Topics:

- Review: Hermitian/unitary matrices, eigenvalues/vectors, single qubit gates/states
- Bloch sphere
- multi-qubit states/gates
- writing multi-qubit gates/states with Qiskit

Review

Hermitian: M is a matrix, $M^{*T} = M^{\dagger} = M$

Hamiltian conjugate ($M^{\dagger}$)

Unitary: M is a matrix, $M^{\dagger} M = \mathbb{1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$

Eigenvalues/vectors: Let A be an $N \times N$ matrix

$\boxed{A|\psi\rangle = \lambda|\psi\rangle}$, λ is an eigenvalue

$|\psi\rangle$ is an eigenvector.

$$\boxed{\det(A - \lambda\mathbb{1}) = 0}$$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $\det(A - \lambda\mathbb{1}) = \det\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 2$

$= \lambda^2 - \lambda - 2$

$= (\lambda-2)(\lambda+1) = 0$

$\lambda = 2, -1$

$\boxed{\lambda = 2}$ $\quad \dots \quad |\psi\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

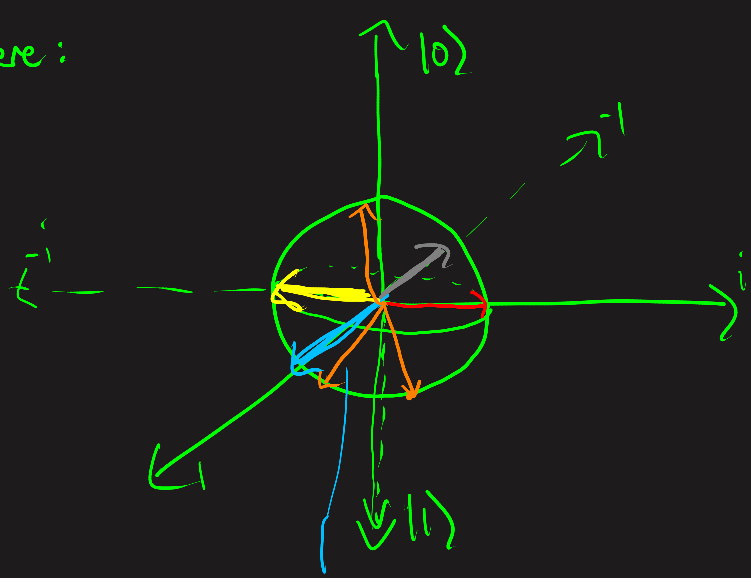
Pauli matrices: (Single qubit gates)

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X, \quad X X^\dagger = \mathbb{1}, \quad Y^\dagger = Y \dots$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \dots$$

Bloch sphere:



$$|\psi\rangle = \left(\frac{1}{\sqrt{5}} |0\rangle + i \sqrt{\frac{4}{5}} |1\rangle \right)$$

$$|\psi\rangle = \left(|0\rangle + i|1\rangle \right) \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \left(|0\rangle - |1\rangle \right) \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \left(|0\rangle - i|1\rangle \right) \frac{1}{\sqrt{2}}$$

Ex $\quad |\varphi_1\rangle = -\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$, 1

To measure something

$$\langle\varphi_1|\mathbb{1}|\varphi_1\rangle = \langle\varphi_1|\varphi_1\rangle =$$

$\uparrow$ $(|\varphi_1\rangle)^\dagger$

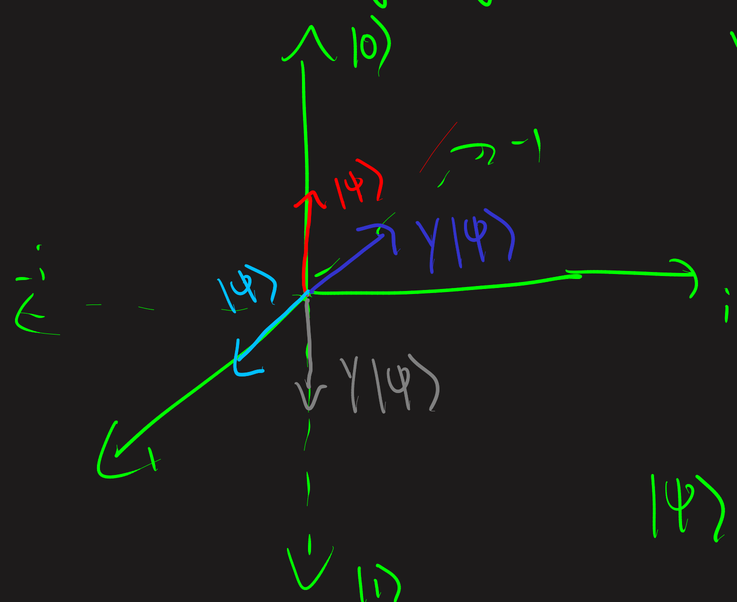
$$|\varphi_1\rangle = e^{i\phi}|\varphi_1\rangle$$

$$\langle\varphi_1| = e^{-i\phi}(|\varphi_1\rangle)^\dagger$$

$$\underbrace{e^{i\phi}e^{-i\phi}}_{1}\langle\varphi_1|\varphi_1\rangle$$

$|\psi = |0\rangle$

Qubit states when we apply gates



$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= i|1\rangle$$

local phase $= i$

$$\underbrace{|\psi\rangle = e^{i\phi}}_{\text{global phase}}|\psi\rangle \quad , \quad |\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)e^{i\phi}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \frac{i}{\sqrt{2}}\left(\overset{|1\rangle}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} - \overset{|0\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\right)$$

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

How does H act on $|0\rangle$ and $|1\rangle$?

— what are the Bloch sphere pictures?

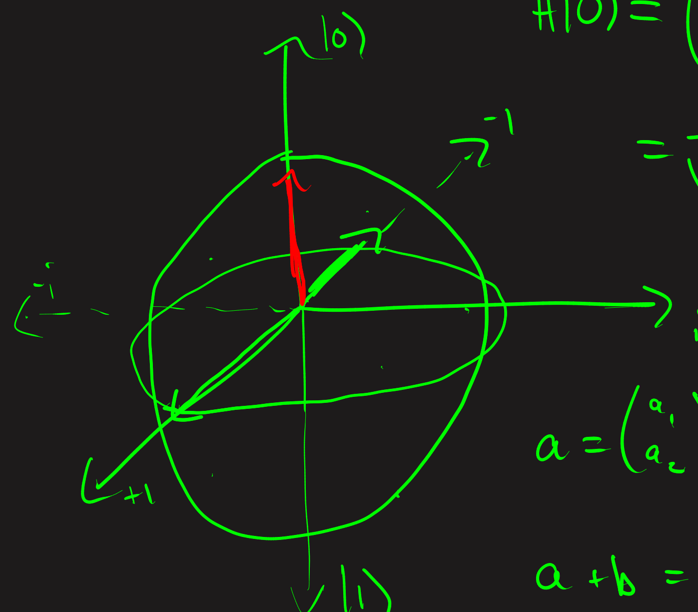
$H|0\rangle, \quad H|1\rangle$

$$H|0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \Bigg(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle} \Bigg)$$

$$H|1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

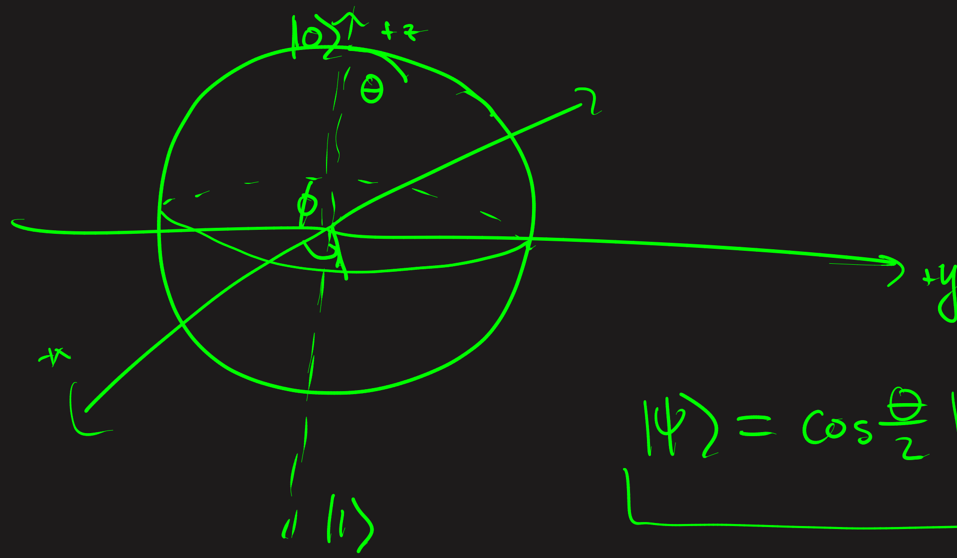
$$= \frac{1}{\sqrt{2}} \Bigg(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} - \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle} \Bigg)$$



$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$$\underbrace{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle}_{\text{most general single qubit state}}$$

$$\theta = 0 \Rightarrow |\psi\rangle = |0\rangle$$

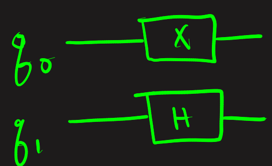
$$\theta = \pi \Rightarrow |\psi\rangle = |1\rangle$$

or equiv. $|\psi\rangle = e^{i\lambda}\left(|\psi\rangle\right)$

Most general single qubit gate:

$$U(\underbrace{\theta,\ \phi}_{\substack{\text{coords}\\ \text{on sphere}}},\ \underbrace{\lambda}_{\substack{\text{global}\\ \text{phase}}}) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$$

$$U(\pi, -\tfrac{\pi}{2}, \tfrac{\pi}{2}) = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(\pi, 0, 0) = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For N qubits, need to have a state space that is 2^N-dimensional.

- gates have to $2^N \times 2^N$ unitary matrices.

$N=2$, 4-dimensions $\begin{array}{c} \\ 4\end{array}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |\psi\rangle$, $\quad \tilde{J} = \begin{array}{cc} & \xrightarrow{4} \\ 4\downarrow & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$

Two qubits: $|\psi_1\rangle = a|0\rangle + b|1\rangle$, $|\psi_2\rangle = c|0\rangle + d|1\rangle$

$|\psi_{tot}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, Basis states for 2-qubit are represented by

$$\underset{\substack{\uparrow \\ \text{1st qubit}}}{|0}\underset{\substack{\uparrow \\ \text{2nd qubit}}}{0\rangle},\ |01\rangle,\ |10\rangle,\ |11\rangle$$

$$|\psi_{tot}\rangle = C_{00}\underline{|00\rangle} + C_{01}\underline{|01\rangle} + C_{10}\underline{|10\rangle} + C_{11}\underline{|11\rangle},\quad |C_{00}|^2 + |C_{01}|^2 + |C_{10}|^2 + |C_{11}|^2 = 1$$

$$V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix},\quad W = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix},\quad \underset{2\text{-d}}{V} \overset{\text{tensor product}}{\otimes} \underset{2\text{-d}}{W} = \begin{pmatrix} V_0 \cdot W \\ V_1 \cdot W \end{pmatrix} = \underbrace{\begin{pmatrix} V_0 W_0 \\ V_0 W_1 \\ V_1 W_0 \\ V_1 W_1 \end{pmatrix}}_{\text{4-dimensional}}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$2^N \times 2^N\text{-dim gates} \overset{N=4}{\Rightarrow} 4 \times 4\text{-dim gates (unitary)}$$

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}, \quad N = \begin{pmatrix} n_{00} & n_{01} \\ n_{10} & n_{11} \end{pmatrix}$$

$$M \otimes N = \begin{pmatrix} m_{00} N & m_{01} N \\ m_{10} N & m_{11} N \end{pmatrix} = \begin{pmatrix} m_{00}n_{00} & m_{00}n_{01} & m_{01}n_{00} & m_{01}n_{01} \\ m_{00}n_{10} & m_{00}n_{11} & m_{01}n_{10} & m_{01}n_{11} \\ m_{10}n_{00} & m_{10}n_{01} & m_{11}n_{00} & m_{11}n_{01} \\ m_{10}n_{10} & m_{10}n_{11} & m_{11}n_{10} & m_{11}n_{11} \end{pmatrix}$$

Not all two-qubit gates can be written as tensor-product.