} &= \Pr\left\{7.51981 + 8.27642\left(1 - e^{-0.06 T_{x+10}}\right) > 14.6139\right\} \\
&= \Pr\{T_{x+10} > 32.4321\} = e^{-0.01\cdot 32.4321} = \boxed{0.7230}
\end{aligned}$$

But this is wrong. Why?

184. (C3 May 2013, #22)
For a whole life insurance of 10,000 on (x), you are given:

(i) Death benefits are payable at the end of the year of death.

(ii) A premium of 30 is payable at the start of each month.

(iii) Commissions are 5% of each premium.

(iv) Expenses of 100 are payable at the start of each year.

(v) $i = 0.05$

(vi) $1000A_{x+10} = 400$

(vii) ${}_{10}V$ is the gross premium reserve at the end of year 10 for this insurance.

Calculate ${}_{10}V$ using the two-term Woolhouse formula for annuities.

Solution:
At $t = 10$ years, given that x has survived:

- the EPV of future benefits is $10,000A_{x+10}$ which is 4000,
- the EPV of future expenses is $100\ddot{a}_{x+10} + 12\cdot(0.05\cdot 30\cdot \ddot{a}^{(12)}_{x+10}) = 100\ddot{a}_{x+10} + 18\ddot{a}^{(12)}_{x+10}$,
- and the EPV of future premiums is $12\cdot 30\cdot \ddot{a}^{(12)}_{x+10} = 360\ddot{a}^{(12)}_{x+10}$

so

$$\begin{aligned}
{}_{10}V &= 4000 + 100\ddot{a}_{x+10} + 18\ddot{a}^{(12)}_{x+10} - 360\ddot{a}^{(12)}_{x+10} \\
{}_{10}V &= 4000 + 100\ddot{a}_{x+10} - 342\ddot{a}^{(12)}_{x+10} \\
{}_{10}V &= 4000 + 100\frac{1-A_{x+10}}{d} - 342\left(\frac{1-A_{x+10}}{d} - 11/24\right) \\
{}_{10}V &= 4000 + 100\frac{1-0.4}{0.05/1.05} - 342\left(\frac{1-0.4}{0.5/1.05} - 11/24\right) = \boxed{1107.55}
\end{aligned}$$

185. (C3, May 2013 #25)

For a fully discrete whole life insurance on (60), you are given:

(i) Mortality follows the Illustrative Life Table

(ii) $i = 0.06$

(iii) The expected company expenses, payable at the beginning of the year, are:

- 50 in the first year
- 10 in years 2 through 10
- 5 in years 11 through 20
- 0 after year 20

Calculate the level annual amount that is actuarially equivalent to the expected company expenses.

Solution:

186. (May 2013, #4)
You are given the following information for two independent lives aged 45 and 65.

(a) Mortality for the life age 45 follows the Illustrative Life Table.
(b) Mortality for the life age 65 follows: $f(y) = 0.2e^{-0.2y}$.

Calculate ${}_{10}p_{\overline{45:65}}$

Solution:

$$\begin{aligned}
{}_{10}p_{\overline{45:65}} &= 1 - {}_{10}q_{\overline{45:65}} = 1 - ({}_{10}q_{45})({}_{10}q_{65}) = 1 - \left(1 - \frac{8640860}{9164050}\right)(1 - e^{-2}) \\
&= 0.950\,635
\end{aligned}$$

187. (C3, Nov 2013 #1)
You are given:

(a) $A_x = 0.30$
(b) $A_{x+n} = 0.40$
(c) $A_{x:\overline{n}|}^{\;1} = 0.35$
(d) $i = 0.05$

Calculate $a_{x:\overline{n}|}$

Solution:

$$\begin{aligned}
a_x &= \ddot{a}_x - 1 = \frac{1 - A_x}{d} - 1 = \frac{1 - 0.3}{0.05/1.05} - 1 = 13.7 \\
a_{x+n} &= \ddot{a}_{x+n} - 1 = \frac{1 - A_{x+n}}{d} - 1 = \frac{1 - 0.4}{0.05/1.05} = 11.6 \\
a_{x:\overline{n}|} &= a_x - \overset{1}{x{:}\overline{n}|}a_{x+n} = 13.7 - 0.35 \cdot 11.6 = \boxed{9.64}
\end{aligned}$$

188. (MLC Nov 2013, #3)
You are given the following extract from a table with select period of 2 years:

(a)

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	x+2
50	99,000	96,000	93,000	52
51	97,000	93,000	89,000	53
52	93,000	88,000	83,000	54
53	90,000	84,000	78,000	55

(b) Deaths are uniformly distributed over each year of age.

Calculate $10,000\ {}_{2.2}q_{[51]+0.5}$

Solution:

$$\begin{aligned}
{}_{2.2}q_{[51]+0.5} &= 1 - \frac{l_{53.7}}{l_{[51]+0.5}} \\
&= 1 - \frac{0.3 \cdot l_{53} + 0.7 \cdot l_{54}}{0.5 \cdot l_{[51]} + 0.5 \cdot l_{[51]+1}} \\
&= 1 - \frac{0.3 \cdot 89,000 + 0.7 \cdot 83,000}{0.5 \cdot 97,000 + 0.5 \cdot 93,000} \\
&= 0.1073684, \qquad \boxed{10,000\ {}_{2.2}q_{[51]+0.5} = 1073.684}
\end{aligned}$$

189. (C3 Nov 2013 #04)
In a homogeneous Markov model with three states: Healthy (H), Sick (S), and Dead (D), you are given:

(i) The monthly transition probabilities are:

$$\begin{array}{c} \text{H} \\ \text{S} \\ \text{D} \end{array} \begin{bmatrix} 0.75 & 0.20 & 0.05 \\ 0.30 & 0.50 & 0.20 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}.$$

(ii)

Initially there are 10 Healthy lives with independent future states.

Calculate the probability that exactly 4 lives will die during the first two months.

Solution:
If we call the transition matrix Q then the $\{1,3\}$ entry of Q^2 gives us the probability an H transitions to D within two months. This value is 0.1275.

Out of 10 individuals, each of whom as a 0.1275 probability of dying, the number N of people who dies follows a Binomial(10,0.1275) disrtibition, and

$$\Pr\{N = 4\} = \binom{10}{4} 0.1275^4 \cdot (1 - 0.1275)^6 = \boxed{0.02448}$$

190. (C3, Nov2013 #5)
Russell, age 40, wins the SOA lottery. He will receive both:

- A deferred life annuity of K per year, payable continuously, starting at age $40 + \mathring{e}_{40}$, and

- An annuity certain of K per year, payable continuously, for $\mathring{e}_{40}$ years.

You are given:

(a) $\mu = 0.02$
(b) $\delta = 0.01$
(c) The actuarial present value of the payments is 10,000.

Calculate K.

Solution:
Since the forces of mortality and interest are constant, we have

$$\begin{aligned}
\mathring{e}_{40} &= \quad = 1/\mu = 50, \\
\bar{a}_{\overline{50}|} &= \frac{1 - e^{0.01 \cdot 50}}{0.01} = 39.3469, \\
\bar{a}_{90} &= \frac{1}{\mu + \delta} = 33.33333 \\
{}_{50}E_{40} &= e^{-(\mu+\delta) \cdot 50} = 0.223130, \\
10,000 &= K \cdot (39.3469 + 0.223130 \cdot 33.33333) \qquad \boxed{K = 213.75}
\end{aligned}$$

191. (C3, Nov 2013 #09)

For a fully discrete 20-year term insurance of 100,000 on (50), you are given:

(i) Gross premiums are payable for 10 years.

(ii) Mortality follows the Illustrative Life Table.

(iii) $i = 0.06$

(iv) Expenses are incurred at the beginning of each year as follows:

	Year 1	Years 2-10	Years 11-20
Commission as % of premium	40%	10%	Not applicable
Premium taxes as % of premium	2%	2%	Not applicable
Maintenance expenses	75	25	25

(v) Gross premiums are calculated using the equivalence principle.

Calculate the gross premium for this insurance.

Solution:

192. (C3, Nov 2013, #10)
For a multiple state model, you are given:

(a) *(diagram showing the following:)* Three states, 0,1,2, which are "Healthy, Disabled, Dead", and it not not possible to transition from "Disabled" back to "Healthy".

(b) The following forces of transition:

   i. $\mu^{01} = 0.02$

   ii. $\mu^{02} = 0.03$

   iii. $\mu^{12} = 0.05$

   Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.

   Solution:

$$\frac{d}{dt}{}_t\mathbf{P} \quad = \quad {}_t\mathbf{P}\ \mathbf{U} = \begin{bmatrix} {}_tp^{00} & {}_tp^{01} & {}_tp^{02} \\ {}_tp^{10} & {}_tp^{11} & {}_tp^{12} \\ {}_tp^{20} & {}_tp^{21} & {}_tp^{22} \end{bmatrix} \begin{bmatrix} -0.05 & 0.02 & 0.03 \\ 0 & -0.05 & 0.05 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \frac{d}{dt}{}_tp^{00} &= \frac{-1}{20}{}_tp^{00}, \qquad {}_0p^{00} = 1, \\ \frac{d}{dt}{}_tp^{01} &= \frac{-1}{20}{}_tp^{01} + \frac{1}{50}{}_tp^{00}, \qquad {}_0p^{01} = 0, \\ \frac{d}{dt}{}_tp^{02} &= \frac{3}{100}{}_tp^{00} + \frac{1}{20}{}_tp^{01}, \qquad {}_0p^{02} = 0. \end{aligned}$$

   We need ${}_{10}p^{00} / \left({}_{10}p^{00} + {}_{10}p^{01}\right)$

$$\begin{aligned} {}_tp^{00} &= e^{-0.05t} \qquad {}_{10}p^{00} = \underline{0.606531} \\ {}_tp^{01} &= \int_0^t e^{-0.05s} \cdot 0.02 \cdot {}_{t-s}p^{00}\, ds = 0.02 \cdot \int_0^t e^{-0.05s} e^{-0.05(t-s)}\, ds = 0.02 \cdot te^{-0.05t} \\ &\qquad {}_{10}p^{01} = \underline{0.121306} \end{aligned}$$

   Ans: $0.606531/(0.606531 + 0.121306) = \boxed{0.8333}$

193. (C3, Nov 2013 #14)
For a special 10-year deferred whole life annuity-due of 50,000 on (62), you are given:

(a) Level annual benefit premiums are payable for 10 years.

(b) A death benefit, payable at the end of the year of death, is provided only over the deferral period and is the sum of the benefit premiums paid without interest.

(c) $\ddot{a}_{62} = 12.2758$

(d) $\ddot{a}_{62:\overline{10|}} = 7.4574$

(e) $A^{1}_{62:\overline{10|}} = 0.0910$

(f) $\displaystyle\sum_{k=1}^{10} A^{1}_{62:\overline{k|}} = 0.4891$

Calculate the benefit premium for this special annuity.

   Solution:
   "Benefit premiums" means premiums arising from the use of the Equivalance Principle, which means they are determined by setting $\mathrm{E}\left[\text{Loss}\right] = 0$, so we need to identify the loss

random variable — let's call it L — for this problem, and let's denote the level annual benefit premium by π.

$$\begin{aligned}
L &= \Big[\text{Money paid by ins. co. }\Big] - \Big[\text{Premiums collected by ins. co. }\Big] \\
L &= \Big[50,000\big({}_{10|}\ddot{Y}_{62}\big) + \pi (IZ)^{\,1}_{62:\overline{10|}}\Big] - \pi \cdot \ddot{Y}_{62:\overline{10|}} \\
\mathrm{E}[L] = 0 &= \Big[50,000\big({}_{10|}\ddot{a}_{62}\big) + \pi (IA)^{\,1}_{62:\overline{10|}}\Big] - \pi \cdot \ddot{a}_{62:\overline{10|}} \\
0 &= 50,000\big({}_{10|}\ddot{a}_{62}\big) + \pi\Big[(IA)^{\,1}_{62:\overline{10|}} - \ddot{a}_{62:\overline{10|}}\Big]
\end{aligned}$$

$$\begin{aligned}
{}_{10|}\ddot{a}_{62} &= \ddot{a}_{62} - \ddot{a}_{62:\overline{10|}} = 12.2758 - 7.4574 = 4.8184 \\
(IA)^{\,1}_{62:\overline{10|}} &= 11A^{\,1}_{62:\overline{10|}} - \sum_{k=1}^{10} A^{\,1}_{62:\overline{k|}} = 11 \cdot 0.0910 - 0.4891 = 0.5119
\end{aligned}$$

$$0 = 50,000 \cdot 4.8184 + \pi \cdot [0.5119 - 7.4574] \quad \Longrightarrow \quad \boxed{\pi = 34,687.21}$$

194. (C3, Nov 2013, #16)
For fully discrete whole life insurances of 1 issued on lives age 50, the annual net premium, P, was calculated using the following:

(a) $q_{50} = 0.0048$
(b) $i = 0.04$
(c) $A_{51} = 0.39788$.

A particular life has a first year mortality rate 10 times the rate used to calculate P. The mortality rates for all other years are the same as the ones used to calculate P.

Calculate the expected present value of the loss at issue random variable for this life, based on the premium P.

Solution:
The loss random variable ("loss at issue") is given by

$$L = Z_{50} - \frac{P}{d}(1 - Z_{50})$$

and since P was computed via the Equivalance Principle (because "net premium"), we can set $\mathrm{E}[L] = 0$ and solve for P, yielding

$$P = (dA_{50}/(1 - A_{50}),$$

so L becomes (after simplifying)

$$L = \frac{Z_{50} - A_{50}}{1 - A_{50}},$$

which equals zero because $\mathrm{E}[Z_{50}] = A_{50}$.

The fact that "a particular life has a first year mortality rate" changes means that Z_{50} changes to, say, $Z^*_{50} \neq Z_{50}$, and $\mathrm{E}[Z^*_{50}] \neq A_{50}$. Instead, since q_{50} has changed to $10q_{50} = 0.048$, we have

$$\mathrm{E}[Z^*_{50}] \;=\; vq^*_{50} + vp^*_{50}A_{51} = \frac{0.048}{1.04} + \frac{1 - 0.048}{1.04} \cdot 0.39788 = 0.4103\,6708.$$

Also, $A_{50} = vq_x + vp_xA_{51} = 0.0048/1.04 + (1 - 0.0048) \cdot 0.39788/1.04 = 0.3853\,5594$. So

$$\mathrm{E}[L] = \mathrm{E}\left[\frac{Z_{50} - A_{50}}{1 - A_{50}}\right] = \frac{0.4103\,6708 - 0.3853\,5594}{1 - 0.3853\,5594} = \boxed{0.04069}$$

195. (C3, Nov 2013 #18)

For a fully discrete whole life insurance of 1000 on (x), you are given:

(i) The following expenses are incurred at the beginning of each year:

	Year 1	Years 2+
Percent of premium	75%	10%
Maintenance expenses	10	2

(ii) An additional expense of 20 is paid when the death benefit is paid.

(iii) The gross premium is determined using the equivalence principle.

(iv) $i = 0.06$

(v) $\ddot{a}_x = 12.0$

(vi) ${}^2A_x = 0.14$

Calculate the variance of the loss at issue random variable.

Solution:
We'll need $A_x = 1 - d\ddot{a}_x = 1 - (0.06/1.06) \cdot 12 = 0.32075$. Now,

$$\begin{aligned}
L &= \overbrace{1020\, v^{K_x+1}}^{\text{death ben. + final exp.}} + \overbrace{8 + 2\ddot{a}_{\overline{K_x+1}|}}^{\text{per policy exp's}} + \overbrace{P \cdot \left(0.65 + 0.1\ddot{a}_{\overline{K_x+1}|}\right)}^{\text{\% prem. exp.}} - \overbrace{P \cdot \ddot{a}_{\overline{K_x+1}|}}^{\text{premium income}} \\
L &= 1020\, v^{K_x+1} + \ddot{a}_{\overline{K_x+1}|}\,(2 - 0.9P) + 8 + 0.65P \\
\mathrm{E}[L] = 0 &= 1020 A_x + \ddot{a}_x\,(2 - 0.9P) + 8 + 0.65P \\
&= 1020 \cdot 0.32075 + 12\,(2 - 0.9P) + 8 + 0.65P \\
P &= 35.3857 \\
L &= 1020\, v^{K_x+1} + \ddot{a}_{\overline{K_x+1}|}\,(2 - 0.9 \cdot 35.3857) + 8 + 0.6 \cdot 535.3857 \\
L &= 1020\, v^{K_x+1} + \left(\frac{1 - v^{K_X+1}}{0.06/1.06}\right)(2 - 0.9 \cdot 35.3857) + 8 + 0.6 \cdot 535.3857 \\
L &= 1547.3 \cdot v^{K_x+1} + [\text{some constant}] \\
\mathrm{Var}[L] &= (1547.3)^2 \left({}^2A_x - (A_x)^2\right) = (1547.3)^2 (0.14 - 0.32075^2) = \boxed{88,861}.
\end{aligned}$$

.

196. (C3, Nov 2013 #19)

For a fully discrete whole life insurance of 10,000 on (45), you are given:

(i) Commissions are 80% of the first year premium and 10% of subsequent premiums. There are no other expenses.

(ii) Mortality follows the Illustrative Life Table.

(iii) $i = 0.06$

(iv) ${}_0L$ denotes the loss at issue random variable.

(v) If $T_{45} = 10.5$, then ${}_0L = 3767$.

Calculate $\mathrm{E}[{}_0L]$.

Solution:

$$L_0 = \overbrace{10,000\,v^{K_{45}+1}}^{\text{Death ben}} + \overbrace{0.7P + 0.1P\ddot{a}_{\overline{T_{45}}|}}^{\text{commissions}} - \overbrace{P\ddot{a}_{\overline{T_{45}}|}}^{\text{premium income}}$$

$$L_0 = 10,000\,v^{K_{45}+1} - P\left(0.9 \cdot \ddot{a}_{\overline{T_{45}}|} - 0.7\right)$$

$$L_0 = 10,000\,v^{K_{45}+1} - P\left(0.9 \cdot \frac{1 - v^{K_{45}+1}}{d} - 0.7\right)$$

If $T_{45} = 10.5$ then $K_{45} + 1 = 11$, and so in this case

$$3767 = 10,000\,1.06^{-11} - P\left(0.9 \cdot \frac{1 - 1.06^{-11}}{0.06/1.06} - 0.7\right)$$

$$P = \underline{219.94}$$

With this value of P, as the SOA solution computes,

$$\mathrm{E}[L_0] = 10,000\,Ax45 - P\,(0.9\ddot{a}_{45} - 0.7) = \boxed{-627.48}$$

197. (C3 Nov 2013 #23)
You are pricing an automobile insurance on (x). The insurance pays 10,000 immediately if (x) gets into an accident within 5 years of issue. The policy pays only for the first accident and has no other benefits. You are given:

(i) You model (x)'s driving status as a multi-state model with the following 3 states:

0 - low risk, without an accident
1 - high risk, without an accident
2 - has had an accident

(ii) (x) is initially in state 0.

(iii) The following transition intensities for $0 \le t \le 5$:

$$\begin{aligned} \mu_{x+t}^{01} &= 0.2 + 0.1t \\ \mu_{x+t}^{02} &= 0.05 + 0.05t \\ \mu_{x+t}^{12} &= 0.15 + 0.01t^2 \end{aligned}$$

No other transitions are possible.

(iv) ${}_3p_x^{01} = 0.4174$

(v) $\delta = 0.02$

(vi) The continuous function $g(t)$ is such that the expected present value of the benefit up to time a equals $\int_0^a g(t)\,dt$, $0 \le a \le 5$, where t is the time of the first accident.

Calculate $g(3)$.

Solution:

The insurance pays when either 3 years goes by or the first accident happens, to the thing to use is $\bar{A}_x^{02}$ which, according to Equation (8.16) on page 248 of AMLCR(ed 1):

$$\bar{A}_x^{ik} = \int_0^\infty \sum_{j\neq k} e^{-\delta t}\,{}_tp_x^{ij}\,\mu_{x+t}^{jk}\,dt.$$

So we have

$$\begin{aligned} g(t) &= 10^4 \sum_{j\neq k} e^{-\delta t}\,{}_tp_x^{ij}\,\mu_{x+t}^{jk} \\ &= 10^4\, e^{-\delta t}\left({}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{12}\mu_{x+t}^{12}\right) \\ g(3) &= 10^4 \cdot e^{-3\cdot 0.02}\left({}_3p_x^{00}\cdot(0.05 + 3\cdot 0.05) + 0.4174\cdot\left(0.15 + 0.01\cdot 3^2\right)\right) \end{aligned}$$

Now

$${}_3p_x^{00} = \exp\left(-\int_0^3 \mu_{x+t}^{01} + \mu_{x+t}^{02}\,dt\right) = \exp\left(-\int_0^3 0.25 + 0.15t\,dt\right) = 0.240508,$$

so

$$\begin{aligned} g(3) &= 10^4 \cdot e^{-3\cdot 0.02}\left(0.240508\cdot(0.05 + 3\cdot 0.05) + 0.4174\cdot\left(0.15 + 0.01\cdot 3^2\right)\right) \\ &= \boxed{1396.43.} \end{aligned}$$

198. (MLC Nov 2013, #25)

You are given:

(a)

x	l_x
60	99,999
61	88,888
62	77,777
63	66,666
64	55,555
65	44,444
66	33,333
67	22,222

(b) $a = {}_{3.4|2.5}q_{60}$, assuming a uniform distribution of deaths over each year of age.

(c) $b = {}_{3.4|2.5}q_{60}$, assuming a constant force of mortality over each year of age.

Calculate $100,000(a-b)$.

Solution:
Under UDD,

$$\begin{aligned} a = {}_{3.4|2.5}q_{60} &= \frac{\ell_{63.4}-\ell_{65.9}}{\ell_{60}} \\ &= \frac{(0.6\cdot\ell_{63}+0.4\cdot\ell_{64})-(0.1\ell_{65}+0.9\ell_{66})}{\ell_{60}} \\ &= \frac{(0.6\cdot 66,666+0.4\cdot 55,555)-(0.1\cdot 44,444+0.9\cdot 33,333)}{99,999} = \underline{0.2\overline{777}} \end{aligned}$$

Under Constant Force,

$$\begin{aligned} b = {}_{3.4|2.5}q_{60} &= \frac{\ell_{63.4}-\ell_{65.9}}{\ell_{60}} \\ &= \frac{\ell_{63}^{0.6}\ell_{64}^{0.4}-\ell_{65}^{0.1}\ell_{66}^{0.9}}{\ell_{60}} \\ &= \frac{66,666^{0.6}\cdot 55,555^{0.4}-44,444^{0.1}\cdot 33,333^{0.9}}{99,999} = \underline{0.27671612} \end{aligned}$$

So $100,000\cdot(a-b) = 100,000\cdot(0.2\overline{777}-0.27671612) = \boxed{106.166}$.

199. (MLC Nov 2013, #25b)
You are given:

(a)

x	50	51	52	53	54	55	56	57
C ℓ_x	99,395	89,247	78,592	66,910	56,072	44,807	33,181	22,369

(b) Under the "constant force" assumption, $\beta = {}_{3.4|2.5}q_{50}$.

(c) Under the UDD assumption, $\alpha = {}_{3.4|2.5}q_{50}$.

Calculate α/β.

Solution:
Under UDD,

$$\begin{aligned} \alpha = {}_{3.4|2.5}q_{50} &= \frac{\ell_{53.4}-\ell_{55.9}}{\ell_{50}} \\ &= \frac{(0.6\cdot\ell_{53}+0.4\cdot\ell_{54})-(0.1\ell_{55}+0.9\ell_{56})}{\ell_{50}} \\ &= \frac{(0.6\cdot 66,910+0.4\cdot 56,072)-(0.1\cdot 44,807+0.9\cdot 33,181)}{99,395} = \underline{0.28403} \end{aligned}$$

Under Constant Force,

$$\begin{aligned} \beta = {}_{3.4|2.5}q_{50} &= \frac{\ell_{53.4}-\ell_{55.9}}{\ell_{50}} \\ &= \frac{\ell_{53}^{0.6}\ell_{54}^{0.4}-\ell_{55}^{0.1}\ell_{56}^{0.9}}{\ell_{50}} \\ &= \frac{66,910^{0.6}\cdot 56,072^{0.4}-44,807^{0.1}\cdot 33,181^{0.9}}{99,395} = \underline{0.283223} \end{aligned}$$

So $\alpha/\beta = 0.28403/0.283223 = \boxed{1.002849}$

200. (MLC May 2014, #1)
You are given the following information from a life table:

(a)

x	l_x	d_x	p_x	q_x
95				0.40
96			0.20	
97		72		1.00

(b) $l_{90} = 1000$ and $l_{93} = 825$.

(c) Deaths are uniformly distributed over each year of age.

Calculate the probability that (90) dies between ages 93 and 95.5.

Solution:
We're asked to compute ${}_{3|2.5}q_{90}$. With the UDD assumption we have

$$\begin{aligned}
{}_{3|2.5}q_{90} &= {}_3p_{90} \cdot {}_{2.5}q_{93} = {}_3p_{90}\left[{}_2q_{93} + {}_2p_{93} \cdot {}_{0.5}q_{95}\right] \\
&= {}_3p_{90}\left[{}_2q_{93} + {}_2p_{93} \cdot (0.5 \cdot q_{95})\right] \quad \text{using UDD} \\
&= \frac{l_{93}}{l_{90}}\left[\left(1 - \frac{l_{95}}{l_{93}}\right) + \frac{l_{95}}{l_{93}} \cdot (0.5 \cdot 0.4)\right] = \frac{825}{1000}\left[1 - \frac{l_{95}}{825} + \frac{l_{95}}{825} \cdot 0.2\right]
\end{aligned}$$

so $\underline{\text{we need } l_{95}}$.

From the table:

(a) Since $q_{97} = 1.0$, then everyone alive at age 97 dies, so $l_{97} = d_{97} = 72$.

(b) Since $l_{97} = l_{96} \cdot p_{96}$, we obtain $l_{96} = 360$.

(c) Since $l_{96} = l_{95} \cdot (1 - q_{95})$, we obtain $l_{95} = 600$.

So

$$\begin{aligned}
{}_{3|2.5}q_{90} &= \frac{825}{1000}\left[1 - \frac{l_{95}}{825} + \frac{l_{95}}{825} \cdot 0.2\right] \\
&= \frac{825}{1000}\left[1 - \frac{600}{825} + \frac{600}{825} \cdot 0.2\right] = \boxed{0.345}
\end{aligned}$$

201. (MLC May 2014, #2)
You are given the following extract from a table with a 3-year select period:

(a)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	x+3
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(b) $e_{64} = 5.10$.

Calculate $e_{[61]}$.

Solution:

$$
\begin{aligned}
e_{[61]} &= \sum_{k=1}^{\infty} {}_kp_{[61]} \\
&= p_{[61]} + {}_2p_{[61]} + {}_3p_{[61]} + \sum_{k=4}^{\infty} {}_kp_{[61]} \\
&= p_{[61]} + {}_2p_{[61]} + {}_3p_{[61]} + \sum_{j=1}^{\infty} {}_{j+3}p_{[61]} \\
&= p_{[61]} + {}_2p_{[61]} + {}_3p_{[61]} + {}_3p_{[61]} \sum_{j=1}^{\infty} {}_jp_{64} \\
&= p_{[61]} + {}_2p_{[61]} + {}_3p_{[61]}\,(1 + e_{64}) \\
&= (1 - q_{[61]}) + (1 - q_{[61]})(1 - q_{[61]+1}) + (1 - q_{[61]})(1 - q_{[61]+1})(1 - q_{[61]+2})(1 + e_{64}) \\
&= 0.9 + 0.9(0.88) + 0.9(0.88)(0.86)(6.1) \\
&= 0.9\Big[1 + 0.88(1 + 0.86 \cdot 6.1)\Big] = \boxed{5.8468}
\end{aligned}
$$

202. (C3 May 2014 #03)
A continuous Markov process is modeled by the following multiple state diagram:

State 0 State 1 State 2

You are given the following constant transition intensities:

(i) $\mu^{01} = 0.08$

(ii) $\mu^{02} = 0.04$

(iii) $\mu^{10} = 0.10$

(iv) $\mu^{12} = 0.05$

For a person in State 1, calculate the probability that the person will continuously remain in State 1 for the next 15 years.

Solution:

$$
\begin{aligned}
\frac{d}{dt}{}_t\mathbf{P} &= {}_t\mathbf{P}\ \mathbf{U} = \begin{bmatrix} {}_tp^{00} & {}_tp^{01} & {}_tp^{02} \\ {}_tp^{10} & {}_tp^{11} & {}_tp^{12} \\ {}_tp^{20} & {}_tp^{21} & {}_tp^{22} \end{bmatrix} \begin{bmatrix} -0.12 & 0.08 & 0.04 \\ 0.10 & -0.15 & 0.05 \\ 0 & 0 & 0 \end{bmatrix} \\
\frac{d}{dt}{}_tp^{11} &= -0.15 \cdot {}_tp^{11}, \qquad {}_0p^{11} = 1, \\
{}_tp^{\overline{11}} &= e^{-0.15 \cdot t} \\
{}_{15}p^{\overline{11}} &= e^{-0.15 \cdot 15} = \boxed{0.105399}
\end{aligned}
$$

203. (C3 May 2014, #6)
For a group of 100 lives age x with independent future lifetimes, you are given:

(a) Each life is to be paid 1 at the beginning of each year, if alive.

(b) $A_x = 0.45$

(c) ${}^2A_x = 0.22$

(d) $i = 0.05$

Y is the present value random variable of the aggregate payments.

Using the normal approximation to Y, calculate the initial size of the fund needed in order to be 95% certain of being able to make the payments for these life annuities.

Solution:

Denoting by Y_i the amount to one of the (x)'s, we have $Y = \sum_{i=1}^{100} Y_i$, and the Y_i's are independent and identically distributed.

Call the amount in the fund (which we seek) F, which is the 95^{th} percentile of Y if we model Y as a normal R.V. This means that

$$F = \mathrm{E}[Y] + 1.644 \cdot \mathrm{Std}[Y] \tag{13}$$

so we need the mean and standard deviation of Y.

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_{i=1}^{n} \mathrm{E}[Y_i] = 100\ddot{a}_x = 100 \cdot \left(\frac{1-A_x}{d}\right) = \frac{100(1-0.45)}{0.05/1.05} = 1155 \\
\mathrm{Var}[Y] &= \sum_{i=1}^{n} \mathrm{Var}[Y_i] = 100 \cdot \left(\frac{{}^2A_x - A_x^2}{d^2}\right) \\
\mathrm{Std}[Y] &= \sqrt{\mathrm{Var}[Y]} = \frac{10}{d}\sqrt{{}^2A_x - A_x^2} = \frac{10}{0.05/1.05}\sqrt{0.22 - 0.45^2} = 27.78039
\end{aligned}$$

$$F = 1155 + 1.644 \cdot 27.78039 = \boxed{1200.695}$$

Here's a review of where equation (13) comes from. We want the amount F for which $\boxed{\Pr\{Y \le F\} = 0.95}$, which is the same as

$$\Pr\left\{\frac{Y - \mathrm{E}[Y]}{\mathrm{Std}[Y]} \le \frac{F - \mathrm{E}[Y]}{\mathrm{Std}[Y]}\right\} = 0.95.$$

If Y is normal (which is approximately true since Y is the sum of 100 iid R.V.'s), then this is the same as

$$\begin{aligned}
\Pr\left\{Z \le \frac{F - \mathrm{E}[Y]}{\mathrm{Std}[Y]}\right\} &= 0.95, \qquad \text{where } Z \sim \mathrm{Normal}(0,1) \\
\frac{F - \mathrm{E}[Y]}{\mathrm{Std}[Y]} &= \Phi^{-1}(0.95) \approx 1.644
\end{aligned}$$

which leads to (13).

204. (C3, May 2014, #9)
For a fully discrete whole life insurance of 1000 on (x) with net premiums payable quarterly, you are given:

(i) $i = 0.06$

(ii) $\ddot{a}_x = 3.4611$

(iii) $P^{(W)}$ and $P^{(UDD)}$ are the annualized net premiums calculated using the 2-term Woolhouse (W) and the uniform distribution of deaths (UDD) assumptions, respectively.

Calculate $\dfrac{P^{(UDD)}}{P^{(W)}}$.

Solution:
From the equivalence principle,

$$\begin{aligned} 0 &= 1000A_x - P\ddot{a}_x^{(4)} \\ P &= \frac{1000A_x}{\ddot{a}_x^{(4)}} \\ \frac{P^{(UDD)}}{P^{(W)}} &= \frac{\ddot{a}_x - 3/8}{\alpha(4)\ddot{a}_x - \beta(4)} \end{aligned}$$

Now $\alpha(4)$ and $\beta(4)$ are in the Tables but they depend on $i^{(4)}$ and $d^{(4)}$:

$$\begin{aligned} i^{(4)} &= 4(1.06^{1/4} - 1) = 0.058695385, \\ d^{(4)} &= 4(1 - 1.06^{-1/4}) = 0.057846553 \\ \alpha(4) &= \frac{id}{i^{(m)}d^{(m)}} = \frac{0.06^2/1.06}{0.058695385 \cdot 0.057846553} = 1.0002653, \\ \beta(4) &= \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} = \frac{0.06 - 0.058695385}{0.058695385 \cdot 0.057846553} = 0.3842385, \end{aligned}$$

so

$$\frac{P^{(UDD)}}{P^{(W)}} = \frac{3.4611 - 3/8}{1.0002653 \cdot 3.4611 - 0.3842385} = \boxed{1.002703}$$

205. (C3, May 2014 #10)

For a fully discrete 20-year endowment insurance of 100,000 on (30), you are given:

(i) $d = 0.05$

(ii) Expenses, payable at the beginning of each year, are:

	First Year		Renewal Years	
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	4%	0	4%	0
Sales Commission	35%	0	2%	0
Policy Maintenance	0%	250	0%	50

(iii) The net premium is 2143.

Calculate the gross premium using the equivalence principle.

Solution:

206. (C3 May 2014 #20)
For a fully discrete 3-year endowment insurance of 100 on (50), you are given:

(i) The following double decrement table, where decrement d refers to death and decrement w refers to withdrawal:

x	$\ell_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$
50	1000	20	35
51	945	25	25
52	895	30	0

(ii) There are no benefits upon withdrawal.

(iii) $i = 0.05$

Calculate the annual net premium for this policy.

Solution:
Denote by A the EPV of the insurance benefit, and denote by P the annual premium, and denote by a the EPV premium annuity. Then $P = A/a$, and

$$\begin{aligned} A &= 100 \cdot \left(vp_{50}^{0d} + v^2 p_{50}^{00} p_{51}^{0d} + v^3 {}_2p_{50}^{00}\right) \\ &= 100 \cdot \left(\frac{20}{1.05 \cdot 1000} + \frac{945 \cdot 25}{1.05^2 \cdot 1000 \cdot 945} + \frac{895}{1.05^3 \cdot 1000}\right) = 81.4858 \end{aligned}$$

$$\begin{aligned} a &= P \cdot \left(1 + vp_{50}^{00} + v^2 {}_2p_{50}^{00}\right) \\ &= P \cdot \left(1 + \frac{945}{1.05 \cdot 1000} + \frac{895}{1.05^2 \cdot 1000}\right) = 2.71179 \end{aligned}$$

$$\boxed{P = 30.049}.$$

207. (C3 Nov 2014 #02)
For a double decrement table, you are given:

(i) $q_x^{'(1)} = 0.1$

(ii) $q_x^{(2)} = 0.2$

(iii) Each decrement is uniformly distributed over each year of age in its associated single decrement table.

Calculate $q_x^{(1)}$

Solution:
Since each decrement is uniformly distributed over each year of age in its associated single decrement table, we know from an unnumbered equation on page 278 of AMLCR(2) that

$$q_x^{(2)} = p_x^{02} = q_x^{*(2)}\left(1 - \frac{1}{2} \cdot q_x^{*(1)}\right).$$

From this we find

$$0.2 = q_x^{*(2)}\left(1 - \frac{1}{2} \cdot 0.1\right) \qquad \Longrightarrow \qquad \underline{q_x^{*(2)} = 4/19 = 0.2105263}$$

So in the associated single decrement models, for $0 \le t \le 1$ we have

$$\mu_{x+t}^{(1)} = \frac{0.1}{1 - 0.1 \cdot t} = \frac{1}{10 - t} \qquad \text{and} \qquad \mu_{x+t}^{(2)} = \frac{4/19}{1 - t/19} = \frac{1}{19/4 - t}$$

$${}_tp_x^{'(1)} = \frac{10 - t}{10} \qquad \text{and} \qquad {}_tp_x^{'(2)} = \frac{19/4 - t}{19/4}$$

$$p_x^{'(1)} = \frac{9}{10} \qquad \text{and} \qquad p_x^{'(2)} = \frac{15}{19}$$

$$q_x^{'(1)} = \frac{1}{10} \qquad \text{and} \qquad q_x^{'(2)} = \frac{4}{19}$$

$$
\begin{aligned}
p_x^{(01)} = q_x^{(1)} &= q_x^{*(1)}\left(1 - \frac{1}{2} q_x^{*(2)}\right) \\
&= \frac{1}{10}\left(1 - \frac{1}{2}\frac{4}{19}\right) = \boxed{\frac{17}{190} \approx 0.08947368}
\end{aligned}
$$

Or, a longer way:

$$
\begin{aligned}
{}_tp_x^{00} &= {}_tp_x^{'(1)}\, {}_tp_x^{'(2)} = \frac{10-t}{10} \cdot \frac{19/4 - t}{19/4} \\
p_x^{(01)} = q_x^{(1)} &= \int_0^1 {}_tp_x^{00} \cdot \mu_{x+t}^{(1)}\, dt = \int_0^1 \frac{19/4 - t}{10 \cdot 19/4}\, dt = \boxed{0.0894737}
\end{aligned}
$$

208. (C3 Nov 2014 #03)
Patients are classified as Sick (S), Critical (C), or Discharged (D). Transitions occur daily according to the following Markov transition matrix:

$$
\begin{array}{c}
(S) \\ (C) \\ (D)
\end{array}
\begin{bmatrix}
0.60 & 0.20 & 0.20 \\
0.10 & 0.50 & 0.40 \\
0.00 & 0.00 & 1.00
\end{bmatrix}.
$$

Calculate the probability that a patient who is classified as Sick today will be classified as Sick three days later.

Solution:

$$
\begin{bmatrix}
0.60 & 0.20 & 0.20 \\
0.10 & 0.50 & 0.40 \\
0.00 & 0.00 & 1.00
\end{bmatrix}^3 =
\begin{bmatrix}
0.25 & 0.186 & 0.564 \\
0.093 & 0.157 & 0.75 \\
0 & 0 & 1
\end{bmatrix}
$$

so the probability that a patient who is classified as Sick today will be classified as Sick three days later is $\boxed{0.25}$

Or another way:

Path	Prob{Path}
$\mathrm{S} \to \mathrm{S} \to S \to S$	$0.6^3 = 0.216$
$\mathrm{S} \to \mathrm{S} \to C \to S$	$0.6 \cdot 0.2 \cdot 0.1 = 0.012$
$\mathrm{S} \to \mathrm{C} \to S \to S$	$0.2 \cdot 0.1 \cdot 0.6 = 0.012$
$\mathrm{S} \to \mathrm{C} \to C \to S$	$0.2 \cdot 0.5 \cdot 0.1 = 0.01$
Sum:	0.25

209. (C3 Nov 2014 #6)
For a 20-year deferred whole life annuity-due with annual payments of 30,000 on (40), you are given:

(a) The single net premium is refunded without interest at the end of the year of death if death occurs during the deferral period.

(b) Mortality follows the Standard Ultimate Life Table.

(c) $i = 0.05$

Calculate the single net premium for this annuity.

Solution:

$$\begin{aligned}
0 &= 30,000 \cdot {}_{20}E_{40}\, \ddot{a}_{60} + \pi A^{\;1}_{40:\overline{20}|} - \pi \\
\pi &= \frac{30,000 \cdot {}_{20}E_{40}\, \ddot{a}_{60}}{1 - A^{\;1}_{40:\overline{20}|}} = \frac{30,000 \cdot 0.36663 \cdot 14.9041}{1 - (0.38126 - 0.36663)} = \boxed{166362.59}
\end{aligned}$$

210. (C3, Nov 2014 #07)

For a fully discrete whole life insurance of 1 on (50), you are given:

(i) Expenses of 0.20 at the start of the first year and 0.01 at the start of each renewal year are incurred.

(ii) Mortality follows the Illustrative Life Table.

(iii) $i = 0.06$

Gross premiums are determined using the equivalence principle.

Calculate the variance of L_0, the gross loss-at-issue random variable.

Solution:

211. (C3, Nov 2014 #11)

For fully discrete 30-payment whole life insurance policies on (x), you are given:

(i) The following expenses payable at the beginning of the year:

	1st Year	Years 2 – 15	Years 16 – 30	Years 31 and after
Per policy	60	30	30	30
Percent of premium	80%	20%	10%	0%

(ii) $\ddot{a}_x = 15.3926$

(iii) $\ddot{a}_{x:\overline{15}|} = 10.1329$

(iv) $\ddot{a}_{x:\overline{30}|} = 14.0145$

(v) Annual gross premiums are calculated using the equivalence principle.

(vi) The annual gross premium is expressed as $kF + h$, where F is the death benefit and k and h are constants for all F.

Calculate h.

Solution:

212. (C3 Nov 2014 #12)
You are analyzing the sensitivity of some of the assumptions used in setting the premium rate for a sickness policy. You are basing your calculations on a multiple state model as diagrammed below:

Healthy Sick Dead

You are given:

(i) Level premiums are paid continuously by Healthy policyholders.

(ii) Level sickness benefits are paid continuously to Sick policyholders.

(iii) There is no death benefit.

Which one of the following changes to the assumptions will be certain to increase the premium rate?

(A) A lower rate of interest and a higher recovery rate from the Sick state.

(B) A lower mortality rate for those in the Healthy state and a lower mortality rate for those in the Sick state.

(C) A higher mortality rate for those in the Healthy state and a higher mortality rate for those in the Sick state.

(D) A lower recovery rate from the Sick state and a lower mortality rate for those in the Sick state.

(E) A higher rate of interest and a lower mortality rate for those in the Healthy state.

Solution:
D

213. (MLC, Nov. 2014, #20)
For a mortality table with a select period of two years, you are given:

(a)

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.0050	0.0063	0.0080	52
51	0.0060	0.0073	0.0090	53
52	0.0070	0.0083	0.0100	54
53	0.0080	0.0093	0.0110	55

(b) The force of mortality is constant between integral ages.

Calculate 1000 ${}_{2.5}q_{[50]+0.4}$.

Solution:
Constant force $\Longrightarrow$ $\underline{{}_sp_{[x]} = (p_{[x]})^s \text{ for } s \in [0,1]}$

$$\begin{aligned} {}_{2.5}p_{[50]+0.4} = \frac{{}_{2.9}p_{[50]}}{{}_{0.4}p_{[50]}} &= \frac{(\,p_{[50]}\,)\cdot(p_{[50]+1})\cdot(\,{}_{0.9}p_{[50]+2}\,)}{{}_{0.4}p_{[50]}} \\ &= \frac{(\,p_{[50]}\,)\cdot(p_{[50]+1})\cdot(p_{52})^{0.9}}{(p_{[50]})^{0.4}} \\ &= \frac{(1-0.0050)\cdot(1-0.0063)\cdot(1-0.0080)^{0.9}}{(1-0.0050)^{0.4}} = \underline{0.98358} \\ 1000\ {}_{2.5}q_{[50]+0.4} &= 1000\cdot(1-0.98358) = \boxed{16.42010} \end{aligned} \tag{14}$$

214. (C3 May 2015 #02)
You are pricing a type of ~~disability~~LIFE? insurance using the following model:

Healthy(0) Disabled(1) Dead(2)

The insurance will pay a benefit only if, by age 65, the insured had been disabled for a period of at least one year. You are given the following forces of transition:

(i) $\mu^{01} = 0.02$
(ii) $\mu^{02} = 0.03$
(i) $\mu^{12} = 0.11$

Calculate the probability that a benefit will be paid for a Healthy life aged 50 who purchases this insurance.

Solution:
The insurance only pays if (50) becomes disabled within 14 years *and* then that disabled 64 year old does not dies within one year. The probability for this is ${}_{14}p_{50}^{01}\cdot p_{64}^{\overline{11}}$ We have

$$\begin{aligned} {}_tp_{50}^{00} &= e^{-0.05t} \\ {}_{14}p_{50}^{01} &= 0.02\cdot\int_0^{14} e^{-0.03s}\,{}_{14-s}p_{50}^{00}\,ds = 0.02\int_0^{14} e^{-0.03s}e^{-0.05(14-s)}\,ds = \underline{0.160462} \\ p_{64}^{\overline{11}} &= e^{-0.11} = \underline{0.895834} \end{aligned}$$

$${}_{14}p_{50}^{01}\cdot p_{64}^{\overline{11}} = \boxed{0.14375}$$

Wrong.

215. (C3 May 2015 #06)
A life insurance company uses the following 3-state Markov model to calculate premiums for a 3-year sickness policy issued to Healthy lives.

Healthy(H) Sick(S) Dead (D)

The company will pay a benefit of 20,000 at the end of each year if the policyholder is Sick at that time. The insurance company uses the following transition probabilities, applicable in each of the three years: three years:

	H	S	D
H	0.950	0.025	0.025
S	0.300	0.600	0.100
D	0.000	0.000	1.000

Calculate the expected present value at issue of sickness benefit payments using an interest rate of 6%.

Solution:

$$\begin{aligned} Q &= \begin{bmatrix} 0.95 & 0.025 & 0.025 \\ 0.3 & 0.6 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}, \\ Q^2 &= \begin{bmatrix} 0.91 & 0.03875 & 0.05125 \\ 0.465 & 0.3675 & 0.1675 \\ 0 & 0 & 1 \end{bmatrix}, \\ Q^3 &= \begin{bmatrix} 0.87612 & 0.046 & 0.077875 \\ 0.552 & 0.23212 & 0.21588 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

k	Probability that (H) is sick at the end of k^{th} year
1	0.025
2	0.03875
3	0.046

You can get these probabilities more directy with the path-path approach.
Anyhow:

$$\text{EPV} = 20,000 \cdot \left(\frac{0.025}{1.06} + \frac{0.03875}{1.06^2} + \frac{0.046}{1.06^3} \right) = \boxed{1933.90}$$

216. (C3 May 2015, #7)
You are given

(a) $A_{35} = 0.188$
(b) $A_{65} = 0.498$
(c) ${}_{30}p_{35} = 0.883$
(d) $i = 0.04$

Calculate $1000\,\ddot{a}^{(2)}_{35:\overline{30|}}$ using the two-term Woolhouse approximation.

Solution:
The two-term Woolhouse approximation is:

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

which in this problem becomes

$$\ddot{a}^{(2)}_{35:\overline{30}|} \approx \ddot{a}_{35:\overline{30}|} - \frac{1}{4}(1 - {}_nE_x)$$

From the problem data:

$$\begin{aligned}
\ddot{a}_{35} &= \frac{1-A_{35}}{d} = \frac{1-0.188}{0.04/1.04} = 21.112 \\
\ddot{a}_{65} &= \frac{1-A_{65}}{d} = \frac{1-0.498}{0.04/1.04} = 13.052 \\
{}_{30}E_{35} &= 1.04^{-30} \cdot 0.883 = 0.2722454 \\
\ddot{a}_{35:\overline{30}|} &= \ddot{a}_{35} - {}_{30}E_{35}\,\ddot{a}_{35} = 17.55865
\end{aligned}$$

So $1000\,\ddot{a}^{(2)}_{35:\overline{30}|} \approx 17.55865 - 0.25 \cdot (1 - 0.2722454) = \boxed{17,376.71}$.

217. (C3, May 2015, #8)
For a fully continuous whole life insurance of 1 on (x), you are given:

(a) L is the present value of the loss at issue random variable if the premium rate is determined by the equivalence principle.
(b) L^* is the present value of the loss at issue random variable if the premium rate is 0.06.
(c) $\delta = 0.07$
(d) $\bar{A}_x = 0.30$
(e) $\mathrm{Var}[L] = 0.18$

Calculate $\mathrm{Var}[L^*]$.

(Possible ambiguity: where it says “the loss at issue random variable if the premium rate is 0.06”, the rate here is *not* the interest rate. It is the annual rate at which money flows in the annuity which constitutes the premium paymenrs. E.g, the APV of the premium payments is $0.06\bar{a}_x$.)

Solution:
Let $\bar{P}$ be the annual premium rate computed under the Equiv. Princ. Then:

$$\begin{aligned}
L &= \bar{Z}_x - \bar{P}\bar{Y}_x \\
0 &= \bar{A}_x - \bar{P}\bar{a}_x \quad \Longrightarrow \quad \bar{P} = \frac{\bar{A}_x}{\bar{a}_x} = \frac{\delta\bar{A}_x}{1-\bar{A}_x} = \frac{0.07\cdot 0.30}{1-0.30} \quad \Longrightarrow \quad \underline{\bar{P} = 0.03}
\end{aligned}$$

$$\begin{aligned}
L = \bar{Z}_x - \bar{P}\bar{Y}_x &= \left(1 + \frac{\bar{P}}{\delta}\right)\bar{Z}_x - \frac{\bar{P}}{\delta} = \left(1 + \frac{0.03}{0.07}\right)\bar{Z}_x - \frac{3}{7} \\
\mathrm{Var}[L] = 0.18 &= \left(1 + \frac{\pi}{\delta}\right)^2 \mathrm{Var}\left[\bar{Z}_x\right] \quad \Longrightarrow \quad \underline{\mathrm{Var}\left[\bar{Z}_x\right] = 0.0882}
\end{aligned}$$

$$\mathrm{Var}[L^*] = \left(1 + \frac{0.06}{0.07}\right)^2 \mathrm{Var}[0.0882] \quad \Longrightarrow \quad \boxed{\mathrm{Var}[L^*] = 0.3042}$$

218. (C3, May 2015 #12)

For a fully discrete 5-payment whole life insurance of 1000 on (40), you are given:

(i) Expenses incurred at the beginning of the first five policy years are as follows:

	Year 1		Years 2-5	
	Percent of premium	Per policy	Percent of premium	Per policy
Sales Commissions	20%	0	5%	0
Policy Maintenance	0%	10	0%	5

(ii) No expenses are incurred after Year 5.

(iii) Mortality follows the Illustrative Life Table.

(iv) $i = 0.06$

Calculate the gross premium using the equivalence principle.

Solution:

219. (C3, May 2015 #17)

For a fully discrete whole life insurance policy on (61), you are given:

(i) The annual gross premium using the equivalence principle is 500.

(ii) Initial expenses, incurred at policy issue, are 15% of the premium.

(iii) Renewal expenses, incurred at the beginning of each year after the first, are 3% of the premium.

(iv) Mortality follows the Illustrative Life Table.

(v) $i = 0.06$

Calculate the amount of the death benefit.

Solution:

220. (C3 Nov 2015 #03)
Johnny Vegas performs motorcycle jumps throughout the year and has injuries in the course of his shows according to the following three-state Markov model:

State 0: No injuries

State 1: Exactly one injury

State 2: At least two injuries

You are given:

(i) Transition intensities between States are per year.

(ii) $\mu_t^{01} = 0.03 + 0.06 \cdot 2^t$

(iii) $\mu_t^{02} = 2.718\mu_t^{01}$

(iv) $\mu_t^{12} = 0.025$

Calculate the probability that Johnny, who currently has no injuries, will sustain at least one injury in the next year.

Solution:

$$\text{ANS} = 1 - p^{\overline{00}} = 1 - \exp\left(-\int_0^1 (1 + 2.718) \cdot \left(0.03 + 0.06 \cdot 2^t\right)\, dt\right) = 0.351684$$

Note: The sum of those hazard rates is a form of Makeham's law and so the result could be calculated using the tables provided on the Exam.

221. (C3, Nov 2015 #07)

Cathy purchases a fully discrete whole life insurance policy of 100,000 on her 35th birthday.

You are given:

(i) The annual gross premium, calculated using the equivalence principle, is 1770.

(ii) The expenses in policy year 1 are 50% of premium and 200 per policy.

(iii) The expenses in policy years 2 and later are 10% of premium and 50 per policy.

(iv) All expenses are incurred at the beginning of the policy year.

(v) $i = 0.035$

Calculate $\ddot{a}_{35}$.

Solution:

222. (C3, Nov 2015 #08)

For a fully discrete whole life insurance of 100 on (x), you are given:

(i) The first year expense is 10% of the gross annual premium.

(ii) Expenses in subsequent years are 5% of the gross annual premium.

(iii) The gross premium calculated using the equivalence principle is 2.338.

(iv) $i = 0.04$

(v) $\ddot{a}_x = 16.50$

(vi) ${}^2A_x = 0.17$

Calculate the variance of the loss at issue random variable.

Solution:

223. (MLC Nov 2015,#9)
For a 2-year deferred, 2-year term insurance of 2000 on [65], you are given:

(a) The following select and ultimate mortality table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
65	0.08	0.10	0.12	0.14	68
66	0.09	0.11	0.13	0.15	69
67	0.10	0.12	0.14	0.16	70
68	0.11	0.13	0.15	0.17	71
69	0.12	0.14	0.16	0.18	72

(b) $i = 0.04$

(c) The death benefit it payable at the end of the year of death.

Calculate the actuarial present value of this insurance.

Solution:

$$2000\left({}_{2|}A^{\;1}_{[65]:\overline{2}|}\right) = 2000 \cdot \left(\frac{{}_{2|}q_{[65]}}{1.04^3} + \frac{{}_{3|}q_{[65]}}{1.04^4}\right)$$

$$\begin{aligned} {}_{2|}q_{[65]} &= p_{[65]}p_{[65]+1}q_{[65]+2} = (1-0.08)\cdot(1-0.10)\cdot 0.12 = 0.099360, \\ {}_{3|}q_{[65]} &= p_{[65]}p_{[65]+1}p_{[65]+2}q_{[65]+3} = (1-0.08)\cdot(1-0.10)\cdot(1-0.12)\cdot 0.14 = 0.1020096 \end{aligned}$$

$$2000\left({}_{2|}A^{\;1}_{[65]:\overline{2}|}\right) = 2000 \cdot \left(\frac{0.099360}{1.04^3} + \frac{0.1020096}{1.04^4}\right) = 351.06$$

224. (based on: MLC Nov 2015,#9)
You are given the following select and ultimate mortality table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
65	0.08	0.10	0.12	0.14	68
66	0.09	0.11	0.13	0.15	69
67	0.10	0.12	0.14	0.16	70
68	0.11	0.13	0.15	0.17	71
69	0.12	0.14	0.16	0.18	72

(a) Calculate ${}_{2|}q_{[65]}$

(b) Calculate the probability that a person *selected* on the day she turns 65 dies between the day she turns 68 and the day she turns 69.

Solution:

(a)

$$\begin{aligned} {}_{2|}q_{[65]} &= {}_{2}p_{[65]} \cdot q_{[65]+2} = p_{[65]} \cdot p_{[65]+1} \cdot q_{[65]+2} \\ &= (1 - q_{[65]}) \cdot (1 - q_{[65]+1}) \cdot q_{[65]+2} \\ &= (1 - 0.08) \cdot (1 - 0.10) \cdot 0.12 = \boxed{0.09936} \end{aligned}$$

(b) The problem is asking us for ${}_{3|}q_{[65]}$.

$$\begin{aligned} {}_{3|}q_{[65]} &= {}_{3}p_{[65]} \cdot q_{[65]+2} = p_{[65]} \cdot p_{[65]+1} \cdot p_{[65]+2} \cdot q_{[65]+3} \\ &= (1 - q_{[65]}) \cdot (1 - q_{[65]+1}) \cdot (1 - q_{[65]+2}) \cdot q_{68} \\ &= (1 - 0.08) \cdot (1 - 0.10) \cdot (1 - 0.12) \cdot 0.14 = \boxed{0.10201} \end{aligned}$$

225. (Spring 2016, #1)
You are using a Markov model for the future repair status of televisions. The three states are:

Fully Functional (F),

Requires Repairs (R), and

Beyond Repair (B).

You are given:

(a) The following annual probability transition matrix: $\begin{bmatrix} 0.82 & 0.10 & 0.08 \\ 0.60 & 0.05 & 0.35 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}$

(b) There are now five televisions that are Fully Functional.

(c) The status of each television is independent of the status of the others.

Calculate the probability that exactly two of these five televisions will be Fully Functional at the end of two years.

Solution:
The number of TV's functional after two years follows a Binomial(5,p) distribution where p is given by the top left entry of the 1-year transition matrix squared.
That entry is: $0.82^2 + 0.10 \cdot 0.6 + 0.08 \cdot 0 = 0.7324$ so this is the value of p.

$$\Pr\{N = 2\} = 0.102791.$$

226. (MLC May 2016, #2)

You are given the survival function:

$$S_0(x) = \left(1 - \frac{x}{60}\right)^{1/3}, \qquad 0 \le x \le 60.$$

Calculate $1000\,\mu_{35}$

Solution:

$$\begin{aligned}
\frac{-d}{dx}S_0(x) &= \frac{1}{3 \cdot 60}\left(1 - \frac{x}{60}\right)^{-2/3} \\
\mu_x &= \frac{\frac{-d}{dx}S_0(x)}{S_0(x)} = \frac{1}{3 \cdot 60}\left(1 - \frac{x}{60}\right)^{-2/3-1/3} = \frac{1}{3 \cdot 60 \cdot (1 - x/60)} \\
& \boxed{\mu_x = \frac{1}{3 \cdot (60 - x)}} \\
\mu_{35} &= \frac{1}{3 \cdot 25} = \frac{1}{75} = \boxed{0.013333333}
\end{aligned}$$

227. (C3, May 2016, #4)
A 5-year sickness insurance policy is based on a Markov model

(diagram showing the usual "healthy-sick-dead" model)

with:

State number	0	1	2
Meaning	Healthy	Sick	Dead

You are given the following constant forces of transition:

(a) $\mu^{01} = 0.05$
(b) $\mu^{10} = 0.02$
(c) $\mu^{02} = 0.01$
(d) $\mu^{12} = 0.06$

Calculate the probability that a Healthy life will become Sick exactly once during the 5 years and remain continuously Sick from that point until the end of the 5 years.

Solution:
Use equation [DHW(2nd ed), equation (8.11), p. 252)] which for this problem reads as

$${}_5p_x^{01} = \int_0^5 e^{-6t/100} \cdot \frac{5}{100} \cdot e^{-8(5-t)/100}\, dt = 0.1762454$$

In the usual "healthy-sick-dead" model, transitions from state 1 to state 0 are possible. However, in this problem they are not: the problem asks for *"the probability that a Healthy life will become Sick exactly once during the 5 years and remain continuously Sick ...."*
Equation (8.11) pertains to the "permanent disability" model, in which state 0 can never be re-entered after it's been left, so it's appropriate.
To really understand this solution we have to really understand the derivation of (8.11).

228. (C3 May 2016, #5)
For a select and ultimate mortality model with a one-year select period, you are given:

(a) $p_{[x]} = (1+k)p_x$, for some constant k.

(b) $\ddot{a}_{x:\overline{n}|} = 21.854$

(c) $\ddot{a}_{[x]:\overline{n}|} = 22.167$

Calculate k.

Solution:

$\ddot{a}_{x:\overline{n}|} = 1 + vp_x \ddot{a}_{x+1:\overline{n-1}|}$

Since the select period is only one year, ${}_kE_{[x]} = {}_kE_x$ for all k except $k=1$. So,

$$\begin{aligned}
\ddot{a}_{[x]:\overline{n}|} = \sum_{j=0}^{n-1} {}_jE_{[x]} &= 1 + \sum_{j=1}^{n-1} {}_jE_{[x]} = 1 + \sum_{i=0}^{n-2} {}_{i+1}E_{[x]} \\
&= 1 + vp_{[x]} \sum_{i=0}^{n-2} {}_iE_{[x]+1} \\
&= 1 + vp_{[x]} \sum_{i=0}^{n-2} {}_iE_{x+1} = 1 + vp_{[x]}\ddot{a}_{x+1:\overline{n-1}|} = 1 + vp_x(1+k)\ddot{a}_{x+1:\overline{n-1}|}
\end{aligned}$$

$$\underline{\ddot{a}_{[x]:\overline{n}|} = 1 + vp_x(1+k)\ddot{a}_{x+1:\overline{n-1}|}}$$

$$\begin{aligned}
\ddot{a}_{[x]:\overline{n}|} &= 1 + vp_x(1+k)\ddot{a}_{x+1:\overline{n-1}|} \\
&= (1 + vp_x\ddot{a}_{x+1:\overline{n-1}|}) + k(\ddot{a}_{x:\overline{n}|} - 1) \\
\ddot{a}_{[x]:\overline{n}|} &= \ddot{a}_{x:\overline{n}|} + k(\ddot{a}_{x:\overline{n}|} - 1)
\end{aligned}$$

$$k = \frac{\ddot{a}_{[x]:\overline{n}|} - \ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|} - 1} = \boxed{0.01545}$$

229. (C3 May 2016 #8)

For a fully continuous 20-year term insurance policy of 100,000 on (50), you are given:

(i) Gross premiums, calculated using the equivalence principle, are payable at an annual rate of 4500.

(ii) Expenses at an annual rate of R are payable continuously throughout the life of the policy.

(iii) $\mu_{50+t} = 0.04$, for $t > 0$

(iv) $\delta = 0.08$

Calculate R.

Solution:

From the Equiv. Princ. we have

$$0 = \overbrace{100{,}000\bar{A}^{1}_{50:\overline{20}|}}^{\text{death ben.}} + \overbrace{R\bar{a}_{50:\overline{20}|}}^{\text{expenses}} - \overbrace{4500\bar{a}_{50:\overline{20}|}}^{\text{premiums}}$$

Using the constant forces,

$$\begin{aligned}
\bar{A}^{\,1}_{50:\overline{25}|} &= \bar{A}_{50} - {}_{20}E_{50}\bar{A}_{70} = \frac{\mu}{\mu+\delta} - e^{-20(\delta+\mu)}\frac{\mu}{\mu+\delta} \\
&= \frac{0.04}{0.04+0.08} - e^{-20(0.08+0.04)}\frac{0.04}{0.04+0.08} \\
&= 0.303094
\end{aligned}$$

$$\begin{aligned}
\bar{a}_{50:\overline{20}|} &= \frac{1}{\mu+\delta} - e^{-20(\delta+\mu)}\frac{1}{\mu+\delta} \\
&= = \frac{1}{0.04+0.08} - e^{-20(0.08+0.04)}\frac{1}{0.04+0.08} \\
&= 7.57735,
\end{aligned}$$

so that

$$\begin{aligned}
0 &= 100,000\bar{A}^{\,1}_{50:\overline{20}|} + R\bar{a}_{50:\overline{20}|} - 4500\bar{a}_{50:\overline{20}|} \\
0 &= 100,000 \cdot 0.303094 - 4500 \cdot 7.57735 + R \cdot 7.57735 \\
& \boxed{R = 500}
\end{aligned}$$

230. (C3 May 2016, #9)

For a fully discrete whole life insurance policy of 50,000 on (35), with premiums payable for a maximum of 10 years, you are given:

(i) Expenses of 100 are payable at the end of each year including the year of death.

(ii) Mortality follows the Illustrative Life Table.

(iii) $i = 0.06$

Calculate the annual gross premium using the equivalence principle.

Solution:
The expenses comprise a 100/year life annuity immediate on (35). If the gross annual premium is P, then

$$\begin{aligned}
0 &= \mathrm{E}\left[L_0^g\right] = 50,000A_{35} + 100a_{35} - P\ddot{a}_{35:\overline{10}|} \\
0 &= 50,000A_{35} + 100(\ddot{a}_{35} - 1) - P(\ddot{a}_{35} - {}_{10}E_{35}\ddot{a}_45) \\
0 &= 50,000 \cdot 0.12872 + 100(15.3926 - 1) - P(15.3926 - 0.54318 \cdot 14.1121) \\
& \boxed{P = 1019.61}
\end{aligned}$$

231. (C3, May 2016, #17)
For a special fully discrete whole life insurance, you are given:

(a) The death benefit is $1000(1.03)^k$ for death in policy year k , for $k = 1, 2, 3, \ldots$.

(b) $q_x = 0.05$

(c) $i = 0.06$

(d) $\ddot{a}_{x+1} = 7.00$

(e) The annual net premium for this insurance at issue age x is 110.

Calculate the annual net premium for this insurance at issue age $x + 1$.

Solution:
Denote the annual net premium for this insurance at issue ages x and $x+1$ by "P_x" and "P_{x+1}", respectively, and the APV of the geometrically increasing death benefit by $1000(GA)_x$. Then

$$(GA)_x = \mathrm{E}\left[(1.03\,v)^{K_x+1}\right] = \mathrm{E}\left[v_*^{K_x+1}\right] = A^*_x,$$

where A^* is APV of a unit whole-life insurance, computed with $v(= 1/1.06)$ replaced by $\underline{v_* = 1.03/1.06}$.

This says that the APV A^*_x of the geometrically increasing death benefit is mathematically the same as the APV A_x of a fixed-amount death benefit, but with a different interest rate.

Now:

(a) Use recursion to get $\ddot{a}_x$ from $\ddot{a}_{x+1}$ (given).

$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1} = 1 + \frac{1-0.05}{1+0.06} \cdot 7$$
$$\ddot{a}_x = 7.273585$$

(b) Use $\ddot{a}_x$ and P_x (given) in the Equivalence Principle to get A^*_x.

$$\mathrm{E}\,[\text{Loss}] = 0 \quad = \quad 1000 \cdot A^*_x - P_x\ddot{a}_x = 1000 \cdot A^*_x - 110 \cdot 7.273585$$
$$A^*_x = 0.80009$$

(c) Use recursion to get A^*_{x+1} from A^*_x.

$$A^*_x = v_*q_x + v_*p_xA^*_{x+1}$$
$$A^*_{x+1} = 0.814103$$

(d) Use A^*_{x+1} and $\ddot{a}_{x+1}$ (given) in the Equivalence Principle to get P_{x+1}

$$\mathrm{E}\,[\text{Loss}] = 0 \quad = \quad 1000 \cdot A^*_{x+1} - P_{x+1}\ddot{a}_{x+1}$$
$$\boxed{P_{x+1} = 116.30046}$$

232. (MLC Nov 2016, # 1)
For the country of Bienna, you are given:

(a) Bienna publishes mortality rates in biennial form, that is, mortality rates are of the form: ${}_2q_{2x}$ for $x = 0, 1, 2, \ldots$.

(b) Deaths are assumed to be uniformly distributed between ages $2x$ and $2x + 2$, for $x = 0, 1, 2, \ldots$.

(c) ${}_2q_{50} = 0.02$

(d) ${}_2q_{52} = 0.04$

Calculate the probability that (50) dies during the next 2.5 years.

Solution:

$$\begin{aligned} {}_{2.5}q_{50} &= {}_2q_{50} + ({}_2p_{50})({}_{0.5}q_{52}) \\ &= {}_2q_{50} + ({}_2p_{50})(0.25 \cdot {}_2q_{52}) \quad \text{under 2-year UDD; } \tfrac{1}{2} \text{ is } \tfrac{1}{4} \text{ of a 2 year period} \\ &= 0.02 + 0.98 \cdot 0.25 \cdot 0.04 = \underline{0.0298} \end{aligned}$$

233. (MLC Nov 2016, # 2)
You are given the following survival function of a newborn:

$$S_0(x) = \begin{cases} 1 - \dfrac{x}{250} & \text{for } x \in [0, 40), \\ 1 - \left(\dfrac{x}{100}\right)^2 & \text{for } x \in [40, 100]. \end{cases}$$

Calculate the probability that (30) dies within the next 20 years.

Solution:

$$\begin{aligned} {}_{20}p_{30} &= \frac{S_0(50)}{S_0(30)} = \frac{\frac{3}{4}}{\frac{220}{250}} = \frac{75}{88} \\ {}_{20}q_{30} &= 1 - {}_{20}p_{30} = 1 - \frac{75}{88} = \frac{13}{88} = \underline{0.14772} \end{aligned}$$

234. (MLC Nov 2016, # 3)
In a population initially consisting of 75% females and 25% males, you are given:

(a) For a female, the force of mortality is constant and equals μ.

(b) For a male, the force of mortality is constant and equals 1.5μ.

(c) At the end of 20 years, the population consists of 85% females and 15% males.

Calculate the probability that a female survives one year.

Solution:

Hint: Here's a way to think about it. We know the initial male/female ratio, and we also know that ratio after 20 years. The ratio changes over time because the males have a higher force of mortality, with a given relationship between the forces of mortality for males and females. and that relationship is expressed with a single parameter, μ.

So if you can work out *an expression for the male/female ratio as it changes over time.* It should depend on time and on μ. Once you get this expression, you can use the remaining problem data, I think.

The probability of a female surviving one year is $e^{-\mu}$, so this is the number we need.

If F is the initial number of females and M is the initial number of males, then initially $\dfrac{M}{F} = \dfrac{1}{3}$. Over the twenty year period, $Fe^{-20\mu}$ of the females survive, and $Me^{-30\mu}$ of the males survive, and so:

$$\begin{aligned} \frac{15}{85} &= \frac{Me^{-30\mu}}{Fe^{-20\mu}} = \frac{M}{F}\frac{e^{-30\mu}}{e^{-20\mu}} = \frac{1}{3}e^{-10\mu} \\ \frac{9}{17} &= e^{-10\mu} \\ \left(\frac{9}{17}\right)^{1/10} &= e^{-\mu} = \boxed{0.93838} \end{aligned}$$

235. (C3 Nov 2016, #9)
For a special fully discrete 2-year term insurance on (x), you are given:

(a) $q_x = 0.01$

(b) $q_{x+1} = 0.02$

(c) $i = 0.05$

(d) The death benefit in the first year is 100,000.

(e) Both the benefits and premiums increase by 1% in the second year.

Calculate the annual net premium in the first year.

Solution:
From the equivalence principle:

$$\begin{aligned}
0 &= 100,000 \cdot \left(v \cdot {}_{0|}q_x + 1.01 \cdot v^2 \cdot {}_{1|}q_x\right) - P \cdot (1 + 1.01vp_x) \\
0 &= 100,000 \cdot \left(\frac{0.01}{1.05} + \frac{1.01 \cdot 0.0198}{1.05^2}\right) - P \cdot \left(1 + \frac{1.01(1-0.01)}{1.05}\right) \\
0 &= 2766.26 - 1.9428571P \qquad \Longrightarrow \underline{P = 1416.933},
\end{aligned}$$

where we've used the fact that ${}_{1|}q_x = (1-q_x)q_{x+1} = 0.0198$.

236. (C3 Nov 2016 #11)

For a fully discrete, 5-payment 10-year term insurance of 200,000 on (30), you are given:

(i) Mortality follows the Illustrative Life Table.

(ii) The following expenses are incurred at the beginning of each respective year:

	Year 1		Years 2 - 10	
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	5%	0	5%	0
Commissions	30%	0	10%	0
Maintenance	0%	8	0%	4

(iii) $i = 0.06$

(iv) $\ddot{a}_{30:\overline{5}|} = 4.4516$

(v) $\ddot{a}_{30:\overline{10}|} = 7.7465$

Calculate the annual gross premium using the equivalence principle.

Solution:

If P is the annual gross premium then the Equivalence Principle gives:

$$
\begin{aligned}
0 &= \overbrace{200,000A^{1}_{30:\overline{10}|}}^{\text{death ben.}} + \overbrace{4 + 4\ddot{a}_{30:\overline{10}|}}^{\text{maint. exp.}} + \overbrace{0.2P + 0.1P\ddot{a}_{30:\overline{5}|}}^{\text{commissions}} + \overbrace{0.05\ddot{a}_{30:\overline{5}|}}^{\text{taxes}} - \overbrace{P\ddot{a}_{30:\overline{5}|}}^{\text{premium pmts}} \\
0 &= 200,000A^{1}_{30:\overline{10}|} + 4 + 4\ddot{a}_{30:\overline{10}|} - P\cdot(0.85\ddot{a}_{30:\overline{5}|} - 0.2) \\
P &= \frac{200,000A^{1}_{30:\overline{10}|} + 4 + 4\ddot{a}_{30:\overline{10}|}}{0.85\ddot{a}_{30:\overline{5}|} - 0.2}
\end{aligned}
$$

237. (C3 Nov 2016, #18)
For a member of a defined contribution plan who is age 65, you are given:

(i) The member's current salary is 250,000, and there will be no future salary increases.

(ii) The member's current retirement fund is 1,500,000 and there are no expected future contributions prior to retirement.

(iii) The retirement fund grows at a rate of 8% annually.

(iv) Upon retirement, the member will use her retirement fund as the single net premium for an annuity due that makes quarterly payments.

(v) The single net premium is calculated using the Illustrative Life Table and $i = 0.06$.

Calculate, using the 2-term Woolhouse approximation, the member's replacement ratio upon retirement at age 66.

Solution:
The retirement fund will grow to $1.08 \cdot 1,500,000 = 1,620,000$ by the time the member retires. This will be used to pay for a life annuity with quarterly payments of, say, P year, so

$$1,620,000 \quad = \quad P\ddot{a}^{(4)}_{66} = P\overbrace{(\ddot{a}_{66} - 3/8)}^{\text{Woolhouse}} = P\cdot(9.63619 - 3/8) = P\cdot 9.26119$$

so $P = 174,923.54$. So her replacement ratio is $174,923.54/250,000 = \boxed{0.69969}$

238. (C3 May 2017 #01)
You are given the following extract from a triple decrement table:

x	$\ell_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
40	15,000	0.01	0.04	0.05
41	____	0.04	0.08	0.10

After the table was prepared, you discover that $q_{40}^{(1)}$ should have been 0.02, and that all other numerical values shown above are correct.

Calculate the resultant change in $d_{41}^{(3)}$

Solution:
Old values:

$$
\begin{aligned}
1 - \frac{\ell_{41}^{(\tau)}}{\ell_{40}^{(\tau)}} &= q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = 0.01 + 0.04 + 0.05 \qquad \Longrightarrow \qquad \underline{\ell_{40}^{(\tau)} = 13,500} \\
d_{41}^{(3)} &= \ell_{41}^{(\tau)} \cdot q_x^{(3)} = 13,500 \cdot 0.1 = 1,350
\end{aligned}
$$

New values:

$$
\begin{aligned}
1-\frac{\ell_{41}^{(\tau)}}{\ell_{40}^{(\tau)}} &= q_x^{(1)}+q_x^{(2)}+q_x^{(3)} = 0.02+0.04+0.05 \qquad \Longrightarrow \qquad \underline{\ell_{40}^{(\tau)} = 13,350} \\
d_{41}^{(3)} &= \ell_{41}^{(\tau)}\cdot q_x^{(3)} = 13,350\cdot 0.1 = 1,335
\end{aligned}
$$

Change in $d_{41}^{(3)}$ is $\boxed{\text{-15.}}$

239. (C3, May 2017, #3)
You are given the following Markov model:

diagram showing a 4-state model with arrows:

$$
\begin{gathered}
0\to 1, \quad 1\to 2, \\
0\to 3, \quad 1\to 3, \quad 2\to 3
\end{gathered}
$$

The forces of transition are the following:

- $\mu^{01} = 0.01$
- $\mu^{03} = 0.02$
- $\mu^{12} = 0.30$
- $\mu^{13} = 0.40$
- $\mu^{23} = 0.70$

Calculate the probability that a person in *(State 0)* today will be in *(State 1)* at the end of 5 years.

Solution:

Set $\mu^{00} = -\sum_{j=1}^{3}\mu^{0j} = 0.03$ and $\mu^{11} = -\sum_{\substack{j=0\\ j\neq 1}}\mu^{1j} = -0.7$. All other μ^{ij} not given are zero. So,

$$
\begin{aligned}
\frac{d}{dt}\,{}_tp_x^{00} &= \sum_{k=0}^{3}{}_tp_x^{0k}\,\mu^{k0} = -0.03\,{}_tp_x^{00} \qquad \text{with} \qquad {}_0p_x^{00} = 1 \\
\Longrightarrow \qquad & \underline{{}_tp_x^{00} = e^{-0.03t}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\,{}_tp_x^{01} &= \sum_{k=0}^{3}{}_tp_x^{0k}\,\mu^{k1} = 0.01\,{}_tp_x^{00} - 0.7\,{}_tp_x^{01} \qquad \text{with} \qquad {}_0p_x^{01} = 0 \\
\Longrightarrow {}_tp_x^{01} &= \int_0^t e^{-0.7(t-s)}\big(0.01\,{}_sp_x^{00}\big)\,ds \\
&= \int_0^t e^{-0.7(t-s)}\big(0.01e^{-0.03s}\big)\,ds \\
&= 0.01e^{-0.7t}\int_0^t e^{0.67s}\,ds \\
\Longrightarrow {}_5p_x^{01} &= 0.01e^{-3.5}\int_0^5 e^{0.67s}\,ds = \boxed{0.0123957}
\end{aligned}
$$

240. (C3 May 2017 #04)
You are given the following Markov model

(i) Annual transition probabilities are as follows:

	Healthy	Sick	Terminated
Healthy	0.90	0.05	0.05
Sick	0.30	0.60	0.10
Terminated	0.00	0.00	1.00

(ii) The annual health care costs each year, paid at the middle of the year, for each state, are:

Healthy: 500 Sick: 5000 Terminated:0

(iii) Transitions occur at the end of the year.

(iv) $\delta = 0.04$

Calculate the actuarial present value of health care costs over the next three years for an individual who is currently Healthy.

Solution:
An individual who is currently Healthy will still be classified as Healthy at mid-year, so the EPV of the first year's medical costs is $\underline{500 \cdot e^{-0.02} = \underline{490.09934}}$ ($v^{1/2}$ since the med. costs are paid at midyear).

At midway thru the 2nd year, an individual who is currently Healthy will then classified as

(H→H) Healthy with probability 0.9, and

(H→S) Sick with probability 0.05,

so the EPV of the second year's medical costs is $\underline{e^{-0.06}\,(0.9 \cdot 500 + 0.05 \cdot 5000){=}659.235}$

At midway thru the 3nd year, an individual who is currently Healthy will then classified as

(H→H→H) + (H→S→H) Healthy with probability $0.9^2 + 0.05 \cdot 0.30 = 0.825$, and

(H→H→S) + (H→S→S) Sick with probability $0.9 \cdot 0.05 + 0.05 \cdot 0.60 = 0.075$,

so the EPV of the second year's medical costs is $\underline{e^{-0.1}\,(0.825 \cdot 500 + 0.075 \cdot 5000){=}\ 712.559,}$

So the EPV of medical costs in the first 3 years is:

$$490.09934 + 659.235 + 712.559 = \boxed{1861.89}$$

241. (C3 May 2017, #6)
For an annual life annuity-due of 1 with a 5-year certain period on (55), you are given:

(a) Mortality follows the Illustrative Life Table

(b) $i = 0.06$

Calculate the probability that the sum of the undiscounted payments actually made under this annuity will exceed the expected present value, at issue, of the annuity.

Solution:
Let "a" denote the EPV of this annuity,

$$a \;=\; a_{\overline{5}|} + {}_5E_{55}\,\ddot{a}_{60} = \frac{1 - 1.06^{-5}}{0.06/1.06} + 0.70810 \cdot 11.1454 = 12.35716$$

So in order for the (undiscounted) sum of the payments to exeeed this value, there must be 13 payment, which means that (55) has survive at least 12 years, and the probability of this is

$${}_{12}p_{55} \;=\; \frac{7,201,634}{8,640,860} = \boxed{0.83344}$$

242. (C3 May 2017, #11)

For a fully discrete 10-payment whole life insurance of H on (45), you are given:

(i) Expenses payable at the beginning of each year are as follows:

Expense Type	First Year	Years 2-10	Years 11+
Per policy	100	20	10
% of Premium	105%	5%	0%

(ii) Mortality follows the Illustrative Life Table.

(iii) $i = 0.06$

(iv) $\ddot{a}_{45:\overline{10}|} = 7.649$

(v) The gross annual premium, calculated using the equivalence principle, is of the form:

$G = gH + f$

where g is the premium rate per 1 of insurance and f is the per policy fee.

Calculate f.

Solution:

$$\begin{aligned}
0 = \mathrm{E}\left[L_0^g\right] &= H \cdot (A_{45} + \overbrace{80 + 10 \cdot (\ddot{a}_{45:\overline{10}|} + \ddot{a}_{45}))}^{\text{per policy exp.}} + \overbrace{G \cdot (1 + 0.05 \cdot \ddot{a}_{45:\overline{10}|})}^{\text{\% prem exp.}} - G \cdot \ddot{a}_{45:\overline{10}|} \\
0 &= H \cdot A_{45} - G \cdot (0.95\ddot{a}_{45:\overline{10}|} - 1) + (80 + 10 \cdot (\ddot{a}_{45:\overline{10}|} + \ddot{a}_{45})) \\
G &= \frac{H \cdot A_{45} + (80 + 10 \cdot (\ddot{a}_{45:\overline{10}|} + \ddot{a}_{45}))}{0.95\ddot{a}_{45:\overline{10}|} - 1} \\
f &= \frac{80 + 10 \cdot (\ddot{a}_{45:\overline{10}|} + \ddot{a}_{45})}{0.95\ddot{a}_{45:\overline{10}|} - 1} \\
f &= \frac{80 + 10 \cdot (7.649 + 14.1121)}{0.95 \cdot 7.649 - 1} = \boxed{47.492.}
\end{aligned}$$

243. (C3 Nov 2017 #02)
You are given the following excerpt from a double decrement table:

x	$\ell_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$
53	____	0.025	0.030
54	5000	____	0.040
55	4625	0.055	0.050

Calculate ${}_2q_{53}^{(1)}$

Solution:

$$\begin{aligned}
\frac{5000}{\ell_{53}^{(\tau)}} = \frac{\ell_{54}^{(\tau)}}{\ell_{53}^{(\tau)}} &= 1-\left(q_{53}^{(1)}+q_{53}^{(2)}\right) = 1-(0.025+0.030) = 0.945 \quad \Longrightarrow \quad \underline{\ell_{53}^{(\tau)} = 5291} \\
\frac{4625}{5000} = \frac{\ell_{55}^{(\tau)}}{\ell_{54}^{(\tau)}} &= 1-\left(q_{54}^{(1)}+q_{54}^{(2)}\right) = 1-\left(q_{54}^{(1)}+0.04\right) \quad \Longrightarrow \quad \underline{q_{54}^{(1)} = 0.035}
\end{aligned}$$

$$\begin{aligned}
d_{53}^{(1)} &= 5291 \cdot 0.025 = \underline{132.275} \\
d_{54}^{(1)} &= 5000 \cdot 0.035 = \underline{175} \\
{}_2q_{53}^{(1)} &= \frac{d_{53}^{(1)}+d_{54}^{(1)}}{\ell_{53}^{(\tau)}} = \frac{132.275+175}{5291} = \boxed{0.058075}
\end{aligned}$$

244. (C3 Nov 2017 #03)
You are given the following Markov chain model:

(i) Annual transition probabilities between the states Healthy, Sick and Dead, of an organism are as follows:

	Healthy	Sick	Dead
Healthy	0.64	0.16	0.20
Sick	0.36	0.24	0.40
Dead	0	0	1

(ii) Transitions occur at the end of the year.

A population of 1000 organisms starts in the Healthy state. Their future states are independent.

Using the normal approximation without the continuity correction, calculate the probability that there will be at least 625 organisms alive (Healthy or Sick) at the beginning of the third year.

Solution:
The probability that there will be at least 625 organisms alive at the beginning of the third year is the same as the probability that at most 375 organisms have died by the beginning of the third year. If we let N denote the number of organisms that have died by the beginning of the third year, then we see $\Pr\{N \leq 375\}$. Since the organisms' lives are independent, $N \sim \text{Binomial}(1000, p)$ for some p.
To compute p, we consider a single Healthy organism, for which the probabilityof being Dead at the beginning of the third year is

$$\begin{array}{rl}
\text{H} \to \text{H} \to \text{D} & 0.64 \cdot 0.2 \\
\text{H} \to \text{S} \to \text{D} & 0.16 \cdot 0.4 \\
\text{H} \to \text{D} & 0.2 \\
\hline
 & p = 0.392
\end{array}$$

For a Binomial(1000,0.392) random variable,

$$\begin{aligned}
\mu &= np = 392, \\
\sigma &= \sqrt{np(1-p)} = 15.4381
\end{aligned}$$

Since $n = 1000$ is reasonably large, the normal approximation makes sense: If

$$Z = \frac{N - \mu}{\sigma}$$

then Z is approximately Normal$(0, 1)$-distributed, and

$$\begin{aligned} \Pr\{N \le 375\} &\approx \Pr\{Z \le z\} = \Phi(z), \qquad \text{where} \qquad z = \frac{372 - \mu}{\sigma} = -1.1012 \\ &= \Phi(-1.1012) = \boxed{0.1354} \end{aligned}$$

245. (C3 Nov 2017, #4)
For an annuity-due that pays 100 at the beginning of each year that (45) is alive, you are given:

(a) Mortality for standard lives follows the Illustrative Life Table.
(b) The force of mortality for standard lives age $45 + t$ is represented as μ_{45+t}^{ILT}
(c) The force of mortality for substandard lives age $45 + t$, μ_{45+t}^{S}, is defined by

$$\mu_{45+t}^{S} = \begin{cases} \mu_{45+t}^{ILT} + 0.05 & \text{for } 0 \le t < 1, \\ \mu_{45+t}^{ILT} & \text{for } t \ge 1 \end{cases}$$

(d) $i = 0.06$

Calculate the actuarial present value of this annuity for a substandard life age 45.

Solution:
The key is to recognize that the only difference in mortality rates between standard is and substandard is in the first year past age 45, $45 + t$ for $0 \le t < 1$.

$$\begin{aligned} p_{45}^{S} &= \exp\left(-\int_0^1 \mu_{45+t}^{S}\,dt\right) = \exp\left(-\int_0^1 \mu_{45+t}^{ILT} + 0.05\,dt\right) = e^{-0.05}\exp\left(-\int_0^1 \mu_{45+t}^{ILT}\,dt\right) \\ p_{45}^{S} &= e^{-0.05} p_{45}^{ILT} \end{aligned}$$

Now $\ddot{a}_{46}^{S} = a_{46}^{ILT}$ because neither of these depend on p_{45}. So,

$$\ddot{a}_{45}^{S} = 1 + vp_{45}^{S}\ddot{a}_{46}^{S} = 1 + e^{-0.05}vp_{45}^{ILT}\ddot{a}_{46}^{ILT} = 1 + e^{-0.05}\frac{0.996}{1.06} \cdot 13.954588$$

$$\boxed{100\ddot{a}_{45}^{S} = 1347.26}$$

246. (C3, Nov 2017 #08)

For a 10-year deferred whole life annuity-due with payments of 100,000 per year on (70), you are given:

(i) Annual gross premiums of G are payable for 10 years.

(ii) First year expenses are 75% of premium.

(iii) Renewal expenses for years 2 and later are 5% of premium during the premium paying period.

(iv) Mortality follows the Illustrative Life Table.

(v) $i = 0.06$

Calculate G using the equivalence principle.

Solution:

$$\begin{aligned} 0 = \mathrm{E}\left[L_0^g\right] &= \overbrace{100,000 \cdot {}_{10|}\ddot{a}_{70}}^{\text{EPV of annuity pmts}} + \overbrace{0.7G + 0.05G \cdot \ddot{a}_{70:\overline{10}|}}^{\text{EPV of expenses}} - \overbrace{G \cdot \ddot{a}_{70:\overline{10}|}}^{\text{EPV of premiums}} \\ 0 &= 100,000 \cdot {}_{10|}\ddot{a}_{70} - (0.95\ddot{a}_{70:\overline{10}|} - 0.7)G \end{aligned}$$

The following values are from the ILT:

$$\ddot{a}_{70} = 8.5693, \qquad \ddot{a}_{80} = 5.9050, \qquad {}_{10}E_{70} = 0.33037.$$

From these we can compute

$$\begin{aligned} {}_{10|}\ddot{a}_{70} &= {}_{10}E_{70} \cdot \ddot{a}_{80} = 0.33037 \cdot 5.9050 = 1.95083 \qquad \text{and} \\ \ddot{a}_{70:\overline{10}|} &= a_{70} - {}_{10|}\ddot{a}_{70} = 8.5693 - 1.95083 = 6.61847 \end{aligned}$$

so that

$$0 = 100,000 \cdot 1.95083 - (0.95 \cdot 6.61847 - 0.7)G,$$

so $\boxed{\text{G=34914}}$.

247. (C3 Nov 2017, #9)
For a special fully discrete 5-year deferred 3-year term insurance of 100,000 on (x), you are given:

(a) There are two premium payments, each equal to P . The first is paid at the beginning of the first year and the second is paid at the end of the 5-year deferral period.

(b) The following probabilities:

$${}_5p_x = 0.95, \qquad q_{x+5} = 0.02, \qquad q_{x+6} = 0.03, \qquad q_{x+7} = 0.04.$$

(c) $i = 0.06$

Calculate P using the equivalence principle.

Solution:

$$0 = 10^5\Big[v^6\ {}_{5|}q_x + v^7\ {}_{6|}q_x + v^8\ {}_{7|}q_x\Big] - P\Big[1 + {}_5E_x\Big]$$

$$\begin{aligned}
{}_{5|}q_x &= {}_5p_x \cdot q_{x+5} = 0.019 \\
{}_{6|}q_x &= {}_5p_x \cdot p_{x+5} \cdot q_{x+6} = 0.02793 \\
{}_{7|}q_x &= {}_5p_x \cdot p_{x+5} \cdot p_{x+6} \cdot q_{x+7} = 0.0361228
\end{aligned}$$

$$\begin{aligned}
0 &= 10^5\left[\frac{0.019}{1.06^6} + \frac{0.02793}{1.06^7} + \frac{0.0361228}{1.06^8}\right] - P \cdot \left[1 + \frac{0.95}{1.06^5}\right] \\
& P = 10^5 \cdot \frac{0.0546\,3319}{1.7098\,9526} = \boxed{3195.12}
\end{aligned}$$

248. (C3, Nov 2017 #10)

For a special whole life insurance of 100,000 on (40), you are given:

(i) The death benefit is payable at the moment of death.

(ii) Level gross premiums are payable monthly for a maximum of 20 years.

(iii) Mortality follows the Illustrative Life Table.

(iv) $i = 0.06$

(v) Deaths are uniformly distributed over each year of age.

(vi) Initial expenses are 200.

(vii) Renewal expenses are 4% of each premium including the first.

(viii) Gross premiums are calculated using the equivalence principle.

Calculate the monthly gross premium.

Solution:
If P is the monthly gross premium then the Equivalence Principle gives:

$$0 = \mathrm{E}\left[L_0^g\right] = \overbrace{100,000\bar{A}_{40}}^{\text{death ben.}} + \overbrace{200 + 0.04 \cdot 12P\ddot{a}^{(12)}_{40:\overline{20}|}}^{\text{expenses}} - \overbrace{12P\ddot{a}^{(12)}_{40:\overline{20}|}}^{\text{premiums}}$$

$$0 = 100,000\bar{A}_{40} + 200 - 0.96 \cdot 12P\ddot{a}^{(12)}_{40:\overline{20}|}$$

Under UDD,

$$
\begin{aligned}
\bar{A}_{40} &= \frac{i}{\delta} A_{40} = \frac{0.06}{\log(1.06)} \cdot 0.161324 = 0.166117 \\
\ddot{a}^{(12)}_{40:\overline{20}|} &= \ddot{a}^{(12)}_{40} - {}_{20}E_{40} \cdot \ddot{a}^{12}_{60} \\
&= \Big(\alpha(12)\ddot{a}_{40} - \beta(12)\Big) - {}_{20}E_{40} \cdot \Big(\alpha(12)\ddot{a}_{60} - \beta(12)\Big) \\
&= 11.4247
\end{aligned}
$$

so that

$$
\begin{aligned}
0 &= 100,000 \cdot 0.166117 + 200 - 0.96 \cdot 12P \cdot 11.4247 \\
& \boxed{P = 127.74}
\end{aligned}
$$

249. (C3, Nov 2018 #13)

An insurer issues a special 10-year deferred whole life annuity-due to (55). You are given:

(i) The annual annuity payment is 50,000.

(ii) There is an additional lump sum benefit of 200,000 payable on survival to age 80.

(iii) Level annual premiums are payable during the deferred period.

(iv) First year commissions are 20% of premium.

(v) Renewal commissions for the second and subsequent years are 5% of premium.

(vi) Mortality follows the Standard Ultimate Life Table.

(vii) Premiums are calculated using the equivalence principle.

(viii) $i = 0.05$

Calculate the gross premium.

Solution:

$$0 = \overbrace{50,000 \cdot {}_{10|}\ddot{a}_{55}}^{\text{Annuty pmt}} + \overbrace{200,000 \cdot {}_{25}E_{55}}^{\text{Survival ben.}} + \overbrace{P(0.15 + 0.05 \cdot \ddot{a}_{55:\overline{10}|})}^{\text{Commissions}} - \overbrace{P \cdot \ddot{a}_{55:\overline{10}|}}^{\text{Premium income}}$$

$$0 = 50,000 \cdot {}_{10|}\ddot{a}_{55} + 200,000 \cdot {}_{25}E_{55} - P(0.95 \cdot \ddot{a}_{55:\overline{10}|} - 0.15)$$

$$P = \frac{50,000 \cdot {}_{10|}\ddot{a}_{55} + 200,000 \cdot {}_{25}E_{55}}{0.95 \cdot \ddot{a}_{55:\overline{10}|} - 0.15}$$

$$\vdots$$

$$P = 59,950 \qquad \text{value from SOA's solution}$$

250. (C3 Nov 2018, #15)
For a special 20-year term insurance of 200,000 with level monthly premiums issued to (65), you are given:

(i) The death benefit is paid six months after the death of the policyholder.
(ii) Mortality follows the Standard Ultimate Life Table.
(iii) Woolhouse's formula with two terms is used to calculate $(1/m)$thly and continuous annuities, and to calculate insurance functions from the annuity values.
(iv) The monthly net premium is 274.
(v) $i = 0.05$

Calculate the net premium reserve at time 10.

Solution:
The APV of future benefits at time $t = 10$ is $200,000 \cdot v^{1/2} \bar{A}^{\,1}_{75:\overline{10}|}$ (the $v^{1/2}$ here reflects the 6-month delay in payment) and the APV of future premiums at time $t = 10$ is $12 \cdot 274 \ddot{a}^{(12)}_{75:\overline{10}|}$, so

$${}_{10}V = 200,000 \cdot v^{1/2} \bar{A}^{\,1}_{75:\overline{10}|} - 12 \cdot 274 \ddot{a}^{(12)}_{75:\overline{10}|}.$$

Now

$$\begin{aligned}
\bar{A}^{\,1}_{75:\overline{10}|} &= \bar{A}_{75:\overline{10}|} - {}_{10}E_{75} = \Big(1 - \delta \bar{a}_{75:\overline{10}|}\Big) - {}_{10}E_{75} = \Big(1 - \delta\left(\bar{a}_{75} - {}_{10}E_{75}\bar{a}_{85}\right)\Big) - {}_{10}E_{75} \\
&\approx 1 - \delta\left((\ddot{a}_{75} - 1/2) - {}_{10}E_{75}\left(\ddot{a}_{85} - 1/2\right)\right) - {}_{10}E_{75} \\
&= 1 - \ln(1.05)\left((10.3178 - 1/2)\right) - 0.44085\,(6.7993 - 1/2)) - 0.44085 = \underline{0.215630},
\end{aligned}$$

and

$$\begin{aligned}
\ddot{a}^{(12)}_{75:\overline{10}|} = \ddot{a}^{(12)}_{75} - {}_{10}E_{75}\ddot{a}^{(12)}_{85} &\approx (\ddot{a}_{75} - 11/24) - {}_{10}E_{75}\left(\ddot{a}_{85} - 11/24\right) \\
&= (10.3178 - 11/24) - 0.44085\,(6.7993 - 11/24) = \underline{7.064052}
\end{aligned}$$

so

$$\begin{aligned}
{}_{10}V &= 200,000 \cdot v^{1/2} \bar{A}^{\,1}_{75:\overline{10}|} - 12 \cdot 274 \ddot{a}^{(12)}_{75:\overline{10}|} \\
&= \frac{200,000}{1.05^{1/2}} \cdot 0.215630 - 12 \cdot 274 \cdot 7.064052 = \boxed{18860.06}
\end{aligned}$$

251. (LTAM, May 2019, #3)
You are given:

- A life table uses a Makeham mortality model with parameters $A = 0.00022, B = 2.7 \times 10^{-6}$, and $c = 1.124$.
- ${}_{10}p_{50} = 0.9803$.

Calculate $\dfrac{d}{dt}{}_tq_{50}$ at $t = 10$.

Solution:
We're asked to compute the derivative of the CDF of T_{50} at a $t = 10$, which is the PDF of T_{50} at $t = 10$, which we know is given by ${}_{10}p_{50}\mu_{60}$. So,

$$\frac{d}{dt}{}_tq_{50}\bigg|_{10} = 0.9803 \cdot \frac{d}{dt}\mu_{x+t}\bigg|_{x+t=60}$$

For the Makeham model,

$$\begin{aligned} \mu_{x+t} &= A(x+t) + Bc^{x+t} \\ \mu_{60}\bigg|_{z=60} &= 0.00022 + (2.7 \times 10^{-6}) \cdot 1.124^{60} = 0.00322153 \end{aligned}$$

So $\dfrac{d}{dt}{}_tq_{50}\bigg|_{10} = 0.9803 \cdot 0.00322153 = \boxed{0.003168}$

252. (C3, May 2019 #13)

QMX Life issues a fully discrete 20-year deferred life annuity-due to (45). You are given:

(i) Premiums are payable at the beginning of each year during the deferral period.

(ii) Annuity payments are 75,000 annually.

(iii) Expenses are as follows:

Expense Type	Year 1	Years 2-20	Years 21+
Percent of Premium	35%	5%	0%
Per Policy	2,000	50	150

(iv) Mortality follows the Standard Ultimate Life Table.

(v) $i = 0.05$

Calculate the gross premium for this policy using the equivalence principle.

Solution:

253. (C3, May 2019 #14)

For a fully discrete 10-year endowment insurance of 300,000 on (50), you are given:

(i) The gross annual premium is 26,470.

(ii) Initial non-commission expenses are 17,500.

(iii) Commissions are 5% of all premiums.

(iv) Mortality follows the Standard Ultimate Life Table.

(v) $i = 0.05$

(vi) L_0 denotes the loss-at-issue random variable.

Calculate $\Pr[L_0 < 0]$.

Solution:

254. (LTAM, Nov 2019, #1)
You are given that ${}_tp_{40} = \exp\left(-0.06 \cdot (1.12^t - 1)\right)$ for all $t \geq 0$. Calculate the probability that a life age 45 survives 10 years.

Solution:

$$
\begin{aligned}
{}_{15}p_{40} = {}_5p_{40} \cdot {}_{10}p_{45} \quad \Longrightarrow \quad {}_{10}p_{45} &= \frac{{}_{15}p_{40}}{{}_5p_{40}} \\
&= \frac{\exp\left(-0.06 \cdot (1.12^{15} - 1)\right)}{\exp\left(-0.06 \cdot (1.12^5 - 1)\right)} \\
&= \boxed{0.800376}
\end{aligned}
$$

255. (C3 Nov 2019, #8)
You are given the following information:

(i) Woolhouse's 3-term formula is used to calculate values for $(1/m)$-thly functions.

(ii) Mortality follows the Standard Ultimate Life Table.

(iii) In the Standard Ultimate Life Table: $\mu_{80} = 0.030162$ and $\mu_{90} = 0.096590$.

(iv) $i = 0.05$

Calculate $20,000\ddot{a}^{(4)}_{80:\overline{10}|}$.

Solution:

$$\begin{aligned}
\ddot{a}^{(4)}_{80:\overline{10}|} &= \ddot{a}^{(4)}_{80} - {}_{10}E_{80}\,\ddot{a}^{(4)}_{90} \\
&= \left(\ddot{a}_{80} - \frac{3}{8} - \frac{15}{192}(\delta + \mu_{80})\right) - {}_{10}E_{80}\left(\ddot{a}_{90} - \frac{3}{8} - \frac{15}{192}(\delta + \mu_{90})\right) \\
&= \left(8.5484 - \frac{3}{8} - \frac{15}{192}(\ln(1.05) + 0.030162)\right) \\
&\quad -0.33952\left(5.1835 - \frac{3}{8} - \frac{15}{192}(\ln(1.05) + 0.096590)\right) \\
&= 8.167231862 - 0.33952 \cdot 4.797142175 = \underline{6.538506}
\end{aligned}$$

so $20,000\ddot{a}^{(4)}_{80:\overline{10}|} = 20,000 \cdot 6.538506 = \boxed{130,770.12}$

256. (C3, Nov 2019 #11)

An insurer issues a 10-year term insurance policy with a sum insured of 1,000,000 to (50).

You are given the following information:

(i) The sum insured is paid at the end of the month of death.

(ii) Premiums are payable monthly.

(iii) Initial expenses, payable at the start of the contract, are 60% of the annualized premium.

(iv) Maintenance expenses are 5% of premiums including the first.

(v) Mortality follows the Standard Ultimate Life Table.

(vi) Deaths are uniformly distributed between integer ages.

(vii) $i = 0.05$

(viii) Premiums are determined using the equivalence principle.

Calculate the monthly premium for this policy.

Solution:
Let P be the monthly premium.

$$0 = \overbrace{10^6 \cdot A^{(12)}_{\substack{1\\50:\overline{10}|}}}^{\text{Death Ben.}} + \overbrace{0.6 \cdot 12P + 0.05 \cdot 12P\ddot{a}^{(12)}_{50:\overline{10}|}}^{\text{\% Premium exp.}} - \overbrace{12P\ddot{a}^{(12)}_{50:\overline{10}|}}^{\text{Premium Income}} \tag{15}$$

Pay careful attention to the way we model item (iii).

From (15),

$$\begin{aligned}
0 &= 10^6 \cdot A^{(12)}_{\substack{1\\50:\overline{10}|}} + 0.6 \cdot 12P + 0.05 \cdot 12P\ddot{a}^{(12)}_{50:\overline{10}|} - 12P\ddot{a}^{(12)}_{50:\overline{10}|} \\
12P &= \frac{10^6 \cdot A^{(12)}_{\substack{1\\50:\overline{10}|}}}{0.95 \cdot \ddot{a}^{(12)}_{50:\overline{10}|} - 0.6} \\
&= \ldots \qquad \text{(SOA's calculations.)} \\
P &= \boxed{181.05}
\end{aligned}$$

257. (LTAM May 2020, #3)
For a special fully discrete 3-year term insurance with level premiums on (x), you are given:

(i) The death benefits, b_{k+1}, and mortality rates are:

k	b_{k+1}	q_{x+k}
0	100,000	0.03
1	200,000	0.05
2	300,000	0.07

(ii) $i = 0.06$

Calculate the net premium reserve at the end of the first policy year.

Solution:
We have to determine the premium using the equivalence principle, and since the benefit is varying we have to use the basic definition of expected value Z of the PV of the benefit, $\mathrm{E}[Z] = \sum_{k=0}^{2} b_{k+1} v^{k+1} {}_{k|}q_x$. So we need the values of ${}_{k|}q_x$, then compute $\mathrm{E}[Z]$ and also the expected present value $\mathrm{E}[Y]$ of the premiums, and then apply the equivalence principle.

$$\begin{aligned}
{}_{0|}q_x &= q_x = 0.03 \\
{}_{1|}q_x &= p_x q_{x+1} = 0.97 \cdot 0.05 = 0.0485 \\
{}_{2|}q_x &= {}_2p_x q_{x+2} = 0.97 \cdot 0.95 \cdot 0.07 = 0.064505 \\
\mathrm{E}[Z] &= 100,000 \cdot \left(\frac{0.03}{1.06} + 2\frac{0.0485}{1.06^2} + 3\frac{0.064505}{1.06^3} \right) = 27,711.05 \\
\mathrm{E}[Y] &= \sum_{k=0}^{2} {}_kE_x = 1 + \frac{(1-0.03)}{1.06} + \frac{(1-0.03)(1-0.05)}{1.06^2} = 2.73522606, \\
0 &= \mathrm{E}[Z] - P\mathrm{E}[Y] \qquad \Longrightarrow \qquad \underline{P = 10,131.20}
\end{aligned}$$

Now we can compute ${}_1V$ from the defintion, ${}_1V = \mathrm{E}[{}_1!L]$, but it's faster to use the recursion formula:

$$\begin{aligned}
({}_0V + P) \cdot 1.06 &= q_x \cdot 100,000 + p_x \cdot ({}_1V) \\
(0 + 10,131.20) \cdot 1.06 &= 0.03 \cdot 100,000 + (1 - 0.03)\,({}_1V) \qquad \Longrightarrow \qquad \boxed{{}_1V = 7978.42}
\end{aligned}$$

258. (C3, May 2020 #11)

A special 10-year deferred, fully discrete 20-year insurance is issued to (50). A benefit of 2,000,000 is paid on death between ages 60 and 80, and a benefit of 500,000 is paid on survival to age 80. There is no benefit on death before age 60.

You are given:

(i) Premiums are paid annually during the 10-year deferral period.

(ii) Commissions are 5% of premium.

(iii) Expenses of 260 per year are incurred at the start of each year following the deferral period.

(iv) Mortality follows the Standard Ultimate Life Table.

(v) $i = 0.05$

Calculate the annual gross premium using the equivalence principle.

Solution:

259. (LTAM May 2020, #14)
For a fully discrete whole life insurance of 1000 issued to (60), you are given the following information:

(i) Renewal expenses are 5 per year.
(ii) Commissions are 10% of the gross premium.
(iii) Mortality follows the Standard Ultimate Life Table.
(iv) The valuation interest rate is 3% per year for the first 10 years, and 5% per year thereafter.
(v) The gross annual premium is 36.

Calculate the gross premium reserve at the end of the 9th year.

Solution:

$${}_9V = \underline{\text{EPV future benefits}} + \underline{\text{EPV future expenses}} - \underline{\text{EPV future premiums.}}$$

- EPV future benefits. Denote this by A. Be careful to account for the changing interest rate. From the SULT we have A_{70} computed at 5% is 0.4281760, and so

$$\begin{aligned} A &= \frac{1000q_{69}}{1.03} + \frac{1000 \cdot p_{69}}{1.03} 0.4281760 \\ &= \frac{1000(1-0.9907061)}{1.03} + \frac{1000 \cdot 0.9907061}{1.03} \cdot 0.4281760 = \underline{420.8645} \end{aligned}$$

- EPV future expenses. Denote this by E, denote the gross annual premium by P (which is 36), and denote the EPV of the life annuity on (69) by a. Again we have to be careful to account for the changing interest rate. From the SULT we have $\ddot{a}_{70}$ computed at 5% is 12.008303, and so

$$\begin{aligned} E &= (0.1 \cdot P + 5) \cdot a = 8.6 \cdot a \\ & \quad a = 1 + \frac{p_{69}}{1.03} \cdot \ddot{a}_{70} = 1 + \frac{0.9907061}{1.03} \cdot 12.008303 == \underline{12.550193} \\ E &= 8.6 \cdot 12.550193 = \underline{107.93166} \end{aligned}$$

- EPV future Premiums: Denote this by Y, then $Y = P \cdot a = 36 \cdot 12.550193 = \underline{451.806948}$

Then

$$\underline{{}_9V = 420.8645 + 107.93166 - 451.806948 = \boxed{76.989}}$$

SHORTER SOLUTION via the the recursive formula

$$({}_9V + P - \text{Expenses}) \cdot (1+i) = q_{69}(\text{EPV of Benefit}) + p_{69} \cdot ({}_{10}V),$$

which is shorter because ${}_{10}V$ is relatively easy to compute:

$$\begin{aligned}
{}_{10}V &= 1000 \cdot A_{70} + (0.1 \cdot 36 + 5)\ddot{a}_{70} - 36\ddot{a}_{70} = 1000 \cdot A_{70} - 27.4\ddot{a}_{70} \\
&= 1000 \cdot 0.4281760 - 27.4 \cdot 12.008303 = \underline{99.14850} \\
{}_9V &= \frac{1}{1.03}\Big[1000 q_{69} 1000 + p_{69} \cdot ({}_{10}V)\Big] - P + (0.1P + 5) \\
{}_9V &= \frac{1}{1.03}\left(1000(1 - 0.9907061) + 0.9907061 \cdot 99.14850\right) - 27.4 = \boxed{76.989}
\end{aligned}$$

260. (LTAM May 2020, #16)
An insurer issues fully discrete whole life policies of 100,000 to a group of lives age (60). The annual premium for each policy is 3765.

At the end of year 5, there were 100 policies still in force. In the following year:

(i) One policyholder died;
(ii) Expenses of 11% of each premium paid were incurred;
(ii) Actual interest earned was 5%.

You are also given that reserves per policy at the end of the 5^{th} and 6^{th} years are ${}_5V = 11,190$ and ${}_6V = 13,529$, respectively.

Calculate the profit on this group of policies in the 6th year.

Solution:
Reserves in consecutive years are related via the recursion formula

$$\underbrace{\Big((\text{Old reserve}) + (\text{Prem. collected}) - (\text{Exp. Incurred})\Big)(1+i)}_{\text{funds on hand at end of } 6^{th} \text{ year}} = \begin{cases} \text{Death Benefit} & \text{if (x) dies} \\ \text{New reserve} & \text{if (x) lives.} \end{cases}$$

In this case, for policies in which the insured dies in the 6^{th} years, this yields

$$\Big(11,190 + 3765(1 - 0.11)\Big)1.05 - 100,000 = -84,732.11,$$

and for policies in which the insured survives the 6^{th} years, this yields

$$\Big(11,190 + 3765(1 - 0.11)\Big)1.05 - 13,529 = 1738.89,$$

so if one person dies and 99 survives, then the profit that year is

$$99 \cdot 1738.89 - 84,732.11 = \boxed{87,418.00}$$

261. (C3, Nov 2020 #12)

Roger, (65), purchases a special fully discrete 10-year term insurance policy. You are given the following information:

(i) The insurance amount is 100,000.

(ii) Premiums are returned without interest if the policyholder is alive at the end of the 10-year contract.

(iii) Initial expenses are 2,000 and 50% of the first premium.

(iv) Renewal expenses are 2.5% of the second and subsequent premiums.

(v) Expenses of 500 are incurred at the end of the year of death, or on the payment of the return of premium benefit.

(vi) Gross premiums are calculated using the equivalence principle.

(vii) Mortality follows the Standard Ultimate Life Table.

(viii) $i = 0.05$

Calculate the gross premium.

Solution:

$$
\begin{aligned}
0 &= \overbrace{100,000A^{1}_{65:\overline{10}|}}^{\text{Ins. Ben.}} + \overbrace{10P\,{}_{10}E_{65}}^{\text{Premium refund}} + \overbrace{2000 + 500A_{65:\overline{10}|}}^{\text{Per policy exp.}} + \overbrace{P(0.475 + 0.025\ddot{a}_{65:\overline{10}|})}^{\text{\% Premium Exp.}} - \overbrace{P\ddot{a}_{65:\overline{10}|}}^{\text{Premium Income}} \\
0 &= 100,000A^{1}_{65:\overline{10}|} + 2000 + 500A_{65:\overline{10}|} - P\Big(0.975\ddot{a}_{65:\overline{10}|} - 0.475 - 10\cdot {}_{10}E_{65}\Big) \\
P &= \frac{100,000A^{1}_{65:\overline{10}|} + 2000 + 500A_{65:\overline{10}|}}{0.975\ddot{a}_{65:\overline{10}|} - 0.475 - 10\cdot {}_{10}E_{65}} \\
&= \ldots \qquad \text{(SOA's computations)} \\
P &= \boxed{5882.32}
\end{aligned}
$$

262. (C3 May 2021, #2)
In the first year of operations, each machine in a factory is subjected to a constant force of failure, μ. The probability that any given machine fails within the year is 0.1.

In the second year, the force of failure will increase linearly over the year from μ to 2μ. In the third year, the force of failure will be constant at 2μ.

At the start of the first year the factory has 20,000 machines, all new. Calculate the expected number of machines still functioning at the end of year 3.

263. (LTAM May 2021, #6)
On the basis of the Standard Ultimate Life Table and an effective annual interest rate of 5%, caclulate

the expected present value of a whole life annuity-due on (30), paying 1 per year annually, deferred for years.

Solution:

$$
\begin{aligned}
{}_{30|}\ddot{a}_{30} &= \frac{\ell_{60}}{\ell_{30}} v^{30}\,\ddot{a}_{60} \\
&= \frac{96634}{99338}\,\frac{1}{1.05^{30}} \cdot 14.90407 = \boxed{3.3546}
\end{aligned}
$$

264. (C3, May 2021 #12)

A life insurer issues a fully discrete whole life insurance with a face amount of 100,000 to (45).

You are given the following information:

(i) Initial expenses are 100 and 50% of the first premium.

(ii) Renewal expenses are 5% of the premium.

(iii) Claim expenses of 1,000 are incurred when the benefit is paid.

(iv) The annual gross premium is 1,050.

(v) Mortality follows the Standard Ultimate Life Table.

(vi) $i = 0.05$

Calculate the standard deviation of the gross future loss at issue for this policy.

Solution:
Since we're asked for the standard deviation, we need to start with the loss random variable per se, and not just EPV's. Let G be the premium.

$$
\begin{aligned}
L_0^g &= \overbrace{100,000\,v^{K_{45}+1}}^{\text{PV of benefit}} + \overbrace{(100 + 0.45G + 0.05G a_{\overline{K_{45}+1}|})}^{\text{per premium exp's}} + \overbrace{1000 v^{K_{45}+1}}^{\text{Claim exp.}} - G a_{\overline{K_{45}+1}|} \\
L_0^g &= 101,000\,v^{K_{45}+1} + 100 + 0.45G - 0.95G a_{\overline{K_{45}+1}|} \\
L_0^g &= 101,000\,v^{K_{45}+1} + 100 + 0.45G - 0.95G\left(\frac{1 - v^{K_{45}+1}}{d}\right) \\
L_0^g &= \left(101,000 + \frac{0.95G}{d}\right) v^{K_{45}+1} + 100 + 0.45G \\
\operatorname{Std}[L_0^g] &= \left(101,000 + \frac{0.95G}{d}\right)\sqrt{{}^2A_{45} - (A_{45})^2} \\
\operatorname{Std}[L_0^g] &= \left(101,000 + \frac{0.95 \cdot 1050}{0.05/1.05}\right)\sqrt{0.034633 - 0.15161^2} = \boxed{13,161}
\end{aligned}
$$

265. (LTAM May 2022 #12)
Calculate the expected present value for a 10-year deferred annuity due paying five thousand per month issued to (55), using an effective annual interest rate of 5%, the Standard Ultimate Life Table, and the 2-term Woolhouse approximation.

Solution:

$$\begin{aligned}
\ddot{a}_x^{(m)} &\approx \ddot{a}_x - \frac{m-1}{2m} \\
\ddot{a}_{65}^{(12)} &\approx \ddot{a}_{65} - \frac{11}{24} = 13.54979 - \frac{11}{24} = 13.09146 \\
{}_{10|}\ddot{a}_{55}^{(12)} &= {}_{10}E_{55}\,\ddot{a}_{65}^{(12)} = 0.59342 \cdot 13.09146 = 7.76873 \\
\text{ANS} &= 12 \cdot 5000 \cdot 7.76873 = \boxed{466,124}
\end{aligned}$$