1. Recursion :: Definition and Example

"In order to understand recursion, one must first understand recursion."

re•cur•sion

/ri' ker zhun/

noun

1. See *recursion*.

In mathematics and computing, we say a definition (or a function) is **recursive** if it is defined in terms of itself. As our first example of recursion, let's consider the problem of calculating the factorial of n. In mathematics, the factorial of an integer $n \geq 0$ is the product of the integers 1, 2, 3, ..., n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n\text{-}1) \times n \quad \text{for } n > 0 \qquad \text{Eqn. 1}$$
$$n! = 1 \quad \text{for } n = 0 \qquad \text{Eqn. 2}$$

For this section discussion we will ignore that $0! = 1$ and will only work with $n > 0$. Since multiplication is commutative, we rewrite Eqn. 1 as:

$$n! = n \times (n\text{-}1) \times \cdots 3 \times 2 \times 1,\ n > 0$$

For example,

$$5! = 5 \times \underline{4 \times 3 \times 2 \times 1} = 120$$

1. Recursion :: Example :: The Factorial Function

Now, in the calculation of 5! what is $4 \times 3 \times 2 \times 1$? I hope you see that it is 4! which means that we can **rephrase** the original problem as:

$$P(5) = 5! = 5 \times 4!$$

where I am using the notation P(5) to refer to the **P**roblem to be solved for $n = \mathbf{5}$. So, all we need to do to solve P(5) is to determine 4! and then multiply 5 by 4!. Easy enough, what is 4!?

$$P(4) = 4! = 4 \times \underline{3 \times 2 \times 1}$$

But what is $3 \times 2 \times 1$? I hope you see that it is 3! which means that we can rephrase P(4) as:

$$P(4) = 4! = 4 \times 3!$$

So, all we need to do to solve P(4) is to determine 3! and then multiply 4 by 3!. What is 3!?

$$P(3) = 3! = 3 \times \underline{2 \times 1}$$

But what is 2×1? I hope you see that it is 2! which means that we can rephrase P(3) above as:

$$P(3) = 3! = 3 \times 2!$$

So, all we need to do to solve P(3) is to determine 2! and then multiply 3 by 2!. What is 2!?

$$P(2) = 2! = 2 \times \underline{1}$$

1. Recursion :: Example :: The Factorial Function (continued)

But what is 1? Hint: you and I both know it is 1, but it is also 1! which means that we can rephrase P(2) above as:

$P(2) = 2! = 2 \times 1!$

So all we need to do to solve P(2) is to determine 1! and then multiply 2 by 1!. What is 1!?

$P(1) = 1! = 1$ (by the definition that the factorial of 1 is the product of the integers from 1 to 1).

At this point **we can stop** creating new problems because in our definition of factorial we specified that n must be greater than 0.

Are we done? Not yet. Note that all we have achieved up to this point is the solution to P(1):

$P(1) = 1! = \underline{1}$

And remember that we were interested in the solution to P(1) because we wanted to solve P(2) which we defined as:

$P(2) = 2 \times 1!$

Therefore:

$P(2) = 2 \times \underline{P(1)} = 2 \times \underline{1} = \underline{2}$

1. Recursion :: Example :: The Factorial Function (continued)

And remember that we were interested in the solution to P(2) because we wanted to solve P(3) which we defined as:

$P(3) = 3 \times 2!$

Therefore:

$P(3) = 3 \times \underline{P(2)} = 3 \times \underline{2} = \underline{6}$

And remember that we were interested in the solution to P(3) because we wanted to solve P(4) which we defined as:

$P(4) = 4 \times 3!$

Therefore:

$P(4) = 4 \times \underline{P(3)} = 4 \times \underline{6} = \underline{24}$

And remember that we were interested in the solution to P(4) because we wanted to solve P(5) which we defined as:

$P(5) = 5 \times 4!$

Therefore:

$P(5) = 5 \times \underline{P(4)} = 5 \times \underline{24} = \underline{120}$

Which tells us that $5! = 120$! (no, not 120 factorial; I am just overly excited that I finally found the answer).

1. Recursion :: Example :: The Factorial Function (continued)

Now, let's rewrite what we just did but more succinctly:

$P(5) = 5! = 5 \times 4! = 5 \times P(4)$

$P(4) = 4! = 4 \times 3! = 4 \times P(3)$

$P(3) = 3! = 3 \times 2! = 3 \times P(2)$

$P(2) = 2! = 2 \times 1! = 2 \times P(1)$

$P(1) = 1! = 1$

Therefore: $P(2) = 2 \times P(1) = 2 \times 1 = 2$

Therefore: $P(3) = 3 \times P(2) = 3 \times 2 = 6$

Therefore: $P(4) = 4 \times P(3) = 4 \times 6 = 24$

Therefore: $P(5) = 5 \times P(4) = 5 \times 24 = 120$

Therefore we have shown (its not a mathematical proof, but close enough) that:

$$fact(n) = n \times fact(n - 1)$$

which is a **recursive definition** (the definition of *fact*() depends on the definition of *fact*()) and the process we used to find P(5)—or *fact*(5)—is called **recursion**.