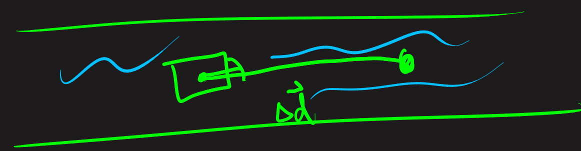
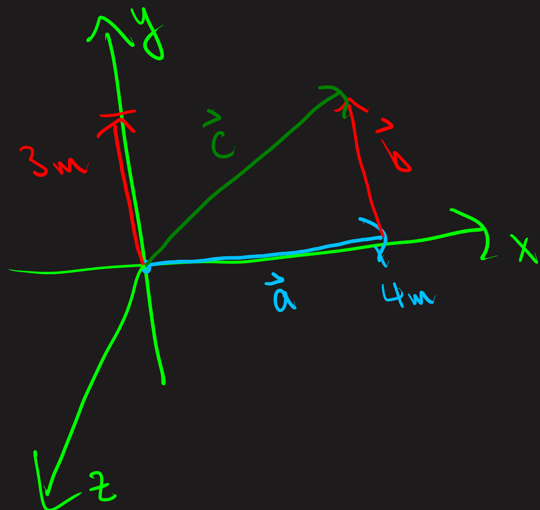
Linear Algebra



Vector — magnitude, direction

Examples of Vectors:
- displacement
- velocity
- acceleration



To add two vectors, place them head to tail. then connect via the hypotenuse.

$$|\vec{c}| = \sqrt{(|\vec{a}|)^2 + (|\vec{b}|)^2}$$

Denote magnitude of vector $\vec{V}$ by $V = |\vec{V}|$

$$\vec{V} = 1\hat{x} + 2\hat{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$V = |\vec{V}| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$

$$\vec{V} = V_x\hat{x} + V_y\hat{y} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} \Rightarrow \boxed{|\vec{V}| = \sqrt{V_x^2 + V_y^2}}$$

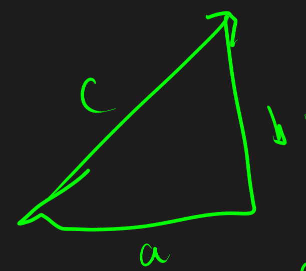
$$\vec{V}_1 = 2\hat{x} + 3\hat{y} \quad , \quad \vec{V}_2 = 4\hat{x} - 2\hat{y}$$

$$\vec{V}_1 + \vec{V}_2 = (2+4)\hat{x} + (3-2)\hat{y} = 6\hat{x} + 1\hat{y}$$

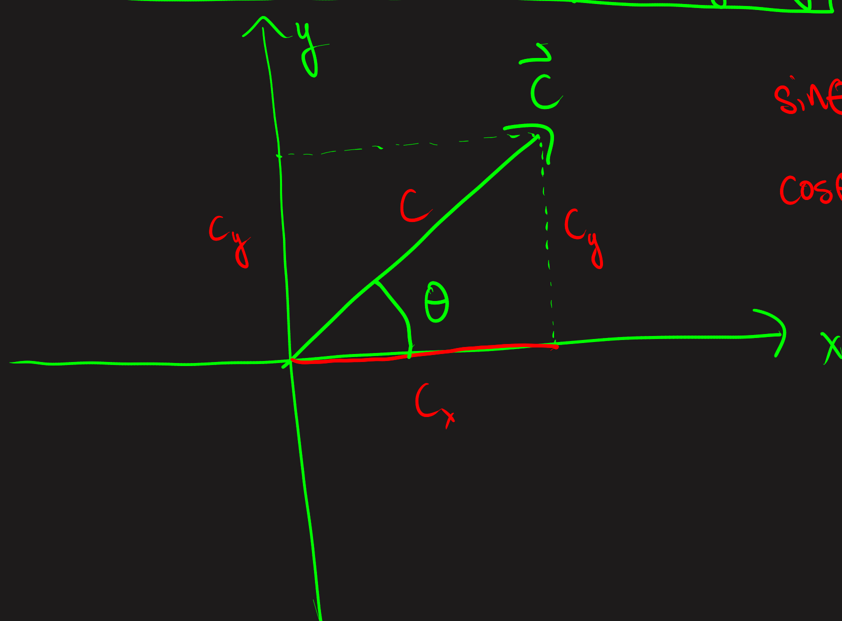
$$\vec{V}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} , \quad \vec{V}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \Rightarrow \vec{V}_1 + \vec{V}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$\vec{V}_1 = V_{1,x}\hat{x} + V_{1,y}\hat{y} \quad , \quad \vec{V}_2 = V_{2,x}\hat{x} + V_{2,y}\hat{y}$$

$$\boxed{\vec{V}_1 + \vec{V}_2 = (V_{1,x} + V_{2,x})\hat{x} + (V_{1,y} + V_{2,y})\hat{y}}$$



$$a^2 + b^2 = c^2$$



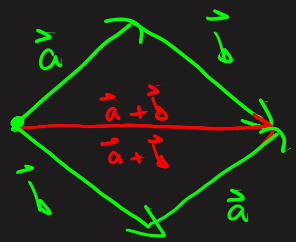
$$\sin\theta = \frac{c_y}{c} \Rightarrow c_y = c \cdot \sin\theta$$

$$\cos\theta = \frac{c_x}{c} \Rightarrow c_x = c \cdot \cos\theta$$

Important properties of vector addition:

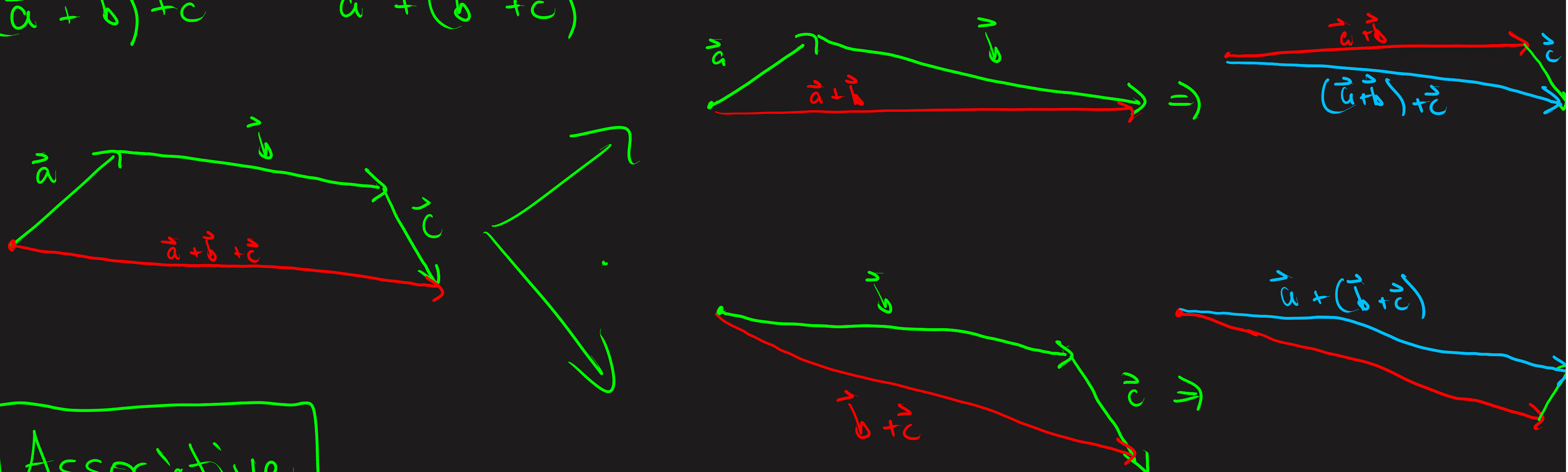
$\vec{a}, \vec{b}$ $\qquad$ $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ $\qquad$ $\boxed{\text{Commutative}}$

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$



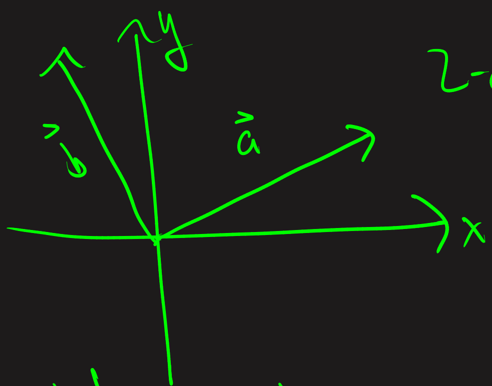
$$\Rightarrow \vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}, \quad \vec{b} + \vec{a} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}$$

$(\vec{a} + \vec{b}) + \vec{c}$ $\qquad$ $\vec{a} + (\vec{b} + \vec{c})$



$\boxed{\text{Associative}}$

$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ Column vector , $\vec{a} = (a_x \ \ a_y)$ Row vector



2-dimensional

3-dimensional

$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$, $\vec{a} = (a_x \ \ a_y \ \ a_z)$

$\vec{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{pmatrix}$ N-dimensional column vector , $\vec{V} = (V_1 \ \ V_2 \ \ V_3 \ \ldots \ V_N)$ N-dimensional row vector

$|a\rangle$ or $\langle a|$

Dirac Bra-Ket notation

$|a\rangle$ is same as column vector

Hermitian conjugate

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{pmatrix}, \quad \langle a| = \left(|a\rangle^*\right)^T = \left(|a\rangle\right)^+$$

$i = \sqrt{-1} = j$

$z_1 = 1 + i$, $z_2 = 50 - 30i$, $z_3 = -\frac{1}{2} i$

Complex conjugate:

$$z_1^* = (1+i)^* = 1 - i$$

Transpose:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (\vec{a})^T = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$|a\rangle = \begin{pmatrix} 1+2i \\ 3i \end{pmatrix}, \quad (|a\rangle)^T = \begin{pmatrix} 1+2i & 3i \end{pmatrix}$$

$$\neq$$

$$, \quad (|a\rangle^*)^T = \begin{pmatrix} 1-2i & -3i \end{pmatrix}$$

$$|b\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{Prove that}$$

$$(|a\rangle)^T = (|a\rangle^*)^T$$

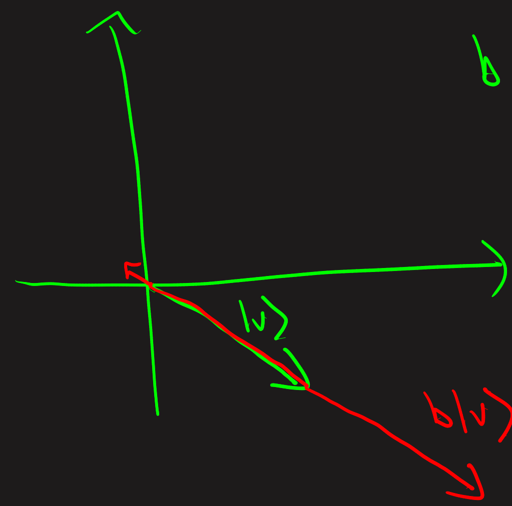
Vector Multiplication

$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $|w\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, b is a complex number

Scalar Multiplication:

$$b\,|v\rangle = \begin{pmatrix} b \cdot v_1 \\ b \cdot v_2 \end{pmatrix}$$

number · vector = another vector



Inner product ("Dot" Product):

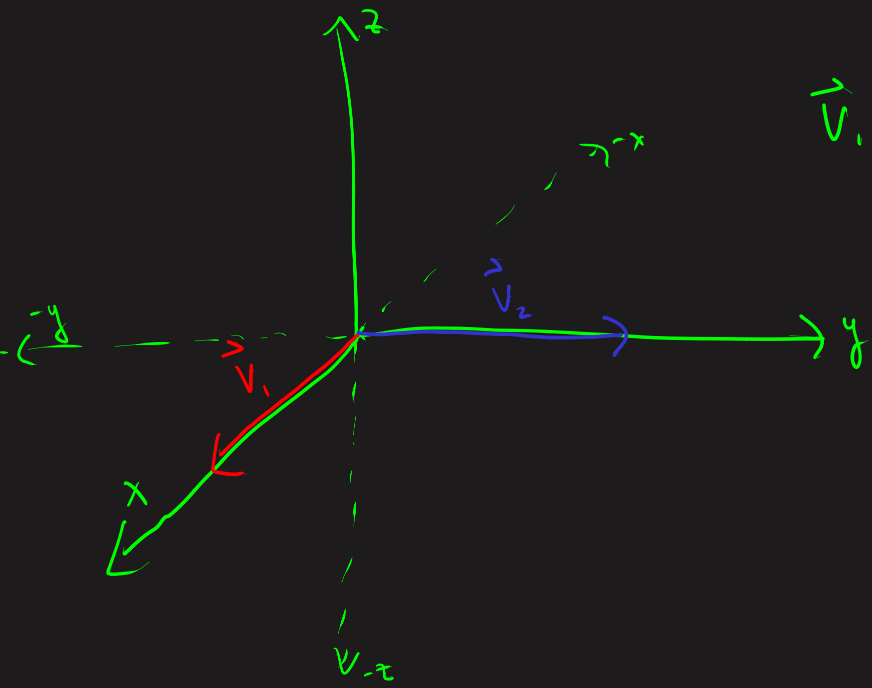
$\vec{V}_1 = \begin{pmatrix} V_{1,x} \\ V_{1,y} \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} V_{2,x} \\ V_{2,y} \end{pmatrix}$

$\vec{V}_1 \cdot \vec{V}_2 =$ number , $\vec{V}_1 \cdot \vec{V}_2 = (V_{1,x} \cdot V_{2,x}) + (V_{1,y} \cdot V_{2,y}) = 2$

Vector Product ("Cross" Product)

$\vec{V}_1 \times \vec{V}_2 =$ another vector

$$\vec{V}_1 = \begin{pmatrix} V_{1,x} \\ V_{1,y} \\ V_{1,z} \end{pmatrix}, \quad \vec{V}_2 = \begin{pmatrix} V_{2,x} \\ V_{2,y} \\ V_{2,z} \end{pmatrix}$$

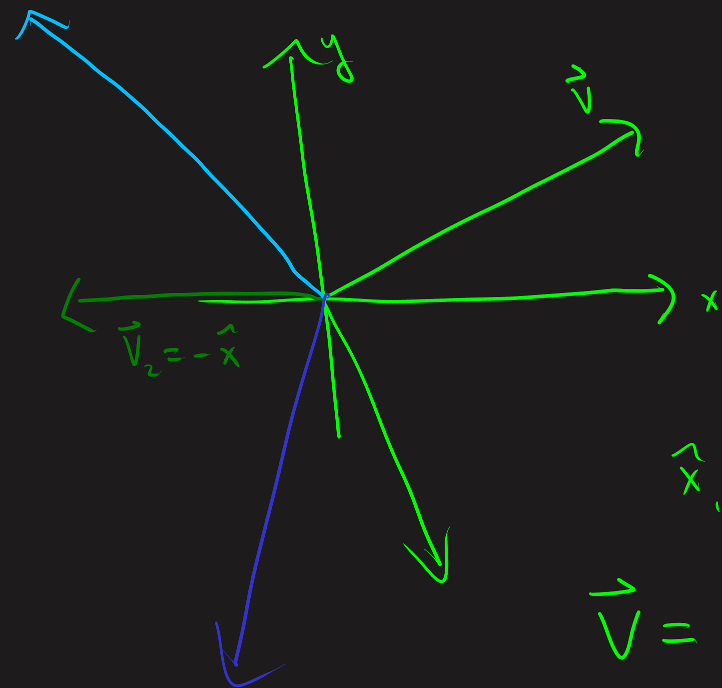


$$\vec{V}_1 \times \vec{V}_2 = \overbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_{1,x} & V_{1,y} & V_{1,z} \\ V_{2,x} & V_{2,y} & V_{2,z} \end{vmatrix}}^{\text{determinant}}$$

$$= \left(V_{1,y}V_{2,z} - V_{1,z}V_{2,y}\right)\hat{x} - \left(V_{1,x}V_{2,z} - V_{1,z}V_{2,x}\right)\hat{y}$$

$$+ \left(V_{1,x}V_{2,y} - V_{1,y}V_{2,x}\right)\hat{z}$$

Basis set



$$\vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} = -3\hat{x} + 2\hat{y}$$

$$\boxed{\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$\hat{x}, \hat{y}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis for 2-dimensions

$$\vec{V} = V_{1,x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + V_{1,y} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

What about 3-dimensions?

$$\vec{V} = V_{1,x} \overset{\hat{x}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + V_{1,y} \overset{\hat{y}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + V_{1,z} \overset{\hat{z}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\hat{x} \cdot \hat{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0, \quad \hat{x} \cdot \hat{z} = 0, \quad \hat{y} \cdot \hat{z} = 0$$

Orthonormal Basis

orthogonal + unit magnitude

perpendicular

$$|\hat{x}| = |\hat{y}| = |\hat{z}| = 1, \quad \hat{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |\hat{x}| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

$$\underbrace{\hat{x}' = \begin{pmatrix}2\\0\\0\end{pmatrix}, \; \hat{y}' = \begin{pmatrix}0\\3\\0\end{pmatrix}, \; \hat{z}' = \begin{pmatrix}0\\0\\4\end{pmatrix}}$$

still a basis set but not unit magnitude

$$|\hat{x}'| = \sqrt{(2)^2 + (0)^2 + (0)^2} = 2 \neq 1$$

Normalize the vectors: Find magnitude of each vector and divide that vector by magnitude

$$\frac{\hat{x}'}{|\hat{x}'|} = \frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \frac{\hat{y}'}{|\hat{y}'|} = \frac{1}{3}\begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \frac{\hat{z}'}{|\hat{z}'|} = \frac{1}{4}\begin{pmatrix}0\\0\\4\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Matrices

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad , \quad (\vec{v})^T = (v_1 \quad v_2)$$

$$m = \begin{array}{cc} & \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \end{array}$$

Matrix $\overset{\text{rows}}{\overset{\downarrow}{2}} \times \overset{\text{columns}}{\overset{\downarrow}{2}}$ dimensions

$M \times N$

$$m = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \cdots & N \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ M \end{array} & \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & \ddots & & \\ \vdots & & \ddots & \\ m_{M1} & & & \end{pmatrix} \end{array}$$

Scalar multiplication:

$$\underset{\text{scalar}}{\underset{\uparrow}{a}}\,\underset{\text{matrix}}{\underset{\uparrow}{m}} = \begin{pmatrix} a \cdot m_{11} & a \cdot m_{12} \\ a \cdot m_{21} & a \cdot m_{22} \end{pmatrix}$$

Matrix Transpose:

$$(M \times N) \to (N \times M)$$

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad m^T = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

2×2 $\qquad$ 2×2

$$n = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{pmatrix}, \quad n^T = \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \\ n_{13} & n_{23} \end{pmatrix}$$

2×3 matrix $\qquad$ 3×2

Matrix complex conjugate:

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad m^* = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix}$$

Matrix Hermitian conjugate:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M^{+} = (M^{*})^{T} = \begin{pmatrix} M_{11}^{*} & M_{21}^{*} \\ M_{12}^{*} & M_{22}^{*} \end{pmatrix}$$

Multiply vector by matrix:

$$\vec{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$M\vec{V} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} M_{11}V_x + M_{12}V_y \\ M_{21}V_x + M_{22}V_y \end{pmatrix}$$

$$2 \times \underline{2} \qquad \underline{2} \times 1 \qquad 2 \times 1$$

$$\vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Does not work

$$M\vec{V} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$2 \times 2 \qquad 3 \times 1$$

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad n = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$$

$$mn = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} m_{11}n_{11} + m_{12}n_{21} & m_{11}n_{12} + m_{12}n_{22} \\ m_{21}n_{11} + m_{22}n_{21} & m_{21}n_{12} + m_{22}n_{22} \end{pmatrix}$$

$$m = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad n = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$mn = \begin{pmatrix} 2+2 & 3+8 \\ 6+4 & 9+16 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 10 & 25 \end{pmatrix}.$$