FALL 2006 EXAM M
SOLUTIONS

Question #1
Key: A

$P\ddot{a}_{x:\overline{3}|} = APV$ (stunt deaths)

$$P\left[\frac{2500+2486/1.08+2466/(1.08)^2}{2500}\right]=500000\left(\frac{4/1.08+5/(1.08)^2+6/(1.08)^3}{2500}\right)$$

$P(2.77)=2550.68$

$\Rightarrow p=921$

Question #2
Key: D

$$\mathring{e}_{30:\overline{50}|}=\frac{\int_{30}^{80}s(x)\,dx}{s(30)}=\frac{\int_{30}^{80}\left(1-\frac{x^2}{10,000}\right)dx}{1-\left(\frac{30}{100}\right)^2}$$

$$=\frac{\left(x-\frac{x^3}{30,000}\right)\Big|_{30}^{80}}{0.91}$$

$$=\frac{33.833}{0.91}$$

$$=37.18$$

Question #3
Key: B

$$A_{60} = v \times \left(p_{60} \times A_{61} + q_{60} \right)$$
$$= \left(1/1.06\right) \times \left(0.98 \times 0.440 + 0.02\right)$$
$$= 0.42566$$
$$\ddot{a}_{60} = \left(1 - A_{60}\right)/d$$
$$= \left(1 - 0.42566\right)/\left(0.06/1.06\right)$$
$$= 10.147$$
$$1000\,{}_{10}V_{50} = 1000A_{60} - 1000P_{50} \times \ddot{a}_{60}$$
$$= 425.66 - 10.147 \times 25$$
$$= 172$$

Question #4
Key: E

Let Portfolio be the present value random variable for the aggregate payments.

Let $Y_{65} =$ present value random variable for an annuity due of one on one life age 65.
Thus $E(Y_{65}) = \ddot{a}_{65}$
Let $Y_{75} =$ present value random variable for an annuity due of one on one life age 75.
Thus $E(Y_{75}) = \ddot{a}_{75}$

Let X represent the 95th percentile.

$$\text{E (Portfolio)} = 50(2)\ddot{a}_{65} + 30(1)\ddot{a}_{75}$$
$$= 100(9.8969) + 30(7.217) = 1206.20$$

$$\text{Var (Portfolio)} = 50\times 2^2 Var[Y_{65}] + 30(1)^2 Var[Y_{75}] = 200(13.2996) + 30(11.5339) = 3005.94$$

where $Var[Y_{65}] = \dfrac{1}{d^2}\left({}^2A_{65} - A_{65}^2\right) = \dfrac{1}{(0.05660)^2}\left[0.23603 - (0.4398)^2\right] = 13.2996$

and $Var[Y_{75}] = \dfrac{1}{d^2}\left({}^2A_{75} - A_{75}^2\right) = \dfrac{1}{(0.05660)^2}\left[0.38681 - (0.59149)^2\right] = 11.5339$

$$\Pr\left[\left(\frac{X - E(\text{Portfolio})}{\sqrt{Var(\text{Portfolio})}}\right) \le 1.645\right] = 0.95 \Rightarrow X = E(\text{Portfolio}) + 1.645\sqrt{Var[\text{Portfolio}]}$$
$$= 1206.20 + 1.645(54.826)$$
$$= 1296.39$$

Question #5
Key: C

$$\bar{a} = \int_0^\infty e^{-\delta t} \times e^{-\mu t} dt = \frac{1}{\delta + \mu}$$

$$APV = 50,000 \times \frac{1}{0.5}\int_{0.5}^{1} \frac{1}{\delta + \mu} d\mu = 100,000 \times \left[\ln(\delta + 1) - \ln(\delta + 0.5)\right]$$
$$= 100,000 \times \ln\left(\frac{0.045 + 1}{0.045 + 0.5}\right)$$
$$= 65,099$$

Question #6
Key: B

Pays 80% of loss over 20, with cap of payment at 60, hence cap of loss of
$\frac{60}{0.8}+20=95=u$

Th 5.13 E(Y per loss) = $\alpha\left[E\left(X\wedge 95\right)-E\left(X\wedge 20\right)\right]$

$=0.8\left[\int_0^{95}S\left(x\right)dx-\int_0^{20}S\left(x\right)dx\right]$

$=0.8\int_{20}^{95}S\left(x\right)dx=0.8\int_{20}^{95}\left(1-\frac{x^2}{10,000}\right)dx=0.8\left(x-\frac{x^3}{30,000}\Big|_{20}^{95}\right)=\left(0.8\right)\left(46.6875\right)=37.35$

E(Y per payment) = $\frac{E\left(Y\text{ per loss}\right)}{1-F\left(20\right)}=\frac{37.35}{0.96}=38.91$

Question #7
Key: D

Let S = aggregate claims, $I_5=$ claims covered by stop loss

S	I_5	
0	0	
5	0	
x	x- 5	(i.e., for any $S>5,\ I_5=S-5$)

$E\left[I_5\right]=E\left[S\right]-5+5\Pr\left(0\text{ claims}\right)$

$E\left[S\right]=5\times\left(0.6\times 5+0.4\times k\right)=15+2k$

$\Pr\left(0\text{ claims}\right)=e^{-5}$

$E\left[I_5\right]=15+2k-5+5e^{-5}=28.03$

$10.034+2k=28.03$

$2k=18$

$k=9$

Question #8
Key: E

The process described, where a key feature is the exponential time between events, is a Poisson process with $\lambda = 1/5$ per minute.
The number of claims in any interval of length n minutes has a Poisson distribution with mean $\lambda n = n/5$.

Here $n = 10$. So parameter = 10/5 = 2

$$\Pr(N \geq 2) = 1 - \Pr(N = 0) - \Pr(N = 1)$$
$$= 1 - e^{-2} - e^{-2}2$$
$$= 1 - 0.135 - 0.271 = 0.594$$

Question #9
Key: D

The payouts in any time period of length t have a Poisson distribution with parameter $5t$.

The payouts can be grouped by size. For each i, the number of payouts of size i is a Poisson random variable with mean $5t/2^i$, and these random variables are independent.
Since they are independent Poisson random variables, the sum of the payouts of size 1, 2 or 3 is a Poisson random variable with mean $\left(\frac{5t}{2} + \frac{5t}{4} + \frac{5t}{8}\right) = \frac{35t}{8}$

For $t = 1/3$ hour, the mean is $\frac{35}{8} \times \frac{1}{3} = 1.4583$

$f(0) = e^{-1.4583} = 0.23$

Question #10
Key: D

How long was wait during first 45 minutes? In that interval, wait is exponential with $\theta = 30$, so

$$E\left(X \wedge 45\right) = 30\left(1 - e^{\frac{-45}{30}}\right) = 23.31$$

Expected trains $= \dfrac{45}{30} = 1.5$, so $f\left(0 \text{ trains}\right) = \dfrac{e^{-1.5}\left(1.5\right)^0}{0!} = 0.223$

If 0, wait an additional 15 minutes (expected) so

Total expected wait $= 23.31 + \left(0.223\right)\left(15\right) = 26.65$

Note that this problem is equivalent to calculate $\overset{\circ}{e}_x$

where $\mu_x\left(t\right) = \begin{cases} 1/30, & 0 \le t < 45 \\ 1/15, & t \ge 45 \end{cases}$

and solution is $\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{45}|} + {}_{45}p_x \, \overset{\circ}{e}_{x+45}$

Question #11
Key: A

Let π be the benefit premium at issue.

$$\pi = 10,000 \frac{A_{45:\overline{20|}}}{\ddot{a}_{45:\overline{20|}}} = 10,000 \frac{\left[0.20120 - 0.25634(0.43980) + 0.25634\right]}{14.1121 - 0.25634(9.8969)}$$
$$= 297.88$$

The expected prospective loss at age 60 is
$$10,000\,{}_{15}V_{45:\overline{20|}} = 10,000 A_{60:\overline{5|}} - 297.88\,\ddot{a}_{60:\overline{5|}}$$
$$= 10,000 \times 0.7543 - 297.88 \times 4.3407$$
$$= 6250$$

where $A^{1}_{60:\overline{5|}} = 0.36913 - 0.68756(0.4398) = 0.06674$
$A_{60:\overset{\;1}{\overline{5|}}} = 0.68756$
$A_{60:\overline{5|}} = 0.06674 + 0.68756 = 0.7543$
$\ddot{a}_{60:\overline{5|}} = 11.1454 - 0.68756 \times 9.8969 = 4.3407$

After the change, expected prospective loss = $10,000 A^{1}_{60:\overline{5|}}$ + (Reduced Amount) $A_{60:\overset{\;1}{\overline{5|}}}$
Since the expected prospective loss is the same
$6250 = (10,000)(0.06674) + (\text{Reduced Amount})(0.68756)$
Reduced Amount = 8119

Question #12
Key: D

$$\bar{A}_x = \bar{A}^{1}_{x:\overline{5}|} + {}_5E_x\, \bar{A}^{\;1}_{x+5:\overline{7}|} + {}_{12}E_x\, \bar{A}_{x+12}$$

where

$${}_5E_x = e^{-5(0.04+0.02)} = 0.7408$$

$$\bar{A}^{1}_{x:\overline{5}|} = \frac{0.04}{0.04+0.02}\times(1-0.7408) = 0.1728$$

$${}_7E_{x+5} = e^{-7(0.05+0.02)} = 0.6126$$

$$\bar{A}^{\;1}_{x+5:\overline{7}|} = \left(\frac{0.05}{0.05+0.02}\right)(1-0.6126) = 0.2767$$

$${}_{12}E_x = {}_5E_x \times {}_7E_{x+5} = 0.7408\times 0.6126 = 0.4538$$

$$\bar{A}_{x+12} = \frac{0.05}{0.05+0.03} = 0.625$$

$$\bar{A}_x = 0.1728 + (0.7408)(0.2767) + (0.4538)(0.625)$$

$$= 0.6614$$

Question #13
Key: A

APV of Accidental death benefit and related settlement expense =

$$(2000\times 1.05)\times\frac{0.004}{0.004+0.04+0.05} = 89.36$$

APV of other DB and related settlement expense = $(1000\times 1.05)\times\frac{0.04}{0.094} = 446.81$

APV of Initial expense = 50

APV of Maintenance expense $= \frac{3}{0.094} = 31.91$

APV of future premiums $= \frac{100}{0.094} = 1063.83$

APV of ${}_0L_e = 89.36 + 446.81 + 50 + 31.91 - 1063.83$

$$= -445.75$$

Question #14
Key: C

Compute the probabilities of moving from healthy to NH. There are three paths.

H to H to NH: (0.8)(0.05) = 0.04
H to HHC to NH: (0.15)(0.05) = 0.0075
H to NH to NH: (0.05)(1) = 0.05

Summing, we get 0.0975 as the probability for each member.
Variance for m members = mpq, here = 50*(0.0975)(0.9025) = 4.40

Question #15
Key: C

$$Q_0 = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_0 \times Q_1 = \begin{pmatrix} 0.36 & 0.18 & 0.46 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_0 \times Q_1 \times Q_2 = \begin{pmatrix} 0 & 0.108 & 0.892 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{APV}(\text{Premiums}) = 1 + 0.9v + 0.54v^2 + 0.108v^3 = 2.35$$

$$\text{APV}(\text{Benefits}) = 4\left(0.3v + 0.18v^2 + 0.108v^3\right) = 2.01$$

Difference = 2.35 – 2.01 = 0.34

In the formula for APV (Premiums), states 0 and 1 are combined. For example, the $0.54\,v^2$ term represents a 0.36 probability of being in state 0 plus a 0.18 probability of being in state 1.

Alternatively, the same effort here but often shorter when everyone is in the same initial state:

$$\begin{pmatrix} 1.00 & 0.00 & 0.00 \end{pmatrix} \times Q_0 = \begin{pmatrix} 0.6 & 0.3 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0.60 & 0.30 & 0.10 \end{pmatrix} \times Q_1 = \begin{pmatrix} 0.36 & 0.18 & 0.46 \end{pmatrix}$$
$$\begin{pmatrix} 0.36 & 0.18 & 0.46 \end{pmatrix} \times Q_2 = \begin{pmatrix} 0 & 0.108 & 0.892 \end{pmatrix}$$

This method just calculates the top row of the cumulative transition matrix. It gives the same elements you use if you calculate the complete cumulative transition matrix, so you finish the problem the same way as before.

Question #16
Key: E

$${}_{0.25|1.5}q_x = {}_{0.25}p_x - {}_{1.75}p_x$$

Let μ be the force of mortality in year 1, so 3μ is the force of mortality in year 2.
Probability of surviving 2 years is 10%

$$\begin{cases} 0.10 = p_x \ p_{x+1} = e^{-\mu}e^{-3\mu} = e^{-4\mu} \\ \mu = \dfrac{\ln(0.1)}{4} = 0.5756 \end{cases}$$

$${}_{0.25}p_x = e^{-\frac{1}{4}(0.5756)} = 0.8660$$

$${}_{1.75}p_x = p_x \times {}_{0.75}p_{x+1} = e^{-\mu}e^{-\frac{3}{4}(3\mu)} = e^{-\frac{13}{4}(0.5756)} = 0.1540$$

$${}_{1.5}q_{x+0.25} = \frac{{}_{0.25|1.5}q_x}{{}_{0.25}p_x} = \frac{{}_{0.25}p_x - {}_{1.75}p_x}{{}_{0.25}p_x} = \frac{0.866 - 0.154}{0.866} = 0.82$$

Question #17
Key: C

The form of l_x for non-smokers matches DeMoivre's law, so

$$\mu_x^{NS} = \frac{1}{110-x}$$
$$= \tfrac{1}{2}\,\mu_x^S \Rightarrow \mu_x^S = \frac{2}{110-x}$$
$$\Rightarrow l_x^S = l_0^S\left(110-x\right)^2 \text{ [see note below]}$$

Thus $\quad {}_t p_{20}^S = \dfrac{l_{20+t}^S}{l_{20}^S} = \dfrac{\left(90-t\right)^2}{90^2}$

$${}_t p_{25}^{NS} = \frac{l_{25+t}^{NS}}{l_{25}^{NS}} = \frac{\left(85-t\right)}{85}$$

$$\overset{\circ}{e}_{20:25} = \int_0^{85} {}_t p_{20:25}\,dt$$
$$= \int_0^{85} {}_t p_{20}^S \; {}_t p_{25}^{NS}\,dt = \int_0^{85} \frac{\left(90-t\right)^2}{\left(90\right)^2}\frac{\left(85-t\right)}{85}\,dt$$
$$= \frac{1}{688{,}500}\int_0^{85}\left(90-t\right)^2\left(90-t-5\right)dt$$
$$= \frac{1}{688{,}500}\left[\int_0^{85}\left(90-t\right)^3 dt - 5\int_0^{85}\left(90-t\right)^2 dt\right]$$
$$= \frac{1}{688{,}500}\left[\frac{-\left(90-t\right)^4}{4} + \frac{5\left(90-t\right)^3}{3}\right]_0^{85}$$
$$= \frac{1}{688{,}500}\left[-156.25 + 208.33 + 16{,}402{,}500 - 1{,}215{,}000\right]$$
$$= 22.1$$

[There are other ways to evaluate the integral, leading to the same result].

Note: the solution above assumes the candidate will recognize that the smoker mortality is modified DeMoivre and can proceed directly to the l_x or $s\left(x\right)$ form. The $s\left(x\right)$ form is derived as $s\left(x\right) = e^{-\int_0^x\left(\frac{2}{110-t}\right)}dt = e^{2\ln\left(110-t\right)\big|_0^x} = \left(\dfrac{110-x}{110}\right)^2$

The l_x form is equivalent.

Question #18
Key: B

$$\ddot{a}_{30:\overline{20|}} = \ddot{a}_{30:\overline{10|}} + {}_{10}E_{30} \times \ddot{a}_{40:\overline{10|}}$$

$$15.0364 = 8.7201 + {}_{10}E_{30} \times 8.6602$$

$${}_{10}E_{30} = (15.0364 - 8.7201) / 8.6602 = 0.72935$$

Actuarial present value (APV) of benefits =

$$= 1000 \times A^{1}_{40:\overline{10|}} + 2000 \times {}_{10}E_{30} \times A^{1}_{50:\overline{10|}}$$

$$= 16.66 + 2 \times 0.72935 \times 32.61 = 64.23$$

APV of premiums = $\pi \times \ddot{a}_{30:\overline{10|}} + 2\pi \times 0.72935 \times \ddot{a}_{40:\overline{10|}}$

$$= \pi \times 8.7201 + 2 \times \pi \times 0.72935 \times 8.6602$$

$$= 21.3527\pi$$

$$\pi = 64.23 / 21.3527 = 3.01$$

Question #19
Key: A

$${}_{15}V_{25} = P_{25}\,\ddot{s}_{25:\overline{15}|} - \frac{A^{1}_{25:\overline{15}|}}{{}_{15}E_{25}}$$ (this is the retrospective reserve calculation)

$$P^{1}_{25:\overline{15}|} = P_{25:\overline{15}|} - P_{25:\overset{1}{\overline{15}|}} = 0.05332 - 0.05107$$
$$= 0.00225$$
$$= \frac{A^{1}_{25:\overline{15}|}}{\ddot{a}_{25:\overline{15}|}}$$

$$0.05107 = P_{25:\overset{1}{\overline{15}|}} = \frac{{}_{15}E_{25}}{\ddot{a}_{25:\overline{15}|}} = \frac{1}{\ddot{s}_{25:\overline{15}|}}$$

$$\frac{A^{1}_{25:\overline{15}|}}{{}_{15}E_{25}} = \frac{A^{1}_{25:\overline{15}|}/\ddot{a}_{25:\overline{15}|}}{{}_{15}E_{25}/\ddot{a}_{25:\overline{15}|}} = \frac{0.00225}{0.05107} = 0.04406$$

$$P_{25}\,\ddot{s}_{25:\overline{15}|} = \frac{0.01128}{0.05107} = 0.22087$$

$$25{,}000\,{}_{15}V_{25} = 25{,}000\left(0.22087 - 0.04406\right) = 25{,}000\left(0.17681\right) = 4420$$

There are other ways of getting to the answer, for example writing

A: the retrospective reserve formula for ${}_{15}V_{25}$.

B: the retrospective reserve formula for ${}_{15}V^{1}_{25:\overline{15}|}$, which = 0

Subtract B from A to get

$$\left(P_{25} - P^{1}_{25:\overline{15}|}\right)\ddot{s}_{25:\overline{15}|} = {}_{15}V_{25}$$

Question #20
Key: B

Let R = Equity index return. $R \sim N\left(\mu = 8\%,\ \sigma = 16\%\right)$

X = 0.75 R $X \sim N\left(\mu = 6\%,\ \sigma = 12\%\right)$

Y = crediting rate = Max (R, 3%)

$Y = X + 3\% - \left(X \wedge 3\%\right)$

(If $X < 3\%$, $X \wedge 3\% = X$ and $Y = X + 3\% - X = 3\%$)

(If $X > 3\%$ $X \wedge 3\% = 3\%$ and $Y = X + 3\% - 3\% = X$)

$$
\begin{aligned}
E(Y) &= E(X + 3\% - (X \wedge 3\%)) \\
&= E\left(X\right) + 3\% - E\left(X \wedge 3\%\right) \\
&= 6\% + 3\% - (-0.43\%) \\
&= 9.43\%
\end{aligned}
$$

Note that $E(X \wedge 3\%)$ is a table lookup in the given information.

Question #21
Key: A

Let S = aggregate losses, X = severity
Since the frequency is Poisson,

$Var(S) = \lambda E\left(X^2\right)$

$E\left(X^2\right) = \dfrac{2^2\Gamma\left(3\right)\Gamma\left(1\right)}{\Gamma\left(3\right)} = 4$ (table lookup)

$Var(S) = 3 \times 4 = 12$

You would get the same result if you used

$Var(S) = E(N)Var(X) + Var(N)(E(X))^2$

Question #22
Key: D

For each member $P(z)=\left[1-1.5(z-1)\right]^{-1}$

so for family of 4 $P(z)=\left[1-1.5(z-1)\right]^{-4}$ negative binomial with $\beta=1.5$ $r=4$

k	p_k
0	0.026
1	0.061
2	0.092
3+	0.821

$E(N \wedge 3)=0\times 0.026+1\times 0.061+2\times 0.092+3\times 0.821=2.71$

$E(N)-E(N \wedge 3)=6-2.71=3.29$

3.29×100 per visit = 329

Alternatively, without using probability generating functions, a geometric distribution is a special case of the negative binomial with r = 1.

Summing four independent negative binomial distributions, each with $\beta=1.5$ and r = 1 gives a negative binomial distribution with β = 1.5 and r = 4. Then continue as above.

Question #23
Key: C

ILT:

We have $p_{70} = 6,396,609/6,616,155 = 0.96682$

$${}_2 p_{70} = 6,164,663/6,616,155 = 0.93176$$

$e_{70:\overline{2|}} = 0.96682 + 0.93176 = 1.89858$

CF: $0.93176 = {}_2 p_{70} = e^{-2\mu} \Rightarrow \mu = 0.03534$

Hence $e_{70:\overline{2|}} = p_{70} + {}_2 p_{71} = e^{-\mu} + e^{-2\mu} = 1.89704$

DM: Since l_{70} and ${}_2 p_{70}$ for the DM model equal the ILT, therefore l_{72} for the DM model also equals the ILT. For DM we have $l_{70} - l_{71} = l_{71} - l_{72} \Rightarrow l_{71}^{(\mathrm{DM})} = 6,390,409$

Hence $e_{70:\overline{2|}} = 6,390,409/6,616,155 + 6,164,663/6,616,155 = 1.89763$

So the correct order is CF < DM < ILT
You could also work with p's instead of l's. For example, with the ILT,

$p_{70} = (1 - 0.03318) = 0.96682$
${}_2 p_{70} = (0.96682)(1 - 0.03626) = 0.93176$

Note also, since $e_{70:\overline{2|}} = p_{70} + {}_2 p_{70}$, and ${}_2 p_{70}$ is the same for all three, you could just order p_{70}.

Question #24
Key: D

$$l_{60}^{(\tau)} = 1000$$
$$l_{61}^{(\tau)} = 1000(0.99)(0.97)(0.90) = 864.27$$
$$d_{60}^{(\tau)} = 1000 - 864.27 = 135.73$$
$$d_{60}^{(3)} = 135.73 \times \frac{-\ln(0.9)}{-\ln\left[(0.99)(0.97)(0.9)\right]} = \frac{0.1054}{0.1459} = 98.05$$
$$l_{62}^{(\tau)} = 864.27(0.987)(0.95)(0.80) = 648.31$$
$$d_{61}^{(\tau)} = 864.27 - 648.31 = 215.96$$
$$d_{61}^{(3)} = 215.96 \times \frac{-\ln(0.80)}{-\ln\left[(0.987)(0.95)(0.80)\right]} = \frac{0.2231}{0.2875} = 167.58$$

So $d_{60}^{(3)} + d_{61}^{(3)} = 98.05 + 167.58 = 265.63$

Question #25
Key: B

$${}_t p_{40} = e^{-0.05t}$$
$${}_t p_{50} = (60 - t)/60$$
$$\mu_{50+t} = 1/(60 - t)$$
$$\int_0^{10} {}_t p_{40:50}\, \mu_{50+t}\, dt = \int_0^{10} \frac{e^{-0.05t}}{60} dt = -\frac{1}{60} \left.\frac{e^{-0.05t}}{(0.05)}\right|_0^{10}$$

$$= \frac{20}{60}\left(1 - e^{-0.5}\right) = 0.13115$$

Question #26
Key: A

Actual payment (in millions) $=\dfrac{3}{1.1}+\dfrac{5}{1.1^2}=6.860$

$q_3=1-\dfrac{0.30}{0.60}=0.5$

$_{1|}q_3=\dfrac{0.30-0.10}{0.60}=0.333$

Expected payment $=10\left(\dfrac{0.5}{1.1}+\dfrac{0.333}{1.1^2}\right)=7.298$

Ratio $\dfrac{6.860}{7.298}=94\%$

Question #27
Key: E

At duration 1

$K(x)$	${}_1L$	Prob
1	$v - P^{1}_{x:\overline{2}\rvert}$	q_{x+1}
>1	$0 - P^{1}_{x:\overline{2}\rvert}$	$1 - q_{x+1}$

So $Var({}_1L) = v^2 q_{x+1}(1 - q_{x+1}) = 0.1296$

That really short formula takes advantage of
$Var(aX + b) = a^2 Var(X)$, if a and b are constants.
Here $a = v$; $b = P^{1}_{x:\overline{2}\rvert}$; X is binomial with $p(X = 1) = q_{x+1}$.

Alternatively, that same formula for *Var* arises from Hattendorf, since
${}_2V = 0$ and $Var({}_2L) = 0$

Alternatively, evaluate $P^{1}_{x:\overline{2}\rvert} = 0.1303$

${}_1L = 0.9 - 0.1303 = 0.7697$ if $K(x) = 1$

${}_1L = 0 - 0.1303 = -0.1303$ if $K(X) > 1$

$E({}_1L) = (0.2)(0.7697) + (0.8)(-0.1303) = 0.0497$

$E({}_1L^2) = (0.2)(0.7697)^2 + (0.8)(-0.1303)^2 = 0.1320$

$Var({}_1L) = 0.1320 - (0.0497)^2 = 0.1295$

Question #28
Key: C

$$\bar{P}\left(\bar{A}_x\right)=\frac{\bar{A}_x}{\bar{a}_x}=\frac{\bar{A}_x}{\left(\frac{1-\bar{A}_x}{\delta}\right)}=\frac{\delta\bar{A}_x}{1-\bar{A}_x}=\frac{(0.1)\left(1/3\right)}{\left(1-1/3\right)}=0.05$$

$$Var(L)=\left(1+\frac{\bar{P}\left(\bar{A}_x\right)}{\delta}\right)^2\left({}^2\bar{A}_x-\bar{A}_x^2\right)$$

$$\frac{1}{5}=\left(1+\frac{0.05}{0.10}\right)^2\left({}^2\bar{A}_x-\bar{A}_x^2\right)$$

$$\left({}^2\bar{A}_x-\bar{A}_x^2\right)=0.08888$$

$$Var[L']=\left(1+\frac{\pi}{\delta}\right)^2\left({}^2\bar{A}_x-\bar{A}_x^2\right)$$

$$\frac{16}{45}=\left(1+\frac{\pi}{0.1}\right)^2(0.08888)$$

$$\left(1+\frac{\pi}{0.1}\right)^2=4$$

$$\pi=0.1$$

Question #29
Key: E

$$\begin{aligned}E(X\wedge 2)&=1\times f(1)+2\left(1-F(1)\right)\\&=1\times f(1)+2\left(1-f(0)-f(1)\right)\\&=1\times 3e^{-3}+2\left(1-e^{-3}-3e^{-3}\right)\\&=2-5e^{-3}\\&=1.75\end{aligned}$$

Cost per loss with deductible $=E(X)-E(X\wedge 2)$

$= 3 - 1.75$

$= 1.25$

Cost per loss with coinsurance $=\alpha E(X)=3\alpha$

Equating cost, $3\alpha=1.25$

$\alpha=0.42$

Question #30
Key: A

Let N be the number of clubs accepted
X be the number of members of a selected club
S be the total persons appearing

N is binomial with m = 1000 q = 0.20

$$E(N) = (1000)(0.20) = 200$$

$$Var(N) = (1000)(0.20)(0.80) = 160$$

$$E(S) = E(N)E(X) = (200)(20) = 4000$$

$$\begin{aligned} Var(S) &= E(N)\,Var(X) + Var(N)\left[E(X)\right]^2 \\ &= (200)(20) + (160)(20)^2 \\ &= 68{,}000 \end{aligned}$$

$$\begin{aligned} \text{Budget } &= 10 \times E(S) + 10 \times \sqrt{Var(S)} \\ &= 10 \times 4000 + 10 \times \sqrt{68{,}000} \\ &= 42{,}610 \end{aligned}$$

Question #31
Key: C

Insurance pays 80% of the portion of annual claim between 6,000 and 1,000, and 90% of the portion of annual claims over 14,000.

The 14,000 breakpoint is where Michael reaches 10,000 that he has paid:

1000 = deductible
1000 = 20% of costs between 1000 and 6000
8000 = 100% of costs between 14,000 and 6,000

$$E[X \wedge x] = \theta\left(1 - \frac{\theta}{x+\theta}\right) = \frac{5000x}{x+5000}$$

x	$E[X \wedge x]$
1000	833.33
6000	2727.27
14000	3684.21
∞	5000

$$0.80\big[E[X \wedge 6000] - E[X \wedge 1000]\big] + 0.90\big[E[X] - E[X \wedge 14000]\big]$$

$$\Rightarrow 0.80[2727.27 - 833.33] + 0.90[5000 - 3684.21]$$

$$= 1515.15 + 1184.21 = 2699.36$$

$$\Rightarrow 2700$$

Question #32
Key: D

We have the following table:

Item	Dist	$E(\ )$	$Var(\ )$
Number claims	$NB(16, 6)$	$16 \times 6 = 96$	$16 \times 6 \times 7 = 672$
Claims amounts	$U(0, 8)$	$8/2 = 4$	$8^2/12 = 5.33$
Aggregate		$4 \times 96 = 384$	$96 \times 5.33 + 672 \times 4^2 = 11,264$

$$\begin{aligned} \text{Premium } &= E(S) + 1.645 * Sqrt(Var(S)) = \\ &= 384 + 1.645 * Sqrt(11,264) \\ &= 559 \end{aligned}$$

1.645 is the 95th percentile of the standard normal distribution.

Question #33
Key: E

Seek g such that $\Pr\left\{\bar{a}_{\overline{T(40)}|} > g\right\} = 0.25$

$\bar{a}_{\overline{T}|}$ is a strictly increasing function of T.

$$\Pr\{T(40) > 60\} = 0.25 \text{ since } {}_{60}p_{40} = \frac{100-40}{120-40} = 0.25$$

$$\therefore \Pr\left\{\bar{a}_{\overline{T(40)}|} > \bar{a}_{\overline{60}|}\right\} = 0.25$$

$$g = \bar{a}_{\overline{60}|} = 19.00$$

Question 34
Key: B

$$A_{51:\overline{9|}} = 1 - d\ddot{a}_{51:\overline{9|}} = 1 - \left(\frac{0.05}{1.05}\right)(7.1) = 0.6619$$

$${}_{11}V = (2000)(0.6619) - (100)(7.1) = 613.80$$

$$\left({}_{10}V + P\right)(1.05) = {}_{11}V + q_{50}\left(2000 - {}_{11}V\right)$$

$$\left({}_{10}V + 100\right)(1.05) = 613.80 + (0.011)(2000 - 613.80)$$

$${}_{10}V = 499.09$$

where $q_{50} = (0.001)(10) + (0.001) = 0.011$

Alternatively, you could have used recursion to calculate $A_{50:\overline{10|}}$ from $A_{51:\overline{9|}}$, then $\ddot{a}_{50:\overline{10|}}$ from $A_{50:\overline{10|}}$, and used the prospective reserve formula for ${}_{10}V$.

Question #35
Key: C

$$1000A_{81} = (1000A_{80})(1+i) - q_{80}(1000 - A_{81})$$

$$689.52 = (679.80)(1.06) - q_{80}(1000 - 689.52)$$

$$q_{80} = \frac{720.59 - 689.52}{310.48} = 0.10$$

$$q_{[80]} = 0.5q_{80} = 0.05$$

$$1000A_{[80]} = 1000vq_{[80]} + vp_{[80]}\,1000A_{81}$$
$$= 1000 \times \frac{0.05}{1.06} + 689.52 \times \frac{0.95}{1.06} = 665.14$$

Question #36
Key: D

	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
42	776	8	16
43	752	8	16

$l_{42}^{(\tau)}$ and $l_{43}^{(\tau)}$ came from $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$

$$APV \text{ Benefits} = \frac{2000\left(8v + 8v^2\right) + 1000\left(16v + 16v^2\right)}{776} = 76.40$$

$$APV \text{ Premiums} = 34\left(\frac{776 + 752v}{776}\right) = (34)(1.92) = 65.28$$

$${}_2V = 76.40 - 65.28 = 11.12$$

Question #37
Key: B

$$p_{xx} = 1 - q_{xx} = 0.96$$

$$p_x = \sqrt{0.96} = 0.9798$$

$$p_{x+1:x+1} = 1 - q_{x+1:x+1} = 0.99$$

$$p_{x+1} = \sqrt{0.99} = 0.995$$

$$\ddot{a}_x = 1 + vp_x + v^2 \times {}_2p_x = 1 + \frac{0.9798}{1.05} + \frac{(0.9798)(0.995)}{1.05^2}$$
$$= 2.8174$$

$$\ddot{a}_{xx} = 1 + vp_{xx} + v^2 \times {}_2p_{xx} = 1 + \frac{0.96}{1.05} + \frac{(0.96)(0.99)}{1.05^2} = 2.7763$$

APV = $2000\,\ddot{a}_x + 2000\,\ddot{a}_x + 6000\,\ddot{a}_{xx}$
$$= (4000)(2.8174) + (6000)(2.7763)$$
$$= 27{,}927$$

Notes: The solution assumes that the future lifetimes are identically distributed. The precise description of the benefit would be a special 3-year temporary life annuity-due.

Question #38
Key: B

$$
\begin{aligned}
&{}_t p_x'^{(1)} \mu_x^{(1)}(t) = q_x'^{(1)} = 0.20 \\
&{}_t p_x'^{(2)} = 1 - t q_x'^{(2)} = 1 - 0.08t \\
&{}_t p_x'^{(3)} = 1 - t q_x'^{(3)} = 1 - 0.125t \\
&q_x^{(1)} = \int_0^1 {}_t p_x^{(\tau)} \mu_x^{(1)}(t)\,dt = \int_0^1 {}_t p_x'^{(2)}\, {}_t p_x'^{(3)}\, {}_t p_x'^{(1)} \mu_x^{(1)}(t)\,dt \\
&\qquad = \int_0^1 (1 - 0.08t)(1 - 0.125t)(0.20)\,dt \\
&\qquad = 0.2\int_0^1 \left(1 - 0.205t + 0.01t^2\right) dt \\
&\qquad = 0.2\left[t - \frac{0.205t^2}{2} + \frac{0.01t^3}{3}\right]_0^1 \\
&\qquad = (0.2)\left[1 - 0.1025 + \frac{0.01}{3}\right] = 0.1802
\end{aligned}
$$

Question #39
Key: E

With probability p, $\quad \text{Prob}(N = 2) = 0.5^2 = 0.25$

With probability $(1 - p)$, $\quad \text{Prob}(N = 2) = Combin(4, 2) * 0.5^4 = 0.375$

$$\text{Prob}(N = 2) = p \times 0.25 + (1 - p) \times 0.375$$

$$0.375 - 0.125\ p$$

Question #40
Key: D

600 can be obtained only 2 ways, from $500+100$ or from 6×100.

Since $\lambda=5$ and $p(100)=0.8,\quad p(500)=0.16$,

$$p(6\text{ claims for }100)=\frac{e^{-5}5^{6}}{6!}(0.8)^{6}=0.03833\text{ or }3.83\%$$

$$p(\overbrace{500+100}^{\text{2 claims}})=\frac{e^{-5}5^{2}}{2!}\left[(0.8)^{1}(0.16)^{1}(2)\right]=0.02156=2.16\%$$

The factor of 2 inside the bracket is because you could get a 500 then 100 or you could get a 100 then 500.

Total $=3.83\%+2.16\%=5.99\%$