Fall 2003 Society of Actuaries
Course 3 Solutions

Question #1
Key: E

$$ {}_{2|}q_{\overline{30:34}} = {}_2p_{\overline{30:34}} - {}_3p_{\overline{30:34}} $$

$$ {}_2p_{30} = (0.9)(0.8) = 0.72 $$
$$ {}_2p_{34} = (0.5)(0.4) = 0.20 $$
$$ {}_2p_{30:34} = (0.72)(0.20) = 0.144 $$
$$ {}_2p_{\overline{30:34}} = 0.72 + 0.20 - 0.144 = 0.776 $$
$$ {}_3p_{30} = (0.72)(0.7) = 0.504 $$
$$ {}_3p_{34} = (0.20)(0.3) = 0.06 $$
$$ {}_3p_{30:34} = (0.504)(0.06) = 0.03024 $$
$$ {}_3p_{\overline{30:34}} = 0.504 + 0.06 - 0.03024 $$
$$ = 0.53376 $$

$$ {}_{2|}q_{\overline{30:34}} = 0.776 - 0.53376 $$
$$ = 0.24224 $$

Alternatively,

$$ {}_{2|}q_{\overline{30:34}} = {}_{2|}q_{30} + {}_{2|}q_{34} - {}_{2|}q_{30:34} $$
$$ = {}_2p_{30}q_{32} + {}_2p_{34}q_{36} - {}_2p_{30:34}(1 - p_{32:36}) $$

= (0.9)(0.8)(0.3) + (0.5)(0.4)(0.7) – (0.9)(0.8)(0.5)(0.4) [1-(0.7)(0.3)]
= 0.216 + 0.140 – 0.144(0.79)
= 0.24224

Alternatively,

$$ {}_{2|}q_{\overline{30:34}} = {}_3q_{30} \times {}_3q_{34} - {}_2q_{30} \times {}_2q_{34} $$
$$ = (1 - {}_3p_{30})(1 - {}_3p_{34}) - (1 - {}_2p_{30})(1 - {}_2p_{34}) $$
$$ = (1 - 0.504)(1 - 0.06) - (1 - 0.72)(1 - 0.20) $$
$$ = 0.24224 $$

(see first solution for ${}_2p_{30}$, ${}_2p_{34}$, ${}_3p_{30}$, ${}_3p_{34}$)

Question #2
Key: E

$$
\begin{aligned}
1000\bar{A}_x &= 1000\left[\bar{A}^{\,1}_{x:\overline{10}|} + {}_{10|}\bar{A}_x\right] \\
&= 1000\left[\int_0^{10} e^{-0.04t}e^{-0.06t}(0.06)dt + e^{-0.4}e^{-0.6}\int_0^{\infty} e^{-0.05t}e^{-0.07t}(0.07)dt\right] \\
&= 1000\left[0.06\int_0^{10} e^{-0.1t}dt + e^{-1}(0.07)\int_0^{\infty} e^{-0.12t}dt\right] \\
&= 1000\left[0.06\left[\frac{-e^{-0.10t}}{0.10}\right]_0^{10} + e^{-1}(0.07)\left[\frac{-e^{-0.12t}}{0.12}\right]_0^{\infty}\right] \\
&= 1000\left[\frac{0.06}{0.10}\left[1-e^{-1}\right] + \frac{0.07}{0.12}e^{-1}\left[1-e^{-1.2}\right]\right] \\
&= 1000\left(0.37927+0.21460\right) = 593.87
\end{aligned}
$$

Because this is a timed exam, many candidates will know common results for constant force and constant interest without integration.

For example $\bar{A}^{\,1}_{x:\overline{10}|} = \dfrac{\mu}{\mu+\delta}\left(1-{}_{10}E_x\right)$

$$
{}_{10}E_x = e^{-10(\mu+\delta)}
$$

$$
\bar{A}_x = \frac{\mu}{\mu+\delta}
$$

With those relationships, the solution becomes

$$
\begin{aligned}
1000\bar{A}_x &= 1000\left[\bar{A}^{\,1}_{x:\overline{10}|} + {}_{10}E_x\ A_{x+10}\right] \\
&= 1000\left[\left(\frac{0.06}{0.06+0.04}\right)\left(1-e^{-(0.06+0.04)10}\right) + e^{-(0.06+0.04)10}\left(\frac{0.07}{0.07+0.05}\right)\right] \\
&= 1000\left[(0.60)\left(1-e^{-1}\right)+0.5833e^{-1}\right] \\
&= 593.86
\end{aligned}
$$

Question #3
Key: A

$$B = \begin{cases} c(400-x) & x < 400 \\ 0 & x \geq 400 \end{cases}$$

$$100 = E(B) = c \bullet 400 - cE(X \wedge 400)$$

$$= c \bullet 400 - c \bullet 300\left(1 - \frac{300}{300+400}\right)$$

$$= c\left(400 - 300 \bullet \frac{4}{7}\right)$$

$$c = \frac{100}{228.6} = 0.44$$

Question #4
Key: C

Let $N = \#$ of computers in department
Let X = cost of a maintenance call
Let S = aggregate cost

$$\mathrm{Var}(X) = [\text{Standard Deviation } (X)]^2 = 200^2 = 40,000$$

$$E(X^2) = \mathrm{Var}(X) + [E(X)]^2$$

$$= 40,000 + 80^2 = 46,400$$

$$E(S) = N \times \lambda \times E(X) = N \times 3 \times 80 = 240N$$

$$\mathrm{Var}(S) = N \times \lambda \times E(X^2) = N \times 3 \times 46,400 = 139,200N$$

We want $0.1 \geq \Pr(S > 1.2E(S))$

$$\geq \Pr\left(\frac{S - E(S)}{\sqrt{139,200N}} > \frac{0.2E(S)}{\sqrt{139,200N}}\right) \Rightarrow \frac{0.2 \times 240N}{373.1\sqrt{N}} \geq 1.282 = \Phi(0.9)$$

$$N \geq \left(\frac{1.282 \times 373.1}{48}\right)^2 = 99.3$$

Question #5
Key: B

If you happen to remember this distribution from the Simulation text (example 4d in third edition), you could use:

$$n = Int\left(\frac{\log(1-u)}{\log q}\right)+1 = Int\frac{\log 0.95}{\log 0.1}+1 = 0+1 = 1$$

For mere mortals, you get the simulated value of *N* from the definition of the inverse transformation method:

$f(1) = F(1) = 0.9$
$0.05 \le 0.9$ so $n = 1$

$$x_1 = \frac{1}{\lambda}\log^{(1-v_1)} = -\frac{1}{0.01}\log 0.7 = 35.67$$

The amount of total claims during the year $= 35.67$

Question #6
Key: D

$\mu_x^{(\tau)} = 0.02 \Rightarrow \text{time of death} = -50\ln(0.35) = 52.491$

$\frac{\mu^{(adb)}}{\mu^{(\tau)}} = 0.25 \Rightarrow \text{Prob}(\text{non adb}) = 0.75$

$0.775 > \text{Prob}(\text{non adb})$ so death is accidental and benefit is 2 (see below for more detail)

$L = 2e^{-\delta T} - 0.025\,\bar{a}_{\overline{T}|}$

$= 2e^{-\delta T} - 0.025\left(\frac{1-e^{-\delta t}}{\delta}\right)$

$= 2e^{-(0.05)(52.491)} - 0.025\left(\frac{1-e^{-(0.05)(52.91)}}{0.05}\right)$

$= 0.1449 - (0.025)(18.55)$

$= -0.319$

Another way of seeing that Prob(non adb) = 0.75.

ADB deaths in time $t = \int_0^t 0.005\, e^{-\mu_x^{(\tau)}(r)}\, dr$

$= 0.25\int_0^t 0.02\, e^{-\mu_x^{(\tau)}(r)}\, dr$

$= 0.25\int_0^t \mu_x^{(\tau)}(r)\, e^{-\mu_x^{(\tau)}(r)} dr$

$= 0.25\times$ all deaths in time t

More detail on why benefit is 2:

Benefit (b)	*f*(b)	*F*(b)
1	0.75	0.75
2	0.25	1

$F(1) < 0.775 < F(2)$ so benefit = 2

Question #7
Key: B

$$\mu_x^{(\tau)} = \mu_x^{(1)} + \mu_x^{(2)} + \mu_x^{(3)} = 0.0001045$$
$${}_t p_x^{(\tau)} = e^{-0.0001045t}$$

$$\text{APV Benefits} = \int_0^\infty e^{-\delta t} 1{,}000{,}000\, {}_t p_x{}^{(\tau)} \mu_x{}^{(1)} dt$$
$$+\int_0^\infty e^{-\delta t} 500{,}000\, {}_t p_x{}^{(\tau)} \mu_x{}^{(2)} dt$$
$$+\int_0^\infty e^{-\delta \tau} 200{,}000\, {}_t p_x{}^{(\tau)} \mu_x{}^{(3)} dt$$
$$= \frac{1{,}000{,}000}{2{,}000{,}000} \int_0^\infty e^{-0.0601045t} dt + \frac{500{,}000}{250{,}000} \int_0^\infty e^{-0.0601045t} dt + \frac{250{,}000}{10{,}000} \int_0^\infty e^{-0.0601045t} dt$$
$$= 27.5\left(16.6377\right) = 457.54$$

Question #8
Key: B

$$APV\ \text{Benefits} = 1000A^{1}_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_kE_{40} 1000 v q_{40+k}$$

$$APV\ \text{Premiums} = \pi \ddot{a}_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_kE_{40} 1000 v q_{40+k}$$

$$\text{Benefit premiums } \Rightarrow \text{Equivalence principle } \Rightarrow$$

$$1000A^{1}_{40:\overline{20|}} + \sum_{k=20}^{\infty} {}_kE_{40} 1000 v q_{40+k} = \pi \ddot{a}_{40:\overline{20|}} + \sum_{20}^{\infty} {}_kE_{40} 1000 v q_{40+k}$$

$$\begin{aligned}\pi &= 1000A^{1}_{40:\overline{20|}} / \ddot{a}_{40:\overline{20|}} \\ &= \frac{161.32 - (0.27414)(369.13)}{14.8166 - (0.27414)(11.1454)} \\ &= 5.11\end{aligned}$$

While this solution above recognized that $\pi = 1000P^{1}_{40:\overline{20|}}$ and was structured to take advantage of that, it wasn't necessary, nor would it save much time. Instead, you could do:

$$APV\ \text{Benefits} = 1000A_{40} = 161.32$$

$$\begin{aligned}APV\ \text{Premiums} &= \pi \ddot{a}_{40:\overline{20|}} + {}_{20}E_{40} \sum_{k=0}^{\infty} {}_kE_{60} 1000 v q_{60+k} \\ &= \pi \ddot{a}_{40:\overline{20|}} + {}_{20}E_{40} 1000A_{60} \\ &= \pi\left[14.8166 - (0.27414)(11.1454)\right] + (0.27414)(369.13) \\ &= 11.7612\pi + 101.19\end{aligned}$$

$$11.7612\pi + 101.19 = 161.32$$

$$\pi = \frac{161.32 - 101.19}{11.7612} = 5.11$$

Question #9
Key: C

$$A_{70} = \frac{\delta}{i}\bar{A}_{70} = \frac{\ln(1.06)}{0.06}(0.53) = 0.5147$$

$$\ddot{a}_{70} = \frac{1-A_{70}}{d} = \frac{1-0.5147}{0.06/1.06} = 8.5736$$

$$\ddot{a}_{69} = 1 + vp_{69}\ddot{a}_{70} = 1 + \left(\frac{0.97}{1.06}\right)(8.5736) = 8.8457$$

$$\begin{aligned}\ddot{a}_{69}^{(2)} &= \alpha(2)\ddot{a}_{69} - \beta(2) = (1.00021)(8.8457) - 0.25739\\ &= 8.5902\end{aligned}$$

Note that the approximation $\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{(m-1)}{2m}$ works well (is closest to the exact answer, only off by less than 0.01). Since $m = 2$, this estimate becomes $8.8457 - \frac{1}{4} = 8.5957$

Question #10
Key: C

The following steps would do <u>in this multiple-choice context</u>:

1. From the answer choices, this is a recursion for an insurance or pure endowment.
2. Only C and E would satisfy $u(70) = 1.0$.
3. It is <u>not</u> E. The recursion for a pure endowment is simpler: $u(k) = \dfrac{1+i}{p_{k-1}}\, u(k-1)$
4. Thus, it must be C.

More rigorously, transform the recursion to its backward equivalent, $u(k-1)$ in terms of $u(k)$:

$$u(k) = -\left(\frac{q_{k-1}}{p_{k-1}}\right) + \left(\frac{1+i}{p_{k-1}}\right) u(k-1)$$
$$p_{k-1} u(k) = -q_{k-1} + (1+i)\, u(k-1)$$
$$u(k-1) = vq_{k-1} + vp_{k-1}\, u(k)$$

This is the form of (a), (b) and (c) on page 119 of Bowers with $x = k-1$. Thus, the recursion could be:

$$A_x = vq_x + vp_x A_{x+1}$$

or $$A^{1}_{x:\overline{y-x}|} = vq_x + vp_x \; A^{\;1}_{x+1:\overline{y-x-1}|}$$

or $$A_{x:\overline{y-x}|} = vq_x + vp_x \; A_{x+1:\overline{y-x-1}|}$$

Condition (iii) forces it to be answer choice C

$u(k-1) = A_x$ fails at $x = 69$ since it is not true that

$$A_{69} = vq_{69} + (vp_{69})(1)$$

$u(k-1) = A^{1}_{x:\overline{y-x}|}$ fails at $x = 69$ since it is not true that

$$A^{1}_{69:\overline{1}|} = vq_{69} + (vp_{69})(1)$$

$u(k-1) = A_{x:\overline{y-x}|}$ is OK at $x = 69$ since

$$A_{69:\overline{1}|} = vq_{69} + (vp_{69})(1)$$

Note: While writing recursion in backward form gave us something exactly like page 119 of Bowers, in its original forward form it is comparable to problem 8.7 on page 251. Reasoning from that formula, with $\pi_h = 0$ and $b_{h+1} = 1$, should also lead to the correct answer.

Question #11
Key: A

You arrive first if both (A) the first train to arrive is a local and (B) no express arrives in the 12 minutes after the local arrives.
$P(A) = 0.75$
Expresses arrive at Poisson rate of $(0.25)(20) = 5$ per hour, hence 1 per 12 minutes.

$$f(0) = \frac{e^{-1}1^0}{0!} = 0.368$$

A and B are independent, so
$P(A \text{ and } B) = (0.75)(0.368) = 0.276$

Question #12
Key: E

(This is covered in Probability Models, section 4.6 in the eighth edition).

Let states be 1, 2, 3 for acutely ill, in remission, cured/dead.

The transition matrix is:

$$\begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

States 1 and 2 are transient. The matrix for transitions from transient states to transient states is.

$$P^T = \begin{pmatrix} 0.6 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$$

$$I - P^T = \begin{pmatrix} 0.4 & -0.3 \\ -0.2 & 0.5 \end{pmatrix} \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is the identity matrix}$$

$$S = \left(I - P^T\right)^{-1} = \begin{pmatrix} 0.5/0.14 & 0.3/0.14 \\ 0.2/0.14 & 0.4/0.14 \end{pmatrix}$$

In that matrix, the entry s_{ij} is the expected time in state *j*, given that you started in state *i*. In this problem, we started in state 1, so the relevant expected times are s_{11} and s_{12}.

Total cost = (10$\times$ expected time in state 1) + (1$\times$ expected time in state 2)

$$\begin{aligned} &= 10s_{11} + s_{12} \\ &= (10)\left(\frac{0.5}{0.14}\right) + \left(\frac{0.3}{0.14}\right) \\ &= 37.84 \end{aligned}$$

Question #13
Key: C

Since all claims are for 1000,
claims rate $= 1000\lambda = 800$.
With relative security loading 0.25, premium rate = (800)(1.25) = 1000.

Per period	*x*	*f*(*x*)	*F*(*x*)	1-*F*(*x*)
	0	0.4493	0.4493	0.5507
	1	0.3595	0.8088	0.1912
	2	0.1438	0.9526	0.0474

Ruin if 2 or more claims by $t = 1$ or 3 or more claims by $t = 2$

2 or more claims by $t = 1$ is ruin since we must pay 2000 or more when we started with 1000 and have collected less than 1000 in premium.
3 or more by $t = 2$ is similar: we can't pay 3000 or more with the initial 1000 and less than 2000 in premium.
Evaluate the probability of ruin by summing the probabilities of three separate cases:

1. Two or more claims before time 1.
2. No claims by time 1, and three or more between 1 and 2.
3. Exactly one claim by time 1, and two more between 1 and 2.

P(ruin) = 1-F(1)		= 0.1912
+ f(0)(1-F(2))	(0.4493)(0.0474) =	0.0213
+ f(1)(1-F(1))	(0.3595)(0.1912) =	0.0687
		0.2812

Question #14
Key: E

d = 0.05 $\rightarrow$ v = 0.095

At issue

$$A_{40} = \sum_{k=0}^{49} v^{k+1} {}_{k|}q_{40} = 0.02\left(v^1 + ... + v^{50}\right) = 0.02v\left(1 - v^{50}\right)/d = 0.35076$$

and $\ddot{a}_{40} = \left(1 - A_{40}\right)/d = \left(1 - 0.35076\right)/0.05 = 12.9848$

so $P_{40} = \dfrac{1000A_{40}}{\ddot{a}_{40}} = \dfrac{350.76}{12.9848} = 27.013$

$$E\left({}_{10}L \middle| K(40) \geq 10\right) = 1000A_{50}^{\text{Revised}} - P_{40}\ddot{a}_{50}^{\text{Revised}} = 549.18 - (27.013)(9.0164) = 305.62$$

where

$$A_{50}^{\text{Revised}} = \sum_{k=0}^{24} v^{k+1} {}_{k|}q_{50}^{\text{Revised}} = 0.04\left(v^1 + ... + v^{25}\right) = 0.04v\left(1 - v^{25}\right)/d = 0.54918$$

and $\ddot{a}_{50}^{\text{Revised}} = \left(1 - A_{50}^{\text{Revised}}\right)/d = \left(1 - 0.54918\right)/0.05 = 9.0164$

Question #15
Key: E

Let NS denote non-smokers and S denote smokers.

The shortest solution is based on the conditional variance formula

$$\mathrm{Var}(X) = E\left(\mathrm{Var}(X|Y)\right) + \mathrm{Var}\left(E(X|Y)\right)$$

Let Y = 1 if smoker; Y = 0 if non-smoker

$$E\left(\bar{a}_{\overline{T}|}|Y=1\right) = \bar{a}_x^S = \frac{1-\bar{A}_x^S}{\delta}$$

$$= \frac{1-0.444}{0.1} = 5.56$$

Similarly $E\left(\bar{a}_{\overline{T}|}|Y=0\right) = \frac{1-0.286}{0.1} = 7.14$

$$E\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right) = E\left(E\left(\bar{a}_{\overline{T}|}|0\right)\right)\times \mathrm{Prob}(\mathrm{Y=0}) + E\left(E\left(\bar{a}_{\overline{T}|}|1\right)\right)\times \mathrm{Prob}(\mathrm{Y=1})$$

$$= (7.14)(0.70) + (5.56)(0.30)$$

$$= 6.67$$

$$E\left[\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right)^2\right] = \left(7.14^2\right)(0.70) + \left(5.56^2\right)(0.30)$$

$$= 44.96$$

$$\mathrm{Var}\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right) = 44.96 - 6.67^2 = 0.47$$

$$E\left(\mathrm{Var}\left(\bar{a}_{\overline{T}|}|Y\right)\right) = (8.503)(0.70) + (8.818)(0.30)$$

$$= 8.60$$

$$\mathrm{Var}\left(\bar{a}_{\overline{T}|}\right) = 8.60 + 0.47 = 9.07$$

Alternatively, here is a solution based on

$\mathrm{Var}(Y) = E\left(Y^2\right) - \left[E(Y)\right]^2$, a formula for the variance of any random variable. This can be transformed into $E\left(Y^2\right) = \mathrm{Var}(Y) + \left[E(Y)\right]^2$ which we will use in its conditional form

$$E\left(\left(\bar{a}_{\overline{T}|}\right)^2|\mathrm{NS}\right) = \mathrm{Var}\left(\bar{a}_{\overline{T}|}|\mathrm{NS}\right) + \left[E\left(\bar{a}_{\overline{T}|}|\mathrm{NS}\right)\right]^2$$

$$\mathrm{Var}\left[\bar{a}_{\overline{T}|}\right] = E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right] - \left(E\left[\bar{a}_{\overline{T}|}\right]\right)^2$$

$$E\left[\bar{a}_{\overline{T}|}\right] = E\left[\bar{a}_{\overline{T}|}|\mathrm{S}\right]\times \mathrm{Prob}[\mathrm{S}] + E\left[\bar{a}_{\overline{T}|}|\mathrm{NS}\right]\times \mathrm{Prob}[\mathrm{NS}]$$

$$= 0.30\bar{a}_x^{\text{S}} + 0.70\bar{a}_x^{\text{NS}}$$

$$= \frac{0.30\left(1-\bar{A}_x^{\text{S}}\right)}{0.1} + \frac{0.70\left(1-\bar{A}_x^{\text{NS}}\right)}{0.1}$$

$$= \frac{0.30(1-0.444)+0.70(1-0.286)}{0.1} = (0.30)(5.56)+(0.70)(7.14)$$

$$= 1.67 + 5.00 = 6.67$$

$$E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right] = E\left[\bar{a}_{\overline{T}|}^{\,2}\,|\text{S}\right]\times\text{Prob}[\text{S}] + E\left[\bar{a}_{\overline{T}|}^{\,2}\,|\text{NS}\right]\times\text{Prob}[\text{NS}]$$

$$= 0.30\left(\text{Var}\left(\bar{a}_{\overline{T}|}|\text{S}\right) + \left(E\left[\bar{a}_{\overline{T}|}|\text{S}\right]\right)^2\right)$$

$$+0.70\left(\text{Var}\left(\bar{a}_{\overline{T}|}|\text{NS}\right) + E\left(\bar{a}_{\overline{T}|}|\text{NS}\right)^2\right)$$

$$= 0.30\left[8.818+(5.56)^2\right] + 0.70\left[8.503+(7.14)^2\right]$$

$$11.919 + 41.638 = 53.557$$

$$\text{Var}\left[\bar{a}_{\overline{T}|}\right] = 53.557 - (6.67)^2 = 9.1$$

Alternatively, here is a solution based on $\bar{a}_{\overline{T}|} = \dfrac{1-v^T}{\delta}$

$$\text{Var}\left(\bar{a}_{\overline{T}|}\right) = \text{Var}\left(\frac{1}{\delta} - \frac{v^T}{\delta}\right)$$

$$= \text{Var}\left(\frac{-v^T}{\delta}\right) \text{ since } \text{Var}(X+\text{constant}) = \text{Var}(X)$$

$$= \frac{\text{Var}\left(v^T\right)}{\delta^2} \text{ since } \text{Var}(\text{constant}\times X) = \text{constant}^2\times\text{Var}(X)$$

$$= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2} \text{ which is Bowers formula 5.2.9}$$

This could be transformed into ${}^2A_x = \delta^2\,\text{Var}\left(\bar{a}_{\overline{T}|}\right) + \bar{A}_x^2$, which we will use to get ${}^2A_x^{\text{NS}}$ and ${}^2A_x^{\text{S}}$.

$$
\begin{aligned}
{}^2A_x &= E\left[v^{2T}\right] \\
&= E\left[v^{2T}\,|\mathrm{NS}\right]\times\mathrm{Prob}(\mathrm{NS}) + E\left[v^{2T}\,|\mathrm{S}\right]\times\mathrm{Prob}(\mathrm{S}) \\
&= \left[\delta^2\mathrm{Var}\left(\bar{a}_{\overline{T}|}\,|\mathrm{NS}\right) + \left(\bar{A}_x^{\mathrm{NS}}\right)^2\right]\times\mathrm{Prob}(\mathrm{NS}) \\
&+ \left[\delta^2\mathrm{Var}\left(\bar{a}_{\overline{T}|}\,|\mathrm{S}\right) + \left(\bar{A}_x^{\mathrm{S}}\right)^2\right]\times\mathrm{Prob}(\mathrm{S}) \\
&= \left[(0.01)(8.503) + 0.286^2\right]\times 0.70 \\
&+ \left[(0.01)(8.818) + 0.444^2\right]\times 0.30 \\
&= (0.16683)(0.70) + (0.28532)(0.30) \\
&= 0.20238
\end{aligned}
$$

$$
\begin{aligned}
\bar{A}_x &= E\left[v^{T}\right] \\
&= E\left[v^{T}\,|\mathrm{NS}\right]\times\mathrm{Prob}(\mathrm{NS}) + E\left[v^{T}\,|\mathrm{S}\right]\times\mathrm{Prob}(\mathrm{S}) \\
&= (0.286)(0.70) + (0.444)(0.30) \\
&= 0.3334
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(\bar{a}_{\overline{T}|}\right) &= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2} \\
&= \frac{0.20238 - 0.3334^2}{0.01} = 9.12
\end{aligned}
$$

Question #16
Key: E

$$_{2}p_0 = 0.7$$
$$_{3}p_0 = 0.4$$

Since hyperbolic,

$$\begin{aligned}
\frac{1}{s(2.25)} &= (0.25)\frac{1}{s(3)} + (0.75)\frac{1}{s(2)} \\
&= (0.25)/(0.4) + (0.75)/(0.7) \\
&= 1.69643 \\
s(2.25) = {}_{2.25}p_0 &= 0.58947
\end{aligned}$$

Alternatively, since hyperbolic

$$\begin{aligned}
p_2 &= \frac{0.4}{0.7} = 0.57143 \\
{}_{0.25}p_2 &= \frac{p_2}{1-(1-0.25)q_2} \\
&= \frac{0.57143}{1-(0.75)(1-0.57143)} \\
&= 0.84211 \\
{}_{2.25}p_0 &= (0.7)(0.84211) = 0.58948
\end{aligned}$$

No matter which way we got ${}_{2.25}p_0$ (ignoring rounding in last digit)

${}_{2.25}q_0 = 1 - 0.58947 = 0.41053$

Probe not transmitting means all three failed

$\text{Prob} = (0.41053)^3 = 0.0692$

Question #17
Key: A

To be a density function, the integral of *f* must be 1 (i.e., everyone dies eventually). The solution is written for the general case, with upper limit ∞. Given the distribution of $f_2(t)$, we could have used upper limit 100 here.

Preliminary calculations from the Illustrative Life Table:

$$\frac{l_{50}}{l_0} = 0.8951$$

$$\frac{l_{40}}{l_0} = 0.9313$$

$$\begin{aligned}
1 = \int_0^\infty f_T(t)\,dt &= \int_0^{50} k\, f_1(t)\,dt + \int_{50}^\infty 1.2\, f_2(t)dt \\
&= k\int_0^{50} f_1(t)\,dt + 1.2\int_{50}^\infty f_2(t)dt \\
&= kF_1(50) + 1.2\left(F_2(\infty) - F_2(50)\right) \\
&= k\left(1 - {}_{50}p_0\right) + 1.2(1-0.5) \\
&= k(1-0.8951) + 0.6 \\
k &= \frac{1-0.6}{1-0.8951} = 3.813
\end{aligned}$$

For $x \le 50, F_T(x) = \int_0^x 3.813 f_1(t)\,dt = 3.813F_1(x)$

$$F_T(40) = 3.813\left(1 - \frac{l_{40}}{l_0}\right) = 3.813(1-0.9313) = 0.262$$

$$F_T(50) = 3.813\left(1 - \frac{l_{50}}{l_0}\right) = 3.813(1-0.8951) = 0.400$$

$${}_{10}p_{40} = \frac{1-F_T(50)}{1-F_T(40)} = \frac{1-0.400}{1-0.262} = 0.813$$

Question #18
Key: D

Let NS denote non-smokers, S denote smokers.

$$\begin{aligned}\text{Prob}(T<t) &= \text{Prob}(T<t \mid \text{NS})\times\text{Prob}(\text{NS})+\text{Prob}(T<t \mid \text{S})\times\text{Prob}(\text{S})\\ &= \left(1-e^{-0.1t}\right)\times 0.7+\left(1-e^{-0.2t}\right)\times 0.3\\ &= 1-0.7e^{-0.1t}-0.3e^{-0.2t}\end{aligned}$$

$S(t)=0.3e^{-0.2t}+0.7e^{-0.1t}$

Want $\hat{t}$ such that $0.75=1-S(\hat{t})$ or $0.25=S(\hat{t})$

$$0.25=0.3e^{-2\hat{t}}+0.7e^{-0.1\hat{t}}=0.3\left(e^{-0.1\hat{t}}\right)^2+0.7e^{-0.1\hat{t}}$$

Substitute: let $x=e^{-0.1\hat{t}}$

$0.3x^2+0.7x-0.25=0$

This is quadratic, so $x=\dfrac{-0.7\pm\sqrt{0.49+(0.3)(0.25)4}}{2(0.3)}$

$x=0.3147$

$e^{-0.1\hat{t}}=0.3147$ so $\hat{t}=11.56$

Question #19
Key: D

The modified severity, X*, represents the conditional payment amount given that a payment occurs. Given that a payment is required (X > *d*), the payment must be uniformly distributed between 0 and $c \cdot (b-d)$.

The modified frequency, N*, represents the number of losses that result in a payment. The deductible eliminates payments for losses below *d*, so only $1-F_x(d)=\frac{b-d}{b}$ of losses will require payments. Therefore, the Poisson parameter for the modified frequency distribution is $\lambda \cdot \frac{b-d}{b}$. (Reimbursing c% after the deductible affects only the payment amount and not the frequency of payments).

Question #20
Key: C

Let N = number of sales on that day
S = aggregate prospective loss at issue on those sales
K = curtate future lifetime

$$N \sim \text{Poisson}(0.2*50) \qquad \Rightarrow E[N] = \text{Var}[N] = 10$$

$$_0L = 10{,}000v^{K+1} - 500\ddot{a}_{\overline{K+1}|} \qquad \Rightarrow E[_0L] = 10{,}000A_{65} - 500\ddot{a}_{65}$$

$$_0L = \left(10{,}000 + \frac{500}{d}\right)v^{K+1} - \frac{500}{d} \qquad \Rightarrow Var[_0L] = \left(10{,}000 + \frac{500}{d}\right)^2 \left[{}^2A_{65} - (A_{65})^2\right]$$

$$S = {}_0L_1 + {}_0L_2 + \ldots + {}_0L_N$$

$$E[S] = E[N] \cdot E[_0L]$$

$$Var[S] = Var[_0L] \cdot E[N] + (E[_0L])^2 \cdot \text{Var}[N]$$

$$\Pr(S < 0) = \Pr\left(Z < \frac{0 - E[S]}{\sqrt{Var[S]}}\right)$$

Substituting d = 0.06/(1+0.06), ${}^2A_{65} = 0.23603$, $A_{65} = 0.43980$ and $\ddot{a}_{65} = 9.8969$ yields

$$E[_0L] = -550.45$$

$$\text{Var}[_0L] = 15{,}112{,}000$$

$$E[S] = -5504.5$$

$$\text{Var}[S] = 154{,}150{,}000$$

Std Dev (S) = 12,416

$$\Pr(S < 0) = \Pr\left(\frac{\text{S} + 5504.5}{12{,}416} < \frac{5504.5}{12{,}416}\right)$$

$$= \Pr(Z < 0.443)$$

$$= 0.67$$

With the answer choices, it was sufficient to recognize that:

$$0.6554 = \phi(0.4) < \phi(0.443) < \phi(0.5) = 0.6915$$

By interpolation, $\phi(0.443) \approx (0.43)\phi(0.5) + (0.57)\phi(0.4)$

$$= (0.43)(0.6915) + (0.57)(0.6554)$$

$$= 0.6709$$

Question #21
Key: A

$$1000P_{40} = \frac{A_{40}}{\ddot{a}_{40}} = \frac{161.32}{14.8166} = 10.89$$

$$1000\,{}_{20}V_{40} = 1000\left(1 - \frac{\ddot{a}_{60}}{\ddot{a}_{40}}\right) = 1000\left(1 - \frac{11.1454}{14.8166}\right) = 247.78$$

$${}_{21}V = \frac{\left({}_{20}V + 5000P_{40}\right)(1+i) - 5000q_{60}}{P_{60}}$$

$$= \frac{\left(247.78 + (5)(10.89)\right)\times 1.06 - 5000\left(0.01376\right)}{1 - 0.01376} = 255$$

[Note: For this insurance, ${}_{20}V = 1000\,{}_{20}V_{40}$ because retrospectively, this is identical to whole life]

Though it would have taken <u>much</u> longer, you can do this as a prospective reserve. The prospective solution is included for educational purposes, not to suggest it would be suitable under exam time constraints.

$1000P_{40} = 10.89$ as above

$$1000A_{40} + 4000\,{}_{20}E_{40}A^{1}_{60:\overline{5}|} = 1000P_{40} + 5000P_{40} \times {}_{20}E_{40}\,\ddot{a}_{60:\overline{5}|} + \pi\,{}_{20}E_{40} \times {}_{5}E_{60}\,\ddot{a}_{65}$$

where $A^{1}_{60:\overline{5}|} = A_{60} - {}_{5}E_{60}\,A_{65} = 0.06674$

$$\ddot{a}_{40:\overline{20}|} = \ddot{a}_{40} - {}_{20}E_{40}\,\ddot{a}_{60} = 11.7612$$

$$\ddot{a}_{60:\overline{5}|} = \ddot{a}_{60} - {}_{5}E_{60}\,\ddot{a}_{65} = 4.3407$$

$$1000\left(0.16132\right) + \left(4000\right)\left(0.27414\right)\left(0.06674\right) =$$

$$= \left(10.89\right)\left(11.7612\right) + \left(5\right)\left(10.89\right)\left(0.27414\right)\left(4.3407\right) + \pi\left(0.27414\right)\left(0.68756\right)\left(9.8969\right)$$

$$\pi = \frac{161.32 + 73.18 - 128.08 - 64.79}{1.86544}$$

$$= 22.32$$

Having struggled to solve for π, you could calculate ${}_{20}V$ prospectively then (as above) calculate ${}_{21}V$ recursively.

$${}_{20}V = 4000A^{1}_{60:\overline{5}|} + 1000A_{60} - 5000P_{40}\,\ddot{a}_{60:\overline{5}|} - \pi\,{}_{5}E_{60}\,\ddot{a}_{65}$$

$$= \left(4000\right)\left(0.06674\right) + 369.13 - \left(5000\right)\left(0.01089\right)\left(4.3407\right) - \left(22.32\right)\left(0.68756\right)\left(9.8969\right)$$

$$= 247.86 \text{ (minor rounding difference from } 1000\,{}_{20}V_{40}\text{)}$$

Or we can continue to ${}_{21}V$ prospectively

$$ {}_{21}V = 5000A^{1}_{61:\overline{4}|} + 1000\,{}_4E_{61}\,A_{65} - 5000P_{40}\,\ddot{a}_{61:\overline{4}|} - \pi\,{}_4E_{61}\,\ddot{a}_{65} $$

where ${}_4E_{61} = \dfrac{l_{65}}{l_{61}} v^4 = \left(\dfrac{7,533,964}{8,075,403}\right)(0.79209) = 0.73898$

$$ \begin{aligned} A^{1}_{61:\overline{4}|} = A_{61} - {}_4E_{61}\,A_{65} &= 0.38279 - 0.73898 \times 0.43980 \\ &= 0.05779 \end{aligned} $$

$$ \begin{aligned} \ddot{a}_{61:\overline{4}|} = \ddot{a}_{61} - {}_4E_{61}\,\ddot{a}_{65} &= 10.9041 - 0.73898 \times 9.8969 \\ &= 3.5905 \end{aligned} $$

$$ \begin{aligned} {}_{21}V &= (5000)(0.05779) + (1000)(0.73898)(0.43980) \\ &\quad - (5)(10.89)(3.5905) - 22.32(0.73898)(9.8969) \\ &= 255 \end{aligned} $$

Finally. A moral victory. Under exam conditions since prospective benefit reserves must equal retrospective benefit reserves, calculate whichever is simpler.

Question #22
Key: C

$$ \mathrm{Var}(Z) = {}^2A_{41} - (A_{41})^2 $$

$$ \begin{aligned} A_{41} - A_{40} = 0.00822 &= A_{41} - (vq_{40} + vp_{40}A_{41}) \\ &= A_{41} - (0.0028/1.05 + (0.9972/1.05)\,A_{41}) \\ &\Rightarrow A_{41} = 0.21650 \end{aligned} $$

$$ \begin{aligned} {}^2A_{41} - {}^2A_{40} = 0.00433 &= {}^2A_{41} - \left(v^2q_{40} + v^2p_{40}\,{}^2A_{41}\right) \\ &= {}^2A_{41} - (0.0028/1.05^2 + \left(0.9972/1.05^2\right)^2 A_{41}) \\ &{}^2A_{41} = 0.07193 \end{aligned} $$

$$ \begin{aligned} \mathrm{Var}(Z) &= 0.07193 - 0.21650^2 \\ &= 0.02544 \end{aligned} $$

Question #23
Key: A

Let L, be the amount by which surplus first drops below 40, given that it does drop below 40.

$$PR\left[40-L_1<35\right]=PR\left[5<L_1\right]=\int_5^{\infty} f_{L_1}(y)dy$$
$$=\int_5^{\infty}\frac{1}{p_1}\left[1-P(y)\right]dy$$

The last step above is equation 13.5.3 in Bowers (page 416)

For claim size distributed uniformly over (0, 10)

$$p_1=5$$
$$\text{and } P(y)=\begin{cases}0 & y<0\\ y/10 & 0<y<10\\ 1 & y>10\end{cases}$$

$$PR=\int_5^{10}\frac{1}{5}\left(1-y/10\right)dy=\frac{-1}{5}\times\frac{1}{2}\times 10\times\left(1-\frac{y}{10}\right)^2\Big|_5^{10}=0.25$$

Question #24
Key: D

This solution looks imposing because there is no standard notation. Try to focus on the big picture ideas rather than starting with the details of the formulas.

Big picture ideas:

1. We can express the present values of the perpetuity recursively.
2. Because the interest rates follow a Markov process, the present value (at time t) of the future payments at time t depends only on the state you are in at time t, not how you got there.
3. Because the interest rates follow a Markov process, the present value of the future payments at times t_1 and t_2 are equal if you are in the same state at times t_1 and t_2.

Method 1: Attack <u>without</u> considering the special characteristics of this transition matrix.

Let $s_k =$ state you are in at time k $\left(\text{thus } s_k = 0, 1 \text{ or } 2\right)$

Let $Y_k =$ present value, at time k, of the future payments.
Y_k is a random variable because its value depends on the pattern of discount factors, which are random. The expected value of Y_k is not constant; it depends on what state we are in at time *k*.

Recursively we can write

$Y_k = v \times \left(1 + Y_{k+1}\right)$, where it would be better to have notation that indicates the *v*'s are not constant, but are realizations of a random variable, where the random variable itself has different distributions depending on what state we're in. However, that would make the notation so complex as to mask the simplicity of the relationship.

Every time we are in state 0 we have

$$E\left[Y_k \middle| s_k = 0\right] = 0.95 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 0\right]\right)$$

$$= 0.95 \times \left(1 + \left\{\left(E\left[Y_{k+1} \middle| s_{k+1} = 0\right]\right)\right) \times \Pr\text{ob}\left(\mathrm{s}_{\mathrm{k+1}} = 0 \middle| s_k = 0\right]\right)$$

$$+ \left(E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right) \times \Pr\text{ob}\left(s_{k+1} = 1 \middle| s_k = 0\right]\right)$$

$$= \left(E\left[Y_{k+1} \middle| s_{k+1} = 2\right]\right) \times \Pr\text{ob}\left(s_{k+1} = 2 \middle| s_k = 0\right]\right)\right\}$$

$$= 0.95 \times \left(1 + E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right)$$

That last step follows because from the transition matrix if we are in state 0, we always move to state 1 one period later.

Similarly, every time we are in state 2 we have

$$E\left[Y_k \middle| s_k = 2\right] = 0.93 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 2\right]\right)$$
$$= 0.93 \times \left(1 + E\left[Y_{k+1} \middle| s_{k+1} = 1\right]\right)$$

That last step follows because from the transition matrix if we are in state 2, we always move to state 1 one period later.

Finally, every time we are in state 1 we have

$$E\left[Y_k \middle| s_k = 1\right] = 0.94 \times \left(1 + E\left[Y_{k+1} \middle| s_k = 1\right]\right)$$
$$= 0.94 \times \left(1 + \left\{E\left[Y_{k+1} \middle| s_{k+1} = 0\right] \times \Pr\left[s_{k+1} = 0 \middle| s_k = 1\right] + E\left[Y_{k+1} \middle| s_{k+1} = 2\right] \times \Pr\left[s_{k+1} = 2 \middle| s_k = 1\right]\right\}\right)$$
$$= 0.94 \times \left(1 + \left\{E\left[Y_{k+1} \middle| s_{k+1} = 0\right] \times 0.9 + E\left[Y_{k+1} \middle| s_{k+1} = 2\right] \times 0.1\right\}\right).$$

Those last two steps follow from the fact that from state 1 we always go to either state 0 (with probability 0.9) or state 2 (with probability 0.1).

Now let's write those last three paragraphs using this shorter notation: $x_n = E\left[Y_k \middle| s_k = n\right]$. We can do this because (big picture idea #3), the conditional expected value is only a function of the state we are in, not when we are in it or how we got there.

$$x_0 = 0.95(1 + x_1)$$
$$x_1 = 0.94(1 + 0.9x_0 + 0.1x_2)$$
$$x_2 = 0.93(1 + x_1)$$

That's three equations in three unknowns. Solve (by substituting the first and third into the second) to get $x_1 = 16.82$.

That's the answer to the question, the expected present value of the future payments given in state 1.

The solution above is almost exactly what we would have to do with <u>any</u> 3×3 transition matrix. As we worked through, we put only the non-zero entries into our formulas. But if for example the top row of the transition matrix had been $\begin{pmatrix} 0.4 & 0.5 & 0.1 \end{pmatrix}$, then the first of our three equations would have become $x_0 = 0.95(1 + 0.4x_0 + 0.5x_1 + 0.1x_2)$, similar in structure to our actual equation for x_1. We would still have ended up with three linear equations in three unknowns, just more tedious ones to solve.

Method 2: Recognize the patterns of changes for <u>this particular transition matrix</u>.

<u>This particular</u> transition matrix has a recurring pattern that leads to a much quicker solution. We are starting in state 1 and are guaranteed to be back in state 1 two steps later, with the same prospective value then as we have now.

Thus,

$$E[Y] = E[Y|\text{first move is to } 0] \times \Pr[\text{first move is to } 0] + E[Y|\text{first move is to } 2] \times \Pr[\text{first move is to } 2]$$

$$= 0.94 \times \left[\left(1 + 0.95 \times \left(1 + E[Y]\right)\right)\right] \times 0.9 + \left[0.94 \times \left[\left(1 + 0.93 \times \left(1 + E[Y]\right)\right) \times 0.1\right]\right]$$

(Note that the equation above is exactly what you get when you substitute x_0 and x_2 into the formula for x_1 in Method 1.)

$$= 1.6497 + 0.8037E[Y] + 0.1814 + 0.0874E[Y]$$

$$E[Y] = \frac{1.6497 + 0.1814}{(1 - 0.8037 - 0.0874)}$$

$$= 16.82$$

Question #25
Key: A

Let state s = number of stocks with market price > strike price.

$s = 0,1,2$

Transition probability matrix:

$$\begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{16} & \frac{6}{16} & \frac{9}{16} \end{array}$$

$\pi_0 =$ limiting probability of 0 stocks with strike price < market price.

$\pi_1 =$ limiting probability of 1 stock with strike price < market price.

$\pi_2 =$ limiting probability of 2 stocks with strike price < market price.

$$\pi_0 = \frac{9}{16}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{16}\pi_2$$

$$\pi_1 = \frac{3}{8}\pi_0 + \frac{1}{2}\pi_1 + \frac{3}{8}\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving algebraically the three equations in three unknowns gives us $\pi_2 = 2/7$.

Question #26
Key: E

The number of problems solved in 10 minutes is Poisson with mean 2.
If she solves exactly one, there is 1/3 probability that it is #3.
If she solves exactly two, there is a 2/3 probability that she solved #3.
If she solves #3 or more, she got #3.

$f(0) = 0.1353$
$f(1) = 0.2707$
$f(2) = 0.2707$

$$P = \left(\frac{1}{3}\right)(0.2707) + \left(\frac{2}{3}\right)(0.2707) + (1 - 0.1353 - 0.2707 - 0.2707) = 0.594$$

Question #27
Key: D

$$\mu_x^{(\tau)} = \mu_x^{(1)}(t) + \mu_x^{(2)}(t)$$

$$= 0.2\mu_x^{(\tau)}(t) + \mu_x^{(2)}(t)$$

$$\Rightarrow \mu_x^{(2)}(t) = 0.8\mu_x^{(\tau)}(t)$$

$$q_x^{'(1)} = 1 - p_x^{'(1)} = 1 - e^{-\int_0^1 0.2k\, t^2\, dt} = 1 - e^{-0.2\frac{k}{3}} = 0.04$$

$$k/3 \Rightarrow \ln(1-0.04)/(-0.2) = 0.2041$$
$$k = 0.6123$$

$${}_2q_x^{(2)} = \int_0^2 {}_tp_x^{(\tau)}\mu_x^{(2)}\ dt = 0.8\int_0^2\ {}_tp_x^{(\tau)}\ \ \mu_x^{(\tau)}(t)\ dt$$

$$= 0.8\ {}_2q_x^{(\tau)} = 0.8\left(1 - {}_2p_x^{(\tau)}\right)$$

$${}_2p_x^{(\tau)} = e^{-\int_0^2 \mu_x(t)dt}$$

$$= e^{-\int_0^2 kt^2\, dt}$$

$$= e^{\frac{-8k}{3}}$$

$$= e^{\frac{-(8)(0.6123)}{3}}$$

$$= 0.19538$$

$${}_2q_x^{(2)} = 0.8(1 - 0.19538) = 0.644$$

Question #28
Key: A

k	$k \wedge 3$	$f(k)$	$f(k)\times(k \wedge 3)$	$f(k)\times(k \wedge 3)^2$
0	0	0.1	0	0
1	1	(0.9)(0.2) = 0.18	0.18	0.18
2	2	(0.72)(0.3) = 0.216	0.432	0.864
3+	3	1-0.1-0.18-0.216 = 0.504	<u>1.512</u>	<u>4.536</u>
			2.124	5.580

$$E(K \wedge 3) = 2.124$$
$$E\left((K \wedge 3)^2\right) = 5.580$$
$$\mathrm{Var}(K \wedge 3) = 5.580 - 2.124^2 = 1.07$$

Note that $E[K \wedge 3]$ is the temporary curtate life expectancy, $e_{x:\overline{3}|}$ if the life is age x. Problem 3.17 in Bowers, pages 86 and 87, gives an alternative formula for the variance, basing the calculation on ${}_k p_x$ rather than ${}_{k|}q_x$.

Question #29
Key: E

$$\begin{aligned} f(x) &= 0.01, \quad 0 \le x \le 80 \\ &= 0.01 - 0.00025(x-80) = 0.03 - 0.00025x, \quad 80 < x \le 120 \end{aligned}$$

$$\begin{aligned} E(x) &= \int_0^{80} 0.01x\,dx + \int_{80}^{120}\left(0.03x - 0.00025x^2\right)dx \\ &= \frac{0.01x^2}{2}\Big|_0^{80} + \frac{0.03x^2}{2}\Big|_{80}^{120} - \frac{0.00025x^3}{3}\Big|_{80}^{120} \\ &= 32 + 120 - 101.33 = 50.66667 \end{aligned}$$

$$\begin{aligned} E(X-20)_+ &= E(X) - \int_0^{20} x f(x)\,dx - 20(1 - \int_0^{20} f(x)dx) \\ &= 50.6667 - \frac{0.01x^2}{2}\Big|_0^{20} - 20\left(1 - 0.01x\Big|_0^{20}\right) \\ &= 50.6667 - 2 - 20(0.8) = 32.6667 \end{aligned}$$

Loss Elimination Ratio = $1 - \dfrac{32.6667}{50.6667} = 0.3553$

Question #30
Key: D

Let q_{64} for Michel equal the standard q_{64} plus *c*. We need to solve for *c*.
Recursion formula for a standard insurance:

$$ {}_{20}V_{45} = \left({}_{19}V_{45} + P_{45}\right)(1.03) - q_{64}\left(1 - {}_{20}V_{45}\right) $$

Recursion formula for Michel's insurance

$$ {}_{20}V_{45} = \left({}_{19}V_{45} + P_{45} + 0.01\right)(1.03) - \left(q_{64} + c\right)(1 - {}_{20}V_{45}) $$

The values of ${}_{19}V_{45}$ and ${}_{20}V_{45}$ are the same in the two equations because we are told Michel's benefit reserves are the same as for a standard insurance.

Subtract the second equation from the first to get:

$$ \begin{aligned} 0 &= -(1.03)(0.01) + c(1 - {}_{20}V_{45}) \\ c &= \frac{(1.03)(0.01)}{\left(1 - {}_{20}V_{45}\right)} \\ &= \frac{0.0103}{1 - 0.427} \\ &= 0.018 \end{aligned} $$

Question #31
Key: B

K is the curtate future lifetime for one insured.
L is the loss random variable for one insurance.
L_{AGG} is the aggregate loss random variables for the individual insurances.
σ_{AGG} is the standard deviation of L_{AGG}.
M is the number of policies.

$$L = v^{K+1} - \pi \ddot{a}_{\overline{K+1}|} = \left(1 + \frac{\pi}{d}\right) v^{K+1} - \pi/d$$

$$E[L] = \left(A_x - \pi \ddot{a}_x\right) = A_x - \pi \frac{(1 - A_x)}{d}$$

$$= 0.24905 - 0.025\left(\frac{0.75095}{0.056604}\right) = -0.082618$$

$$\text{Var}[L] = \left(1 + \frac{\pi}{d}\right)^2 \left({}^2A_x - A_x^2\right) = \left(1 + \frac{0.025}{0.056604}\right)^2 \left(0.09476 - (0.24905)^2\right) = 0.068034$$

$$E[L_{AGG}] = M\,E[L] = -0.082618M$$

$$\text{Var}[L_{AGG}] = M\ \text{Var}[L] = M(0.068034) \Rightarrow \sigma_{AGG} = 0.260833\sqrt{M}$$

$$\Pr[L_{AGG} > 0] = \left[\frac{L_{AGG} - E[L_{AGG}]}{\sigma_{AGG}} > \frac{-E(L_{AGG})}{\sigma_{AGG}}\right]$$

$$\approx \Pr\left(N(0,1) > \frac{0.082618M}{\sqrt{M}\,(0.260833)}\right)$$

$$\Rightarrow 1.645 = \frac{0.082618\sqrt{M}}{0.260833}$$

$$\Rightarrow M = 26.97$$

$\Rightarrow$ minimum number needed = 27

Question #32
Key: D

Annuity benefit: $Z_1 = 12{,}000\dfrac{1-v^{K+1}}{d}$ for $K = 0,1,2,...$

Death benefit: $Z_2 = Bv^{K+1}$ for $K = 0,1,2,...$

New benefit: $Z = Z_1 + Z_2 = 12{,}000\dfrac{1-v^{K+1}}{d} + Bv^{K+1}$

$$= \frac{12{,}000}{d} + \left(B - \frac{12{,}000}{d}\right)v^{K+1}$$

$$\mathrm{Var}(Z) = \left(B - \frac{12{,}000}{d}\right)^2 \mathrm{Var}\left(v^{K+1}\right)$$

$$\mathrm{Var}(Z) = 0 \text{ if } B = \frac{12{,}000}{0.08} = 150{,}000\,.$$

In the first formula for $\mathrm{Var}(Z)$, we used the formula, valid for any constants *a* and *b* and random variable *X*,

$$\mathrm{Var}(a + bX) = b^2\mathrm{Var}(X)$$

Question #33
Key: B

First restate the table to be CAC's cost, after the 10% payment by the auto owner:

Towing Cost, *x*	*p*(*x*)
72	50%
90	40%
144	10%

Then $E(X) = 0.5*72 + 0.4*90 + 0.1*144 = 86.4$

$E(X^2) = 0.5*72^2 + 0.4*90^2 + 0.1*144^2 = 7905.6$

$\text{Var}(X) = 7905.6 - 86.4^2 = 440.64$

Because Poisson, $E(N) = \text{Var}(N) = 1000$

$E(S) = E(X)E(N) = 86.4*1000 = 86,400$

$\text{Var}(S) = E(N)\text{Var}(X) + E(X)^2\text{Var}(N) = 1000*440.64 + 86.4^2*1000 = 7,905,600$

$$\Pr(S > 90,000) + \Pr\left(\frac{S - E(S)}{\sqrt{\text{Var}(S)}} > \frac{90,000 - 86,400}{\sqrt{7,905,600}}\right) = \Pr(Z > 1.28) = 1 - \Phi(1.28) = 0.10$$

Since the frequency is Poisson, you could also have used

$\text{Var}(S) = \lambda E(X^2) = (1000)(7905.6) = 7,905,600$

That way, you would not need to have calculated $\text{Var}(X)$.

Question #34
Key: C

$$\text{LER} = \frac{E\left(X \wedge d\right)}{E\left(X\right)} = \frac{\theta\left(1-e^{-d/\theta}\right)}{\theta} = 1-e^{-d/\theta}$$

Last year $\quad 0.70 = 1-e^{-d/\theta} \Rightarrow -d = \theta \log\ 0.30$

Next year: $\quad -d_{new} = \theta \log(1-\text{LER}_{new})$

Hence $\theta \log\left(1-\text{LER}_{\text{new}}\right) = -d_{\text{new}} = \frac{4}{3}\theta \log\ 0.30$

$$\log\left(1-\text{LER}_{new}\right) = -1.6053$$

$$\left(1-\text{LER}_{new}\right) = e^{-1.6053} = 0.20$$

$$\text{LER}_{new} = 0.80$$

Question #35
Key: E

$E\left(X\right) = e\left(d\right)S\left(d\right) + E\left(X \wedge d\right)$ [Klugman Study Note, formula 3.10]

$$62 = \overset{\circ}{e}_{40} \times {}_{40}p_0 + E\left(T \wedge 40\right)$$

$$62 = (\overset{\circ}{e}_{40})(0.6) + 40 - (0.005)(40^2)$$

$$= 0.6\overset{\circ}{e}_{40} + 32$$

$$\overset{\circ}{e}_{40} = \frac{\left(62-32\right)}{0.6} = 50$$

The first equation, in the notation of Bowers, is $\overset{\circ}{e}_0 = \overset{\circ}{e}_{40} \times {}_{40}p_0 + \overset{\circ}{e}_{0:\overline{40}|}$. The corresponding formula, with $i > 0$, is a very commonly used one:

$$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x\, \bar{a}_{x+n}$$

Question #36
Key: C

$$\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_t p_x dt$$

$$q_{90} = 0.18877$$

For U $\quad {}_t p_x = 1 - t \cdot q_x \quad$ so $\quad \overset{\circ}{e}_{90:\overline{1}|} = 1 - \tfrac{1}{2}(0.18877) = 0.9056$

For C $\quad {}_t p_x = (p_x)^t \quad$ so $\quad \overset{\circ}{e}_{90:\overline{1}|} = \dfrac{q_{90}}{-\log p_{90}} = \dfrac{0.18877}{-\log(0.81123)} = 0.9023$

For H $\quad {}_t p_x = \dfrac{p_x}{p_x + tq_x} \quad$ so $\quad \overset{\circ}{e}_{90:\overline{1}|} = \dfrac{p_{90}}{q_{90}}\left(-\log\ p_{90}\right) = \dfrac{0.81123}{0.18877}\left(-\log 0.81123\right) = 0.8990$

Alternatively

For U, $\mu(x+z) = \dfrac{q_x}{1 - tq_x}$ is increasing for $0 < t < 1$

For C, $\mu(x+t) = -\log\ p_x$ is constant for $0 < t < 1$

For H, $\mu(x+t) = \dfrac{q_x}{1-(1-t)q_x}$ is decreasing for $0 < t < 1$

With ${}_1 p_x$ the same for all three, we must have ${}_t p_x^U > {}_t p_x^C > {}_t p_x^H$ for 0<t<1
and $\int_0^1 {}_t p_{90}^U\ dt > \int_0^1 {}_t p_{90}^C\ dt > \int_0^1 {}_t p_{90}^H\ dt$

Alternatively (not rigorously)

$$_{0.5} p_{90}^U = 0.9056$$
$$_{0.5} p_{90}^C = 0.9007$$
$$_{0.5} p_{90}^H = 0.8958$$

That should strongly suggest that $\int_0^1 {}_t p_{90}^U\ dt > \int_0^1 {}_t p_{90}^C\ dt > \int_0^1 {}_t p_{90}^H\ dt$
You could compare additional values of p for greater comfort.

Question #37
Key: B

$d = 0.05 \Rightarrow v = 0.95$

Step 1 Determine p_x from Kevin's work:

$$608 + 350vp_x = 1000vq_x + 1000v^2 p_x\left(p_{x+1} + q_{x+1}\right)$$
$$608 + 350(0.95)\,p_x = 1000(0.95)(1 - p_x) + 1000(0.9025)\,p_x(1)$$
$$608 + 332.5p_x = 950(1 - p_x) + 902.5p_x$$
$$p_x = 342/380 = 0.9$$

Step 2 Calculate $1000P_{x:\overline{2|}}$, as Kira did:

$$608 + 350(0.95)(0.9) = 1000P_{x:\overline{2|}}\left[1 + (0.95)(0.9)\right]$$
$$1000P_{x:\overline{2|}} = \frac{[299.25 + 608]}{1.855} = 489.08$$

The first line of Kira's solution is that the actuarial present value of Kevin's benefit premiums is equal to the actuarial present value of Kira's, since each must equal the actuarial present value of benefits. The actuarial present value of benefits would also have been easy to calculate as $(1000)(0.95)(0.1) + (1000)\left(0.95^2\right)(0.9) = 907.25$

Question #38
Key: E

Because no premiums are paid after year 10 for (*x*), ${}_{11}V_x = A_{x+11}$

Rearranging 8.3.10 from Bowers, we get ${}_{h+1}V = \dfrac{\left({}_hV + \pi_h\right)(1+i) - b_{h+1}q_{x+h}}{p_{x+h}}$

$${}_{10}V = \frac{(32,535 + 2,078)\times(1.05) - 100,000\times 0.011}{0.989} = 35,635.642$$

$${}_{11}V = \frac{(35,635.642 + 0)\times(1.05) - 100,000\times 0.012}{0.988} = 36,657.31 = A_{x+11}$$

Question #39
Key: B

For De Moivre's law where $s(x) = \left(1 - \dfrac{x}{\omega}\right)$:

$$\overset{\circ}{e}_x = \frac{\omega - x}{2} \text{ and } {}_tp_x = \left(1 - \frac{t}{\omega - x}\right)$$

$$\overset{\circ}{e}_{45} = \frac{105 - 45}{2} = 30$$

$$\overset{\circ}{e}_{65} = \frac{105 - 65}{2} = 20$$

$$\overset{\circ}{e}_{45:65} = \int_0^{40} {}_tp_{45:65}\,dt = \int_0^{40} \frac{60-t}{60}\times\frac{40-t}{40}\,dt$$

$$= \frac{1}{60\times 40}\left(60\times 40\times t - \frac{60+40}{2}t^2 + \frac{1}{3}t^3\right)\Bigg|_0^{40}$$

$$= 15.56$$

$$\overset{\circ}{e}_{\overline{45:65}} = \overset{\circ}{e}_{45} + \overset{\circ}{e}_{65} - \overset{\circ}{e}_{45:65}$$

$$= 30 + 20 - 15.56 = 34$$

In the integral for $\overset{\circ}{e}_{45:65}$, the upper limit is 40 since 65 (and thus the joint status also) can survive a maximum of 40 years.

Question #40
Key: B

$F(0) = 0.8$
$F(t) = 0.8 + 0.00025(t\text{-}1000), \quad 1000 \le t \le 5000$

$0.75 \Rightarrow$ 0 found since $F(0) \ge 0.75$

$0.85 \Rightarrow$ 2000 found since $F(2000) = 0.85$

Average of those two outcomes is 1000.