

THE ESSENTIALS OF

Computer Organization *and* Architecture

THIRD EDITION

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Chapter 3

Boolean Algebra and Digital Logic

Chapter 3 Objectives

- Understand the relationship between Boolean logic and digital computer circuits.
- Learn how to design simple logic circuits.
- Understand how digital circuits work together to form complex computer systems.

3.1 Introduction

- In the latter part of the nineteenth century, George Boole incensed philosophers and mathematicians alike when he suggested that logical thought could be represented through mathematical equations.
 - *How dare anyone suggest that human thought could be encapsulated and manipulated like an algebraic formula?*
- Computers, as we know them today, are implementations of Boole's *Laws of Thought*.
 - John Atanasoff and Claude Shannon were among the first to see this connection.

3.1 Introduction

- In the middle of the twentieth century, computers were commonly known as “thinking machines” and “electronic brains.”
 - Many people were fearful of them.
- Nowadays, we rarely ponder the relationship between electronic digital computers and human logic. Computers are accepted as part of our lives.
 - Many people, however, are still fearful of them.
- In this chapter, you will learn the simplicity that constitutes the essence of the machine.

3.2 Boolean Algebra

- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
 - In formal logic, these values are “true” and “false.”
 - In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low.”
- Boolean expressions are created by performing operations on Boolean variables.
 - Common Boolean operators include AND, OR, and NOT.

3.2 Boolean Algebra

- A Boolean operator can be completely described using a truth table.
- The truth table for the Boolean operators AND and OR are shown at the right.
- The AND operator is also known as a Boolean product. The OR operator is the Boolean sum.

X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

3.2 Boolean Algebra

- The truth table for the Boolean NOT operator is shown at the right.
- The NOT operation is most often designated by an overbar. It is sometimes indicated by a prime mark (') or an “elbow” (\neg).

NOT x	
x	\overline{x}
0	1
1	0

3.2 Boolean Algebra

- A Boolean function has:
 - At least one Boolean variable,
 - At least one Boolean operator, and
 - At least one input from the set $\{0,1\}$.
- It produces an output that is also a member of the set $\{0,1\}$.

Now you know why the binary numbering system is so handy in digital systems.

3.2 Boolean Algebra

- The truth table for the Boolean function:

$$F(x, y, z) = x\bar{z} + y$$

is shown at the right.

- To make evaluation of the Boolean function easier, the truth table contains extra (shaded) columns to hold evaluations of subparts of the function.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	\bar{z}	$x\bar{z}$	$x\bar{z} + y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

3.2 Boolean Algebra

- As with common arithmetic, Boolean operations have rules of precedence.
- The NOT operator has highest priority, followed by AND and then OR.
- This is how we chose the (shaded) function subparts in our table.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	\bar{z}	$x\bar{z}$	$x\bar{z} + y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

3.2 Boolean Algebra

- Digital computers contain circuits that implement Boolean functions.
- The simpler that we can make a Boolean function, the smaller the circuit that will result.
 - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

3.2 Boolean Algebra

- Most Boolean identities have an AND (product) form as well as an OR (sum) form. We give our identities using both forms. Our first group is rather intuitive:

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$x\bar{x} = 0$	$x + \bar{x} = 1$

3.2 Boolean Algebra

- Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity Name	AND Form	OR Form
Commutative Law	$xy = yx$	$x+y = y+x$
Associative Law	$(xy)z = x(yz)$	$(x+y)+z = x+(y+z)$
Distributive Law	$x+yz = (x+y)(x+z)$	$x(y+z) = xy+xz$

3.2 Boolean Algebra

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law	$x(x+y) = x$	$x + xy = x$
DeMorgan's Law	$\overline{(xy)} = \bar{x} + \bar{y}$	$\overline{(x+y)} = \bar{x}\bar{y}$
Double Complement Law	$\overline{(\bar{x})} = x$	

3.2 Boolean Algebra

- We can use Boolean identities to simplify:

$$F(X, Y, Z) = (X+Y) (X+\bar{Y}) (\bar{X}\bar{Z})$$

as follows:

FUNCTION	RULE
$(X + Y)(X + \bar{Y}) (\bar{X}\bar{Z})$	
$(X + Y)(X + \bar{Y}) (\bar{X} + \bar{\bar{Z}})$	DeMorgan's
$(X + Y)(X + \bar{Y}) (\bar{X} + \bar{\bar{Z}})$	Double Complement
$(\bar{X}\bar{X} + \bar{X}\bar{Y} + Y\bar{X} + Y\bar{Y}) (\bar{X} + Z)$	Distributive and Commutative
$((\bar{X}\bar{X} + Y\bar{Y}) + (\bar{X}\bar{Y} + Y\bar{X})) (\bar{X} + Z)$	Associative
$((\bar{X}\bar{X} + Y\bar{Y}) + \bar{X}(\bar{Y} + Y)) (\bar{X} + Z)$	Distributive
$((\bar{X} + Y\bar{Y}) + \bar{X}(\bar{Y} + Y)) (\bar{X} + Z)$	Idempotent

3.2 Boolean Algebra

- We can use Boolean identities to simplify:

$$F(X, Y, Z) = (X + Y)(X + \bar{Y})(\bar{X}Z)$$

as follows:

FUNCTION	RULE
$((X + Y\bar{Y}) + X(\bar{Y} + Y))(\bar{X} + Z)$... continued
$((X + 0) + X(1))(\bar{X} + Z)$	Inverse
$(X + X)(\bar{X} + Z)$	Identity
$X(\bar{X} + Z)$	Idempotent
$X\bar{X} + XZ$	Distributive
$0 + XZ$	Inverse
XZ	Identity

3.2 Boolean Algebra

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- Recall DeMorgan's law states:

$$\overline{(xy)} = \bar{x} + \bar{y} \quad \text{and} \quad \overline{(x+y)} = \bar{x}\bar{y}$$

3.2 Boolean Algebra

- DeMorgan's law can be extended to any number of variables.
- Replace each variable by its complement and change all ANDs to ORs and all ORs to ANDs.
- Thus, we find the the complement of:

$$F(X, Y, Z) = (XY) + (\overline{X}Y) + (X\overline{Z})$$

is:

$$\begin{aligned}\overline{F}(X, Y, Z) &= \overline{(XY) + (\overline{X}Y) + (X\overline{Z})} \\ &= \overline{(XY)} \overline{(\overline{X}Y)} \overline{(X\overline{Z})} \\ &= (\overline{X} + \overline{Y})(X + \overline{Z})(\overline{Y} + Z)\end{aligned}$$

3.2 Boolean Algebra

- Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.
 - These “synonymous” forms are *logically equivalent*.
 - Logically equivalent expressions have identical truth tables.
- In order to eliminate as much confusion as possible, designers express Boolean functions in *standardized* or *canonical* form.

3.2 Boolean Algebra

- There are two canonical forms for Boolean expressions: sum-of-products and product-of-sums.
 - Recall the Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the sum-of-products form, ANDed variables are ORed together.
 - For example: $F(x, y, z) = xy + xz + yz$
- In the product-of-sums form, ORed variables are ANDed together:
 - For example: $F(x, y, z) = (x+y)(x+z)(y+z)$

3.2 Boolean Algebra

- It is easy to convert a function to sum-of-products form using its truth table.
- We are interested in the values of the variables that make the function true (=1).
- Using the truth table, we list the values of the variables that result in a true function value.
- Each group of variables is then ORed together.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$x\bar{z} + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

3.2 Boolean Algebra

- The sum-of-products form for our function is:

$$F(x, y, z) = (\bar{x}\bar{y}\bar{z}) + (\bar{x}yz) + (x\bar{y}\bar{z}) + (x\bar{y}z)$$

We note that this function is not in simplest terms. Our aim is only to rewrite our function in canonical sum-of-products form.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$x\bar{z} + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Section Break

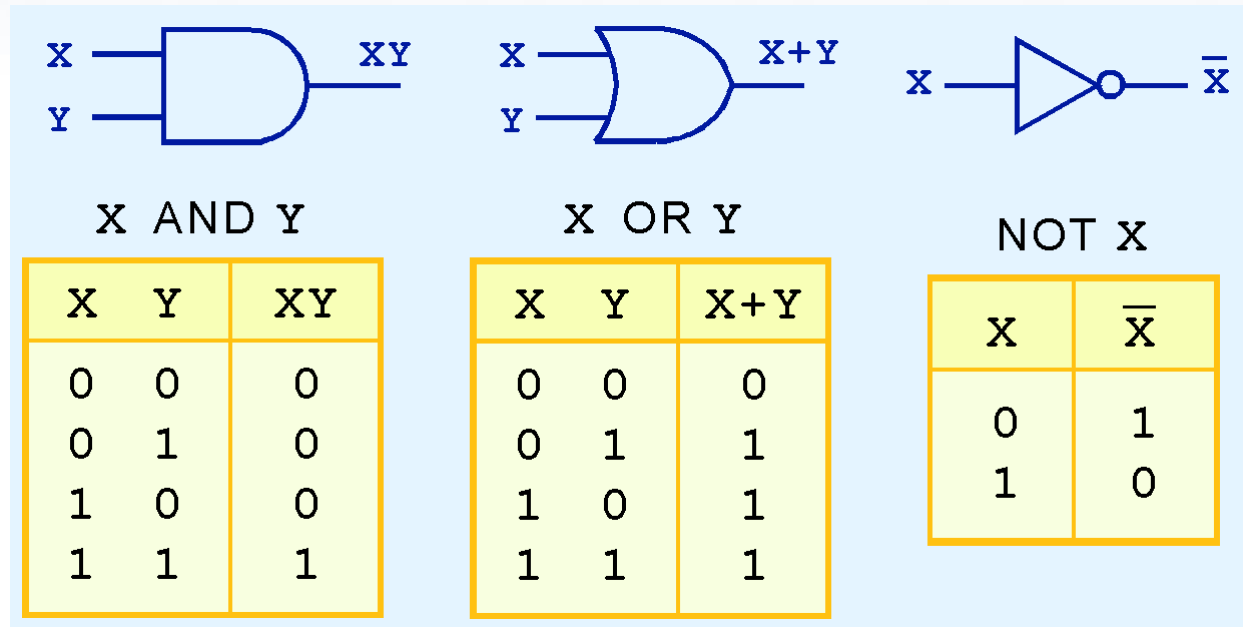
Let's take a look at Karnaugh Maps

3.3 Logic Gates

- We have looked at Boolean functions in abstract terms.
- In this section, we see that Boolean functions are implemented in digital computer circuits called gates.
- A gate is an electronic device that produces a result based on two or more input values.
 - In reality, gates consist of one to six transistors, but digital designers think of them as a single unit.
 - Integrated circuits contain collections of gates suited to a particular purpose.

3.3 Logic Gates

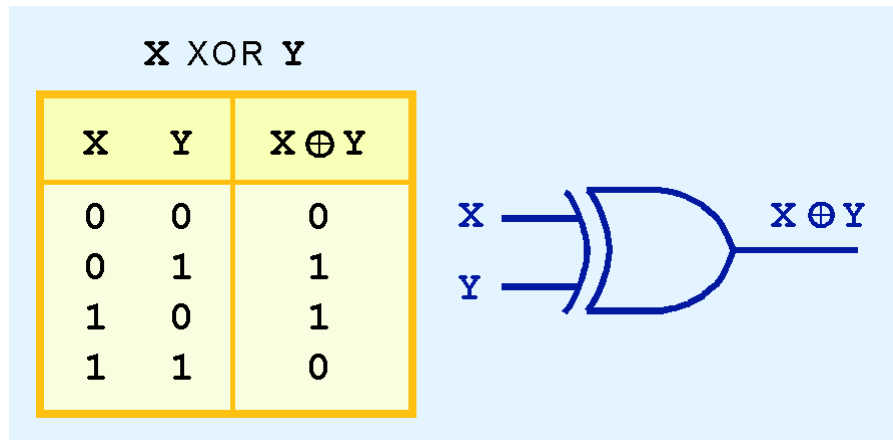
- The three simplest gates are the AND, OR, and NOT gates.



- They correspond directly to their respective Boolean operations, as you can see by their truth tables.

3.3 Logic Gates

- Another very useful gate is the exclusive OR (XOR) gate.
- The output of the XOR operation is true only when the values of the inputs differ.



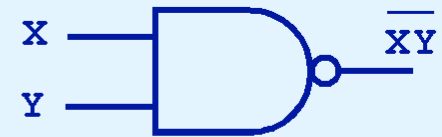
Note the special symbol \oplus for the XOR operation.

3.3 Logic Gates

- NAND and NOR are two very important gates. Their symbols and truth tables are shown at the right.

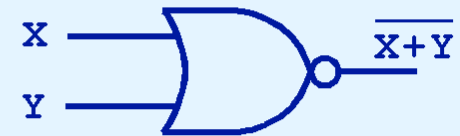
X NAND Y

X	Y	X NAND Y
0	0	1
0	1	1
1	0	1
1	1	0



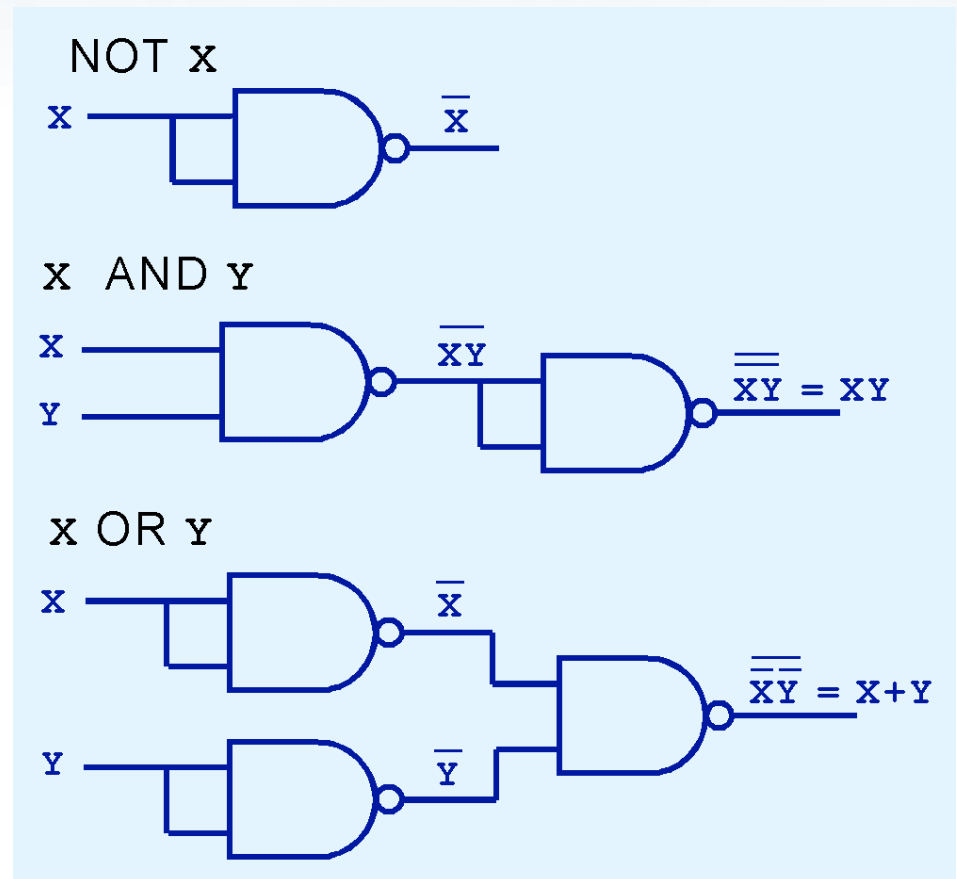
X NOR Y

X	Y	X NOR Y
0	0	1
0	1	0
1	0	0
1	1	0



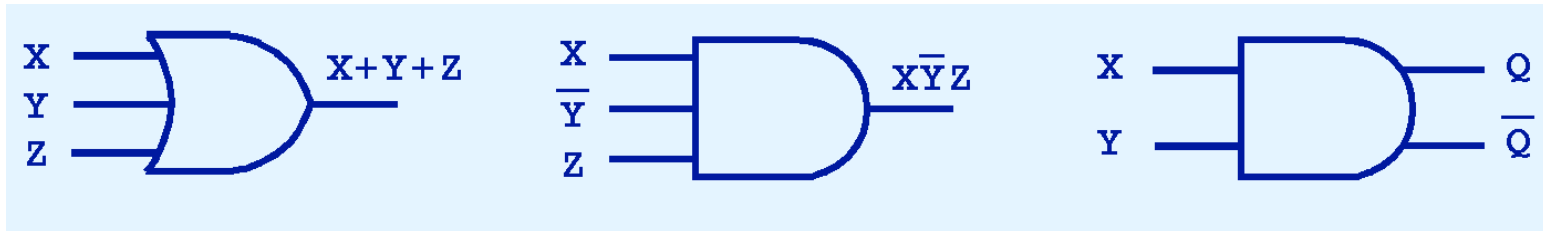
3.3 Logic Gates

- NAND and NOR are known as *universal gates* because they are inexpensive to manufacture and any Boolean function can be constructed using only NAND or only NOR gates.



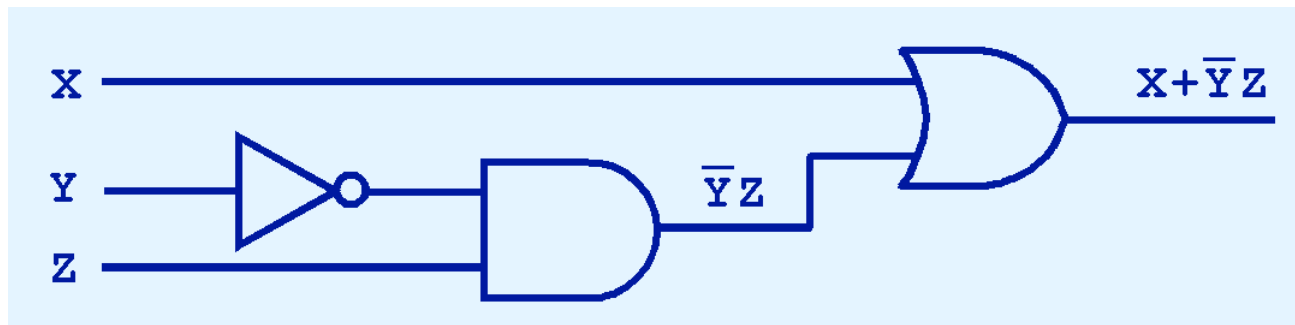
3.3 Logic Gates

- Gates can have multiple inputs and more than one output.
 - A second output can be provided for the complement of the operation.
 - We'll see more of this later.



3.3 Logic Gates

- The main thing to remember is that combinations of gates implement Boolean functions.
- The circuit below implements the Boolean function: $F(X, Y, Z) = X + \bar{Y}Z$



We simplify our Boolean expressions so that we can create simpler circuits.

3.5 Combinational Circuits

- We have designed a circuit that implements the Boolean function:

$$F(X, Y, Z) = X + \overline{Y}Z$$

- This circuit is an example of a *combinational logic* circuit.
- Combinational logic circuits produce a specified output (almost) at the instant when input values are applied.
 - In a later section, we will explore circuits where this is not the case.

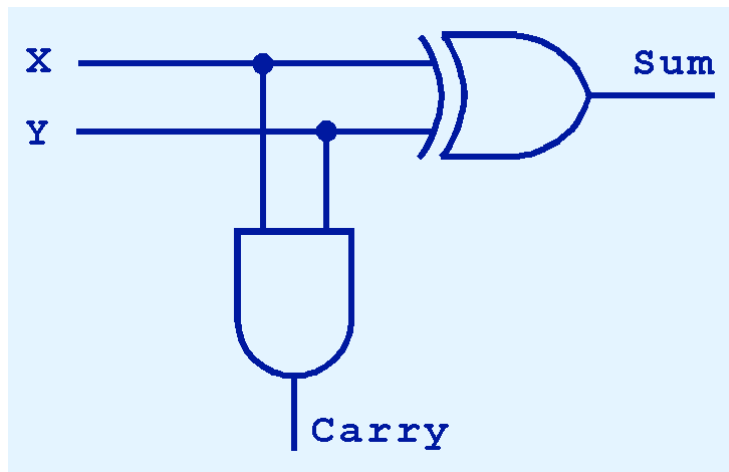
3.5 Combinational Circuits

- Combinational logic circuits give us many useful devices.
- One of the simplest is the *half adder*, which finds the sum of two bits.
- We can gain some insight as to the construction of a half adder by looking at its truth table, shown at the right.

Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

3.5 Combinational Circuits

- As we see, the sum can be found using the XOR operation and the carry using the AND operation.



Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

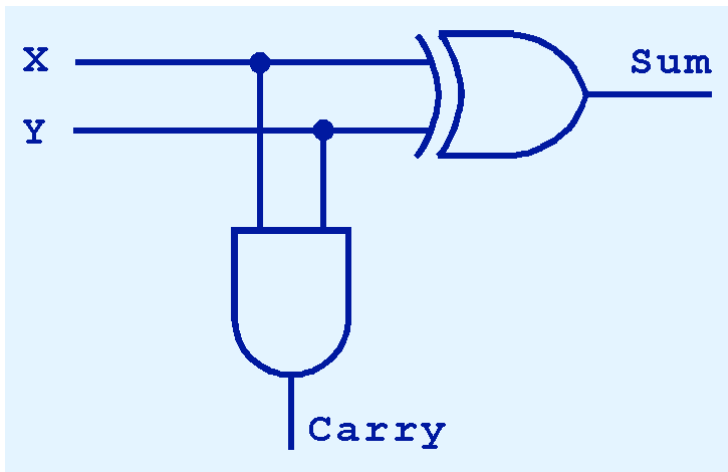
3.5 Combinational Circuits

- We can change our half adder into to a full adder by including gates for processing the carry bit.
- The truth table for a full adder is shown at the right.

Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

3.5 Combinational Circuits

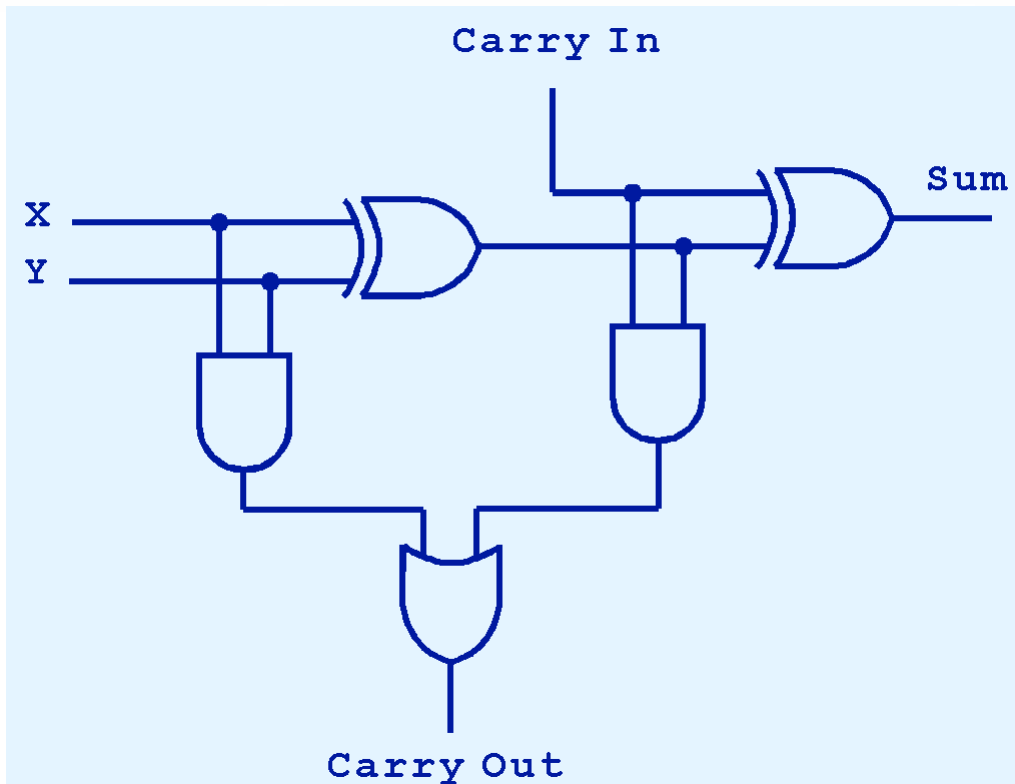
- How can we change the half adder shown below to make it a full adder?



Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

3.5 Combinational Circuits

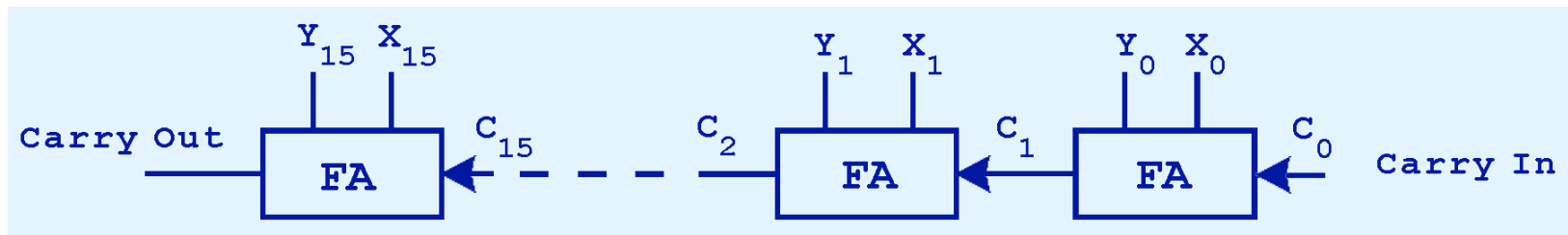
- Here's our completed full adder.



Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

3.5 Combinational Circuits

- Just as we combined half adders to make a full adder, full adders can be connected in series.
- The carry bit “ripples” from one adder to the next; hence, this configuration is called a *ripple-carry adder*.

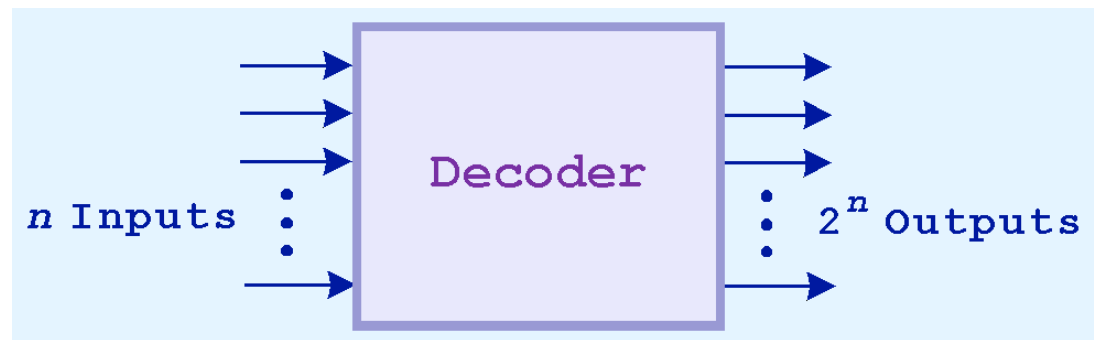


Today's systems employ more efficient adders.

3.5 Combinational Circuits

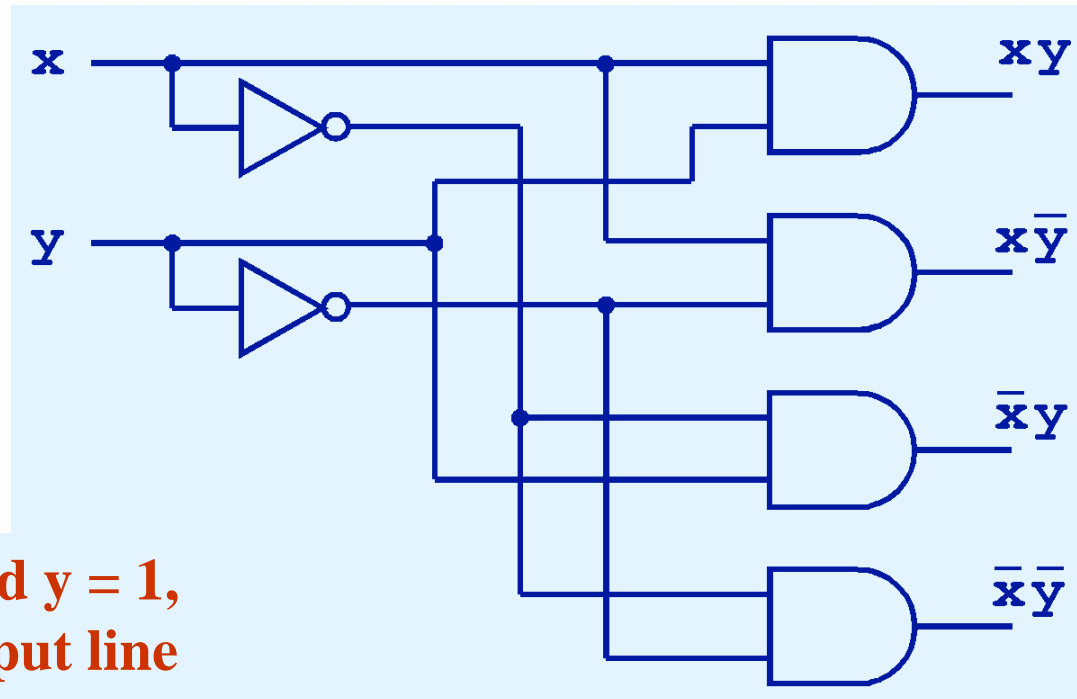
- Decoders are another important type of combinational circuit.
- Among other things, they are useful in selecting a memory location according a binary value placed on the address lines of a memory bus.
- Address decoders with n inputs can select any of 2^n locations.

This is a block diagram for a decoder.



3.5 Combinational Circuits

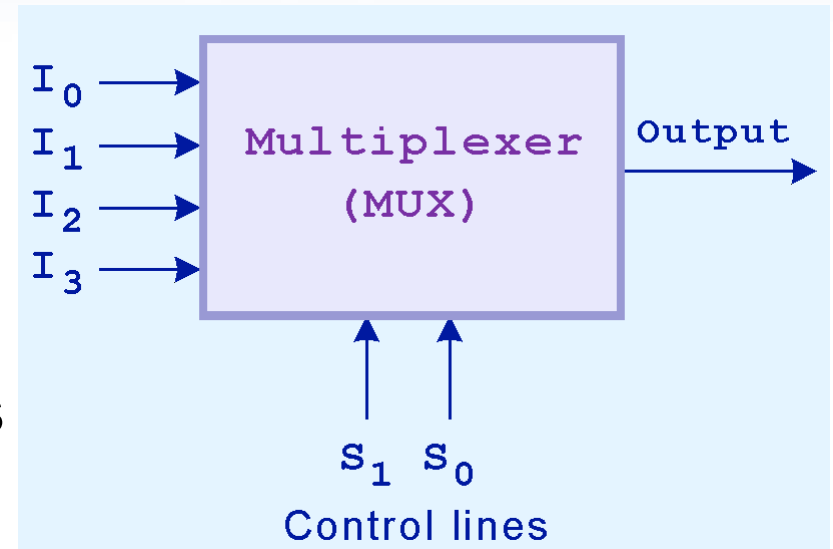
- This is what a 2-to-4 decoder looks like on the inside.



**If $x = 0$ and $y = 1$,
which output line
is enabled?**

3.5 Combinational Circuits

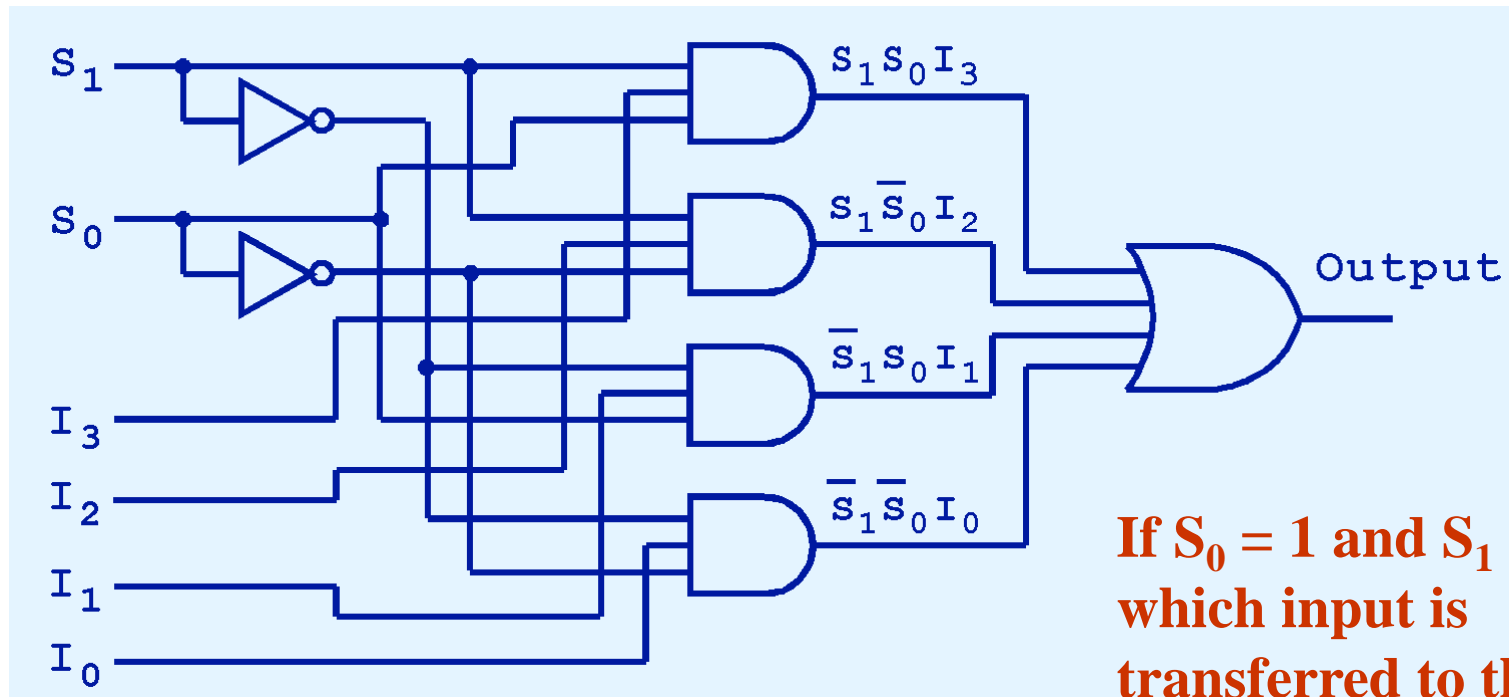
- A multiplexer does just the opposite of a decoder.
- It selects a single output from several inputs.
- The particular input chosen for output is determined by the value of the multiplexer's control lines.
- To be able to select among n inputs, $\log_2 n$ control lines are needed.



This is a block diagram for a multiplexer.

3.5 Combinational Circuits

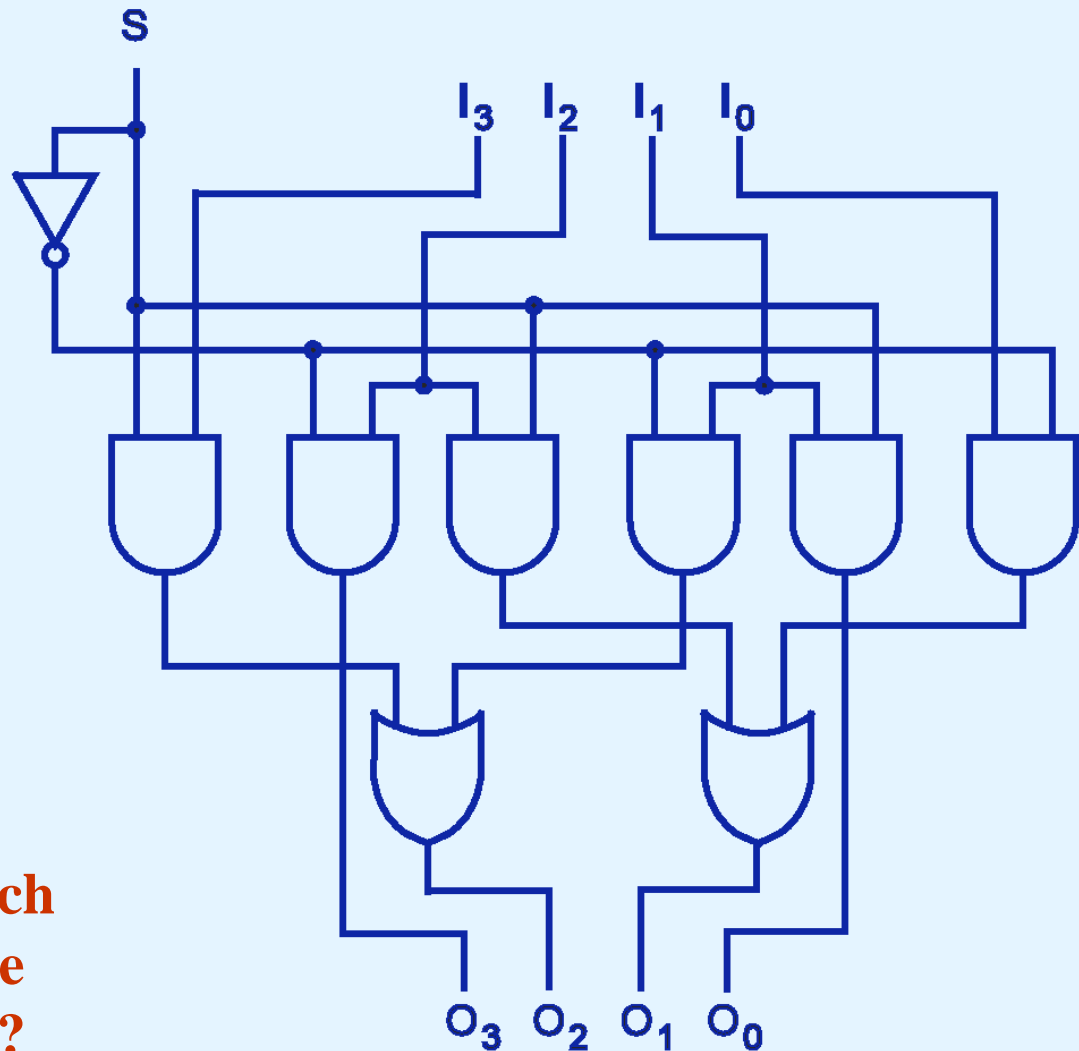
- This is what a 4-to-1 multiplexer looks like on the inside.



**If $S_0 = 1$ and $S_1 = 0$,
which input is
transferred to the
output?**

3.5 Combinational Circuits

- This shifter moves the bits of a nibble one position to the left or right.



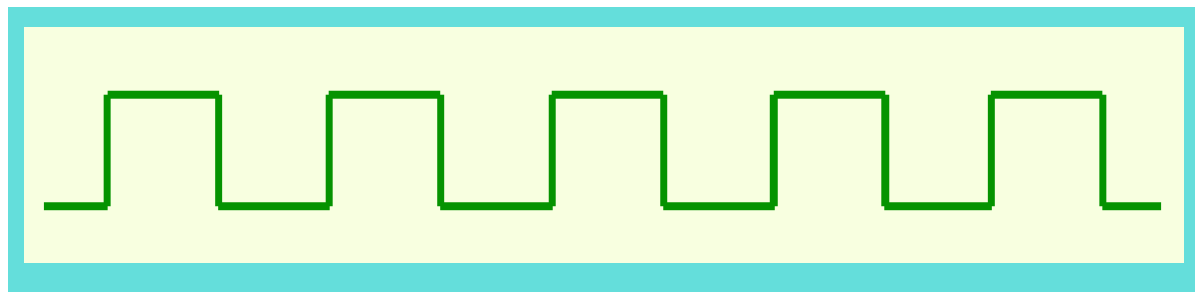
If $S = 0$, in which direction do the input bits shift?

3.6 Sequential Circuits

- Combinational logic circuits are perfect for situations when we require the immediate application of a Boolean function to a set of inputs.
- There are other times, however, when we need a circuit to change its value with consideration to its current state as well as its inputs.
 - These circuits have to “remember” their current state.
- *Sequential logic circuits* provide this functionality for us.

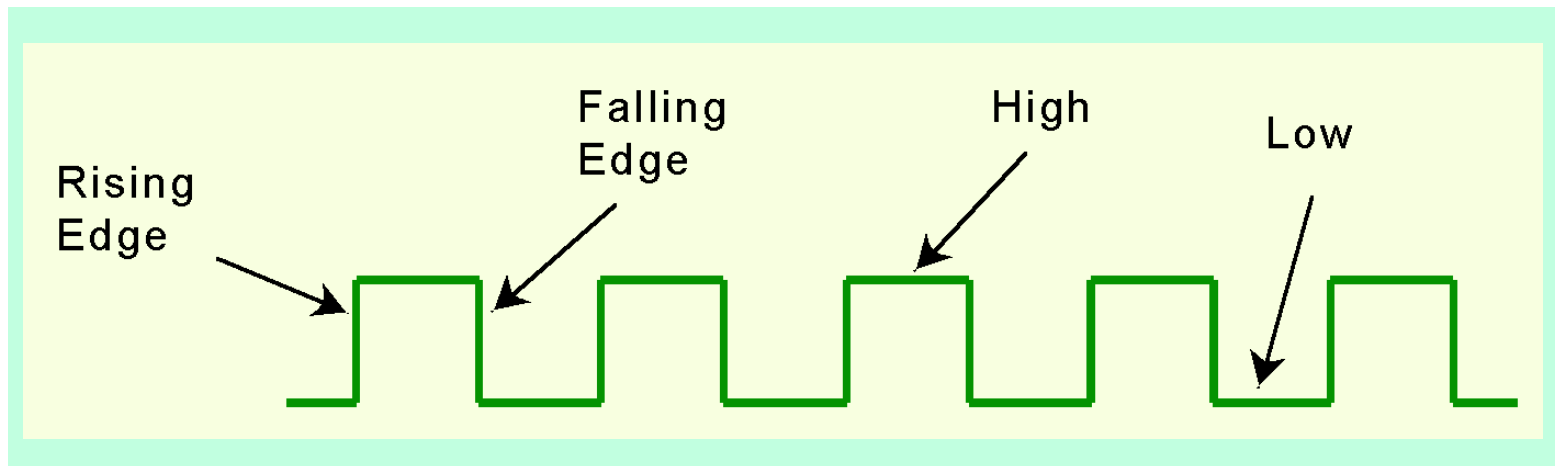
3.6 Sequential Circuits

- As the name implies, sequential logic circuits require a means by which events can be sequenced.
- State changes are controlled by clocks.
 - A “clock” is a special circuit that sends electrical pulses through a circuit.
- Clocks produce electrical waveforms such as the one shown below.



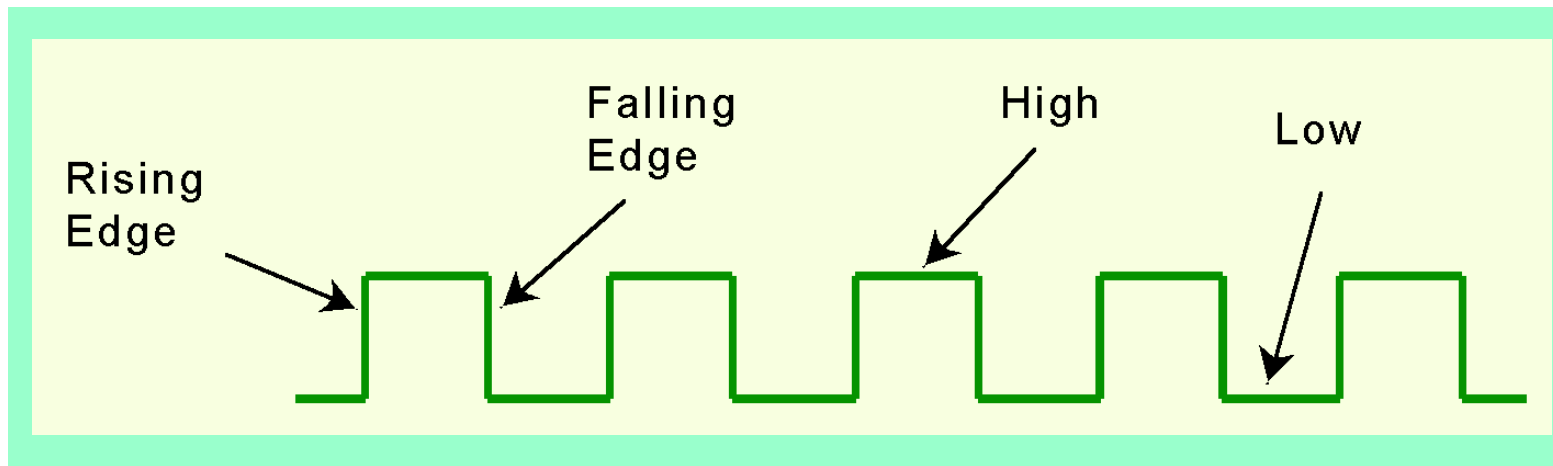
3.6 Sequential Circuits

- State changes occur in sequential circuits only when the clock ticks.
- Circuits can change state on the rising edge, falling edge, or when the clock pulse reaches its highest voltage.



3.6 Sequential Circuits

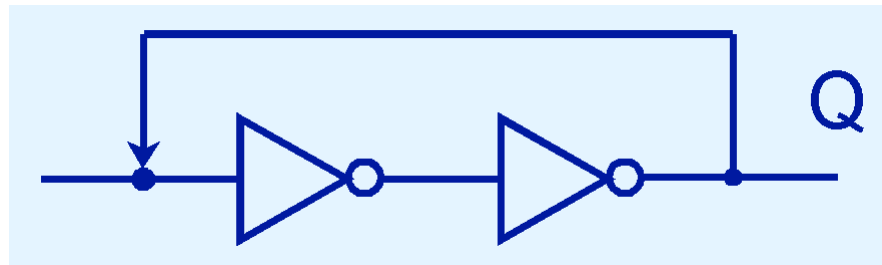
- Circuits that change state on the rising edge, or falling edge of the clock pulse are called *edge-triggered*.
- *Level-triggered circuits* change state when the clock voltage reaches its highest or lowest level.



3.6 Sequential Circuits

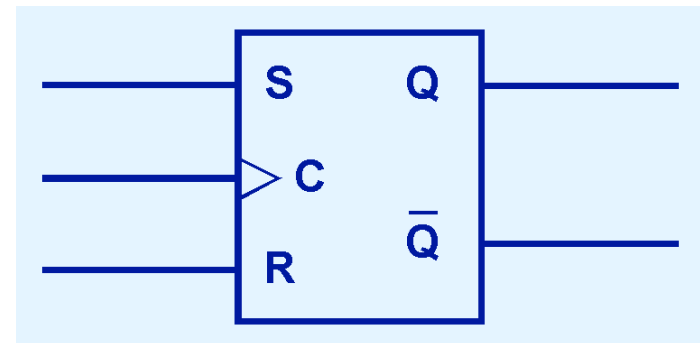
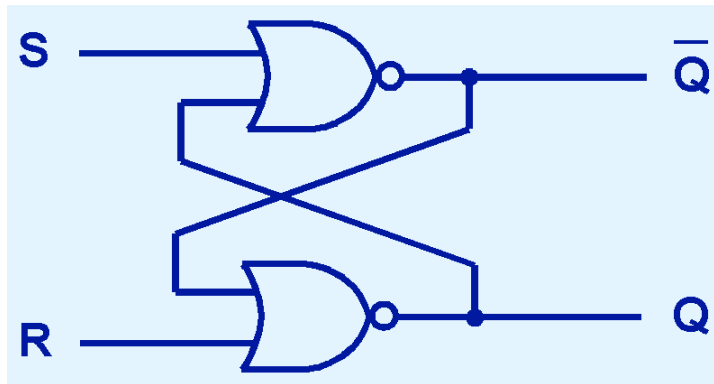
- To retain their state values, sequential circuits rely on *feedback*.
- Feedback in digital circuits occurs when an output is looped back to the input.
- A simple example of this concept is shown below.
 - If Q is 0 it will always be 0, if it is 1, it will always be 1.

Why?



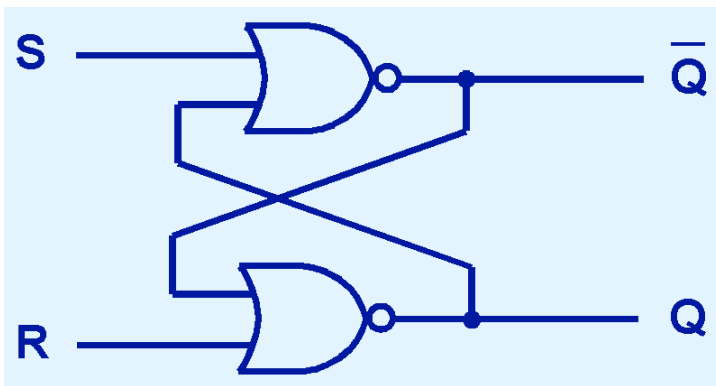
3.6 Sequential Circuits

- You can see how feedback works by examining the most basic sequential logic components, the SR flip-flop.
 - The “SR” stands for set/reset.
- The internals of an SR flip-flop are shown below, along with its block diagram.



3.6 Sequential Circuits

- The behavior of an SR flip-flop is described by a characteristic table.
- $Q(t)$ means the value of the output at time t .
 $Q(t+1)$ is the value of Q after the next clock pulse.



S	R	$Q(t+1)$
0	0	$Q(t)$ (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	undefined

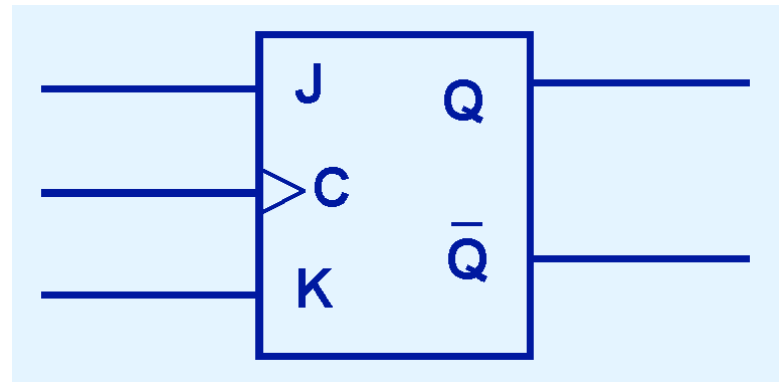
3.6 Sequential Circuits

- The SR flip-flop actually has three inputs: S, R, and its current output, Q.
- Thus, we can construct a truth table for this circuit, as shown at the right.
- Notice the two undefined values. When both S and R are 1, the SR flip-flop is unstable.

Present State			Next State
S	R	Q (t)	Q (t+1)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	undefined
1	1	1	undefined

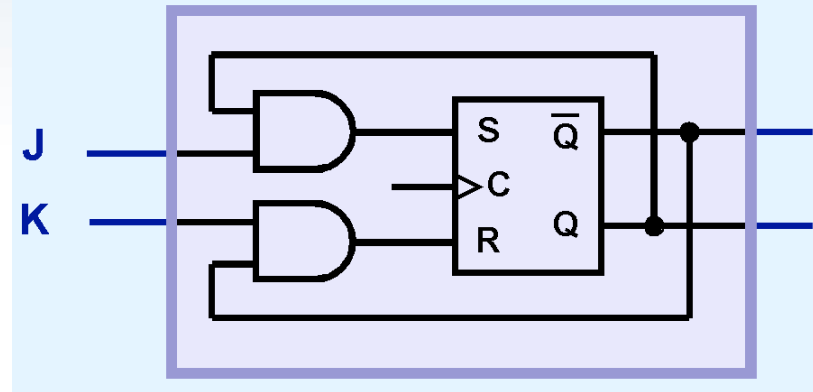
3.6 Sequential Circuits

- If we can be sure that the inputs to an SR flip-flop will never both be 1, we will never have an unstable circuit. This may not always be the case.
- The SR flip-flop can be modified to provide a stable state when both inputs are 1.
- This modified flip-flop is called a JK flip-flop, shown at the right.
 - The “JK” is in honor of Jack Kilby.



3.6 Sequential Circuits

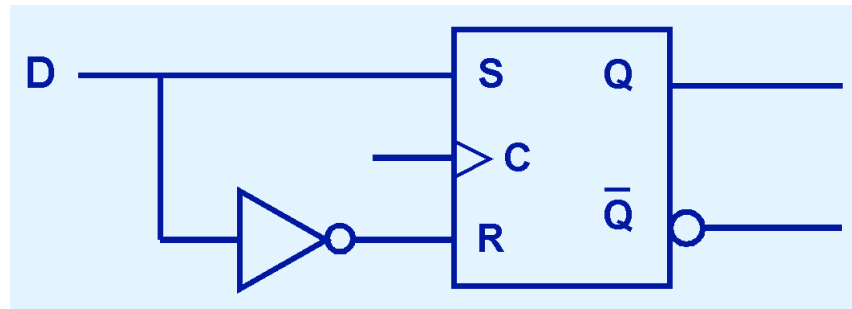
- At the right, we see how an SR flip-flop can be modified to create a JK flip-flop.
- The characteristic table indicates that the flip-flop is stable for all inputs.



J	K	$Q(t+1)$
0	0	$Q(t)$ (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	$\bar{Q}(t)$

3.6 Sequential Circuits

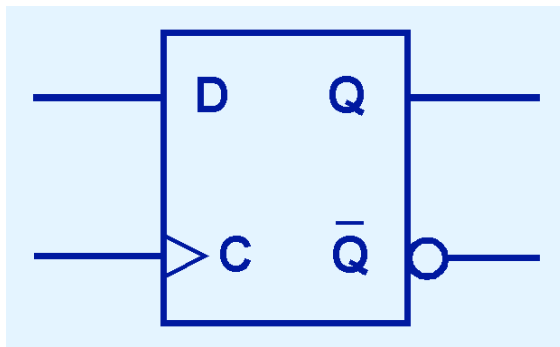
- Another modification of the SR flip-flop is the D flip-flop, shown below with its characteristic table.
- You will notice that the output of the flip-flop remains the same during subsequent clock pulses. The output changes only when the value of D changes.



D	$Q(t+1)$
0	0
1	1

3.6 Sequential Circuits

- The D flip-flop is the fundamental circuit of computer memory.
 - D flip-flops are usually illustrated using the block diagram shown below.
- The characteristic table for the D flip-flop is shown at the right.



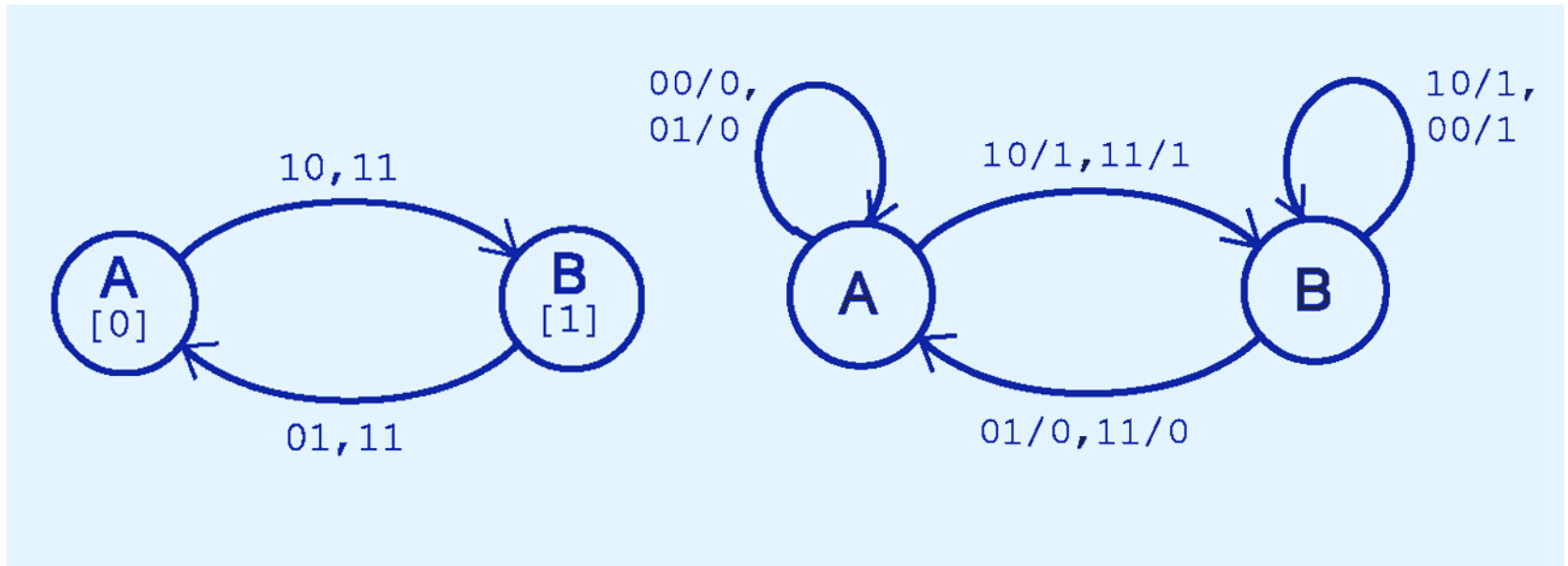
D	$Q(t+1)$
0	0
1	1

3.6 Sequential Circuits

- The behavior of sequential circuits can be expressed using characteristic tables or finite state machines (FSMs).
 - FSMs consist of a set of nodes that hold the states of the machine and a set of arcs that connect the states.
- Moore and Mealy machines are two types of FSMs that are equivalent.
 - They differ only in how they express the outputs of the machine.
- Moore machines place outputs on each node, while Mealy machines present their outputs on the transitions.

3.6 Sequential Circuits

- The behavior of a JK flop-flop is depicted below by a Moore machine (left) and a Mealy machine (right).



3.6 Sequential Circuits

- It is difficult to express the complexities of actual implementations using only Moore and Mealy machines.
 - For one thing, they do not address the intricacies of timing very well.
 - Secondly, it is often the case that an interaction of numerous signals is required to advance a machine from one state to the next.

3.6 Sequential Circuits

- Sequential circuits are used anytime that we have a “stateful” application.
 - A stateful application is one where the next state of the machine depends on the current state of the machine and the input.
- A stateful application requires both combinational and sequential logic.
- The following slides provide several examples of circuits that fall into this category.

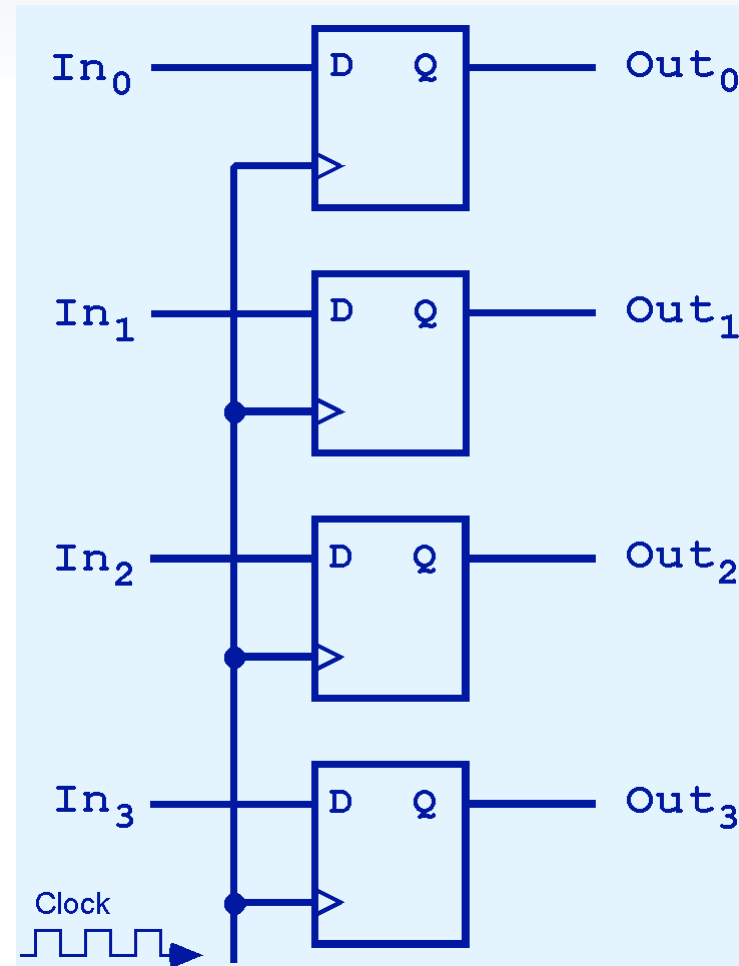
Can you think of others?

3.6 Sequential Circuits

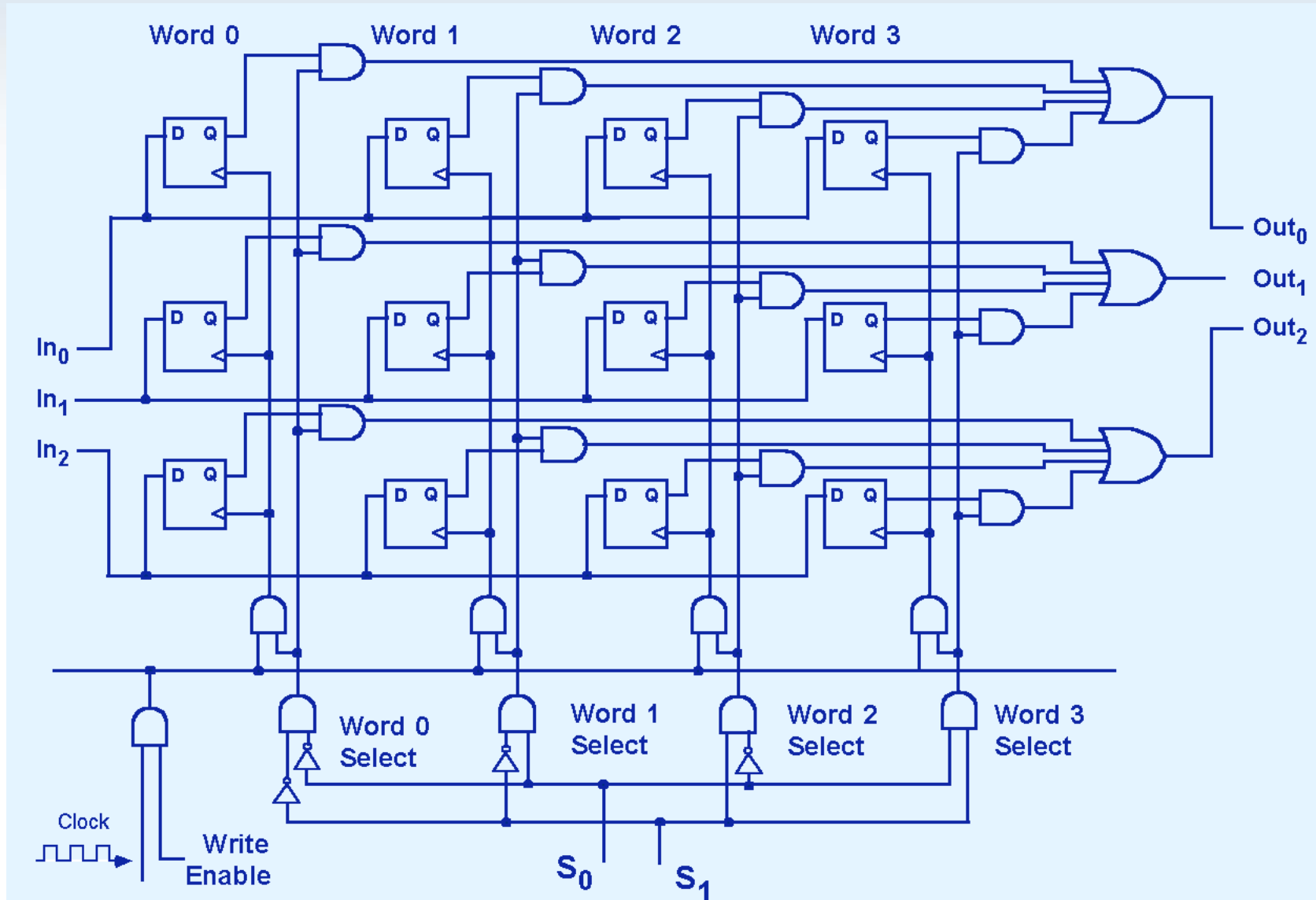
- This illustration shows a 4-bit register consisting of D flip-flops. You will usually see its block diagram (below) instead.



A larger memory configuration is shown on the next slide.

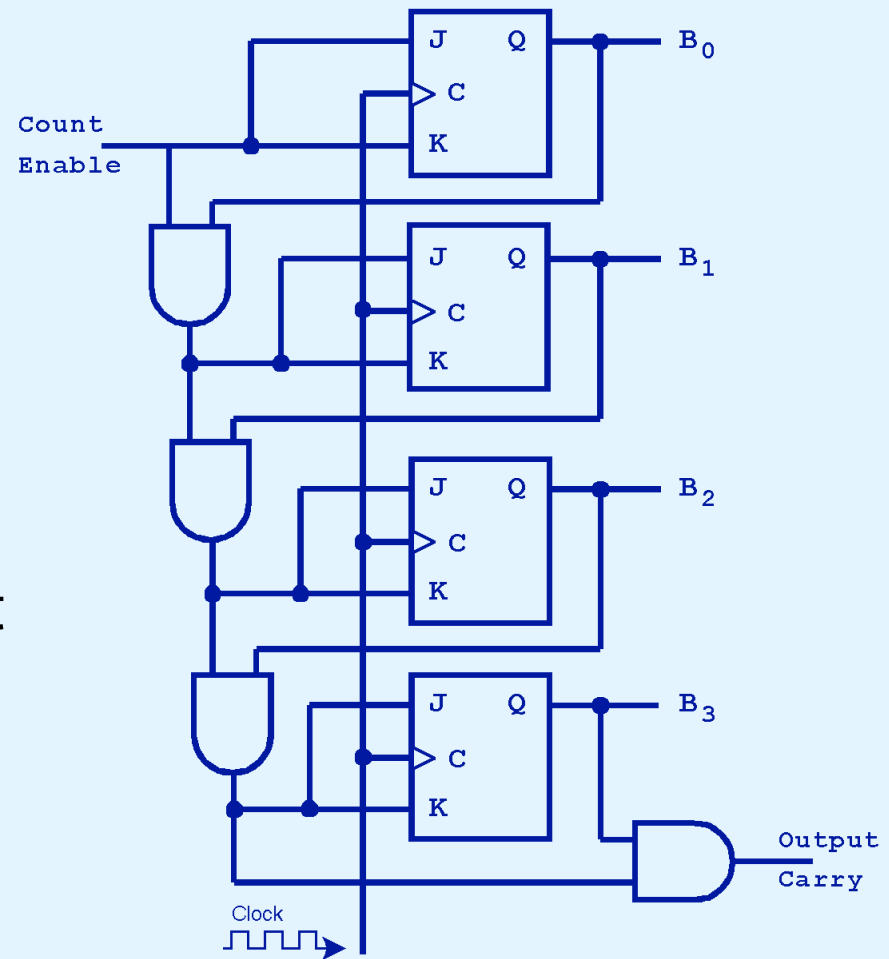


3.6 Sequential Circuits



3.6 Sequential Circuits

- A binary counter is another example of a sequential circuit.
- The low-order bit is complemented at each clock pulse.
- Whenever it changes from 0 to 1, the next bit is complemented, and so on through the other flip-flops.



3.7 Designing Circuits

- We have seen digital circuits from two points of view: digital analysis and digital synthesis.
 - *Digital analysis* explores the relationship between a circuit's inputs and its outputs.
 - *Digital synthesis* creates logic diagrams using the values specified in a truth table.
- Digital systems designers must also be mindful of the physical behaviors of circuits to include minute propagation delays that occur between the time when a circuit's inputs are energized and when the output is accurate and stable.

3.7 Designing Circuits

- Digital designers rely on specialized software to create efficient circuits.
 - Thus, software is an enabler for the construction of better hardware.
- Of course, software is in reality a collection of algorithms that could just as well be implemented in hardware.
 - Recall the Principle of Equivalence of Hardware and Software.

3.7 Designing Circuits

- When we need to implement a simple, specialized algorithm and its execution speed must be as fast as possible, a hardware solution is often preferred.
- This is the idea behind *embedded systems*, which are small special-purpose computers that we find in many everyday things.
- Embedded systems require special programming that demands an understanding of the operation of digital circuits, the basics of which you have learned in this chapter.

Chapter 3 Conclusion

- Computers are implementations of Boolean logic.
- Boolean functions are completely described by truth tables.
- Logic gates are small circuits that implement Boolean operators.
- The basic gates are AND, OR, and NOT.
 - The XOR gate is very useful in parity checkers and adders.
- The “universal gates” are NOR, and NAND.

Chapter 3 Conclusion

- Computer circuits consist of combinational logic circuits and sequential logic circuits.
- Combinational circuits produce outputs (almost) immediately when their inputs change.
- Sequential circuits require clocks to control their changes of state.
- The basic sequential circuit unit is the flip-flop: The behaviors of the SR, JK, and D flip-flops are the most important to know.

Chapter 3 Conclusion

- The behavior of sequential circuits can be expressed using characteristic tables or through various finite state machines.
- Moore and Mealy machines are two finite state machines that model high-level circuit behavior.
- Examples of sequential circuits include memory, counters, and Viterbi encoders and decoders.

End of Chapter 3