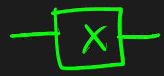
Topics

- Review matrices
- Hermitian / unitary
- eigenvalues / eigenvectors
- single qubit gates / states
- python and Qiskit



Review matrices:

$M \times N$ dimensional object

$$\begin{array}{c|c} & \overset{\longleftarrow N \longrightarrow}{} \\ \Big\uparrow M \Big\downarrow & \begin{pmatrix} m_{00} & m_{01} & m_{02} & \cdots \\ m_{10} & & & \\ m_{20} & & & \\ \vdots & & & \end{pmatrix} \end{array}$$

Vector is just

$$\overset{\text{column vector}}{\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix}}, \quad \overset{\text{row vector}}{\begin{pmatrix} v_0 & v_1 & v_2 & \cdots \end{pmatrix}}, \quad \vec{v} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

Specialized notation for vectors: Dirac Bra-ket Notation

$$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \langle\varphi_2| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad \overset{\text{ket}}{|\psi\rangle} = \text{column vector}, \quad \overset{\text{bra}}{\langle\psi|} = \text{row vector}$$

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \overset{1}{\cancel{1 \cdot 1}} + \overset{2}{\cancel{1 \cdot 2}} + \overset{3}{\cancel{1 \cdot 3}} = 6$$

$$\langle\varphi_2|\varphi_1\rangle = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6$$

$$\left(|\psi\rangle\right)^{*T} = \left(|\psi\rangle\right)^{\dagger} \leftarrow \text{Hermitian conjugate}$$

$$4^* = 4, \quad (4+i)^* = 4-i, \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad |\psi_1\rangle^* = \begin{pmatrix} 1^* \\ 2^* \\ 3^* \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = |\psi_1\rangle$$

$$|\psi_3\rangle = \begin{pmatrix} 1+i \\ 2-3i \\ 3+4i \end{pmatrix}, \quad |\psi_3\rangle^* = \begin{pmatrix} 1-i \\ 2+3i \\ 3-4i \end{pmatrix}$$

$$|\psi_3\rangle^{\dagger} = \begin{pmatrix} 1-i \\ 2+3i \\ 3-4i \end{pmatrix}^T = \begin{pmatrix} 1-i & 2+3i & 3-4i \end{pmatrix}$$

$$\left(|\psi_3\rangle^{\dagger}\right)^{\dagger} = |\psi_3\rangle$$

Scalar x matrix:

$$2, \quad M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow 2 \cdot M = \begin{pmatrix} 2\cdot 1 & 2\cdot 2 \\ 2\cdot 3 & 2\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

matrix x matrix:

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad MN = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 2\cdot 1 & 1\cdot 1 + 2\cdot 2 \\ 3\cdot 2 + 4\cdot 1 & 3\cdot 1 + 4\cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 10 & 11 \end{pmatrix}$$

$$NM = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 7 & 10 \end{pmatrix}$$

$$MN \neq NM \,, \quad \underbrace{[M, N]}_{\text{Commutator}} = MN - NM = 0$$

Vector × matrix:

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$vM = \cancel{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$Mv = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$Nw = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 2\cdot 1 \\ 1\cdot 2 + 3\cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Hermitian / Unitary Matrices

Both square matrices $(M \times M)$

$$\begin{array}{cc} & \longleftarrow M \longrightarrow \\ \begin{array}{c}\uparrow\\ M\\ \downarrow\end{array} & \begin{pmatrix} \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{pmatrix} \end{array}$$

Hermitian: $A^\dagger = A$, Ex: Pauli Matrices $(\sigma_x, \sigma_y, \sigma_z, \mathbb{1})$

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$\sigma_y = Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Y^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = Y$$

$$\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{1}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Unitary: $A^\dagger A = \mathbb{1} \Rightarrow A^\dagger = A^{-1}, \quad A^{-1} A = \mathbb{1}$

$$\mathbb{1} M = M, \quad 1 \times 4 = 4$$

$$X \cdot X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$Y \cdot Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$i = \sqrt{-1}, \quad i \cdot i = \sqrt{-1}^2 = -1$$

$$Z \cdot Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Turn a Hermitian matrix into a unitary matrix:

$$H = H^\dagger$$

$$U = e^{iH}, \quad U^\dagger = \left(e^{iH}\right)^\dagger = e^{-iH^\dagger} = e^{-iH}$$

$$UU^\dagger = e^{iH} e^{-iH} = e^{iH-iH} = e^0 = \mathbb{1}$$

Hermitian matrices describe measurements on a system

Spectral Theorem: Says Hermitian matrices have real <u>eigenvalues</u>.

Unitary matrices describe evolution of a system.

Spectral Theorem: Says unitary matrices have eigenvalues with unit absolute value

$|a| = 1$.

proper, character

Eigenvectors / Eigenvalues:

column vector

Let A is $N \times N$ matrix, $|\psi\rangle$ is an eigenvector of A with eigenvalue λ

if $|\psi\rangle \neq 0$ and $\underline{A|\psi\rangle = \lambda|\psi\rangle}$.

To solve the eigensystem write

$$\boxed{(A - \mathbb{1}\lambda)|\psi\rangle = 0}$$

$$\cancel{|\psi\rangle = 0} \quad \text{or} \quad (A - \mathbb{1}\lambda) = 0$$

if $\cancel{(A - \mathbb{1}\lambda)^{-1}} \, \cancel{(A - \mathbb{1}\lambda)} |\psi\rangle = \cancel{(A - \mathbb{1}\lambda)^{-1}} \cdot 0 \rightarrow 0$

$\mathbb{1}$

$\Rightarrow \quad |\psi\rangle = 0$

Only exist solutions with non zero $|\varphi\rangle$ if $(A - \mathbb{1}\lambda)$ cannot be inverted.

If $\det(\text{matrix}) = 0$ then cannot be inverted.

$$\det(A - \lambda \mathbb{1}) = 0 \quad \text{Characteristic Equation}$$

Example for 2x2:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \det\left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = 0$$

$$\det\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = AD - BC$$

$$\Rightarrow \det\left(\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} \right) = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \boxed{\lambda = 2, -1}$$

$$\boxed{\lambda = 2}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{(A - \mathbb{1}\cdot\lambda)|\psi\rangle} = 0$$

$$\left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \begin{pmatrix} 1-\lambda & 2-0 \\ 1-0 & 0-\lambda \end{pmatrix} \overset{\lambda=2}{=} \boxed{\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}}$$

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -\psi_1 + 2\psi_2 \\ \psi_1 - 2\psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{-\psi_1 + 2\psi_2 = 0,} \text{ and } \psi_1 - 2\psi_2 = 0$$

$$\Rightarrow 2\psi_2 = \psi_1$$

$$\boxed{|\psi\rangle = \text{const.} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$A|\psi\rangle = \lambda|\psi\rangle \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 \\ 2\cdot 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \checkmark$$

$\underline{\lambda = -1}$

$$(A - \lambda \mathbb{1})|\psi\rangle = 0$$

$$\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2\psi_1 + 2\psi_2 \\ \psi_1 + \psi_2 \end{pmatrix} = 0$$

$\Rightarrow \quad 2\psi_1 + 2\psi_2 = 0$ and $\psi_1 + \psi_2 = 0$

$$\Rightarrow \psi_1 = -\psi_2 \quad \Rightarrow \boxed{|\psi\rangle = \text{const.} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ with eigenvalues $\lambda = 2, -1$ and eigenvectors $|\psi\rangle = \text{const.} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$|0\rangle \, , \; |1\rangle$$

$$X|0\rangle = |1\rangle \, , \quad X|1\rangle = |0\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \, , \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \quad X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Hadamard

$$-\boxed{H}- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \, ,$$

Phase

$$-\boxed{S}- = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$\pi/8$

$$, \quad -\boxed{T}- = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{1}_{\text{eigenvalue}}\underbrace{|0\rangle}_{\text{eigenvector}}$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1|1\rangle$$

$$X|\psi\rangle = \lambda|\psi\rangle \quad , \quad |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$X|\psi\rangle = 1|\psi\rangle$$