Fall 2005
Exam M Solutions

Question #1
Key: C

$$Var[Z] = E\left[Z^2\right] - E[Z]^2$$

$$E[Z] = \int_0^\infty \left(v^t b_t\right)_t p_x \mu_x(t)\,dt = \int_0^\infty e^{-0.08t} e^{0.03t} e^{-0.02t}(0.02)\,dt$$

$$= \int_0^\infty (0.02) e^{-0.07t} dt = \frac{0.02}{0.07} = 2/7$$

$$E\left[Z^2\right] = \int_0^\infty \left(v_t\, b_t\right)^2 {}_t p_x \mu_x(t)\,dt = \int_0^\infty \left(e^{-0.05t}\right)^2 e^{-0.02t}(0.02)\,dt$$

$$= \int_0^\infty 0.02\, e^{-0.12t} \mu_x(t)\,dt = 2/12 = 1/6$$

$$Var[Z] = \frac{1}{6} - \left(2/7\right)^2 = \frac{1}{6} - \frac{4}{49} = 0.08503$$

Question #2
Key: C

From $A_x = 1 - d\ddot{a}x$ we have $A_x = 1 - \frac{0.1}{1.1}(8) = 3/11$

$$A_{x+10} = 1 - \frac{0.1}{1.1}(6) = 5/11$$

$$\bar{A}_x = A_x \times i/\delta$$

$$\bar{A}_x = \frac{3}{11} \times \frac{0.1}{\ln(1.1)} = 0.2861$$

$$\bar{A}_{x+10} = \frac{5}{11} \times \frac{0.1}{\ln(1.1)} = 0.4769$$

$$_{10}V_x = \bar{A}_{x+10} - P\left(\bar{A}_x\right) \times \ddot{a}_{x+10}$$

$$= 0.4769 - \left(\frac{0.2861}{8}\right)6$$

$$= 0.2623$$

There are many other equivalent formulas that could be used.

Question #3
Key: C

Regular death benefit $= \int_0^{\infty} 100,000 \times e^{-0.06t} \times e^{-0.001t} 0.001 dt$

$$= 100,000\left(\frac{0.001}{0.06+0.001}\right)$$

$$= 1639.34$$

Accidental death $= \int_0^{10} 100,000\, e^{-0.06t} e^{-0.001t}\left(0.0002\right) dt$

$$= 20\int_0^{10} e^{-0.061t} dt$$

$$= 20\left[\frac{1-e^{-0.61}}{0.061}\right] = 149.72$$

Actuarial Present Value $= 1639.34 + 149.72 = 1789.06$

Question #4
Key: D

Once you are dead, you are dead. Thus, you never leave state 2 or 3, and rows 2 and 3 of the matrix must be (0 1 0) and (0 0 1).

Probability of dying from cause 1 within the year, given alive at age 61, is 160/800 = 0.20.

Probability of dying from cause 2 within the year, given alive at age 61, is 80/800 = 0.10

Probability of surviving to 62, given alive at 61, is 560/800 = 0.70
(alternatively, 1 – 0.20 – 0.10), so correct answer is D.

Question #5
Key: C

This first solution uses the method on the top of page 9 of the study note.

Note that if the species is it is not extinct after Q_3 it will never be extinct.
This solution parallels the example at the top of page 9 of the Daniel study note. We want the second entry of the product $\left(Q_1 \times Q_2 \times Q_3\right)e_3$ which is equal to $Q_1 \times \left(Q_2 \times \left(Q_3 \times e_3\right)\right)$.

$$Q_3\begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0.1 \\ 1 \end{vmatrix}$$

$$Q_2\begin{vmatrix} 0 \\ 0.1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0.01 \\ 0.27 \\ 1 \end{vmatrix}$$

$$Q_1\begin{vmatrix} 0.01 \\ 0.27 \\ 1 \end{vmatrix} = \begin{vmatrix} 0.049 \\ 0.489 \\ 1 \end{vmatrix}$$

The second entry is 0.489; that's our answer.

Alternatively, start with the row matrix (0 1 0) and project it forward 3 years.

```
(0     1     0   )   Q1 = (0.00  0.70  0.30)
(0     0.70  0.30)   Q2 = (0.07  0.49  0.44)
(0.07  0.49  0.44)   Q3 = (0.16  0.35  0.49)
```

Thus, the probability that it is in state 3 after three transitions is 0.49.

Yet another approach would be to multiply $Q_1 \times Q_2 \times Q_3$, and take the entry in row 2, column 3. That would work but it requires more effort.

Question #6
Key: B

Probabilities of being in each state at time *t*:

t	Active	Disabled	Dead	Deaths
0	1.0	0.0	0.0	
1	0.8	0.1	0.1	0.1
2	0.65	0.15	0.2	0.1
3	not needed	not needed	0.295	0.095

We built the Active Disabled Dead columns of that table by multiplying each row times the transition matrix. E.g., to move from $t = 1$ to $t = 2$, (0.8 0.1 0.1) Q = (0.65 0.15 0.2)
The deaths column is just the increase in Dead. E.g., for t = 2, 0.2 – 0.1 = 0.1.

$v = 0.9$

APV of death benefits = $100{,}000 * \left(0.1v + 0.1v^2 + 0.095v^3\right) = 24{,}025.5$

APV of \$1 of premium = $1 + 0.8v + 0.65v^2 = 2.2465$

Benefit premium = $\dfrac{24{,}025.5}{2.2465} = 10{,}695$

Question #7
Key: A

Split into three independent processes:
Deposits, with $\lambda^* = (0.2)(100)(8) = 160$ per day
Withdrawals, with $\lambda^* = (0.3)(100)(8) = 240$ per day
Complaints. Ignore, no cash impact.

For aggregate deposits,

$$E(D) = (160)(8000) = 1,280,000$$

$$Var(D) = (160)(1000)^2 + (160)(8000)^2$$
$$= 1.04 \times 10^{10}$$

For aggregate withdrawals

$$E(W) = (240)(5000) = 1,200,000$$

$$Var(W) = (240)(2000)^2 + (240)(5000)^2$$
$$= 0.696 \times 10^{10}$$

$$E(W - D) = 1,200,000 - 1,280,000 = -80,000$$

$$Var(W - D) = 0.696 \times 10^{10} + 1.04 \times 10^{10} = 1.736 \times 10^{10}$$

$$SD(W - D) = 131,757$$

$$\Pr(W > D) = \Pr(W - D > 0) = \Pr\left(\frac{W - D + 80,000}{131,757} > \frac{80,000}{131,757}\right)$$
$$= 1 - \Phi(0.607)$$
$$= 0.27$$

Question #8
Key: D

Exponential inter-event times and independent implies Poisson process (imagine additional batteries being activated as necessary; we don't care what happens after two have failed).

Poisson rate of 1 per year implies failures in 3 years is Poisson with $\lambda = 3$.

x	$f(x)$	$F(x)$
0	0.050	0.050
1	0.149	0.199

Probe works provided that there have been fewer than two failures, so we want $F(1) = 0.199$.

Alternatively, the sum of two independent exponential $\theta = 1$ random variables is Gamma with $\alpha = 2, \quad \theta = 1$

$$
\begin{aligned}
F(3) = \Gamma(2;3) &= \frac{1}{\Gamma(2)}\int_0^3 t\,e^{-t}dt \\
&= (-t-1)e^{-t}\Big|_0^3 \\
&= 1 - 4e^{-3} \\
&= 0.80 \text{ is probability 2 have occurred}
\end{aligned}
$$

$1 - 0.80 = 0.20$

Question #9
Key: B

$$1000 P_{45} \ddot{a}_{45:\overline{15|}} + \pi\, \ddot{a}_{60:\overline{15|}} \times {}_{15}E_{45} = 1000 A_{45}$$

$$1000 \frac{A_{45}}{\ddot{a}_{45}} \left(\ddot{a}_{45} - {}_{15}E_{45}\, \ddot{a}_{60} \right) + \pi \left(\ddot{a}_{60} - {}_{15}E_{60}\, \ddot{a}_{75} \right) \left({}_{15}E_{45} \right) = 1000 A_{45}$$

$$\frac{201.20}{14.1121} \left(14.1121 - (0.72988)(0.51081)(11.1454)\right)$$
$$+\pi\left(11.1454 - (0.68756)(0.39994)(7.2170)\right) \times (0.72988)(0.51081) = 201.20$$

where ${}_{15}E_x$ was evaluated as ${}_{5}E_x \times {}_{10}E_{x+5}$

$$14.2573(9.9568) + (\pi)(3.4154) = 201.20$$

$$\pi = 17.346$$

Question #10
Key: A

$${}_{1}V = \left({}_{0}V + \pi\right)(1+i) - \left(1000 + {}_{1}V - {}_{1}V\right) q_x$$
$${}_{2}V = \left({}_{1}V + \pi\right)(1+i) - \left(2000 + {}_{2}V - {}_{2}V\right) q_{x+1} = 2000$$
$$\left(\left(\pi(1+i) - 1000 q_x\right) + \pi\right)(1+i) - 2000 q_{x+1} = 2000$$
$$\left(\left(\pi(1.08) - 1000 \times 0.1\right) + \pi\right)(1.08) - 2000 \times 0.1 = 2000$$

$$\pi = 1027.42$$

Question #11
Key: A

Let Y be the present value of payments to 1 person.
Let S be the present value of the aggregate payments.

$$E[Y] = 500\,\ddot{a}_x = 500\frac{(1-A_x)}{d} = 5572.68$$

$$\sigma_Y = \sqrt{\mathrm{Var}[Y]} = \sqrt{(500)^2 \frac{1}{d^2}\left({}^2A_x - A_x^2\right)} = 1791.96$$

$$S = Y_1 + Y_2 + \ldots + Y_{250}$$

$$E(S) = 250E[Y] = 1{,}393{,}170$$

$$\sigma_S = \sqrt{250} \times \sigma_Y = 15.811388\sigma_Y = 28{,}333$$

$$0.90 = \Pr(S \le F) = \Pr\left[\frac{S - 1{,}393{,}170}{28{,}333} \le \frac{F - 1{,}393{,}170}{28{,}333}\right]$$

$$\approx \Pr\left[N(0,1) \le \frac{F - 1{,}393{,}170}{28{,}333}\right]$$

$$0.90 = \Pr\left(N(0,1) \le 1.28\right)$$

$$F = 1{,}393{,}170 + 1.28(28{,}333)$$

=1.43 million

Question #12
Key: A

$$q'_{41}{}^{(1)} = 1 - p'_{41}{}^{(1)} = 1 - \left(p_{41}{}^{(\tau)}\right)^{q_{41}{}^{(1)} / q_{41}{}^{(\tau)}}$$

$$l_{41}{}^{(\tau)} = l_{40}{}^{(\tau)} - d_{40}{}^{(1)} - d_{40}{}^{(2)} = 1000 - 60 - 55 = 885$$

$$d_{41}{}^{(1)} = l_{41}{}^{(\tau)} - d_{41}{}^{(2)} - l_{42}{}^{(\tau)} = 885 - 70 - 750 = 65$$

$$p_{41}{}^{(\tau)} = \frac{750}{885} \qquad \frac{q_{41}{}^{(1)}}{q_{41}{}^{(\tau)}} = \frac{65}{135}$$

$$q'_{41}{}^{(1)} = 1 - \left(\frac{750}{885}\right)^{65/135} = 0.0766$$

Question #13
Key: D

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}$$

$$\mu(x) = \frac{d}{dx}\ \log\ \left(s(x)\right) = \frac{\alpha}{\omega - x}$$

$$\overset{\circ}{e}_x = \int_0^{\omega - x} \left(1 - \frac{t}{\omega - x}\right)^{\alpha} dt = \frac{\omega - x}{\alpha + 1}$$

$$\overset{\circ}{e}{}_0^{\text{new}} = \frac{1}{2} \times \frac{\omega}{\alpha^{\text{old}} + 1} = \frac{\omega}{\alpha^{\text{new}} + 1} \Rightarrow \alpha^{\text{new}} = 2\alpha^{\text{old}} + 1$$

$$\mu_0^{(\text{new})} = \frac{2\alpha^{\text{old}} + 1}{\omega} = \frac{9}{4} \times \frac{\alpha^{\text{old}}}{\omega} \Rightarrow \alpha^{\text{old}} = 4$$

Question #14
Key: C

Constant force implies exponential lifetime

where $\theta = 1/\mu$ using the Loss Models parameterization

$$Var[T] = E\left[T^2\right] - \left(E[T]\right)^2 = \frac{2}{\mu^2} - \left(\frac{1}{\mu}\right)^2 = \frac{1}{\mu^2} = 100$$
$$\mu = 0.1$$
$$E[T \wedge 10] = \int_0^{10} e^{-\mu t} dt = \frac{1 - e^{-10\mu}}{\mu} = 6.3$$

Alternatively, the formula for $E(X \wedge x)$ is given in the tables handout.
Note that since *T* is the future lifetime random variable, $E(T \wedge 10)$ can also be written as

$\mathring{e}_{x:\overline{10}|}$, which for the exponential distribution (constant force of mortality) is independent of *x*.

Question #15
Key: A

$$G\ddot{a}_{x:\overline{15}|} = 100{,}000 A_x + \overbrace{\left(0.08G + 0.02G\ddot{a}_{x:\overline{15}|}\right)}^{\text{\% premium amount for 15 years}} + \underbrace{\left(\left(x-5\right) + 5\ddot{a}_x\right)}_{\text{Per policy for life}}$$

$$4669.95(11.35) = 51{,}481.97 + (0.08)(4669.95) + (0.02)(11.35)(4669.95) + \left((x-5) + 5\ddot{a}_x\right)$$

$$\ddot{a}_x = \frac{1 - Ax}{d} = \frac{1 - 0.5148197}{0.02913} = 16.66$$

$$53{,}003.93 = 51{,}481.97 + 1433.67 + (x-5) + 83.30$$
$$4.99 = (x-5)$$
$$x = 9.99$$

The % of premium expenses could equally well have been expressed as $0.10G + 0.02G\, a_{x:\overline{14}|}$.
The per policy expenses could also be expressed in terms of an annuity-immediate.

Question #16
Key: D

For the density where $T(x) \neq T(y)$,

$$\Pr\left(T(x) < T(y)\right) = \int_{y=0}^{40}\int_{x=0}^{y} 0.0005\,dx\,dy + \int_{y=40}^{50}\int_{x=0}^{40} 0.0005\,dx\,dy$$

$$= \int_{y=0}^{40} 0.0005\,x\Big|_0^y\,dy + \int_{y=40}^{50} 0.0005x\Big|_0^{40}\,dy$$

$$= \int_0^{40} 0.0005y\,dy + \int_{y=40}^{50} 0.02\,dy$$

$$= \frac{0.0005y^2}{2}\Big|_0^{40} + 0.02y\Big|_{40}^{50}$$

$$= 0.40 + 0.20 = 0.60$$

For the overall density,
$\Pr\left(T(x) < T(y)\right) = 0.4 \times 0 + 0.6 \times 0.6 = 0.36$
where the first 0.4 is the probability that $T(x) = T(y)$ and the first 0.6 is the probability that $T(x) \neq T(y)$.

Question #17
Key: D

The following derives the general formula for the statistic to be forgotten by time x. It would work fine, and the equations would look simpler, if you immediately plugged in $x = \frac{1}{2}$, the only value you want. Then the $x + \frac{1}{2}$ becomes 1.

Let *X* be the random variable for when the statistic is forgotten. Then $F_X\left(x|y\right) = 1 - e^{-xy}$

For the unconditional distribution of *X*, integrate with respect to y

$$F_X\left(x\right) = \int_0^\infty \left(1 - e^{-xy}\right)\frac{1}{\Gamma\left(2\right)y}\left(\frac{y}{2}\right)^2 e^{-y/2}dy$$

$$= 1 - \frac{1}{4}\int_0^\infty y\, e^{-y\left(x+1/2\right)}d\,y$$

$$= 1 - \frac{1}{4\left(x + 1/2\right)^2}$$

$$F\left(1/2\right) = 1 - \frac{1}{4\left(1/2 + 1/2\right)^2} = 0.75$$

There are various ways to evaluate the integral in the second line:

1. Calculus, integration by parts
2. Recognize that $\int_0^\infty y\left(x + \frac{1}{2}\right)e^{-y\left(x+\frac{1}{2}\right)}dy$

   is the expected value of an exponential random variable with $\theta = \frac{1}{x + \frac{1}{2}}$

3. Recognize that $\Gamma\left(2\right)\left(x + \frac{1}{2}\right)^2 ye^{-y\left(x+\frac{1}{2}\right)}$ is the density function for a Gamma random variable with $\alpha = 2$ and $\theta = \frac{1}{x + \frac{1}{2}}$, so it would integrate to 1.

(Approaches 2 and 3 would also work if you had plugged in $x = \frac{1}{2}$ at the start. The resulting θ becomes 1).

Question #18
Key: D

State#	Number	Probability of needing Therapy	Mean Number of visits E(X)	E(N)	Var(N)	Var(X)	E(S)	Var(S)
1	400	0.2	2	80	64	6	160	736
2	300	0.5	15	150	75	240	2,250	52,875
3	200	0.3	9	60	42	90	540	8,802
							2,950	62,413

Std Dev $(S) = \sqrt{62413} = 250$

$$\Pr(S > 3000) = \Pr\left(\frac{S-2950}{250} > \frac{50}{250}\right) = 1 - \Phi(0.2) = 0.42$$

The $Var(X)$ column came from the formulas for mean and variance of a geometric distribution.

Using the continuity correction, solving for $\Pr(S > 3000.5)$, is theoretically better but does not affect the rounded answer.

Question #19
Key: B

Frequency is geometric with $\beta = 2$, so
$p_0 = 1/3, \quad p_1 = 2/9, \quad p_2 = 4/27$

Convolutions of $f_X(x)$ needed are

x	f	f^{*2}
5	0.2	0
10	0.3	0.04

so $f_S(0) = 1/3, \quad f_S(5) = 2/9(0.2) = 0.044, \quad f_S(10) = 2/9(0.3) + 4/27(0.04) = 0.073$

$$E(X) = (0.2)(5) + (0.3)(10) + (0.5)(20) = 14$$
$$E[S] = 2E(X) = 28$$
$$\begin{aligned} E[S-15]_+ &= E[S] - 5(1-F(0)) - 5(1-F(5)) - 5(1-F(10)) \\ &= 28 - 5(1-1/3) - 5(1-1/3-0.044) - 5(1-1/3-0.044-0.073) \\ &= 28 - 3.33 - 3.11 - 2.75 = 18.81 \end{aligned}$$

Alternatively,

$$\begin{aligned} E[S-15]_+ &= E[S] - 15 + 15 f_S(0) + 10 f_S(5) + 5 f_S(10) \\ &= 28 - 15 + (15)\left(\frac{1}{3}\right) + 10(0.044) + 5(0.073) \\ &= 18.81 \end{aligned}$$

Question #20
Key: B

The conditional expected value of the annuity, given μ, is $\dfrac{1}{0.01+\mu}$.

The unconditional expected value is

$$\bar{a}_x = 100\int_{0.01}^{0.02} \frac{1}{0.01+\mu} d\mu = 100 \ln\left(\frac{0.01+0.02}{0.01+0.01}\right) = 40.5$$

100 is the constant density of μ on the internal $[0.01, 0.02]$. If the density were not constant, it would have to go inside the integral.

Question #21
Key: E

Recall $\quad \mathring{e}_x = \dfrac{\omega - x}{2}$

$$\mathring{e}_{\overline{x:x}} = \mathring{e}_x + \mathring{e}_x - \mathring{e}_{x:x}$$

$$\mathring{e}_{x:x} = \int_0^{\omega - x} \left(1 - \frac{t}{\omega - x}\right)\left(1 - \frac{t}{\omega - y}\right) dt$$

Performing the integration we obtain

$$\mathring{e}_{x:x} = \frac{\omega - x}{3}$$

$$\mathring{e}_{\overline{x:x}} = \frac{2(\omega - x)}{3}$$

(i) $\quad \dfrac{2(\omega - 2a)}{3} = 3 \times \dfrac{2(\omega - 3a)}{3} \Rightarrow 2\omega = 7a$

(ii) $\quad \dfrac{2}{3}(\omega - a) = k \times \dfrac{2(\omega - 3a)}{3}$

$$3.5a - a = k(3.5a - 3a)$$

$$k = 5$$

The solution assumes that all lifetimes are independent.

Question #22
Key: B

Upon the first death, the survivor receives $10{,}000 \dfrac{\mu}{\mu + \delta} = 10{,}000 \left(\dfrac{0.10}{0.10 + 0.04} \right) = 7143$

The actuarial present value of the insurance of 7143 is

$$7{,}143 \frac{\mu_{xy}}{\mu_{xy} + \delta} = (7{,}143)\left(\frac{0.12}{0.12 + 0.04} \right) = 5357$$

If the force of mortality were not constant during each insurance period, integrals would be required to express the actuarial present value.

Question #23
Key: E

Let ${}_k p_0 =$ Probability someone answers the first k problems correctly.

${}_2 p_0 = (0.8)^2 = 0.64$

${}_2 p_{0:0} = ({}_2 p_0)^2 = 0.64^2 = 0.41$

${}_2 p_{\overline{0:0}} = {}_2 p_0 + {}_2 p_0 - {}_2 p_{0:0} = 0.87$

${}_4 p_0 = (0.8)^4 = 0.41$

${}_4 p_{0:0} = (0.41)^2 = 0.168$

${}_4 p_{\overline{0:0}} = 0.41 + 0.41 - 0.168 = 0.652$

Prob(second child loses in round 3 or 4) = ${}_2 p_{\overline{0:0}} - {}_4 p_{\overline{0:0}}$
= 0.87-0.652
= 0.218

Prob(second loses in round 3 or $4 \mid$ second loses after round 2) = $\dfrac{{}_2 p_{\overline{0:0}} - {}_4 p_{\overline{0:0}}}{{}_2 p_{\overline{0:0}}}$

$= \dfrac{0.218}{0.87} = 0.25$

Question #24
Key: E

If (40) dies before 70, he receives one payment of 10, and Y = 10. Under DeMoivre, the probability of this is (70 – 40)/(110 – 40) = 3/7

If (40) reaches 70 but dies before 100, he receives 2 payments.
$Y = 10 + 20v^{30} = 16.16637$
The probability of this is also 3/7. (Under DeMoivre, all intervals of the same length, here 30 years, have the same probability).

If (40) survives to 100, he receives 3 payments.
$Y = 10 + 20v^{30} + 30v^{60} = 19.01819$
The probability of this is 1 – 3/7 – 3/7 = 1/7
$E(Y) = (3/7)\times 10 + (3/7)\times 16.16637 + (1/7)\times 19.01819 = 13.93104$

$E(Y^2) = (3/7)\times 10^2 + (3/7)\times 16.16637^2 + (1/7)\times 19.01819^2 = 206.53515$

$Var(Y) = E(Y^2) - [E(Y)]^2 = 12.46$

Since everyone receives the first payment of 10, you could have ignored it in the calculation.

Question #25
Key: C

$$\mathrm{E}(Z)=\sum_{k=0}^{2}\left(v^{k+1}b_{k+1}\right)\quad {}_k p_x \quad q_{x+k}$$

$$=\left[v(300)\times 0.02+v^2(350)(0.98)(0.04)+v^3 400(0.98)(0.96)(0.06)\right]$$

$$=36.8$$

$$E\left(Z^2\right)=\sum_{k=0}^{2}\left(v^{k+1}b_{k+1}\right)^2\quad {}_k p_x \quad q_{x+k}$$

$$=v^2(300)^2\times 0.02+v^4(350)^2(0.98)(0.04)+v^6 400^2(0.98)(0.96)0.06$$

$$=11,773$$

$$Var[Z]=E\left(Z^2\right)-E(Z)^2$$

$$=11,773-36.8^2$$

$$=10,419$$

Question #26
Key: E

$$S_X(4)=1-\int_0^4 f_X(x)\,dx=1-\int_0^4 0.02\,x\,dx$$

$$=1-0.01x^2\Big|_0^4$$

$$=0.84$$

$$f_{Y^p}(y)=\frac{f_X(y+4)}{S_X(4)}=\frac{0.02(y+4)}{0.84}=0.0238(y+4)^2$$

$$\mathrm{E}\left(Y^p\right)=\int_0^6 y\left(0.0238(y+4)\right)dy=0.0238\left(\frac{y^3}{3}+\frac{4y^2}{2}\right)\Bigg|_0^6$$

$$=3.4272$$

Question #27
Key: E

By Theorem 4.51 (on page 93 of the second edition of Loss Models), probability of zero claims = pgf of negative binomial applied to the probability that Poisson equals 0.

For the Poisson, $f(0) = e^{-\lambda}$

So $0.067 = \left[1 - \beta\left(e^{-\lambda} - 1\right)\right]^{-r} = \left[1 - 3\left(e^{-\lambda} - 1\right)\right]^{-2}$

Solving gives $\lambda = 3$

Question #28
Key: D

For any deductible *d* and the given severity distribution

$$E\left(X-d\right)_+ = E(X) - E\left(X \wedge d\right)$$
$$= 3000 - 3000\left(1 - \frac{3000}{3000+d}\right)$$
$$= \left(3000\right)\left(\frac{3000}{3000+d}\right)$$

So $P_{2005} = \left(1.2\right)\left(3000\right)\left(\frac{3000}{3600}\right) = 3000$

The following paragraph just clarifies the notation in the rest of the solution:

Let *r* denote the reinsurer's deductible relative to losses (not relative to reinsured claims). Thus if $r = 1000$ (we are about to solve for *r*), then on a loss of 4000, the insured collects 4000 – 600 = 3400, the reinsurer pays 4000 – 1000 = 3000, leaving the primary insurer paying 400.

Another way, exactly equivalent, to express that reinsurance is that the primary company pays the insured 3400. The reinsurer reimburses the primary company for its claims less a deductible of 400 applied to claims. So the reinsurer pays 3400 – 400 = 3000, the same as before.

Expected reinsured claims in 2005

$$= \left(3000\right)\left(\frac{3000}{3000+r}\right) = \frac{9{,}000{,}000}{3000+r}$$

$$R_{2005} = \left(1.1\right)\left(\frac{9{,}000{,}000}{3000+r}\right) = \left(0.55\right)P_{2005}$$
$$\frac{9{,}900{,}000}{3000+r} = \left(0.55\right)\left(3000\right) = 1650$$
$$r = 3000$$

In 2006, after 20% inflation, losses will have a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = \left(1.2\right)\left(3000\right) = 3600$.

The general formula for claims will be

$$E\left(X-d\right)_+ = \left(3600\right)\left(\frac{3600}{3600+d}\right) = \frac{12{,}960{,}000}{3600+d}$$
$$P_{2006} = 1.2\left(\frac{12{,}960{,}000}{3000+600}\right) = 3703$$

$$R_{2006} = 1.1\left(\frac{12{,}960{,}000}{3600+3000}\right) = 2160$$

$$R_{2006} / P_{2006} = 0.5833$$

[If you applied the reinsurer's deductible to the primary insurer's claims, you would solve that the deductible is 2400, and the answer to the problem is the same].

Question #29
Key: A

$$\begin{array}{lllll} & \text{Benefits } + & \text{Expenses} & - \text{ Premiums} \\ {}_0L_e = & 1000v^3 \; + & (0.20G+8)+(0.06G+2)v+(0.06G+2)v^2 & - \; G\,\ddot{a}_{\overline{3|}} \end{array}$$

at $G = 41.20$ and $i = 0.05$,

${}_0L_e$ $(\text{for } K = 2) = 770.59$

Question #30
Key: D

$$P = 1000P_{40}$$

$$(235 + P)(1 + i) - 0.015(1000 - 255) = 255 \quad [A]$$

$$(255 + P)(1 + i) - 0.020(1000 - 272) = 272 \quad [B]$$

Subtract [A] from [B]

$$20(1 + i) - 3.385 = 17$$

$$1 + i = \frac{20.385}{20} = 1.01925$$

Plug into [A] $(235 + P)(1.01925) - 0.015(1000 - 255) = 255$

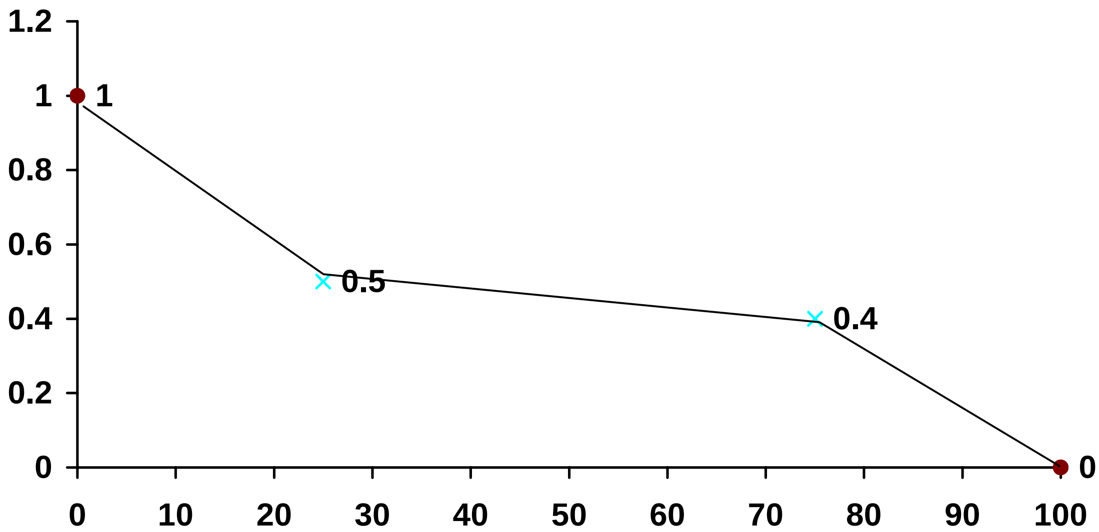
$$235 + P = \frac{255 + 11.175}{1.01925}$$

$$P = 261.15 - 235 = 26.15$$

$$1000\ {}_{25}V_{40} = \frac{(272 + 26.15)(1.01925) - (0.025)(1000)}{1 - 0.025}$$

$$= 286$$

Question #31
Key: A



	Given		Given		Given		Given
x	0	15	25	35	75	90	100
$s(x)$	1	0.70	0.50	0.48	0.4	0.16	0
		Linear Interpolation		Linear Interpolation		Linear Interpolation	

$$_{55}q_{35} = 1 - \frac{s(90)}{s(35)} = 1 - \frac{0.16}{0.48} = \frac{32}{48} = 0.6667$$

$$_{20|55}q_{15} = \frac{s(35) - s(90)}{s(15)} = \frac{0.48 - 0.16}{0.70} = \frac{32}{70} = 0.4571$$

$$\frac{_{20|55}q_{15}}{_{55}q_{35}} = \frac{0.4571}{0.6667} = 0.6856$$

Alternatively,

$$\frac{_{20|55}q_{15}}{_{55}q_{35}} = \frac{_{20}p_{15} \times {}_{55}q_{35}}{_{55}q_{35}} = {}_{20}p_{15} = \frac{s(35)}{s(15)}$$

$$= \frac{0.48}{0.70}$$

$$= 0.6856$$

Question #32
Key: A

$$s(80) = \tfrac{1}{2} * \left(e \wedge (-0.16 * 50) + e \wedge (-0.08 * 50)\right) = 0.00932555$$
$$s(81) = \tfrac{1}{2} * \left(e \wedge (-0.16 * 51) + e \wedge (-0.08 * 51)\right) = 0.008596664$$

$$p_{80} = s(81) / s(80) = 0.008596664 / 0.00932555 = 0.9218$$
$$q_{80} = 1 - 0.9218 = 0.078$$

Alternatively (and equivalent to the above)
For non-smokers, $p_x = e^{-0.08} = 0.923116$
$_{50}p_x = 0.018316$
For smokers, $p_x = e^{-0.16} = 0.852144$
$_{50}p_x = 0.000335$

So the probability of dying at 80, weighted by the probability of surviving to 80, is

$$\frac{0.018316 \times (1 - 0.923116) + 0.000335 \times (1 - 0.852144)}{0.018316 + 0.000335} = 0.078$$

Question #33
Key: B

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	2000	20	60
41	1920	30	50
42	1840	40	

because $2000-20-60=1920$; $1920-30-50=1840$

Let premium = P

APV premiums = $\left(\frac{2000}{2000}+\frac{1920}{2000}\,v+\frac{1840}{2000}\,v^2\right)P=2.749P$

APV benefits = $1000\left(\frac{20}{2000}\,v+\frac{30}{2000}\,v^2+\frac{40}{2000}\,v^3\right)=40.41$

$P=\frac{40.41}{2.749}=14.7$

Question #34
Key: C

Consider Disease 1 and other Diseases as independent Poisson processes with respective $\lambda's = (0.16)\left(\frac{1}{16}\right) = 0.01$ and $(0.16)\left(\frac{15}{16}\right) = 0.15$ respectively. Let $S_1 =$ aggregate losses from Disease 1; S_2 = aggregate losses from other diseases.

$$E(S_1) = 100 \times 0.01 \times 5 = 5$$
$$Var(S_1) = 100 \times 0.01 \times \left(50^2 + 5^2\right) = 2525$$
$$E(S_2) = 100 \times 0.15 \times 10 = 150$$
$$Var(S_2) = 100 \times 0.15 \times \left(20^2 + 10^2\right) = 7500$$

If no one gets the vaccine:
$$E(S) = 5 + 150 = 155$$
$$Var(S) = 2525 + 7500 = 10,025$$
$$\Phi(0.7) = 1 - 0.24$$
$$A = 155 + 0.7\sqrt{10,025} = 225.08$$

If all get the vaccine, vaccine cost = $(100)(0.15) = 15$
No cost or variance from Disease 1

$$B = 15 + 150 + 0.7\sqrt{7500} = 225.62$$
$$A / B = 0.998$$

Question #35
Key: A

For current model $f(x) = \frac{1}{4} e^{-\frac{x}{4}}$

Let $g(x)$ be the new density function, which has

(i) $g(x) = c$, $0 \le x \le 3$

(ii) $g(x) = ke^{-x/4}$, $x > 3$*

(iii) $c = ke^{-3/4}$, since continuous at $x = 3$

Since g is density function, it must integrate to 1.

$$1 = 3c + \int_3^\infty ke^{-x/4}\,dx = 3ke^{-3/4} + 4ke^{-3/4} = 3c + 4c \Rightarrow c = 1/7$$

$$F(3) = \int_0^3 c\,dx = \int_0^3 \frac{1}{7}\,dx = 3/7 = 0.43$$

*This could equally well have been written $g(x) = d \times \left(\frac{1}{4} e^{-x/4}\right)$, then let $k = d/4$, or even carry the $d/4$ throughout.

Question #36
Key: A

$$\bar{a}_{30} = \int_0^{10} e^{-0.08t}\, e^{-0.05} dt + {}_{10}E_x \int_0^{\infty} e^{-0.08t} e^{-0.08t} dt$$
$$= \int_0^{10} e^{-0.13t} dt + e^{-1.3} \int_0^{\infty} e^{-0.16} dt$$
$$\frac{-e^{-0.13t}}{0.13}\Big|_0^{10} + \left(e^{-1.3}\right)\frac{-e^{-0.16t}}{0.16}\Big|_0^{\infty}$$
$$= \frac{-e^{1.3}}{0.13} + \frac{1}{0.13} + \frac{e^{-1.3}}{0.16}$$
$$= 7.2992$$
$$\bar{A}_{30} = \int_0^{10} e^{-0.08t} e^{-0.05t} \left(0.05\right) dt + e^{-1.3} \int_0^{\infty} e^{-0.16t} \left(0.08\right) dt$$
$$= 0.05\left(\frac{1}{0.13} - \frac{e^{-1.3}}{0.13}\right) + \left(0.08\right)\frac{e^{-1.3}}{0.16}$$
$$= 0.41606$$
$$= \bar{P}\left(\bar{A}_{30}\right) = \frac{\bar{A}_{30}}{\bar{a}_{30}} = \frac{0.41606}{7.29923} = 0.057$$
$$\bar{a}_{40} = \frac{1}{0.08 + 0.08} = \frac{1}{0.16}$$
$$\bar{A}_{40} = 1 - \delta\, \bar{a}_{40}$$
$$= 1 - \left(0.08/0.16\right) = 0.5$$
$${}_{10}\bar{V}\left(\bar{A}_{40}\right) = \bar{A}_{40} - \bar{P}\left(\bar{A}_{40}\right)\bar{a}_{40}$$
$$= 0.5 - \frac{\left(0.057\right)}{0.16} = 0.14375$$

Question #37
Key: C

Let T be the future lifetime of Pat, and [T] denote the greatest integer in T. ([T] is the same as K, the curtate future lifetime).

$$L = 100,000\, v^{T} - 1600\, \ddot{a}_{\overline{[T]+1|}} \qquad 0 < T \le 10$$
$$= 100,000 v^{T} - 1600\, \ddot{a}_{\overline{10|}} \qquad 10 < t \le 20$$
$$- 1600\, \ddot{a}_{\overline{10|}} \qquad 20<t$$

Minimum is $-1600\, \ddot{a}_{\overline{10|}}$ when evaluated at i = 0.05

$$= -12,973$$

Question #38
Key: C

Since loss amounts are uniform on (0, 10), 40% of losses are below the deductible (4), and 60% are above. Thus, claims occur at a Poisson rate $\lambda^* = (0.6)(10) = 6$.

Since loss amounts were uniform on (0, 10), claims are uniform on (0, 6).

Let *N* = number of claims; *X* = claim amount; *S* = aggregate claims.

$$E(N) = Var(N) = \lambda^* = 6$$
$$E(X) = (6-0)/2 = 3$$
$$Var(X) = (6-0)^2/12 = 3$$
$$\begin{aligned} Var(S) &= E(N)Var(X) + Var(N)\ [E(X)]^2 \\ &= 6*3 + 6*3^2 \\ &= 72 \end{aligned}$$

Question #39
Key: E

n	p_n	$n \times p_n$	$n^2 \times p_n$
0	0.1	0	0
1	0.4	0.4	0.4
2	0.3	0.6	1.2
3	0.2	0.6	1.8
		$E[N] = 1.6$	$E[N^2] = 3.4$

$$Var(N) = 3.4 - 1.6^2\ = 0.84$$
$$E[X] = \lambda = 3$$
$$Var(X) = \lambda = 3$$

$$\begin{aligned} Var(S) &= \\ &= E[N]Var(X) + E(X)^2 \times Var(N) \\ &= 1.6(3) + (3)^2(0.84) \\ &= 12.36 \end{aligned}$$

Question #40
Key: B

Method 1: as three independent processes, based on the amount deposited. Within each process, since the amount deposited is always the same, $Var(X) = 0$.

Rate of depositing 10 = 0.05 * 22 = 1.1
Rate of depositing 5 = 0.15 * 22 = 3.3
Rate of depositing 1 = 0.80 * 22 = 17.6

Variance of depositing 10 = 1.1 * 10 * 10 = 110
Variance of depositing 5 = 3.3 * 5 * 5 = 82.5
Variance of depositing 1 = 17.6 * 1 *1 = 17.6

Total Variance = 110 + 82.5 + 17.6 = 210.1

Method 2: as a single compound Poisson process

$$E(X) = 0.8\times 1 + 0.15\times 5 + 0.05\times 10 = 2.05$$

$$E\left(X^2\right) = 0.8\times 1^2 + 0.15\times 5^2 + 0.05\times 10^2 = 9.55$$

$$\begin{aligned} Var(S) &= E(N)Var(X) + Var(N)\left(E(X)\right)^2 \\ &= (22)(5.3475) + (22)\left(2.05^2\right) \\ &= 210.1 \end{aligned}$$