

Fall 2002
Course 3 solutions

Question #1

Answer: E

$$m(4) = -s'(4) / s(4)$$

$$= \frac{-(-e^4 / 100)}{1 - e^4 / 100}$$

$$= \frac{e^4 / 100}{1 - e^4 / 100}$$

$$= \frac{e^4}{100 - e^4}$$

$$= 1.202553$$

Question # 2**Answer: A**

$$q_x^{(i)} = q_x^{(t)} \left[\frac{\ln p_x'^{(i)}}{\ln p_x^{(t)}} \right] = q_x^{(t)} \left[\frac{\ln e^{-\mathbf{m}^{(i)}}}{\ln e^{-\mathbf{m}^{(t)}}} \right]$$

$$= q_x^{(t)} \times \frac{\mathbf{m}^{(i)}}{\mathbf{m}^{(t)}}$$

$$\mathbf{m}_x^{(t)} = \mathbf{m}_x^{(1)} + \mathbf{m}_x^{(2)} + \mathbf{m}_x^{(3)} = 15$$

$$q_x^{(t)} = 1 - e^{-\mathbf{m}^{(t)}} = 1 - e^{-1.5}$$

$$= 0.7769$$

$$q_x^{(2)} = \frac{(0.7769)\mathbf{m}^{(2)}}{\mathbf{m}^{(t)}} = \frac{(0.5)(0.7769)}{15}$$

$$= 0.2590$$

Question # 3**Answer: D**

$$\begin{array}{ccccc}
{}_2|_2 A_{[60]} = v^3 & \times & {}_2p_{[60]} & \times & q_{[60]+2} & + \\
\downarrow & & \downarrow & & \downarrow & \\
\text{pay at end} & & \text{live} & & \text{then die} & \\
\text{of year 3} & & \text{2 years} & & \text{in year 3} &
\end{array}$$

$$\begin{array}{ccccc}
+v^4 & \times & {}_3p_{[60]} & \times & q_{60+3} \\
\text{pay at end} & & \text{live} & & \text{then die} \\
\text{of year 4} & & \text{3 years} & & \text{in year 4}
\end{array}$$

$$= \frac{1}{(1.03)^3} (1-0.09)(1-0.11)(0.13) + \frac{1}{(1.03)^4} (1-0.09)(1-0.11)(1-0.13)(0.15)$$

$$= 0.19$$

Question # 4**Answer: B**

$$\bar{a}_x = \bar{a}_{x:\overline{5}|} + {}_5E_x \bar{a}_{x+5}$$

$$\bar{a}_{x:\overline{5}|} = \frac{1 - e^{-0.07(5)}}{0.07} = 4.219, \text{ where } 0.07 = \mathbf{m} + \mathbf{d} \text{ for } t < 5$$

$${}_5E_x = e^{-0.07(5)} = 0.705$$

$$\bar{a}_{x+5} = \frac{1}{0.08} = 12.5, \text{ where } 0.08 = \mathbf{m} + \mathbf{d} \text{ for } t \geq 5$$

$$\therefore \bar{a}_x = 4.219 + (0.705)(12.5) = 13.03$$

Question # 5**Answer: E**

The distribution of claims (a gamma mixture of Poissons) is negative binomial.

$$E(N) = E_{\Lambda}(E(N|\Lambda)) = E_{\Lambda}(\Lambda) = 3$$

$$Var(N) = E_{\Lambda}(Var(N|\Lambda)) + Var_{\Lambda}(E(N|\Lambda))$$

$$= E_{\Lambda}(\Lambda) + Var_{\Lambda}(\Lambda) = 6$$

$$r\mathbf{b} = 3$$

$$r\mathbf{b}(1 + \mathbf{b}) = 6$$

$$(1 + \mathbf{b}) = 6 / 3 = 2 ; \quad \mathbf{b} = 1$$

$$r\mathbf{b} = 3$$

$$r = 3$$

$$p_0 = (1 + \mathbf{b})^{-r} = 0.125$$

$$p_1 = \frac{r\mathbf{b}}{(1 + \mathbf{b})^{r+1}} = 0.1875$$

$$\begin{aligned} \text{Pr ob(at most 1)} &= p_0 + p_1 \\ &= 0.3125 \end{aligned}$$

Question # 6**Answer: A**

$$E(S) = E(N) \times E(X) = 110 \times 1,101 = 121,110$$

$$\begin{aligned} Var(S) &= E(N) \times Var(X) + E(X)^2 \times Var(N) \\ &= 110 \times 70^2 + 1101^2 \times 750 \\ &= 909,689,750 \end{aligned}$$

$$\text{Std Dev}(S) = 30,161$$

$$\begin{aligned} \text{Pr}(S < 100,000) &= \text{Pr}(Z < (100,000 - 121,110) / 30,161) \text{ where } Z \text{ has standard normal distribution} \\ &= \text{Pr}(Z < -0.70) = 0.242 \end{aligned}$$

Question # 7

Answer: C

This is just the Gambler's Ruin problem, in units of 5,000 calories.

Each day, up one with $p = 0.45$; down 1 with $q = 0.55$

Will Allosaur ever be up 1 before being down 2?

$$P_2 = \frac{(1 - (0.55/0.45)^2)}{(1 - (0.55/0.45)^3)} = 0.598$$

Or, by general principles instead of applying a memorized formula:

Let P_1 = probability of ever reaching 3 (15,000 calories) if at 1 (5,000 calories).

Let P_2 = probability of ever reaching 3 (15,000 calories) if at 2 (10,000 calories).

From either, we go up with $p = 0.45$, down with $q = 0.55$

$P(\text{reaching 3}) = P(\text{up}) \times P(\text{reaching 3 after up}) + P(\text{down}) \times P(\text{reaching 3 after down})$

$$P_2 = 0.45 \times 1 + 0.55 \times P_1$$

$$P_1 = 0.45 \times P_2 + 0.55 \times 0 = 0.45 \times P_2$$

$$P_2 = 0.45 + 0.55 \times P_1 = 0.45 + 0.55 \times 0.45 \times P_2 = 0.45 + 0.2475 P_2$$

$$P_2 = 0.45 / (1 - 0.2475) = 0.598$$

Here is another approach, feasible since the number of states is small.

Let states 0,1,2,3 correspond to 0; 5,000; 10,000; ever reached 15,000 calories. For purposes of this problem, state 3 is absorbing, since once the allosaur reaches 15,000 we don't care what happens thereafter.

The transition matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.55 & 0 & 0.45 & 0 \\ 0 & 0.55 & 0 & 0.45 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Starting with the allosaur in state 2;

[0	0	1	0]	at inception
[0	0.55	0	0.45]	after 1
[0.3025	0	0.2475	0.45]	after 2
[0.3025	0.1361	0	0.5614]	after 3
[0.3774	0	0.0612	0.5614]	after 4
[0.3774	0.0337	0	0.5889]	after 5
[0.3959	0	0.0152	0.5889]	after 6

By this step, if not before, Prob(state 3) must be converging to 0.60. It's already closer to 0.60 than 0.57, and its maximum is $0.5889 + 0.0152$

Question # 8**Answer: B**

The investor will receive at least 1.5 if and only if $C(5) \leq 0$.

$C(5) - C(1)$ is normally distributed with mean $(4)(0) = 0$,
variance $= (4)(0.01) = 0.04$, standard deviation $= 0.2$.

$$\begin{aligned}\Pr(C(5) \leq 0 | C(1) = 0.05) &= \Pr(C(5) - C(1) \leq -0.05) = \\ &= \Pr((C(5) - C(1)) / 0.2 \leq -0.05 / 0.2) \\ &= 1 - \Phi(0.25) = 0.4\end{aligned}$$

Question # 9**Answer: D**

Per 10 minutes, find coins worth exactly 10 at Poisson rate $(0.5)(0.2)(10) = 1$

Per 10 minutes,	$f(0) = 0.3679$	$F(0) = 0.3679$
	$f(1) = 0.3679$	$F(1) = 0.7358$
	$f(2) = 0.1839$	$F(2) = 0.9197$
	$f(3) = 0.0613$	$F(3) = 0.9810$

Let Period 1 = first 10 minutes; period 2 = next 10.

Method 1, succeed with 3 or more in period 1; or exactly 2, then one or more in period 2

$$P = (1 - F(2)) + f(2)(1 - F(0)) = (1 - 0.9197) + (0.1839)(1 - 0.3679) \\ = 0.1965$$

Method 2: fail in period 1 if < 2 ;

$$\text{Prob} = F(1) = 0.7358$$

fail in period 2 if exactly 2 in period 1, then 0;

$$\text{Prob} = f(2)f(0) \\ = (0.1839)(0.3679) = 0.0677$$

Succeed if fail neither period;

$$\text{Prob} = 1 - 0.7358 - 0.0677 \\ = 0.1965$$

(Method 1 is attacking the problem as a stochastic process model; method 2 attacks it as a ruin model.)

Question # 10**Answer: C**

Which distribution is it from?

$0.25 < 0.30$, so it is from the exponential.

Given that Y is from the exponential, we want

$$\Pr(Y \leq y) = F(y) = 0.69$$

$$1 - e^{-y/0.5} = 0.69$$

$$1 - e^{-y/0.5} = 0.69 \text{ since mean} = 0.5$$

$$\frac{-y}{0.5} = \ln(1 - 0.69) = -1.171$$

$$y = 0.5855$$

Question # 11**Answer: D**

Use Mod to designate values unique to this insured.

$$\ddot{a}_{60} = (1 - A_{60}) / d = (1 - 0.36933) / [(0.06) / (1.06)] = 11.1418$$

$$1000P_{60} = 1000A_{60} / \ddot{a}_{60} = 1000(0.36933 / 11.1418) = 33.15$$

$$A_{60}^{Mod} = v(q_{60}^{Mod} + p_{60}^{Mod}A_{61}) = \frac{1}{1.06} [0.1376 + (0.8624)(0.383)] = 0.44141$$

$$\ddot{a}_{60}^{Mod} = (1 - A_{60}^{Mod}) / d = (1 - 0.44141) / [0.06 / 1.06] = 9.8684$$

$$E[{}_0L^{Mod}] = 1000(A_{60}^{Mod} - P_{60}\ddot{a}_{60}^{Mod})$$

$$= 1000[0.44141 - 0.03315(9.8684)]$$

$$= 114.27$$

Question # 12**Answer: D**

The prospective reserve at age 60 per 1 of insurance is A_{60} , since there will be no future premiums. Equating that to the retrospective reserve per 1 of coverage, we have:

$$A_{60} = P_{40} \frac{\ddot{s}_{40:\overline{10}|}}{10E_{50}} + P_{50}^{Mod} \ddot{s}_{50:\overline{10}|} - {}_{20}k_{40}$$

$$A_{60} = \frac{A_{40}}{\ddot{a}_{40}} \times \frac{\ddot{a}_{40:\overline{10}|}}{10E_{40} 10E_{50}} + P_{50}^{Mod} \frac{\ddot{a}_{50:\overline{10}|}}{10E_{50}} - \frac{A_{40}^1}{20E_{40}}$$

$$0.36913 = \frac{0.16132}{14.8166} \times \frac{7.70}{(0.53667)(0.51081)} + P_{50}^{Mod} \frac{7.57}{0.51081} - \frac{0.06}{0.27414}$$

$$0.36913 = 0.30582 + 14.8196 P_{50}^{Mod} - 0.21887$$

$$1000 P_{50}^{Mod} = 19.04$$

Alternatively, you could equate the retrospective and prospective reserves at age 50. Your equation would be:

$$A_{50} - P_{50}^{Mod} \ddot{a}_{50:\overline{10}|} = \frac{A_{40}}{\ddot{a}_{40}} \cdot \frac{\ddot{a}_{40:\overline{10}|}}{10E_{40}} - \frac{A_{40}^1}{10E_{40}}$$

$$\begin{aligned} \text{where } A_{40}^1 &= A_{40} - {}_{10}E_{40} A_{50} \\ &= 0.16132 - (0.53667)(0.24905) \\ &= 0.02766 \end{aligned}$$

$$0.24905 - (P_{50}^{Mod})(7.57) = \frac{0.16132}{14.8166} \times \frac{7.70}{0.53667} - \frac{0.02766}{0.53667}$$

$$1000 P_{50}^{Mod} = \frac{(1000)(0.14437)}{7.57} = 19.07$$

Alternatively, you could set the actuarial present value of benefits at age 40 to the actuarial present value of benefit premiums. The change at age 50 did not change the benefits, only the pattern of paying for them.

$$A_{40} = P_{40} \ddot{a}_{40:\overline{10}|} + P_{50}^{Mod} {}_{10}E_{40} \ddot{a}_{50:\overline{10}|}$$

$$0.16132 = \left(\frac{0.16132}{14.8166} \right) (7.70) + (P_{50}^{Mod})(0.53667)(7.57)$$

$$1000 P_{50}^{Mod} = \frac{(1000)(0.07748)}{4.0626} = 19.07$$

Question # 13**Answer: A**

$$d_x^{(2)} = q_x^{(2)} \times l_x^{(t)} = 400$$

$$d_x^{(1)} = 0.45(400) = 180$$

$$q_x'^{(2)} = \frac{d_x^{(2)}}{l_x^{(t)} - d_x^{(1)}} = \frac{400}{1000 - 180} = 0.488$$

$$p_x'^{(2)} = 1 - 0.488 = 0.512$$

Note: The UDD assumption was not critical except to have all deaths during the year so that 1000 - 180 lives are subject to decrement 2.

Question #14**Answer: D**

Use “age” subscripts for years completed in program. E.g., p_0 applies to a person newly hired (“age” 0).

Let decrement 1 = fail, 2 = resign, 3 = other.

$$\begin{aligned}\text{Then } q_0^{(1)} &= 1/4, \quad q_1^{(1)} = 1/5, \quad q_2^{(1)} = 1/3 \\ q_0^{(2)} &= 1/5, \quad q_1^{(2)} = 1/3, \quad q_2^{(2)} = 1/8 \\ q_0^{(3)} &= 1/10, \quad q_1^{(3)} = 1/9, \quad q_2^{(3)} = 1/4\end{aligned}$$

$$\text{This gives } p_0^{(t)} = (1 - 1/4)(1 - 1/5)(1 - 1/10) = 0.54$$

$$p_1^{(t)} = (1 - 1/5)(1 - 1/3)(1 - 1/9) = 0.474$$

$$p_2^{(t)} = (1 - 1/3)(1 - 1/8)(1 - 1/4) = 0.438$$

$$\text{So } l_0^{(t)} = 200, \quad l_1^{(t)} = 200(0.54) = 108, \quad \text{and } l_2^{(t)} = 108(0.474) = 51.2$$

$$q_2^{(1)} = \left[\log p_2^{(1)} / \log p_2^{(t)} \right] q_2^{(t)}$$

$$q_2^{(1)} = \left[\log\left(\frac{2}{3}\right) / \log(0.438) \right] [1 - 0.438]$$

$$= (0.405 / 0.826)(0.562)$$

$$= 0.276$$

$$d_2^{(1)} = l_2^{(t)} q_2^{(1)}$$

$$= (51.2)(0.276) = 14$$

Question #15**Answer: C**Let: N = number X = profit S = aggregate profitsubscripts G = “good”, B = “bad”, AB = “accepted bad”

$$I_G = \left(\frac{2}{3}\right)(60) = 40$$

$I_{AB} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(60) = 10$ (If you have trouble accepting this, think instead of a heads-tails rule, that the application is accepted if the applicant’s government-issued identification number, e.g. U.S. Social Security Number, is odd. It is not the same as saying he automatically alternates accepting and rejecting.)

$$\begin{aligned} \text{Var}(S_G) &= E(N_G) \times \text{Var}(X_G) + \text{Var}(N_G) \times E(X_G)^2 \\ &= (40)(10,000) + (40)(300^2) = 4,000,000 \end{aligned}$$

$$\begin{aligned} \text{Var}(S_{AB}) &= E(N_{AB}) \times \text{Var}(X_{AB}) + \text{Var}(N_{AB}) \times E(X_{AB})^2 \\ &= (10)(90,000) + (10)(-100)^2 = 1,000,000 \end{aligned}$$

 S_G and S_{AB} are independent, so

$$\begin{aligned} \text{Var}(S) &= \text{Var}(S_G) + \text{Var}(S_{AB}) = 4,000,000 + 1,000,000 \\ &= 5,000,000 \end{aligned}$$

If you don’t treat it as three streams (“goods”, “accepted bads”, “rejected bads”), you can compute the mean and variance of the profit per “bad” received.

$$I_B = \left(\frac{1}{3}\right)(60) = 20$$

$$\begin{aligned} \text{If all “bads” were accepted, we would have } E(X_B^2) &= \text{Var}(X_B) + E(X_B)^2 \\ &= 90,000 + (-100)^2 = 100,000 \end{aligned}$$

Since the probability a “bad” will be accepted is only 50%,

$$\begin{aligned} E(X_B) &= \text{Prob(accepted)} \cdot E(X_B | \text{accepted}) + \text{Prob(not accepted)} \cdot E(X_B | \text{not accepted}) \\ &= (0.5)(-100) + (0.5)(0) = -50 \end{aligned}$$

$$E(X_B^2) = (0.5)(100,000) + (0.5)(0) = 50,000$$

Likewise,

$$\begin{aligned} \text{Now } \text{Var}(S_B) &= E(N_B) \times \text{Var}(X_B) + \text{Var}(N_B) \times E(X_B)^2 \\ &= (20)(47,500) + (20)(50^2) = 1,000,000 \end{aligned}$$

 S_G and S_B are independent, so

$$\begin{aligned} \text{Var}(S) &= \text{Var}(S_G) + \text{Var}(S_B) = 4,000,000 + 1,000,000 \\ &= 5,000,000 \end{aligned}$$

Question # 16

Answer: C

Let N = number of prescriptions then $S = N \times 40$

n	$f_N(n)$	$F_N(n)$	$1 - F_N(n)$
0	0.2000	0.2000	0.8000
1	0.1600	0.3600	0.6400
2	0.1280	0.4880	0.5120
3	0.1024	0.5904	0.4096

$$E(N) = 4 = \sum_{j=0}^{\infty} (1 - F(j))$$

$$\begin{aligned} E[(S - 80)_+] &= 40 \times E[(N - 2)_+] = 40 \times \sum_{j=2}^{\infty} (1 - F(j)) \\ &= 40 \times \left[\sum_{j=0}^{\infty} (1 - F(j)) - \sum_{j=0}^1 (1 - F(j)) \right] \\ &= 40(4 - 1.44) = 40 \times 2.56 = 102.40 \end{aligned}$$

$$\begin{aligned} E[(S - 120)_+] &= 40 \times E[(N - 3)_+] = 40 \times \sum_{j=3}^{\infty} (1 - F(j)) \\ &= 40 \times \left[\sum_{j=0}^{\infty} (1 - F(j)) - \sum_{j=0}^2 (1 - F(j)) \right] \\ &= 40(4 - 1.952) = 40 \times 2.048 = 81.92 \end{aligned}$$

Since no values of S between 80 and 120 are possible,

$$E[(S - 100)_+] = \frac{(120 - 100) \times E[(S - 80)_+] + (100 - 80) \times E[(S - 120)_+]}{120} = 92.16$$

Alternatively,

$$E[(S - 100)_+] = \sum_{j=0}^{\infty} (40j - 100)f_N(j) + 100f_N(0) + 60f_N(1) + 20f_N(2)$$

(The correction terms are needed because $(40j - 100)$ would be negative for $j = 0, 1, 2$; we need to add back the amount those terms would be negative)

$$\begin{aligned} &= 40 \sum_{j=0}^{\infty} j f_N(j) - 100 \sum_{j=0}^{\infty} f_N(j) + (100)(0.200) + (60)(0.16) + (20)(0.128) \\ &= 40 E(N) - 100 + 20 + 9.6 + 2.56 \\ &= 160 - 67.84 = 92.16 \end{aligned}$$

Question #17**Answer: B**

$$\begin{aligned}
{}_{10}E_{30:40} &= {}_{10}P_{30} {}_{10}P_{40} v^{10} = ({}_{10}P_{30} v^{10}) ({}_{10}P_{40} v^{10}) (1+i)^{10} \\
&= ({}_{10}E_{30}) ({}_{10}E_{40}) (1+i)^{10} \\
&= (0.54733)(0.53667)(1.79085) \\
&= 0.52604
\end{aligned}$$

The above is only one of many possible ways to evaluate ${}_{10}P_{30} {}_{10}P_{40} v^{10}$, all of which should give 0.52604

$$\begin{aligned}
a_{\overline{30:40:\overline{10}}}| &= a_{30:40} - {}_{10}E_{30:40} a_{30+10:40+10} \\
&= (\ddot{a}_{30:40} - 1) - (0.52604)(\ddot{a}_{40:50} - 1) \\
&= (13.2068) - (0.52604)(11.4784) \\
&= 7.1687
\end{aligned}$$

Question #18**Answer: A**

Equivalence Principle, where \mathbf{p} is annual benefit premium, gives

$$1000(A_{35} + (IA)_{35} \times \mathbf{p}) = \ddot{a}_x \mathbf{p}$$

$$\begin{aligned}
\mathbf{p} &= \frac{1000A_{35}}{(\ddot{a}_{35} - (IA)_{35})} = \frac{1000 \times 0.42898}{(11.99143 - 6.16761)} \\
&= \frac{428.98}{5.82382} \\
&= 73.66
\end{aligned}$$

We obtained \ddot{a}_{35} from

$$\ddot{a}_{35} = \frac{1 - A_{35}}{d} = \frac{1 - 0.42898}{0.047619} = 11.99143$$

Question #19**Answer: D**

Low random \Rightarrow early deaths, so we want ${}_tq_{80} = 0.42$ or $l_{80+t} = 0.58l_{80}$

Using the Illustrative Life Table, $l_{80+t} = (0.58)3,914,365$
 $= 2,270,332$

$$l_{80+5} > l_{80+t} > l_{80+6}$$

so $K =$ curtate future lifetime $= 5$

$$\begin{aligned} {}_{30}L &= 1000v^{k+1} - (\text{Contract premium}) \ddot{a}_{\overline{k+1}|} \\ &= 705 - (20)(5.2124) \\ &= 601 \end{aligned}$$

Question #20**Answer: C**

Time until arrival = waiting time plus travel time.

Waiting time is exponentially distributed with mean $\frac{1}{\lambda}$. The time you may already have been waiting is irrelevant: exponential is memoryless.

You: $E(\text{wait}) = \frac{1}{20} \text{ hour} = 3 \text{ minutes}$
 $E(\text{travel}) = (0.25)(16) + (0.75)(28) = 25 \text{ minutes}$
 $E(\text{total}) = 28 \text{ minutes}$

Co-worker: $E(\text{wait}) = \frac{1}{5} \text{ hour} = 12 \text{ minutes}$
 $E(\text{travel}) = 16 \text{ minutes}$
 $E(\text{total}) = 28 \text{ minutes}$

Question #21**Answer: B**

Bankrupt has 3 states 0, 1, 2, corresponding to surplus = 0, 1, 2

Transition matrix is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0.01 & 0.09 & 0.9 \end{bmatrix}$$

Initial vector $t_0 = [0 \ 0 \ 1]$

$$t_0 M = t_1 = [0.01 \ 0.09 \ 0.90]$$

$$t_1 M = t_2 = [0.028 \ 0.162 \ 0.81]$$

$$t_2 M = t_3 = [0.0523 \ 0.2187 \ 0.729]$$

At end of 2 months, probability ruined = 0.028 < 5%

At end of 3 months, probability ruined = 0.0523 > 5%

Question #22**Answer: D**

This is equivalent to a compound Poisson surplus process. Water from stream is like premiums. Deer arriving to drink is like claims occurring. Water drunk is like claim size.

$$E[\text{water drunk in a day}] = (1.5)(250) = 375$$

$$(1+q) = \frac{500}{375}$$

Prob(surplus level at time t is less than initial surplus, for some t)

$$= \Psi(0) = \frac{1}{1+q} = \frac{375}{500} = 75\%$$

Question #23**Answer: C**

Time of first claim is $T_1 = -1/3 \log(0.5) = 0.23$. Size of claim $= 10^{0.23} = 1.7$

Time of second claim is $T_2 = 0.23 - 1/3 \log(0.2) = 0.77$. Size of claim $= 10^{0.77} = 5.9$

Time of third claim is $T_3 = 0.77 - 1/3 \log(0.1) = 1.54$. Size of claim $= 10^{1.54} = 34.7$

Since initial surplus $= 5 >$ first claim, the first claim does not determine c

$$\text{Test at } T_2: \quad \text{Cumulative assets} = 5 + \int_0^{0.77} ct^4 dt = 5 + 0.054c$$

$$\text{Cumulative claims} = 1.7 + 5.9 = 7.6$$

$$\text{Assets} \geq \text{claims for } c \geq 48.15$$

$$\text{Test at } T_3: \quad \text{Cumulative claims} = 1.7 + 5.9 + 34.7 = 42.3$$

$$\text{Cumulative assets} = 5 + \int_0^{1.54} ct^4 dt = 5 + 1.732c$$

$$\text{Assets} \geq \text{claims for } c \geq 21.54$$

If $c < 48.15$, insolvent at T_2 .

If $c > 48.15$, solvent throughout.

49 is smallest choice > 48.15 .

Question #24**Answer: C**

$$m_{xy} = m_x + m_y = 0.14$$

$$\bar{A}_x = \bar{A}_y = \frac{m}{m+d} = \frac{0.07}{0.07+0.05} = 0.5833$$

$$\bar{A}_{xy} = \frac{m_{xy}}{m_{xy}+d} = \frac{0.14}{0.14+0.05} = \frac{0.14}{0.19} = 0.7368 \text{ and } \bar{a}_{xy} = \frac{1}{m_{xy}+d} = \frac{1}{0.14+0.05} = 5.2632$$

$$P = \frac{\bar{A}_{xy}}{\bar{a}_{xy}} = \frac{\bar{A}_x + \bar{A}_y - \bar{A}_{xy}}{\bar{a}_{xy}} = \frac{2(0.5833) - 0.7368}{5.2632} = 0.0817$$

Question #25**Answer: E**

$$({}_{20}V_{20} + P_{20})(1+i) - q_{40}({}_{1-21}V_{20}) = {}_{21}V_{20}$$

$$(0.49 + 0.01)(1+i) - 0.022(1 - 0.545) = 0.545$$

$$(1+i) = \frac{(0.545)(1 - 0.022) + 0.022}{0.50}$$

$$= 1.11$$

$$({}_{21}V_{20} + P_{20})(1+i) - q_{41}({}_{1-22}V_{20}) = {}_{22}V_{20}$$

$$(0.545 + 0.01)(1.11) - q_{41}(1 - 0.605) = 0.605$$

$$q_{41} = \frac{0.61605 - 0.605}{0.395}$$

$$= 0.028$$

Question #26**Answer: E**

$$1000 P_{60} = 1000 A_{60} / \ddot{a}_{60}$$

$$= 1000 v(q_{60} + p_{60}A_{61}) / (1 + p_{60} v \ddot{a}_{61})$$

$$= 1000(q_{60} + p_{60} A_{61}) / (1.06 + p_{60} \ddot{a}_{61})$$

$$= (15 + (0.985)(382.79)) / (1.06 + (0.985)(10.9041)) = 33.22$$

Question #27**Answer: E**

Method 1:

In each round,

 N = result of first roll, to see how many dice you will roll X = result of for one of the N dice you roll S = sum of X for the N dice

$$E(X) = E(N) = 3.5$$

$$Var(X) = Var(N) = 2.9167$$

$$E(S) = E(N) * E(X) = 12.25$$

$$\begin{aligned} Var(S) &= E(N)Var(X) + Var(N)E(X)^2 \\ &= (3.5)(2.9167) + (2.9167)(3.5)^2 \\ &= 45.938 \end{aligned}$$

Let S_{1000} = the sum of the winnings after 1000 rounds

$$E(S_{1000}) = 1000 * 12.25 = 12,250$$

$$Stddev(S_{1000}) = \sqrt{1000 * 45.938} = 214.33$$

After 1000 rounds, you have your initial 15,000, less payments of 12,500, plus winnings of S_{1000} .

Since actual possible outcomes are discrete, the solution tests for continuous outcomes greater than 15000-0.5. In this problem, that continuity correction has negligible impact.

$$\begin{aligned} &Pr(15000 - 12500 + S_{1000} > 14999.5) = \\ &= Pr((S_{1000} - 12250) / 214.33 > (14999.5 - 2500 - 12250) / 214.33) = \\ &= 1 - \Phi(1.17) = 0.12 \end{aligned}$$

Method 2

Realize that you are going to determine N 1000 times and roll the sum of those 1000 N 's dice, adding the numbers showing.

Let N_{1000} = sum of those N 's

$$E(N_{1000}) = 1000E(N) = (1000)(3.5) = 3500$$

$$Var(N_{1000}) = 1000Var(N) = 2916.7$$

$$E(S_{1000}) = E(N_{1000})E(X) = (3500)(3.5) = 12.250$$

$$\begin{aligned} Var(S_{1000}) &= E(N_{1000})Var(X) + Var(N_{1000})E(X)^2 \\ &= (3500)(2.9167) + (2916.7)(3.5)^2 = 45.938 \end{aligned}$$

$$Stddev(S_{1000}) = 214.33$$

Now that you have the mean and standard deviation of S_{1000} (same values as method 1), use the normal approximation as shown with method 1.

Question #28

Answer: B

$$p_k = \left(a + \frac{b}{k}\right)p_{k-1}$$

$$0.25 = (a + b) \times 0.25 \Rightarrow a + b = 1$$

$$0.1875 = \left(a + \frac{b}{2}\right) \times 0.25 \Rightarrow \left(1 - \frac{b}{2}\right) \times 0.25 = 0.1875$$

$$b = 0.5$$

$$a = 0.5$$

$$p_3 = \left(0.5 + \frac{0.5}{3}\right) \times 0.1875 = 0.125$$

Question #29**Answer: C**

Limiting probabilities satisfy (where $B = \text{Bad} = \text{Poor}$):

$$P = 0.95P + 0.15S$$

$$S = 0.04P + 0.80S + 0.25B$$

$$B = 0.01P + 0.05S + 0.75B$$

$$P + S + B = 1.00$$

Solving, $P = 0.694$

Question #30**Answer: B**

Transform these scenarios into a four-state Markov chain, where the final disposition of rates in any scenario is that they decrease, rather than if rates increase, as what is given.

State	from year $t - 3$ to year $t - 2$	from year $t - 2$ to year $t - 1$	Probability that year t will decrease from year $t - 1$
0	Decrease	Decrease	0.8
1	Increase	Decrease	0.6
2	Decrease	Increase	0.75
3	Increase	Increase	0.9

Transition matrix is

$$\begin{bmatrix} 0.80 & 0.00 & 0.20 & 0.00 \\ 0.60 & 0.00 & 0.40 & 0.00 \\ 0.00 & 0.75 & 0.00 & 0.25 \\ 0.00 & 0.90 & 0.00 & 0.10 \end{bmatrix}$$

$$P_{00}^2 + P_{01}^2 = 0.8 * 0.8 + 0.2 * 0.75 = 0.79$$

For this problem, you don't need the full transition matrix. There are two cases to consider. Case 1: decrease in 2003, then decrease in 2004; Case 2: increase in 2003, then decrease in 2004.

For Case 1: decrease in 2003 (following 2 decreases) is 0.8; decrease in 2004 (following 2 decreases) is 0.8. Prob(both) = $0.8 \times 0.8 = 0.64$

For Case 2: increase in 2003 (following 2 decreases) is 0.2; decrease in 2004 (following a decrease, then increase) is 0.75. Prob(both) = $0.2 \times 0.75 = 0.15$

Combined probability of Case 1 and Case 2 is $0.64 + 0.15 = 0.79$

Question #31**Answer: B**

$$l_x = w - x = 105 - x$$

$$\Rightarrow {}_tP_{45} = l_{45+t} / l_{45} = 60 - t / 60$$

Let K be the curtate future lifetime of (45). Then the sum of the payments is 0 if $K \leq 19$ and is $K - 19$ if $K \geq 20$.

$$\begin{aligned} {}_{20|}\ddot{a}_{45} &= \dot{\mathbf{a}}_{\overline{60}|} \cdot \left(\frac{60 - K}{60} \right) \cdot 1 \\ &= \frac{(40 + 39 + \dots + 1)}{60} = \frac{(40)(41)}{2(60)} = 13.\overline{66} \end{aligned}$$

Hence,

$$\text{Prob}(K - 19 > 13.\overline{66}) = \text{Prob}(K > 32.\overline{66})$$

$$= \text{Prob}(K \geq 33) \text{ since } K \text{ is an integer}$$

$$= \text{Prob}(T \geq 33)$$

$$= {}_{33}p_{45} = \frac{l_{78}}{l_{45}} = \frac{27}{60}$$

$$= 0.450$$

Question #32**Answer: C**

$${}^2\bar{A}_x = \frac{\mathbf{m}}{\mathbf{m} + 2\mathbf{d}} = 0.25 \rightarrow \mathbf{m} = 0.04$$

$$\bar{A}_x = \frac{\mathbf{m}}{\mathbf{m} + \mathbf{d}} = 0.4$$

$$(\bar{IA})_x = \int_0^{\mathbb{Y}} {}_s\bar{A}_x ds$$

$$\int_0^{\infty} {}_sE_x \bar{A}_x ds$$

$$= \int_0^{\mathbb{Y}} (e^{-0.1s})(0.4) ds$$

$$= (0.4) \left(\frac{-e^{-0.1s}}{0.1} \right) \bigg|_0^{\mathbb{Y}} = \frac{0.4}{0.1} = 4$$

Alternatively, using a more fundamental formula but requiring more difficult integration.

$$\begin{aligned} (\bar{IA})_x &= \int_0^{\mathbb{Y}} {}_t p_x \mathbf{m}_x(t) e^{-\mathbf{d}t} dt \\ &= \int_0^{\mathbb{Y}} {}_t e^{-0.04t} (0.04) e^{-0.06t} dt \\ &= 0.04 \int_0^{\mathbb{Y}} {}_t e^{-0.1t} dt \end{aligned}$$

(integration by parts, not shown)

$$\begin{aligned} &= 0.04 \left(\frac{-t}{0.1} - \frac{1}{0.01} \right) e^{-0.1t} \bigg|_0^{\mathbb{Y}} \\ &= \frac{0.04}{0.01} = 4 \end{aligned}$$

Question #33**Answer: E**

Subscripts A and B here just distinguish between the tools and do not represent ages.

We have to find $\overset{o}{e}_{AB}$

$$\overset{o}{e}_A = \int_0^{10} \left(1 - \frac{t}{10}\right) dt = t - \frac{t^2}{20} \Big|_0^{10} = 10 - 5 = 5$$

$$\overset{o}{e}_B = \int_0^7 \left(1 - \frac{t}{7}\right) dt = t - \frac{t^2}{14} \Big|_0^7 = 49 - \frac{49}{14} = 3.5$$

$$\overset{o}{e}_{AB} = \int_0^7 \left(1 - \frac{t}{7}\right) \left(1 - \frac{t}{10}\right) dt = \int_0^7 \left(1 - \frac{t}{10} - \frac{t}{7} + \frac{t^2}{70}\right) dt$$

$$= t - \frac{t^2}{20} - \frac{t^2}{14} + \frac{t^3}{210} \Big|_0^7$$

$$= 7 - \frac{49}{20} - \frac{49}{14} + \frac{343}{210} = 2.683$$

$$\overset{o}{e}_{AB} = \overset{o}{e}_A + \overset{o}{e}_B - \overset{o}{e}_{AB}$$

$$= 5 + 3.5 - 2.683 = 5.817$$

Question #34**Answer: A**

$$m_x^{(t)}(t) = 0.100 + 0.004 = 0.104$$

$${}_t p_x^{(t)} = e^{-0.104t}$$

Actuarial present value (APV) = APV for cause 1 + APV for cause 2.

$$2000 \int_0^5 e^{-0.04t} e^{-0.104t} (0.100) dt + 500,000 \int_0^5 e^{-0.04t} e^{-0.104t} (0.400) dt$$

$$= (2000(0.10) + 500,000(0.004)) \int_0^5 e^{-0.144t} dt$$

$$= \frac{2200}{0.144} (1 - e^{-0.144(5)}) = 7841$$

Question #35**Answer: A**

$$R = 1 - p_x = q_x$$

$$S = 1 - p_x \times e^{(-k)} \text{ since } e^{-\int_0^1 (m_x(t) + k) dt} = e^{-\int_0^1 m_x(t) dt - \int_0^1 k dt}$$

$$= e^{-\int_0^1 m_x(t) dt} e^{-\int_0^1 k dt}$$

$$\text{So } S = 0.75R \Rightarrow 1 - p_x \times e^{-k} = 0.75q_x$$

$$e^{-k} = \frac{1 - 0.75q_x}{p_x}$$

$$e^k = \frac{p_x}{1 - 0.75q_x} = \frac{1 - q_x}{1 - 0.75q_x}$$

$$k = \ln \left[\frac{1 - q_x}{1 - 0.75q_x} \right]$$

Question #36**Answer: E**

$$b = \text{mean} = 4; \quad p_k = b^k / (1 + b)^{k+1}$$

n	$P(N = n)$
0	0.2
1	0.16
2	0.128
3	0.1024

x	$f^{(1)}(x)$	$f^{(2)}(x)$	$f^{(3)}(x)$
0	0	0	0
1	0.25	0	0
2	0.25	0.0625	0
3	0.25	0.125	0.0156

$f^{(k)}(x)$ = probability that, given exactly k claims occur, that the aggregate amount is x .

$f^{(1)}(x) = f(x)$; the claim amount distribution for a single claim

$$f^{(k)}(x) = \sum_{j=0}^x \left(f^{(k-1)}(j) \right) x f(x-j)$$

$f_s(x) = \sum_{k=0}^x P(N = k) \times f^{(k)}(x)$; upper limit of sum is really ∞ , but here with smallest possible

claim size 1, $f^{(k)}(x) = 0$ for $k > x$

$$f_s(0) = 0.2$$

$$f_s(1) = 0.16 * 0.25 = 0.04$$

$$f_s(2) = 0.16 * 0.25 + 0.128 * 0.0625 = 0.048$$

$$f_s(3) = 0.16 * 0.25 + 0.128 * 0.125 + 0.1024 * 0.0156 = 0.0576$$

$$F_s(3) = 0.2 + 0.04 + 0.048 + 0.0576 = 0.346$$

Question #37**Answer: E**

Let L = incurred losses; P = earned premium = 800,000

$$\text{Bonus} = 0.15 \times \left(0.60 - \frac{L}{P} \right) \times P \text{ if positive}$$

$$= 0.15 \times (0.60P - L) \text{ if positive}$$

$$= 0.15 \times (480,000 - L) \text{ if positive}$$

$$= 0.15 \times (480,000 - (L \dot{\cup} 480,000))$$

$$E(\text{Bonus}) = 0.15 (480,000 - E(L \dot{\cup} 480,000))$$

From Appendix A.2.3.1

$$= 0.15 \{ 480,000 - [500,000 \times (1 - (500,000 / (480,000 + 500,000)))] \}$$

$$= 35,265$$

Question #38**Answer: D**

$$\bar{A}_{28:\overline{2}|}^1 = \int_0^2 e^{-dt} \frac{1}{72} dt$$

$$= \frac{1}{72d} (1 - e^{-2d}) = 0.02622 \text{ since } d = \ln(1.06) = 0.05827$$

$$\ddot{a}_{28:\overline{2}|} = 1 + v \left(\frac{71}{72} \right) = 1.9303$$

$${}_3V = 500,000 \bar{A}_{28:\overline{2}|}^1 - 6643 \ddot{a}_{28:\overline{2}|}$$

$$= 287$$

Question #39**Answer: D**

Let \bar{A}_x and \bar{a}_x be calculated with $\mathbf{m}_x(t)$ and $\mathbf{d} = 0.06$

Let \bar{A}_x^* and \bar{a}_x^* be the corresponding values with $\mathbf{m}_x(t)$ increased by 0.03 and \mathbf{d} decreased by 0.03

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\mathbf{d}} = \frac{0.4}{0.06} = 6.667$$

$$\bar{a}_x^* = \bar{a}_x$$

$$\left[\begin{aligned} \text{Proof: } \bar{a}_x^* &= \int_0^{\infty} e^{-\int_0^t (\mathbf{m}_x(s) + 0.03) ds} e^{-0.03t} dt \\ &= \int_0^{\infty} e^{-\int_0^t \mathbf{m}_x(s) ds} e^{-0.03t} e^{-0.03t} dt \\ &= \int_0^{\infty} e^{-\int_0^t \mathbf{m}_x(s) ds} e^{-0.06t} dt \\ &= \bar{a}_x \end{aligned} \right]$$

$$\begin{aligned} \bar{A}_x^* &= 1 - 0.03\bar{a}_x^* = 1 - 0.03\bar{a}_x \\ &= 1 - (0.03)(6.667) \\ &= 0.8 \end{aligned}$$

Question #40**Answer: A**

year	bulb ages				# replaced
	0	1	2	3	
0	10000	0	0	0	-
1	1000	9000	0	0	1000
2	100+2700	900	6300	0	2800
3	280+270+3150				3700

The diagonals represent bulbs that don't burn out.

E.g., of the initial 10,000, $(10,000) (1-0.1) = 9000$ reach year 1.

$(9000) (1-0.3) = 6300$ of those reach year 2.

Replacement bulbs are new, so they start at age 0.

At the end of year 1, that's $(10,000) (0.1) = 1000$

At the end of 2, it's $(9000) (0.3) + (1000) (0.1) = 2700 + 100$

At the end of 3, it's $(2800) (0.1) + (900) (0.3) + (6300) (0.5) = 3700$

$$\begin{aligned} \text{Actuarial present value} &= \frac{1000}{1.05} + \frac{2800}{1.05^2} + \frac{3700}{1.05^3} \\ &= 6688 \end{aligned}$$