Course 3 Solutions
November 2001 Exams

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Society of Actuaries

Question #1
Key: E

Solution:

$$\overset{\circ}{e}_{25:\overline{25}|} = \int_0^{15} {}_t p_{25} dt + {}_{15}p_{25} \int_0^{10} {}_t p_{40} dt$$

$$= \int_0^{15} e^{-.04t} dt + \left(e^{-\int_0^{15} .04\, ds} \right) \int_0^{10} e^{-.05t} dt$$

$$= \frac{1}{.04}\left(1 - e^{-.60}\right) + e^{-.60}\left[\frac{1}{.05}\left(1 - e^{-.50}\right)\right]$$

$$= 11.2797 + 4.3187$$

$$= 15.60$$

Question #2
Key: C

Solution:

$${}_5 p_{[60]+1} =$$

$$\left(1 - q_{[60]+1}\right)\left(1 - q_{[60]+2}\right)\left(1 - q_{63}\right)\left(1 - q_{64}\right)\left(1 - q_{65}\right)$$

$$= (0.89)(0.87)(0.85)(0.84)(0.83)$$

$$= 0.4589$$

Question # 3
Key: E

Solution:

$$12.50 = \bar{a}_x = \frac{1}{m+d} \Rightarrow m+d = 0.08 \Rightarrow m = d = 0.04$$

$$\bar{A}_x = \frac{m}{m+d} = 0.5$$

$${}^2\bar{A}_x = \frac{m}{m+2d} = \frac{1}{3}$$

$$\text{Var}\left(\bar{a}_{\overline{T|}}\right) = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{d^2}$$

$$= \frac{\frac{1}{3} - \frac{1}{4}}{0.0016} = 52.083$$

$$\text{S.D.} = \sqrt{52.083} = 7.217$$

Question # 4
Key: D

Solution:

$v = 0.90 \Rightarrow d = 0.10$

$A_x = 1 - d\ddot{a}_x = 1 - (0.10)(5) = 0.5$

Benefit premium $p = \dfrac{5000A_x - 5000vq_x}{\ddot{a}_x}$

$$= \frac{(5000)(0.5) - 5000(0.90)(0.05)}{5} = 455$$

$${}_{10}V_x = 1 - \frac{\ddot{a}_{x+10}}{\ddot{a}_x}$$

$$0.2 = 1 - \frac{\ddot{a}_{x+10}}{5} \Rightarrow \ddot{a}_{x+10} = 4$$

$A_{x+10} = 1 - d\ddot{a}_{x+10} = 1 - (0.10)(4) = 0.6$

${}_{10}V = 5000A_{x+10} - p\,\ddot{a}_{x+10} = (5000)(0.6) - (455)(4) = 1180$

Question #5
Key: D

Solution:

v is the lowest premium to ensure a zero % chance of loss in year 1 (The present value of the payment upon death is v, so you must collect at least v to avoid a loss should death occur).
Thus v = 0.95.

$$\mathrm{E}(\mathrm{Z}) = vq_x + v^2 p_x q_{x+1} = 0.95 \times 0.25 + (0.95)^2 \times 0.75 \times 0.2$$
$$= 0.3729$$
$$\mathrm{E}(\mathrm{Z}^2) = v^2 q_x + v^4 p_x \, q_{x+1} = (0.95)^2 \times 0.25 + (0.95)^4 \times 0.75 \times 0.2$$
$$= 0.3478$$

$$\mathrm{Var}(\mathrm{Z}) = \mathrm{E}(\mathrm{Z}^2) - (\mathrm{E}(\mathrm{Z}))^2 = 0.3478 - (0.3729)^2 = 0.21$$

Question # 6
Key: D

Solution:

Severity after increase	Severity after increase and deductible
60	0
120	20
180	80
300	200

Expected payment per loss $= 0.25 \times 0 + 0.25 \times 20 + 0.25 \times 80 + 0.25 \times 200$
= 75

Expected payments = Expected number of losses $\times$ Expected payment per loss
= 75×300
= 22,500

Question # 7
Key: A

Solution:

E (S) = E (N) E (X) = 50×200 = 10,000

$$\begin{aligned}\mathrm{Var}(\mathrm{S}) &= \mathrm{E}(\mathrm{N})\mathrm{Var}(X)+\mathrm{E}(X)^2\,\mathrm{Var}(\mathrm{N})\\ &=(50)(400)+\left(200^2\right)(100)\\ &=4{,}020{,}000\end{aligned}$$

$$\begin{aligned}\Pr(S<8{,}000) &= \Pr\left(Z<\frac{8{,}000-10{,}000}{\sqrt{4{,}020{,}000}}\right)\\ &=\Pr(Z<-0.998)\cong 16\%\end{aligned}$$

Question #8
Key: A

Solution:

Let Z be the present value random variable for one life.
Let S be the present value random variable for the 100 lives.

$$\begin{aligned}E(Z) &= 10\int_5^{\infty} e^{dt}e^{mt}m\,dt\\ &=10\frac{m}{d+m}e^{-(d+m)5}\\ &=2.426\end{aligned}$$

$$\begin{aligned}E\left(Z^2\right) &= 10^2\left(\frac{m}{2d+m}\right)e^{-(2d+m)5}\\ &=10^2\left(\frac{0.04}{0.16}\right)\left(e^{-0.8}\right)=11.233\end{aligned}$$

$$\begin{aligned}Var(Z) &= E\left(Z^2\right)-\left(E(Z)\right)^2\\ &=11.233-2.426^2\\ &=5.348\end{aligned}$$

$$\mathrm{E}(\mathrm{S})=100\,\mathrm{E}(\mathrm{Z})=242.6$$

$$\mathrm{Var}(\mathrm{S})=100\,\mathrm{Var}(\mathrm{Z})=534.8$$

$$\frac{F-242.6}{\sqrt{534.8}}=1.645\rightarrow F=281$$

Question #9
Key: D

Solution:

Prob{only 1 survives} = 1-Prob{both survive}-Prob{neither survives}

$$= 1 - {}_3p_{50} \times {}_3p_{[50]} - \left(1 - {}_3p_{50}\right)\left(1 - {}_3p_{[50]}\right)$$

$$= 1 - \underbrace{(0.9713)(0.9698)(0.9682)}_{=0.912012}\underbrace{(0.9849)(0.9819)(0.9682)}_{0.936320} - (1 - 0.912012)(1 - 0.93632)$$

$$= 0.140461$$

Question # 10
Key: C

Solution:

The tyrannosaur dies at the end of the first day if it eats no scientists that day. It dies at the end of the second day if it eats exactly one the first day and none the second day. If it does not die by the end of the second day, it will have at least 10,000 calories then, and will survive beyond 2.5.

$$\text{Prob (ruin)} = f(0) + f(1)f(0)$$
$$= 0.368 + (0.368)(0.368)$$
$$= 0.503$$

since $f(0) = \dfrac{e^{-1}1^0}{0!} = 0.368$

$$f(1) = \frac{e^{-1}1^1}{1!} = 0.368$$

Question #11
Key: B

Solution:

Let X = expected scientists eaten.

For each period, $E\left[X\right]=E\left[X|\text{dead}\right]\times\text{Prob}(\text{already dead})+E\left[X|\text{alive}\right]\times\text{Prob}(\text{alive})$
$$=0\times\text{Prob}(\text{dead})+E\left[X|\text{alive}\right]\times\text{Prob}(\text{alive})$$

Day 1, $\mathrm{E}\left[X_1\right]=1$

$$\text{Prob}(\text{dead at end of day 1})=f(0)=\frac{e^{-1}0^1}{0!}=0.368$$

Day 2, $\mathrm{E}\left[X_2\right]=0\times0.368+1\times(1-0.368)=0.632$

Prob (dead at end of day 2) = 0.503
[per problem 10]

Day 2.5, $\mathrm{E}\left[X_{2.5}\right]=0\times0.503+0.5\times(1-0.503)=0.249$

where $E\left[X_{2.5}|\text{alive}\right]=0.5$ since only ½ day in period.

$$\mathrm{E}\left[X\right]=\mathrm{E}\left[X_1\right]+\mathrm{E}\left[X_2\right]+\mathrm{E}\left[X_{2.5}\right]=1+0.632+0.249=1.881$$
$$\mathrm{E}\left[10{,}000X\right]=18{,}810$$

Question # 12
Key: C

Solution:

This solution applies the equivalence principle to each life. Applying the equivalence principle to the 100 life group just multiplies both sides of the first equation by 100, producing the same result for *P*.

$$APV(\text{Prems})=P=APV(\text{Benefits})=10q_{70}v+10p_{70}q_{71}v^2+Pp_{70}p_{71}v^2$$
$$P=\frac{(10)(0.03318)}{1.08}+\frac{(10)(1-0.03318)(0.03626)}{1.08^2}+\frac{P(1-0.03318)(1-0.03626)}{1.08^2}$$
$$=0.3072+0.3006+0.7988P$$
$$P=\frac{0.6078}{0.2012}=3.02$$

(APV above means Actuarial Present Value).

Question #13
Key: C

Solution:

For a binomial random variable with $n = 100$ and $p = q_{70} = 0.03318$, simulate number of deaths:

$i = 0$: $(1-p)^{100} = 0.03424 = f(0) = F(0)$

Since $0.18 > F(0)$, continue

$$\begin{aligned} i = 1: f(1) &= f(0)(n)(p)/(1-p) \\ &= (0.03424)(100)(0.03318)/(0.96682) \\ &= 0.11751 \end{aligned}$$

$F(1) = F(0) + f(1) = 0.03424 + 0.11751 = 0.15175$
Since $0.18 > F(1)$, continue

$$\begin{aligned} i = 2: f(2) &= f(1)\left[(n-1)/2\right](p)/(1-p) \\ &= (0.11751)(99/2)(0.03318/0.96682) \\ &= 0.19962 \end{aligned}$$

$F(2) = F(1) + f(2) = 0.15175 + 0.19962 = 0.35137$
Since $0.18 < F(2)$, number of claims $= 2$, so claim amount $= 20$.

Question #14
Key: C

Solution:

For the Pareto parameters given, the mean is $\frac{\theta}{\alpha-1}=1000$, so the annual premium is $2(1000)(1.1)=2200$.

Simulating the times of claims, we use the formula $t=-0.5\ln(1-x)$ to get the times between claims to be 0.886, 0.388, 0.327, … so claims occur at times 0.886, 1.274, and 1.601, so we only have to worry about the first claim.

For the Pareto, $F(x)=1-\left(\frac{1000}{x+1000}\right)^2$, so inverting gives $x=\frac{1000}{\sqrt{1-F(u)}}-1000$.

Using this on the first value provided gives a severity of $\frac{1000}{\sqrt{1-0.89}}-1000=2015$.

Clearly this will not cause ruin, since more than 15 of premium has been collected by time 0.886, so final surplus = initial surplus + premium – losses = 2000 + 2200 – 2015 = 2185.

Question #15
Key: E

Solution:

One approach is to recognize an interpretation of formula 7.4.11 or exercise 7.17a:

> Level benefit premiums can be split into two pieces: one piece to provide term insurance
> for *n* years; one to fund the reserve for those who survive.

If you think along those lines, you can derive formula 7.4.11:

$$P_x = P^1_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ 1}\ {}_nV_x$$

And plug in to get

$$0.090 = P^1_{x:\overline{n}|} + (0.00864)(0.563)$$
$$P^1_{x:\overline{n}|} = 0.0851$$

Another approach is to think in terms of retrospective reserves. Here is one such solution:

$$\begin{aligned}
{}_nV_x &= \left(P_x - P^1_{x:\overline{n}|}\right)\ddot{s}_{x:\overline{n}|} \\
&= \left(P_x - P^1_{x:\overline{n}|}\right)\frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} \\
&= \left(P_x - P^1_{x:\overline{n}|}\right)\frac{\ddot{a}_{x:\overline{n}|}}{P_{x:\overline{n}|}^{\ \ 1}\ddot{a}_{x:\overline{n}|}} \\
&= \frac{\left(P_x - P^1_{x:\overline{n}|}\right)}{\left(P_{x:\overline{n}|}^{\ \ 1}\right)}
\end{aligned}$$

$$0.563 = \left(0.090 - P^1_{x:\overline{n}|}\right) / 0.00864$$

$$\begin{aligned}
P^1_{x:\overline{n}|} &= 0.090 - (0.00864)(0.563) \\
&= 0.0851
\end{aligned}$$

Question #16
Key: A

Solution:

$$\delta = \ln(1.05) = 0.04879$$

$$\begin{aligned}\overline{A}_x &= \int_0^{w-x} {}_t p_x m_x(t) e^{-dt}\,dt \\ &= \int_0^{w-x} \frac{1}{w-x} e^{-dt}\,dt \text{ for DeMoivre} \\ &= \frac{1}{w-x}\bar{a}_{\overline{w-x}|}\end{aligned}$$

From here, many formulas for ${}_{10}\overline{V}\left(\overline{A}_{40}\right)$ could be used. One approach is:

Since

$$\overline{A}_{50} = \frac{\bar{a}_{\overline{50}|}}{50} = \frac{18.71}{50} = 0.3742 \text{ so } \bar{a}_{50} = \left(\frac{1-\overline{A}_{50}}{d}\right) = 12.83$$

$$\overline{A}_{40} = \frac{\bar{a}_{\overline{60}|}}{60} = \frac{19.40}{60} = 0.3233 \text{ so } \bar{a}_{40} = \left(\frac{1-\overline{A}_{40}}{d}\right) = 13.87$$

$$\text{so } \overline{P}\left(\overline{A}_{40}\right) = \frac{0.3233}{13.87} = 0.02331$$

$${}_{10}\overline{V}\left(\overline{A}_{40}\right) = \left[\overline{A}_{50} - \overline{P}\left(\overline{A}_{40}\right)\bar{a}_{50}\right] = \left[0.3742 - (0.02331)(12.83)\right] = 0.0751.$$

Question #17
Key: D

Solution:

$$\bar{A}_x = E\left[v^{T(x)}\right] = E\left[v^{T(x)} \middle| NS\right] \times \text{Prob}(\text{NS}) + E\left[v^{T(x)} \middle| S\right] \times \text{Prob}(\text{S})$$
$$= \left(\frac{0.03}{0.03+0.08}\right) \times 0.70 + \left(\frac{0.6}{0.06+0.08}\right) \times 0.30$$
$$= 0.3195$$

Similarly, $^2\bar{A}_x = \left(\frac{0.03}{0.03+0.16}\right) \times 0.70 + \left(\frac{0.06}{0.06+0.16}\right) \times 0.30 = 0.1923.$

$$\text{Var}\left(\bar{a}_{\overline{T(x)}|}\right) = \frac{^2\bar{A}_x - \bar{A}_x^2}{d^2} = \frac{0.1923 - 0.3195^2}{0.08^2} = 14.1.$$

Question #18
Key: B

Solution:

Let S denote aggregate losses before deductible.

$E[S] = 2 \times 2 = 4$, since mean severity is 2.

$f_S(0) = \frac{e^{-2} 2^0}{0!} = 0.1353,$ since must have 0 number to get aggregate losses = 0.

$f_S(1) = \left(\frac{e^{-2} 2}{1!}\right)\left(\frac{1}{3}\right) = 0.0902,$ since must have 1 loss whose size is 1 to get aggregate losses = 1.

$$E(S \wedge 2) = 0 \times f_S(0) + 1 \times f_S + 2 \times (1 - f_S(0) - f_S(1))$$
$$= 0 \times 0.1353 + 1 \times 0.0902 + 2 \times (1 - 0.1353 - 0.0902)$$
$$= 1.6392$$

$$E\left[(S-2)_+\right] = E[S] - E[S \wedge 2]$$
$$= 4 - 1.6392$$
$$= 2.3608$$

Question #19
Key: D

Solution:

Poisson processes are separable. The aggregate claims process is therefore equivalent to two independent processes, one for Type I claims with expected frequency $\left(\frac{1}{3}\right)(3000) = 1000$ and one for Type II claims.

Let $S_I =$ aggregate Type I claims.
$N_I =$ number of Type I claims.
$X_I =$ severity of a Type I claim (here = 10).

Since $X_I = 10$, a constant, $E(X_I) = 10$; $\text{Var}(X_I) = 0$.

$$
\begin{aligned}
\text{Var}(S_I) &= E(N_I)\ \text{Var}(X_I) + \text{Var}(N_I)\big[E(X_I)\big]^2 \\
&= (1000)(0) + (1000)(10)^2 \\
&= 100{,}000
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(S) &= \text{Var}(S_I) + \text{Var}(S_{II}) \text{ since independent} \\
2{,}100{,}000 &= 100{,}000 + \text{Var}(S_{II}) \\
Var(S_{II}) &= 2{,}000{,}000
\end{aligned}
$$

Question #20
Key: A

Solution:

$$
\begin{aligned}
{}_5p_{50}^{(t)} &= {}_5p_{50}^{\prime(1)}\,{}_5p_{50}^{\prime(2)} \\
&= \left(\frac{100-55}{100-50}\right)e^{-(0.05)(5)} \\
&= (0.9)(0.7788) = 0.7009
\end{aligned}
$$

Similarly

$$
\begin{aligned}
{}_{10}p_{50}^{(t)} &= \left(\frac{100-60}{100-50}\right)e^{-(0.05)(10)} \\
&= (0.8)(0.6065) = 0.4852
\end{aligned}
$$

$$
\begin{aligned}
{}_{5|5}q_{50}^{(t)} = {}_5p_{50}^{(t)} - {}_{10}p_{50}^{(t)} &= 0.7009 - 0.4852 \\
&= 0.2157
\end{aligned}
$$

Question #21
Key: C

Solution:

Only decrement 1 operates before $t = 0.7$

${}_{0.7}q_{40}^{\prime(1)} = (0.7)\,q_{40}^{\prime(1)} = (0.7)(0.10) = 0.07$ since UDD

Probability of reaching $t = 0.7$ is 1-0.07 = 0.93

Decrement 2 operates only at $t = 0.7$, eliminating 0.125 of those who reached 0.7

$q_{40}^{(2)} = (0.93)(0.125) = 0.11625$

Question #22
Key: B

Solution:

Let p_1 = long run probability of being in R at the start of a trip
Let p_2 = probability of being in S

		p_1 City R	p_2 City S
p_1	City R	0.2	0.8
p_2	City S	0.3	0.7

$$0.2p_1 + 0.3p_2 = p_1 \Rightarrow 0.8p_1 = 0.3p_2 \quad p_1 = \frac{3}{8}p_2$$

$$p_1 + p_2 = 1 \Rightarrow \frac{3}{8}p_2 + p_2 = 1 \quad \frac{11}{8}p_2 = 1 \quad p_2 = \frac{8}{11} \quad p_1 = \frac{3}{11}$$

E(R) = expected profit per trip if in city R= $0.2 \times 1 + 2 \times 0.8 = 1.8$
E(S) = expected profit per trip if in city S= $0.3 \times 2 + 1.2 \times 0.7 = 1.44$

$\Rightarrow$ In the long run, the profit per trip $= 1.8 \times \frac{3}{11} + 1.44 \times \frac{8}{11} = 1.54$

Question #23
Key: D

Solution:

Let states 0, 1, 2, 3 correspond to surplus of 0, 1, 2, 3.

One year transition matrix

$$T=\begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.40 & 0.25 & 0.15 \\ 0.20 & 0.00 & 0.40 & 0.40 \\ 0.00 & 0.20 & 0.00 & 0.80 \end{bmatrix}$$

$$T^2=\begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.33 & 0.19 & 0.20 & 0.28 \\ 0.28 & 0.08 & 0.16 & 0.48 \\ 0.04 & 0.24 & 0.05 & 0.67 \end{bmatrix}$$

$$T^3=\begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.408 & 0.132 & 0.1275 & 0.3325 \\ 0.328 & 0.128 & 0.084 & 0.460 \\ 0.098 & 0.230 & 0.080 & 0.592 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\times T^3=\begin{bmatrix} 0.098 & 0.230 & 0.080 & 0.592 \end{bmatrix}$$

Prob of starting in 3 and ending in 0 = 0.098
1 – 0.098 = 0.902
or 0.230 + 0.080 + 0.592 = 0.902.

If you anticipate how T^3 will be used, you see it is sufficient to calculate only the fourth row of it, or even only column 1 of row 4.

Alternatively, it is certainly correct, and probably shorter, to calculate $\left[(M\times T)\times T\right]\times T$ rather than $M\times T^3$, where $M=\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$.

See problem 29 for an example of that method.

Question #24
Key: C

Solution:

$$\pi\left(1+{}_2p_{80}v^2\right)=1000A_{80}+\frac{\pi v q_{80}}{2}+\frac{\pi v^3\,{}_2p_{80}q_{82}}{2}$$

$$\pi\left(1+\frac{0.83910}{1.06^2}\right)=665.75+\pi\left(\frac{0.08030}{2(1.06)}+\frac{0.83910\times 0.09561}{2(1.06)^3}\right)$$

$$\pi(1.74680)=665.75+\pi(0.07156)$$
$$\pi(1.67524)=665.75$$
$$\pi=397.41$$

Where ${}_2p_{80}=\dfrac{3{,}284{,}542}{3{,}914{,}365}=0.83910$

Or ${}_2p_{80}=(1-0.08030)(1-0.08764)=0.83910$

Question #25
Key: E

Solution:

At issue, actuarial present value (APV) of benefits

$$= \int_0^{\infty} b_t v^t \, {}_t p_{65} \, \mu_{65}(t)\,dt$$

$$= \int_0^{\infty} 1000\left(e^{0.04t}\right)\left(e^{-0.04t}\right) {}_t p_{65} \, \mu_{65}(t)\,dt$$

$$= 1000\int_0^{\infty} {}_t p_{65} \, \mu_{65}(t)\,dt = 1000 \; {}_{\infty}q_{65} = 1000$$

APV of premiums $= \pi \, \bar{a}_{65} = \pi\left(\dfrac{1}{0.04+0.02}\right) = 16.667\pi$

Benefit premium $\pi = 1000/16.667 = 60$

$${}_2\bar{V} = \int_0^{\infty} b_{2+u} \, v^u \, {}_u p_{67} \, \mu_{65}(2+u)\,du - \pi \, \bar{a}_{67}$$

$$= \int_0^{\infty} 1000\, e^{0.04(2+u)} e^{-0.04u} \; {}_u p_{67} \, \mu_{65}(2+u)\,du - (60)(16.667)$$

$$= 1000 e^{0.08} \int_0^{\infty} {}_u p_{67} \mu_{65}(2+u)\,du - 1000$$

$$= 1083.29 \; {}_{\infty}q_{67} - 1000 = 1083.29 - 1000 = 83.29$$

Question #26
Key: B

Solution:

(1) $a_{x:\overline{20}|} = \ddot{a}_{x:\overline{20}|} - 1 + {}_{20}E_x$

(2) $\ddot{a}_{x:\overline{20}|} = \dfrac{1 - A_{x:\overline{20}|}}{d}$

(3) $A_{x:\overline{20}|} = A^{1}_{x:\overline{20}|} + A_{x:\overset{1}{\overline{20}|}}$

(4) $A_x = A^{1}_{x:\overline{20}|} + {}_{20}E_x \; A_{x+20}$

$0.28 = A^{1}_{x:\overline{20}|} + (0.25)(0.40)$

$A^{1}_{x:\overline{20}|} = 0.18$

Now plug into (3): $A_{x:\overline{20}|} = 0.18 + 0.25 = 0.43$

Now plug into (2): $\ddot{a}_{x:\overline{20}|} = \dfrac{1 - 0.43}{(0.05/1.05)} = 11.97$

Now plug into (1): $a_{x:\overline{20}|} = 11.97 - 1 + 0.25 = 11.22$

Question #27
Key: A

Solution:

$$E[N] = E_\Lambda\left[E[N|\Lambda]\right] = E_\Lambda[\Lambda] = 2$$

$$\begin{aligned}\mathrm{Var}[N] &= E_\Lambda\left[\mathrm{Var}[N|\Lambda]\right] + \mathrm{Var}_\Lambda\left[E[N|\Lambda]\right] \\ &= E_\Lambda[\Lambda] + Var_\Lambda[\Lambda] = 2 + 4 = 6\end{aligned}$$

Distribution is negative binomial.

Per supplied tables:

$$\begin{aligned}&r\beta = 2 \text{ and } r\beta(1+\beta) = 6 \\ &1+\beta = 3 \\ &\quad \beta = 2 \\ &r\beta = 2 \quad r = 1\end{aligned}$$

$$p_1 = \frac{rb^1}{1!(1+b)^{r+1}} = \frac{(1)(2)}{(1)(3)^2} = 0.22$$

Alternatively, if you don't recognize that N will have a negative binomial distribution, derive gamma density from moments (hoping α is an integer).

$$\text{Mean} = \theta\alpha = 2$$

$$\begin{aligned}Var = E\left[\Lambda^2\right] - E[\Lambda]^2 &= \theta^2\left(\alpha^2 + \alpha\right) - \theta^2\alpha^2 \\ &= \theta^2\alpha = 4\end{aligned}$$

$$\theta = \frac{\theta^2\alpha}{\theta\alpha} = \frac{4}{2} = 2$$

$$\alpha = \frac{\theta\alpha}{\theta} = 1$$

$$p_1 = \int_0^\infty (p_1 | l) f(l)\, dl = \int_0^\infty \frac{e^{-l}\, l^1}{1!} \frac{(l/2) e^{-(l/2)}}{l\,\Gamma(1)}\, dl$$

$$= \frac{1}{2} \int_0^\infty l e^{-\frac{a}{2}}\, dl$$

[Integrate by parts; not shown]

$$= \frac{1}{2} \left(-\frac{2}{3} l e^{-\frac{3}{2}l} - \frac{4}{9} e^{-\frac{3}{2}l} \right) \Bigg|_0^\infty$$

$$= \frac{2}{9} = 0.22$$

Question #28
Key: C

Solution:

Limited expected value =

$$\int_0^{1000} (1-F(x))dx = \int_0^{1000} \left(0.8e^{-0.02x} + 0.2e^{-0.001x}\right)dx = \left(-40e^{-0.02x} - 200e^{-0.001x}\right)\Big|_0^{1000} = 40 + 126.4$$

$$= 166.4$$

Question #29
Key: E

Solution:

$$M = \text{Initial state matrix } = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \text{One year transition matrix } = \begin{bmatrix} 0.20 & 0.80 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0.75 & 0 & 0 & 0.25 \\ 1.00 & 0 & 0 & 0 \end{bmatrix}$$

$$M \times T = \begin{bmatrix} 0.20 & 0.80 & 0 & 0 \end{bmatrix}$$

$$(M \times T) \times T = \begin{bmatrix} 0.44 & 0.16 & 0.40 & 0 \end{bmatrix}$$

$$((M \times T) \times T) \times T = \begin{bmatrix} 0.468 & 0.352 & 0.08 & 0.10 \end{bmatrix}$$

Probability of being in state F after three years = 0.468.

$$\text{Actuarial present value} = \left(0.468v^3\right)(500) = 171$$

Notes:

1. Only the first entry of the last matrix need be calculated (verifying that the four sum to 1 is useful "quality control.")
2. Compare this with solution 23. It would be valid to calculate T^3 here, but advancing M one year at a time seems easier.

Question #30
Key: B

Solution:

Let Y_i be the number of claims in the ith envelope.

Let $X(13)$ be the aggregate number of claims received in 13 weeks.

$$E[Y_i] = (1\times 0.2) + (2\times 0.25) + (3\times 0.4) + (4\times 0.15) = 2.5$$
$$E[Y_i^2] = (1\times 0.2) + (4\times 0.25) + (9\times 0.4) + (16\times 0.15) = 7.2$$
$$E[X(13)] = 50\times 13\times 2.5 = 1625$$
$$\text{Var}[X(13)] = 50\times 13\times 7.2 = 4680$$
$$\text{Prob}\{\text{X}(13) \le Z\} = 0.90 = \Phi(1.282)$$
$$\Rightarrow \text{Prob}\left\{\frac{X(13)-1625}{\sqrt{4680}} \le 1.282\right\}$$
$$X(13) \le 1712.7$$

Note: The formula for $\text{Var}[X(13)]$ took advantage of the frequency's being Poisson. The more general formula for the variance of a compound distribution, $\text{Var}(S) = E(N)\ \text{Var}(X) + \text{Var}(N) E(X)^2,$ would give the same result.

Question #31
Key: C

Solution:

$$1+q = \frac{18}{(12)(1)} = 1.5$$
$$q = 0.5$$
$q > 0$ so (C)

Question #32
Key: D

Solution:

No real issues with #1. It matches the real survival process.

#2 is never valid. Even if $x = y$, so that Prob (x dies first) = 0.50, the conditional future lifetime of x would not follow De Moivre.

Question #33
Key: E

Solution:

$$\mu^{M}(60) = \frac{1}{\omega - 60} = \frac{1}{75-60} = \frac{1}{15}$$
$$\mu^{F}(60) = \frac{1}{\omega' - 60} = \frac{1}{15} \times \frac{3}{5} = \frac{1}{25} \Rightarrow \omega' = 85$$
$${}_t p_{65}^{M} = 1 - \frac{t}{10}$$
$${}_t p_{60}^{F} = 1 - \frac{t}{25}$$

Let x denote the male and y denote the female.

$$\overset{\circ}{e}_x = 5 \ \left(\text{mean for uniform distribution over } (0,10)\right)$$
$$\overset{\circ}{e}_y = 12.5 \left(\text{mean for uniform distribution over } (0,25)\right)$$
$$\begin{aligned}\overset{\circ}{e}_{xy} &= \int_0^{10} \left(1 - \frac{t}{10}\right)\left(1 - \frac{t}{25}\right) \cdot dt \\ &= \int_0^{10} \left(1 - \frac{7}{50}t + \frac{t^2}{250}\right) \cdot dt \\ &= \left(t - \frac{7}{100}t^2 + \frac{t^3}{750}\right)\Bigg|_0^{10} = 10 - \frac{7}{100} \times 100 + \frac{1000}{750} \\ &= 10 - 7 + \frac{4}{3} = \frac{13}{3}\end{aligned}$$

$$\overset{\circ}{e}_{\overline{xy}} = \overset{\circ}{e}_x + \overset{\circ}{e}_y - \overset{\circ}{e}_{xy} = 5 + \frac{25}{2} - \frac{13}{3} = \frac{30 + 75 - 26}{6} = 13.17$$

Question #34
Key: B

Solution:

$$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{1}{3}$$
$$^2\bar{A}_x = \frac{\mu}{\mu+2\delta} = \frac{1}{5}$$

$$\bar{P}\left(\bar{A}_x\right) = \mu = 0.04$$

$$\begin{aligned}\operatorname{Var}(L) &= \left(1+\frac{\bar{P}\left(\bar{A}_x\right)}{\delta}\right)^2\left({}^2\bar{A}_x - \bar{A}_x^2\right) \\ &= \left(1+\frac{0.04}{0.08}\right)^2\left(\frac{1}{5}-\left(\frac{1}{3}\right)^2\right) \\ &= \left(\frac{3}{2}\right)^2\left(\frac{4}{45}\right) \\ &= \frac{1}{5}\end{aligned}$$

Question #35
Key: B

Solution:

$$\begin{aligned}\text{Mean excess loss} &= \frac{E(X)-E(X\wedge 100)}{1-F(100)} \\ &= \frac{331-91}{0.8} = 300\end{aligned}$$

$$E(X) = E(X \wedge 1000) \text{ since } F(1000) = 1.0$$

Question #36
Key: E

Solution:

Expected insurance benefits per factory $= E\left[(X-1)_+\right]$
$= 0.2\times1+0.1\times2 = 0.4.$

Insurance premium = (1.1) (2 factories) (0.4 per factory) = 0.88.

Let R = retained major repair costs, then

$$f_R(0) = 0.4^2 = 0.16$$
$$f_R(1) = 2\times0.4\times0.6 = 0.48$$
$$f_R(2) = 0.6^2 = 0.36$$

Dividend $= 3-0.88-R-(0.15)(3)$, if positive
$= 1.67-R$, if positive

$$E(\text{Dividend}) = (0.16)(1.67-0)+(0.48)(1.67-1)+(0.36)(0) = 0.5888$$

[The (0.36)(0) term in the last line represents that with probability 0.36, $(1.67-R)$ is negative so the dividend is 0.]

Question #37
Key: A

Solution:

$$E[X] = \frac{\alpha\theta}{\alpha-1} = \frac{4\alpha}{\alpha-1} = 8 \Rightarrow 4\alpha = 8\alpha-8$$
$$\alpha = 2$$

$$F(6) = 1-\left(\frac{\theta}{6}\right)^{\alpha} = 1-\left(\frac{4}{6}\right)^2$$

$$= 0.555$$

$$s(6) = 1-F(6) = 0.444$$

Question #38
Key: B

Solution:

$${}_2q_{64}^{(1)} = q_{64}^{(1)} + p_{64}^{(\tau)}\; q_{65}^{(1)}$$

$$p_{64}^{(\tau)} = (1-0.025)(1-0.035)(1-0.200) = 0.75270$$

$$\begin{aligned} q_{64}^{(\tau)} &= 1 - p_{64}^{(\tau)} \\ &= 0.24730 \end{aligned}$$

$$p_{64}'^{(1)} = (1-0.025) = 0.975$$

$$q_{64}^{(1)} = \frac{\ln p_{64}'^{(1)}}{\ln p_{64}^{(\tau)}} q_{64}^{(\tau)} = \frac{\ln(0.975)}{\ln(0.75270)}(0.2473) = 0.02204$$

$${}_2q_{64}^{(1)} = 0.02204 + (0.75270)(0.02716) = 0.04248$$

Question #39
Key: B

Solution:

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots = 11.05$$

$$\text{Annuity} = v^3\ {}_3p_x 1000 + v^4\ {}_4p_x \times 1000 \times (1.04) + \ldots$$
$$= \sum_{k=3}^{\infty} 1000(1.04)^{k-3} v^k\ {}_kp_x$$
$$= 1000v^3 \sum_{k=3}^{\infty} {}_k p_x$$
$$= 1000v^3 (e_x - 0.99 - 0.98) = 1000\left(\frac{1}{1.04}\right)^3 \times 9.08 = 8072$$

Let π = benefit premium.

$$\pi\left(1 + 0.99v + 0.98v^2\right) = 8072$$
$$2.8580\pi = 8072$$
$$\pi = 2824$$

Question #40
Key B

Solution:

$$p\ \ddot{a}_{30:\overline{10}|} = 1000A_{30} + P(IA)^{1}_{30:\overline{10}|} + (10p)\left({}_{10|}A_{30}\right)$$

$$p = \frac{1000A_{30}}{\ddot{a}_{30:\overline{10}|} - (IA)^{1}_{30:\overline{10}|} - 10\,{}_{10|}A_{30}}$$
$$= \frac{1000(0.102)}{7.747 - 0.078 - 10(0.088)}$$
$$= \frac{102}{6.789}$$
$$= 15.024$$