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Concerning the effect of compressibility on laminar boundary layers and their separation

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The theory of compressible flow in a laminar boundary layer has been developed for the case when the viscosity is assumed to be proportional to the absolute temperature and the Prandtl number is unity. (These assumptions may be compared with the empirical relations $\mu \propto T^{\frac{8}{9}}$ and $\sigma = 0{\cdot}715$ suggested by Cope.) It is shown that a transformation of the ordinate normal to the layer can lead to a simplified form of equation of motion very similar to the ordinary incompressible equation but modified by a multiplicative factor G in the pressure term. This factor is greater than unity at the boundary and tends to one at the outside of the layer.

Several particular solutions are considered including accelerated flow with a linearly increasing velocity and retarded flow along a flat plate with a linearly decreasing velocity.

The general implications of the theory are discussed and qualitative conclusions are drawn when the mainstream velocity starts from a stagnation point, rises to a maximum and subsequently falls. It is concluded that for such a velocity distribution increasing compressibility will reduce the skin friction, increase the boundary layer thickness and cause earlier separation as compared with the incompressible flow with the same mainstream velocity distribution and the kinematic viscosity corresponding to conditions at the stagnation point.

1. Introduction

The recent work of Cope & Hartree (1948) has made it abundantly clear that a complete study of compressible flow in boundary layers when allowance is made for the empirical temperature variation of viscosity and conductivity is a matter for modern electronic calculating machines. These authors have, in fact, initiated a study of the flow along a flat plate in the presence of a linear retarding pressure gradient by this means.

The empirical relations chosen by Cope (unpublished) for air in the temperature range 90° K to 300° K (the range important in wind tunnel experiments) are

$$\mu \propto T^{\frac{8}{9}} \quad \text{and} \quad \sigma = 0{\cdot}715, \tag{1}$$

where μ is the viscosity, T is the absolute temperature and σ is the Prandtl number defined as $\mu c_p/k$ where k is the thermal conductivity and c_p is the specific heat at constant pressure.

The difficulty in any numerical approach to a problem of this complexity lies in the number of particular solutions required to give a full understanding of the effects of compressibility on boundary layers. It was therefore thought to be worthwhile to develop the theory when the empirical relations are replaced by

$$\mu \propto T, \tag{2}$$

and

$$\sigma = 1, \tag{3}$$

for it appears that in this case considerable simplifications in the theory can be achieved. It is thought that these assumptions are sufficiently close to Cope's empirical relations to make the results at least of qualitative interest.

The assumption $\sigma = 1$ has been made by many previous writers and leads, even in the case of variable viscosity, to stagnation temperature at a heat insulating boundary (Crocco 1946). The assumption $\mu \propto T$ has effectively been made by Crocco for the flow along a flat plate in the absence of a pressure gradient when he assumed $\rho\mu$ constant. The effects of these two assumptions taken together appears, in view of the stagnation temperature at the boundary, to lead to an overestimate of viscous effects in the boundary layer.

Accepting these two basic assumptions the theory will be developed exactly and with particular reference to the effects of pressure gradients on boundary layers along heat insulating boundaries.

2. The equations of motion and continuity

The equations of motion and continuity are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{4}$$

$$0 = \frac{\partial p}{\partial y}, \tag{5}$$

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0. \tag{6}$$

Equation (6) implies that a stream function ψ exists such that

$$\rho u = \rho_s\frac{\partial \psi}{\partial y}, \quad \rho v = -\rho_s\frac{\partial \psi}{\partial x}, \tag{7}$$

where the suffix s is used to denote some standard condition of the gas, at some specified point of the boundary layer or main flow outside. The viscosity will also be defined in terms of the temperature T_s at this point by the relation

$$\mu = \mu_s\frac{T}{T_s}. \tag{8}$$

Let us now change from independent variables x, y to x, Y where

$$Y = \int_0^y \left(\frac{\nu_s}{\nu}\right)^{\frac{1}{2}} dy = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} \int_0^y \frac{T_s}{T} dy \tag{9}$$

and p, T denote the pressure and temperature at any point inside the boundary layer. The pressure p being independent of y by virtue of equation (5) has been taken outside the integral sign. Furthermore let us modify ψ by writing

$$\psi(x,y) = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} \chi(x, Y). \tag{10}$$

Then
$$u = \frac{\rho_s}{\rho}\left(\frac{\partial \psi}{\partial y}\right)_x = \frac{\rho_s}{\rho}\left(\frac{p}{p_s}\right)^{\frac{1}{2}}\left(\frac{\partial \chi}{\partial Y}\right)_x\left(\frac{\partial Y}{\partial y}\right)_x, \tag{11}$$

where subscripts x, y, Y are used to show which quantities remain constant in a partial differentiation. Hence

$$u = \frac{\rho_s}{\rho}\left(\frac{p}{p_s}\right)^{\frac{1}{2}} \frac{\partial \chi}{\partial Y}\left(\frac{p}{p_s}\right)^{\frac{1}{2}} \frac{T_s}{T} = \frac{p\rho_s T_s}{p_s \rho T}\frac{\partial \chi}{\partial Y} = \frac{\partial \chi}{\partial Y} \tag{12}$$

assuming the gas to be perfect. Next

$$v = -\frac{\rho_s}{\rho}\left(\frac{\partial \psi}{\partial x}\right)_y = -\frac{1}{2}\frac{\rho_s}{p_s \rho}\left(\frac{p_s}{p}\right)^{\frac{1}{2}}\frac{\partial p}{\partial x}\chi - \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{\rho_s}{\rho}\left[\left(\frac{\partial \chi}{\partial x}\right)_Y + \left(\frac{\partial \chi}{\partial Y}\right)_x\left(\frac{\partial Y}{\partial x}\right)_y\right], \tag{13}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial u}{\partial x}\right)_Y + \left(\frac{\partial u}{\partial Y}\right)_x\left(\frac{\partial Y}{\partial x}\right)_y = \frac{\partial^2 \chi}{\partial x \partial Y} + \frac{\partial^2 \chi}{\partial Y^2}\left(\frac{\partial Y}{\partial x}\right)_y, \tag{14}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \left(\frac{\partial u}{\partial Y}\right)_x\left(\frac{\partial Y}{\partial y}\right)_x = \frac{\partial^2 \chi}{\partial Y^2}\left(\frac{\partial Y}{\partial y}\right)_x = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{T_s}{T}\frac{\partial^2 \chi}{\partial Y^2}. \tag{15}$$

Hence
$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{\partial \chi}{\partial Y}\frac{\partial^2 \chi}{\partial x \partial Y} + \frac{\partial \chi}{\partial Y}\frac{\partial^2 \chi}{\partial Y^2}\left(\frac{\partial Y}{\partial x}\right)_y - \frac{1}{2p_s}\frac{\rho_s T_s}{\rho T}\chi\frac{\partial^2 \chi}{\partial Y^2}\frac{\partial p}{\partial x} \\ &\quad - \frac{p\rho_s T_s}{p_s \rho T}\left[\frac{\partial \chi}{\partial x}\frac{\partial^2 \chi}{\partial Y^2} + \frac{\partial \chi}{\partial Y}\frac{\partial^2 \chi}{\partial Y^2}\left(\frac{\partial Y}{\partial x}\right)_y\right] \\ &= \frac{\partial^2 \chi}{\partial x \partial Y}\frac{\partial \chi}{\partial Y} - \frac{\partial^2 \chi}{\partial Y^2}\frac{\partial \chi}{\partial x} - \frac{1}{2p}\frac{\partial p}{\partial x}\chi\frac{\partial^2 \chi}{\partial Y^2} \end{aligned} \tag{16}$$

again making use of the perfect gas law.

Now from equation (15)

$$\mu\left(\frac{\partial u}{\partial y}\right)_x = \frac{\mu_s T}{T_s}\left(\frac{\partial u}{\partial y}\right)_x = \mu_s\left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{\partial^2 \chi}{\partial Y^2}, \tag{17}$$

so that
$$\begin{aligned} \frac{1}{\rho}\left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]_x &= \frac{T_s}{\rho T}\left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{\partial}{\partial Y}\left\{\mu_s\left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{\partial^2 \chi}{\partial Y^2}\right\} \\ &= \frac{\mu_s p T_s}{\rho p_s T}\frac{\partial^3 \chi}{\partial Y^3} = \nu_s\frac{\partial^3 \chi}{\partial Y^3}. \end{aligned} \tag{18}$$

If now we use a suffix 1 on ρ and T to denote general values in the mainstream at the edge of the boundary layer

$$\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -U\frac{dU}{dx}, \tag{19}$$

where U is the mainstream velocity. Hence

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\rho_1}{\rho}U\frac{dU}{dx} = -\frac{T}{T_1}U\frac{dU}{dx},$$

since the pressure is constant across a section of the boundary layer. Finally

$$\frac{1}{p}\frac{\partial p}{\partial x} = -\frac{\rho_1}{p}U\frac{dU}{dx} = -\frac{\gamma}{a_1^2}U\frac{dU}{dx}, \tag{20}$$

where a_1 denotes the local velocity of sound at the edge of the boundary layer. Hence equation (4) takes on the form

$$\frac{\partial^2\chi}{\partial x\partial Y}\frac{\partial\chi}{\partial Y}-\frac{\partial^2\chi}{\partial Y^2}\frac{\partial\chi}{\partial x} = U\frac{dU}{dx}\left[\frac{T}{T_1}-\frac{\gamma}{2a_1^2}\chi\frac{\partial^2\chi}{\partial Y^2}\right]+\nu_s\frac{\partial^3\chi}{\partial Y^3}. \tag{21}$$

The boundary conditions are

$$u = v = 0 \text{ at } y = 0, \quad u \to U \text{ as } y \to \infty,$$

and these imply

$$\chi = \frac{\partial\chi}{\partial Y} = 0 \text{ at } Y = 0, \quad \frac{\partial\chi}{\partial Y} \to U \text{ as } Y \to \infty. \tag{22}$$

Equation (21) together with its boundary conditions may be compared with the equation for the stream function in incompressible flow for the same distribution of mainstream velocity

$$\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial y}-\frac{\partial^2\psi}{\partial y^2}\frac{\partial\psi}{\partial x} = U\frac{dU}{\partial x}+\nu\frac{\partial^3\psi}{\partial y^3}, \tag{23}$$

which has boundary conditions

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \text{ at } y = 0, \quad \frac{\partial\psi}{\partial y} \to U \text{ as } y \to \infty. \tag{24}$$

(The condition $\partial\psi/\partial x = 0$ at $y = 0$ could be replaced by $\psi = 0$ at $y = 0$ without loss.) This flow will be termed the associated incompressible flow.

The effects of compressibility may therefore be thought of as summarized in the term in square brackets

$$G = \frac{T}{T_1}-\frac{\gamma}{2a_1^2}\chi\frac{\partial^2\chi}{\partial Y^2}, \tag{25}$$

which multiplies the effective pressure gradient, and in the altered scale in the direction normal to the boundary.

Except in the case where $dU/dx = 0$ about which some comments will be made later we cannot proceed further without a knowledge of the temperature distribution. This is provided by the energy equation of course.†

† Even when $dU/dx = 0$ a knowledge of the temperature distribution is essential before the solution can be completed and y determined from Y.

3. The energy equation

The energy equation may be written

$$\rho J c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - u \frac{\partial p}{\partial x} = J \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2, \tag{26}$$

where k is the conductivity. In terms of the Prandtl number σ

$$k = \rho c_p \kappa = \frac{\mu c_p}{\sigma}.$$

If now we multiply equation (4) by ρu and add to (26) we find

$$\rho \left[u \frac{\partial}{\partial x} (\tfrac{1}{2} u^2 + J c_p T) + v \frac{\partial}{\partial y} (\tfrac{1}{2} u^2 + J c_p T) \right] = \frac{\partial}{\partial y} \left[\mu \frac{\partial}{\partial y} \left(\tfrac{1}{2} u^2 + \frac{J c_p T}{\sigma} \right) \right] \tag{27}$$

assuming c_p and σ are constants. Hence, whatever be the form of the temperature variation of μ, if $\sigma = 1$

$$J c_p T + \tfrac{1}{2} u^2 = \text{const.}$$

is a particular integral of (26) (see Crocco 1946) and is such that

$$\frac{\partial T}{\partial y} = 0 \quad \text{when} \quad u = 0,$$

i.e. it corresponds to a boundary which is thermally insulating. Hence we find

$$J c_p T + \tfrac{1}{2} u^2 = J c_p T_1 + \tfrac{1}{2} U^2, \tag{28}$$

so that

$$\begin{aligned} \frac{T}{T_1} &= 1 + \frac{1}{2 J c_p T_1} (U^2 - u^2) \\ &= 1 + \frac{R \rho_1}{2 J c_p p} (U^2 - u^2) \\ &= 1 + \frac{(\gamma - 1)}{2 a_1^2} (U^2 - u^2). \end{aligned} \tag{29}$$

Hence the term G which multiplies $U(dU/dx)$ in equation (21) takes the form

$$G = 1 + \frac{(\gamma - 1)}{2 a_1^2} \left[U^2 - \left(\frac{\partial \chi}{\partial Y} \right)^2 \right] - \frac{\gamma}{2 a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2}, \tag{30}$$

and that the full equation of motion is

$$\frac{\partial^2 \chi}{\partial x \partial Y} \frac{\partial \chi}{\partial Y} - \frac{\partial^2 \chi}{\partial Y^2} \frac{\partial \chi}{\partial x} = \left[1 + \frac{(\gamma - 1)}{2 a_1^2} \left\{ U^2 - \left(\frac{\partial \chi}{\partial Y} \right)^2 \right\} - \frac{\gamma}{2 a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2} \right] U \frac{dU}{dx} + \nu_s \frac{\partial^3 \chi}{\partial Y^3}. \tag{31}$$

At the boundary $G = 1 + \dfrac{(\gamma - 1)}{2 a_1^2} U^2$ whilst as $Y \to \infty$, $G \to 1$ as is obviously required. The effect of compressibility is therefore to exaggerate the effect of the pressure gradient in the neighbourhood of the boundary as compared with the associated incompressible flow. In one or two examples which will be discussed later it appears that at least for small Mach numbers G becomes less than unity in the outer part of the boundary layer (i.e. G tends to 1 from below) so that there is a corresponding reduction in effect in the outer parts of the layer.

It also appears that, as Illingworth (unpublished) found, no 'similar' solutions of equation (31) other than the one corresponding to $U = \text{const.}$ exist. A number of solutions in series will be considered below but since Pohlhausen's method will also be used we shall now consider the momentum integral equation corresponding to equation (31).

4. Momentum integral equation

Let us denote by δ' the boundary layer thickness measured in terms of Y and write

$$\delta_1' = \int_0^{\delta'} \left(1 - \frac{u}{U}\right) dY, \tag{32}$$

$$\vartheta' = \int_0^{\delta'} \left(1 - \frac{u}{U}\right) \frac{u}{U} dY. \tag{33}$$

Consider first of all the term obtained by integrating G

$$\begin{aligned}\int_0^{\delta'} G\,dY &= \delta' + \frac{(\gamma-1)}{2a_1^2} U^2 \left\{\delta' - \int_0^{\delta'} \frac{u^2}{U^2} dY\right\} - \frac{\gamma}{2a_1^2}\left[\chi \frac{\partial \chi}{\partial Y}\right]_0^{\delta'} + \frac{\gamma}{2a_1^2} \int_0^{\delta'} \left(\frac{\partial \chi}{\partial Y}\right)^2 dY \\ &= \delta' + \frac{U^2}{2a_1^2} \int_0^{\delta'} \frac{u^2}{U^2} dY - \frac{\gamma}{2a_1^2} U \int_0^{\delta'} \frac{\partial \chi}{\partial Y} dY + \frac{(\gamma-1)}{2a_1^2} U^2 \delta' \\ &= \delta' + \frac{U^2}{2a_1^2}[\delta' - \delta_1' - \vartheta'] - \frac{\gamma U^2}{2a_1^2}[\delta' - \delta_1'] + \frac{(\gamma-1)}{2a_1^2} U^2 \delta' \\ &= \delta' - \frac{1}{2}\frac{U^2}{a_1^2}\vartheta' + \frac{(\gamma-1)}{2a_1^2} U^2 \delta_1'. \end{aligned} \tag{34}$$

Hence then exactly as for incompressible flow

$$\frac{\partial}{\partial x}\int_0^{\delta'} u^2 dY - U \frac{\partial}{\partial x}\int_0^{\delta'} u\,dY = U\frac{dU}{dx}\left[\delta' + \frac{U^2}{2a_1^2}\{(\gamma-1)\,\delta_1' - \vartheta'\}\right] - \nu_s \left(\frac{\partial u}{\partial Y}\right)_0, \tag{35}$$

i.e. $$-U^2 \frac{d\vartheta'}{dx} - U\frac{dU}{dx}(2\vartheta' + \delta_1' - \delta') = U\frac{dU}{dx}\left[\delta' + \frac{U^2}{2a_1^2}\{(\gamma-1)\,\delta_1' - \vartheta'\}\right] - \nu_s\left(\frac{\partial u}{\partial Y}\right)_0.$$

Hence $$U^2\frac{d\vartheta'}{dx} + U\frac{dU}{dx}\left[\vartheta'\left(2 - \frac{U^2}{2a_1^2}\right) + \delta'\left\{1 + \frac{(\gamma-1)}{2}\frac{U^2}{a_1^2}\right\}\right] = \nu_s\left(\frac{\partial u}{\partial Y}\right)_0.\dagger \tag{36}$$

This equation may be compared with the associated incompressible form

$$U^2\frac{d\vartheta}{dx} + U\frac{dU}{dx}[2\vartheta + \delta_1] = \nu\left(\frac{\partial u}{\partial y}\right)_0. \tag{37}$$

† This equation may be obtained from first principles when due allowance is made for the altered scale in the y-direction, δ_1' and ϑ' being defined as

$$\delta_1' = \int_0^{\delta'}\left(1 - \frac{u}{U}\right) dY = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} \int_0^{\delta}\left(1 - \frac{u}{U}\right)\frac{T_s}{T} dy = \int_0^{\delta}\left(\frac{\nu_s}{\nu}\right)^{\frac{1}{2}}\left(1 - \frac{u}{U}\right) dy,$$

$$\vartheta' = \int_0^{\delta'} \frac{u}{U}\left(1 - \frac{u}{U}\right) dY = \left(\frac{p}{p_s}\right)^{\frac{1}{2}} \int^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)\frac{T_s}{T} dy = \int_0^{\delta}\left(\frac{\nu_s}{\nu}\right)^{\frac{1}{2}} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy.$$

5. The Kármán-Pohlhausen method

The standard Kármán-Pohlhausen method for incompressible flow is then immediately applicable to equation (36). Let us for the moment write $M_1 = U/a_1$; we notice that if a_0 is the velocity of sound corresponding to rest conditions in the main stream then

$$a_1^2 + \frac{(\gamma-1)}{2} U^2 = a_0^2, \tag{38}$$

so that

$$M_1^2 = \frac{U^2}{a_0^2 - \frac{1}{2}(\gamma-1) U^2} = \frac{M_0^2}{1-\frac{1}{2}(\gamma-1) M_0^2}, \tag{39}$$

and

$$1 + \frac{(\gamma-1)}{2} M_1^2 = \frac{a_0^2}{a_0^2 - \frac{1}{2}(\gamma-1) U^2} = \frac{1}{1-\frac{1}{2}(\gamma-1) M_0^2}, \tag{40}$$

where $M_0 = U/a_0$.

The theory will be developed with the standard quartic form of velocity distribution putting

$$u/U = f(Y/\delta') = F(\eta) + \lambda G(\eta), \tag{41}$$

where

$$\eta = Y/\delta', \quad F(\eta) = 2\eta - 2\eta^3 + \eta^4, \quad G(\eta) = \tfrac{1}{6}\eta(1-\eta)^3, \tag{42}$$

and

$$f(0) = 0, \quad f''(0) = -\lambda, \quad f'(1) = f''(1) = 0. \tag{43}$$

Now

$$f''(0) = \frac{\delta'^2}{U}\left(\frac{\partial^2 u}{\partial Y^2}\right)_0 = \frac{\delta'^2}{U}\left(\frac{\partial^3 \chi}{\partial Y^3}\right)_0 = -\frac{\delta'^2}{U\nu_s}\left[1 + \frac{(\gamma-1)}{2} M_1^2\right] U \frac{dU}{dx}.$$

Hence

$$\lambda = \frac{\delta'^2}{\nu_s}\frac{dU}{dx}\left[1 + \frac{(\gamma-1)}{2} M_1^2\right], \tag{44}$$

and differs from the corresponding quantity for incompressible flow by the factor $[(1+\frac{1}{2}(\gamma-1) M_1^2]$. It still remains true as for incompressible flow that

$$\vartheta' = \frac{\delta'}{315}\left(37 - \frac{\lambda}{3} - \frac{5\lambda^2}{144}\right), \quad \delta_1' = \frac{\delta'}{120}(36-\lambda) \quad \text{and} \quad \left(\frac{\partial u}{\partial Y}\right)_0 = \frac{U}{\delta'}\left(2 + \frac{\lambda}{6}\right). \tag{45}$$

Substituting these values in equation (36) leads to the following differential equation for λ:

$$\frac{d\lambda}{dx} = \frac{d^2U/dx^2}{dU/dx}[\lambda + \lambda^2 h(\lambda)] + \frac{1}{U}\frac{dU}{dx}\left[g(\lambda) + \gamma M_1^2\{\lambda + \lambda^2 h(\lambda)\} + \frac{(\gamma-1)}{2} M_1^2 j(\lambda)\right], \tag{46}$$

where

$$g(\lambda) = \frac{15120 - 2784\lambda + 79\lambda^2 + \frac{5}{3}\lambda^3}{(12-\lambda)(37+\frac{25}{12}\lambda)}, \tag{47}$$

$$h(\lambda) = \frac{8 + \frac{5}{3}\lambda}{(12-\lambda)(37+\frac{25}{12}\lambda)}, \tag{48}$$

$$j(\lambda) = \frac{15120 - 1008\lambda + 63\lambda^2}{(12-\lambda)(37+\frac{25}{12}\lambda)}. \tag{49}$$

The functions g and h are the same as those occurring in incompressible flow and tabulated in *Modern developments of fluid dynamics*, **1**, 160. The function $j(\lambda)$ is tabulated in table 1 together with

$$k(\lambda) = \frac{(\gamma-1)}{2} j(\lambda) + \gamma\{\lambda^2 h(\lambda) + \lambda\}, \tag{50}$$

when $\gamma = 1{\cdot}4$.

Equation (46) may also be put in a form analogous to the one found most useful in incompressible flow problems by writing

$$\zeta = \lambda \bigg/ \frac{dU}{dx}.$$

Then $$\frac{d\zeta}{dx} = \frac{1}{U}\left[g(\lambda) + \gamma M^2\{\lambda^2 h(\lambda) + \lambda\} + \frac{(\gamma-1)}{2} M^2 j(\lambda)\right] + \zeta^2 \frac{d^2U}{dx^2} h(\lambda). \tag{51}$$

Near a stagnation point the equations reduce to the standard incompressible form and since $M^2/U \to 0$ as $U \to 0$ we find

$$\lambda = 7{\cdot}052, \quad \frac{d\zeta}{dx} = -5{\cdot}391 \frac{d^2U}{dx^2} \bigg/ \left(\frac{dU}{dx}\right)^2 \tag{52}$$

for the conditions holding at the stagnation point.

TABLE 1

λ	$j(\lambda)$	$k(\lambda)$	λ	$j(\lambda)$	$k(\lambda)$
12·0	∞	∞	3·0	32·53	11·13
11·0	194·52	128·76	2·0	32·44	9·442
10·0	98·04	63·47	1·0	32·97	8·026
9·0	66·67	41·53	0·0	34·05	6·811
8·0	51·65	30·44	−1·0	35·67	6·753
7·8	49·12	28·74	−2·0	37·83	4·823
7·6	47·74	27·38	−3·0	40·57	3·995
7·4	46·07	26·04	−4·0	43·95	3·255
7·2	44·58	24·81	−5·0	48·09	2·593
7·052	43·57	23·96	−6·0	53·14	2·002
6·8	42·02	22·63	−7·0	59·31	1·480
6·6	40·91	21·02	−8·0	66·92	1·011
6·4	39·91	20·74	−9·0	76·44	0·613
6·2	39·00	19·89	−10·0	88·57	0·297
6·0	38·18	19·09	−11·0	104·44	0·084
5·0	35·11	15·74	−12·0	126·00	0·000
4·0	33·35	13·17			

The condition $\lambda = -12$ is still, of course, the condition which governs separation.

So far we have been concerned with developing the general theory. Its applications to particular examples will now be considered.

6. Flat plate solution

The solution of the flat plate problem has been given by Crocco (1946) but it seems worthwhile to make a few comments for the sake of completeness.

If $dU/dx = 0$ then equations (21) and (31) reduce to

$$\frac{\partial^2\chi}{\partial x\,\partial Y}\frac{\partial\chi}{\partial Y} - \frac{\partial^2\chi}{\partial Y^2}\frac{\partial\chi}{\partial Y} = \nu_s \frac{\partial^3\chi}{\partial Y^3}, \tag{53}$$

and the boundary conditions are

$$\frac{\partial\chi}{\partial Y} = 0,\ \chi = 0 \ \text{at} \ Y = 0, \quad \frac{\partial\chi}{\partial Y} \to U \ \text{as} \ Y \to \infty$$

The solution is known (since it is the same as the associated incompressible solution) to be

$$\chi = (\nu_s U x)^{\frac{1}{2}} f(\eta), \quad u = \frac{U}{2} f'(\eta), \tag{54}$$

where $$\eta = \frac{1}{2}\left(\frac{U}{\nu_s x}\right)^{\frac{1}{2}} Y \quad \text{and} \quad f''' + ff'' = 0,$$

with the boundary conditions

$$f = f' = 0 \text{ at } \eta = 0, \quad f' = 2 \text{ at } \eta = \infty.$$

The skin friction τ at the plate is

$$\mu\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\mu_s T}{T_s}\left(\frac{\partial u}{\partial Y}\right)_0 \frac{T_s}{T} = \mu_s\left(\frac{\partial^2 \chi}{\partial Y^2}\right)_0$$

$$= \frac{\mu_s}{4}\frac{U^{\frac{3}{2}}}{(\nu_s x)^{\frac{1}{2}}} f''(0). \tag{55}$$

We can take the standard conditions to refer to the uniform conditions in the main stream and so obtain, using suffix 1 for mainstream values,

$$\tau = \tfrac{1}{4}\alpha \rho_1 U \left(\frac{\nu_1 U}{x}\right)^{\frac{1}{2}}, \tag{56}$$

where $\alpha = 1{\cdot}32824$. The skin friction coefficient

$$\frac{\tau}{\rho_1 U^2} = \tfrac{1}{4}\alpha\left(\frac{\nu_1}{Ux}\right)^{\frac{1}{2}} \tag{57}$$

is therefore independent of Mach number in this case.

Although the velocity distribution in terms of Y is also independent of Mach number, this is obviously no longer so in terms of the true scale y. Thus we have u as function of η only where

$$y = \int_0^Y \frac{T}{T_1} dY$$

$$= \int_0^Y \left[1 + \frac{(\gamma - 1)}{2a_1^2}(U^2 - u^2)\right] dY$$

$$= \left[1 + \frac{(\gamma-1)}{2} M_1^2\right] Y - \frac{(\gamma-1)}{2} M_1^2 \int_0^Y \frac{u^2}{U^2} dY, \tag{58}$$

so that $$y = \frac{1}{2}\left(\frac{U}{\nu_1 x}\right)^{-\frac{1}{2}}\left[\eta + \frac{(\gamma-1)}{2} M_1^2 \left\{\eta - \frac{1}{4}\int_0^\eta f'^2 d\eta\right\}\right].$$

Now if we integrate the equation for f we find

$$\int_0^\eta f'^2 d\eta = f'' - f''(0) + ff'.$$

Hence $$y = \frac{1}{2}\left(\frac{U}{\nu_1 x}\right)^{-\frac{1}{2}}\left[\eta + \frac{(\gamma-1)}{2} M_1^2 \{\eta - \tfrac{1}{4}f'' - \tfrac{1}{4}ff' + 0{\cdot}34206\}\right], \tag{59}$$

so that if we use y_i to denote the ordinate of the incompressible flow with viscosity ν_1,

$$y_i = \frac{1}{2}\left(\frac{U}{\nu_1 x}\right)^{-\frac{1}{2}} \eta \quad \text{and} \quad \frac{y-y_i}{y_i} = \frac{(\gamma-1)}{2} M_1^2 \left\{1 - \frac{(f''+ff')}{4\eta} + \frac{0\cdot324206}{\eta}\right\}. \tag{60}$$

When $\eta \to 0$ the expression in curly brackets tends to 1. When η is large we find

$$\frac{y-y_i}{y_i} = 0\cdot5962 \frac{(\gamma-1)}{\eta} M_1^2. \tag{61}$$

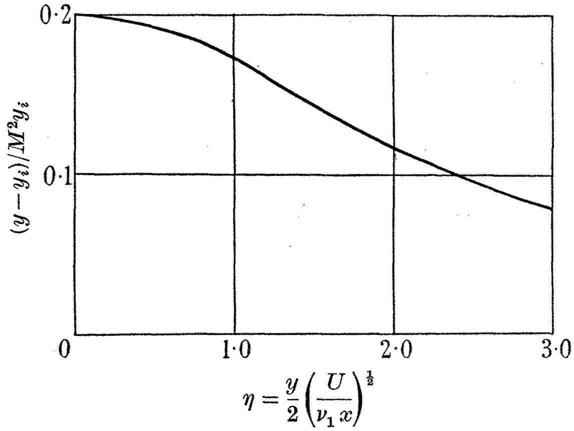

FIGURE 1

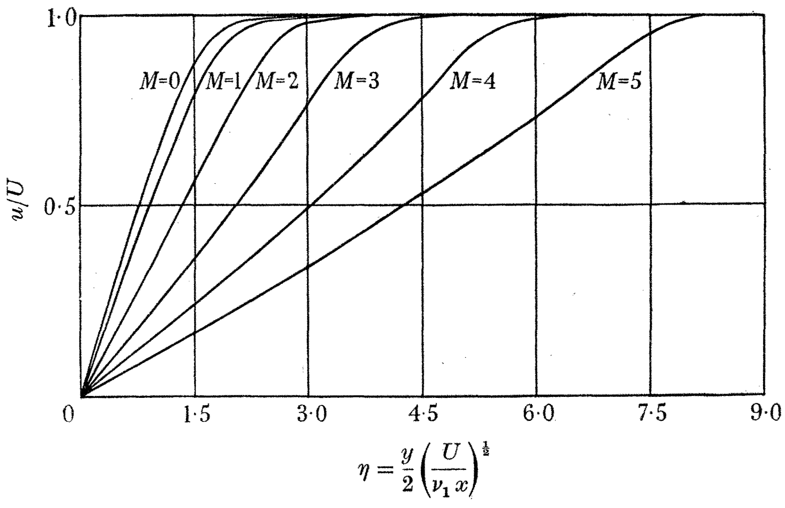

FIGURE 2

To see the significance of this last result it is convenient to imagine a finite boundary layer thickness δ defined by the condition $u/U = 0\cdot999$, say. This corresponds to a value of 3 for η. Hence

$$\frac{\delta-\delta_i}{\delta_i} = 0\cdot1987(\gamma-1) M_1^2$$

$$= 0\cdot0795 M_1^2 \tag{62}$$

when $\gamma = 1{\cdot}4$. Thus there is an 8 % increase in boundary layer thickness when $M_1 = 1$ and a doubling in thickness when $M_1 = 3{\cdot}5$.

The relation defined by equation (60) is shown graphically in figure 1 when $\gamma = 1{\cdot}4$. Figure 2 contains a number of velocity distributions at different Mach numbers.

The forms of the velocity distributions are much the same as those obtained by Kármán & Tsien (1938) who assumed $\mu \propto T^{0{\cdot}76}$. They found as did Brainerd & Emmons (1941, 1942) with a variety of viscosity-temperature variations that the skin friction decreased slowly as the Mach number increased. The constant value found in the present investigation is due, as suggested earlier, to our assumptions effectively overestimating the effects of viscosity.

7. Flow in wakes and jets

Although the problem of viscous compressible flow in wakes and jets may not be very significant practically since the flow is probably turbulent it is nevertheless of some interest to see the effects of compressibility even in this case.

With the usual approximations equation (31) adapted to flow in a wake becomes

$$U\frac{\partial^2\chi}{\partial x\,\partial Y} = \nu_1\frac{\partial^3\chi}{\partial Y^3}, \tag{63}$$

where ν_1 is the kinematic viscosity corresponding to the undisturbed flow, the drag D being given by the integral

$$D = \rho_1 U^2\int_{-\infty}^{\infty}\frac{\rho}{\rho_1}\frac{u}{U}\left(1-\frac{u}{U}\right)dy. \tag{64}$$

Thus
$$D = \rho_1 U^2\int_{-\infty}^{\infty}\frac{\rho T}{\rho_1 T_1}\frac{u}{U}\left(1-\frac{u}{U}\right)dY$$

$$= \rho_1 U^2\int_{-\infty}^{\infty}\frac{u}{U}\left(1-\frac{u}{U}\right)dY. \tag{65}$$

Putting $w = 1-u/U$ we then have

$$D = \rho_1 U^2\int_{-\infty}^{\infty} w\,dY \tag{66}$$

approximately.

The solution in terms of Y is then given by the standard incompressible solution

$$w = w_0 e^{-\frac{1}{2}\eta^2}, \tag{67}$$

where
$$\eta = \left(\frac{U}{2\nu_1 x}\right)^{\frac{1}{2}} Y \quad\text{and}\quad w_0 = \frac{D}{2\rho_1 U^2}\left(\frac{U}{\pi\nu_1 x}\right)^{\frac{1}{2}}. \tag{68}$$

The integral
$$Jc_p T + \tfrac{1}{2}u^2 = \text{const.}$$

of the energy equation is applicable to this problem too since it makes $\partial T/\partial y$ vanish with $\partial u/\partial y$. Hence as in the flat plate problem we have a velocity distribution determined by η, the true scale y being given by

$$y = Y + \frac{(\gamma-1)}{2} M_1^2 \int_0^Y \left(1 - \frac{u^2}{U^2}\right) dY$$

$$= Y + (\gamma - 1) M_1^2 \int_0^Y w\,dY. \tag{69}$$

Hence if Y is sufficiently large

$$y = Y + \frac{(\gamma-1) M_1^2 D}{2\rho_1 U^2}. \tag{70}$$

As for the flow along a flat plate we can define a width of wake by $w/w_0 = 0{\cdot}1\,\%$, say, which gives a value of Y of $\dfrac{4(2\nu_1 x)^{\frac{1}{2}}}{U}$ approximately.

Of course $D/\rho_1 U^2$ is in general an unknown function of both Reynolds and Mach numbers for a given obstacle so that variation in wake size implied by (70) is also not known. One can, however, interpret (70) by saying that for a given value of $D/\rho_1 U^2$ the ordinary incompressible theory with kinematic viscosity ν_1 would underestimate the wake thickness by a constant amount $\dfrac{(\gamma-1)}{2} M_1^2 \dfrac{D}{\rho_1 U^2}$.

In the particular case of the wake behind a flat plate of length l

$$\frac{D}{\rho_1 U^2} = 1{\cdot}328 l \left(\frac{Ul}{\nu_1}\right)^{-\frac{1}{2}}, \tag{71}$$

and
$$\frac{u}{U} = 1 - \frac{0{\cdot}664}{\sqrt{\pi}} \left(\frac{l}{x}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\eta^2}, \tag{72}$$

where
$$\eta = \left(\frac{U}{2\nu_1 x}\right)^{\frac{1}{2}} Y, \tag{73}$$

and
$$y = Y + 0{\cdot}664(\gamma-1) M_1^2 l \left(\frac{Ul}{\nu_1}\right)^{-\frac{1}{2}} \operatorname{erf}\left[\frac{Y}{2}\left(\frac{U}{\nu_1 x}\right)^{\frac{1}{2}}\right], \tag{74}$$

where
$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt.$$

Defining the edge of the wake as in the general case above the implied total width of the wake is

$$8\left(\frac{2\nu_1 x}{U}\right)^{\frac{1}{2}} + 1{\cdot}328(\gamma-1) M_1^2 l \left(\frac{\nu_1}{Ul}\right)^{\frac{1}{2}} = l\left(\frac{\nu_1}{Ul}\right)^{\frac{1}{2}} \left[8\sqrt{2}\left(\frac{x}{l}\right)^{\frac{1}{2}} + 1{\cdot}328(\gamma-1) M_1^2\right]$$

$$= l\left(\frac{\nu_1}{Ul}\right)^{\frac{1}{2}} \left[11{\cdot}314\left(\frac{x}{l}\right)^{\frac{1}{2}} + 0{\cdot}5313 M_1^2\right], \tag{75}$$

when $\gamma = 1{\cdot}4$.

The velocity distribution in the wake when $x = 10l$ for Mach numbers of 0, 5 and 10 is shown graphically in figure 3.

Consider next a two-dimensional jet. The flux of momentum F across each section of the jet is

$$F = 2\int_0^\infty \rho u^2 dy$$

$$= 2\rho_0 \int_0^\infty \frac{\rho}{\rho_0} u^2 dy = 2\rho_0 \int_0^\infty u^2 dY, \tag{76}$$

where the suffix 0 refers to conditions in the gas at rest outside the jet.

The equation of motion derived from (31) is

$$\frac{\partial^2 \chi}{\partial x \partial Y}\frac{\partial \chi}{\partial Y} - \frac{\partial^2 \chi}{\partial Y^2}\frac{\partial \chi}{\partial x} = \nu_0 \frac{\partial^3 \chi}{\partial Y^3}. \tag{77}$$

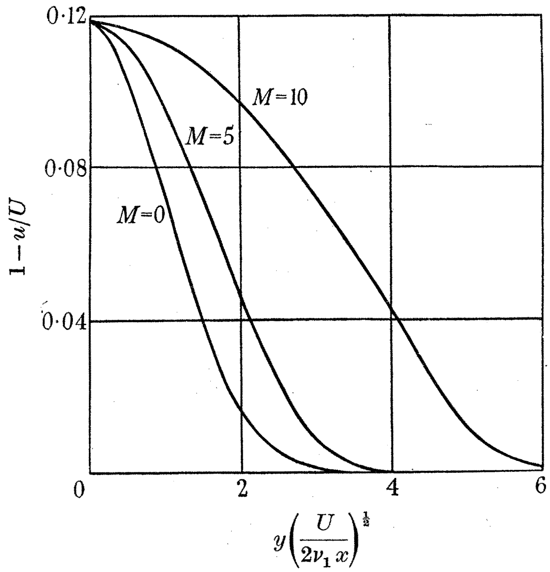

FIGURE 3

The standard incompressible solution can then be used to give

$$u = \frac{\partial \chi}{\partial Y} = 0{\cdot}4543\left(\frac{F^2}{\rho_0^2 \nu_0 x}\right)^{\frac{1}{3}} \operatorname{sech}^2 \xi, \tag{78}$$

where

$$\xi = 0{\cdot}2752\left(\frac{F}{\rho_0 \nu_0^2 x^2}\right)^{\frac{1}{3}} Y. \tag{79}$$

The ordinate y is then given by

$$y = Y - \frac{(\gamma-1)}{2a_0^2}\int_0^Y u^2 dY, \tag{80}$$

where a_0 is the velocity of sound corresponding to rest conditions. For sufficiently large Y we see that $y = Y - \frac{(\gamma-1)}{4a_0^2}\frac{F}{\rho_0}$, so that for a given value of F/ρ_0 the jet width is less by an amount $\frac{(\gamma-1)}{2a_0^2}\frac{F}{\rho_0}$ than that obtained by incompressible theory with kinematic viscosity ν_0.

Of course the considerations set out in the section on wakes and jets do not hold in the neighbourhood of $x = 0$ where neither the assumptions of the boundary layer nor the neglect of pressure gradient are valid.

8. Flow with linearly increasing velocity in the main stream

One of the standard solutions of incompressible flow is given by considering the velocity distribution

$$U = \beta x$$

in the mainstream. It corresponds to flow in the vicinity of a stagnation point. It is obvious of course that even when the flow is compressible the velocity distribution in the boundary layer in the immediate vicinity of the stagnation point must be the same as the incompressible one. If, however, the linear velocity distribution $U = \beta x$ may be supposed to hold for some appreciable distance downstream then compressibility will begin to play a part. To this extent the problem is more artificial than the corresponding incompressible problem but it is worthy of consideration since it may be typical of accelerated flow problems generally.

With $U = \beta x$ equation (31) takes the form

$$\frac{\partial^2\chi}{\partial x\,\partial Y}\frac{\partial\chi}{\partial Y}-\frac{\partial^2\chi}{\partial Y^2}\frac{\partial\chi}{\partial x}=\left[1+\frac{(\gamma-1)}{2a_1^2}\left\{\beta^2x^2-\left(\frac{\partial\chi}{\partial Y}\right)^2\right\}-\frac{\gamma}{2a_1^2}\chi\frac{\partial^2\chi}{\partial Y^2}\right]\beta^2x+\nu_0\frac{\partial^3\chi}{\partial Y^3}, \tag{81}$$

where we use the suffix zero to denote conditions at the stagnation point and treat these as our standard.† We see that

$$a_1^2+\frac{(\gamma-1)}{2}\beta^2x^2 = a_0^2, \tag{82}$$

so that

$$\frac{1}{a_1^2}=\frac{1}{a_0^2}\left[1+\frac{(\gamma-1)}{2}\frac{\beta^2x^2}{a_0^2}+\left(\frac{\gamma-1}{2}\right)^2\frac{\beta^4x^4}{a_0^4}+\dots\right]. \tag{83}$$

We can obtain a series solution in the form

$$\chi = (\nu_0\beta)^{\frac{1}{2}}x\left[f_1(\eta)+\frac{\beta^2x^2}{a_0^2}f_3(\eta)+\frac{\beta^4x^4}{a_0^4}f_5(\eta)+\dots\right], \tag{84}$$

where

$$\eta = \left(\frac{\beta}{\nu_0}\right)^{\frac{1}{2}}Y. \tag{85}$$

The differential equations satisfied by f_1, f_3 and f_5 are

$$f_1'^2-f_1f_1'' = 1+f_1''', \tag{86}$$

$$4f_1'f_3'-3f_1''f_3-f_1f_3'' = \frac{(\gamma-1)}{2}(1-f_1'^2)-\frac{\gamma}{2}f_1f_1''+f_3''', \tag{87}$$

$$\begin{aligned}6f_1'f_5'-5f_1''f_5-f_1f_5'' = {}&\frac{(\gamma-1)^2}{4}(1-f_1'^2)-\gamma\frac{(\gamma-1)}{4}f_1f_1''\\&-(\gamma-1)f_1'f_3'-\frac{\gamma}{2}(f_1f_3''+f_3f_1'')-3(f_3'^2-f_3f_3'')+f_5''',\end{aligned} \tag{88}$$

† The suffix 1 on a is used as before to denote general mainstream values.

and the boundary conditions are

$$f_1 = f_1' = f_3 = f_3' = f_5 = f_5' = 0 \text{ at } \eta = 0, \quad f_1' = 1, f_3' = f_5' = 0 \text{ at } \eta = \infty.$$

As remarked above f_1 is the same function that appears in the solution of the incompressible problem. Taking $\gamma = 1{\cdot}4$ we have

$$4f_1'f_3' - 3f_1''f_3 - f_1f_3'' = 0{\cdot}2(1 - f_1'^2) - 0{\cdot}7f_1f_1'' + f_3''', \tag{89}$$

$$6f_1'f_5' - 5f_1''f_5 - f_1f_5'' = 0{\cdot}04(1 - f_1'^2) - 0{\cdot}14f_1f_1'' - 0{\cdot}4f_1'f_3'$$
$$- 0{\cdot}7(f_1f_3'' + f_3f_1'') - 3(f_3'^2 - f_3f_3'') + f_5'''. \tag{90}$$

The velocity distribution is given by

$$u = \frac{\partial \chi}{\partial Y} = \beta x\left[f_1'(\eta) + \frac{\beta^2x^2}{a_0^2}f_3'(\eta) + \frac{\beta^4x^4}{a_0^4}f_5'(\eta) + \dots\right], \tag{91}$$

where $$\eta = \left(\frac{\beta}{\nu_0}\right)^{\frac{1}{2}} Y$$

and $$y = \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \int_0^Y \frac{T}{T_0}\, dY$$
$$= \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \left[Y - \frac{(\gamma-1)}{2a_0^2}\int_0^Y u^2\, dY\right]$$
$$= \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \left(\frac{\nu_0}{\beta}\right)^{\frac{1}{2}} \left[\eta - \frac{(\gamma-1)}{2a_0^2}\beta^2x^2\int_0^\eta f_1'^2\, d\eta - (\gamma-1)\frac{\beta^4x^4}{a_0^4}\int_0^\eta f_1'f_3'\, d\eta + \dots\right]. \tag{92}$$

The skin friction τ at the wall is given by

$$\tau = \mu_w\left(\frac{\partial u}{\partial y}\right)_w = \frac{\mu_w T_0}{T_w}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\left(\frac{\partial u}{\partial Y}\right)_w, \tag{93}$$

where the suffix w refers to conditions at the wall. Hence

$$\tau = \rho_0\nu_0^{\frac{1}{2}}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\beta^{\frac{3}{2}}x\left[f_1''(0) + \frac{\beta^2x^2}{a_0^2}f_3''(0) + \frac{\beta^4x^4}{a_0^4}f_5''(0) + \dots\right]. \tag{94}$$

The pressure ratio p/p_0 which appears in equations (92) and (94) is given by

$$\frac{p}{p_0} = \left[1 - \frac{(\gamma-1)}{2}\frac{\beta^2x^2}{a_0^2}\right]^{\gamma/(\gamma-1)}, \tag{95}$$

and may be expanded in powers of β^2x^2/a_0^2 to give

$$y = \left(\frac{\nu_0}{\beta}\right)^{\frac{1}{2}}\left[\eta + \frac{\beta^2x^2}{a_0^2}\left\{\frac{\gamma\eta}{4} - \frac{(\gamma-1)}{2}\int_0^\eta f_1'^2\, d\eta\right\}\right.$$
$$\left. + \frac{\beta^4x^4}{a_0^4}\left\{\gamma\frac{(3\gamma-2)}{32}\eta - \gamma\frac{(\gamma-1)}{8}\int_0^\eta f_1'^2\, d\eta - (\gamma-1)\int_0^\eta f_1'f_3'\, d\eta\right\} + \dots\right], \tag{96}$$

and $$\tau = \rho_0\nu_0^{\frac{1}{2}}\beta^{\frac{3}{2}}x\left[f_1''(0) + \frac{\beta^2x^2}{a_0^2}\left\{f_3''(0) - \frac{\gamma}{4}f_1''(0)\right\}\right.$$
$$\left. + \frac{\beta^4x^4}{a_0^4}\left\{f_5''(0) - \frac{\gamma}{4}f_3''(0) + \gamma\frac{(2-\gamma)}{32}f_1''(0)\right\} + \dots\right]. \tag{97}$$

The integration of equation (89) for f_3 has been carried out and the results are shown in table 2; f_3' is shown graphically in figure 4. This function is in fact quite small compared with f_1; $f_3''(0)$ is positive and of the order of 5 % of $f_1''(0)$. The velocity correction implied by f_3' is positive in the inner quarter, say, of the boundary layer with a maximum of just less than 1 % of the maximum value, unity, of f_1' and is negative in the remainder with a maximum numerical value of just over 1 % of the maximum value of f_1'.

TABLE 2

η	f_3	f_3'	f_3''	η	f_3	f_3'	f_3''
0·0	0·000	0·000	0·056	1·6	−0·003	−0·012	0·002
0·1	0·000	0·005	0·036	1·7	−0·004	−0·011	0·005
0·2	0·001	0·007	0·018	1·8	−0·005	−0·011	0·007
0·3	0·002	0·008	0·003	1·9	−0·006	−0·010	0·009
0·4	0·003	0·008	−0·010	2·0	−0·007	−0·009	0·010
0·5	0·003	0·007	−0·018	2·1	−0·008	−0·008	0·010
0·6	0·004	0·004	−0·024	2·2	−0·008	−0·007	0·010
0·7	0·004	0·002	−0·027	2·3	−0·009	−0·006	0·010
0·8	0·004	−0·001	−0·027	2·4	−0·009	−0·005	0·009
0·9	0·004	−0·004	−0·025	2·5	−0·010	−0·004	0·008
1·0	0·003	−0·006	−0·022	2·6	−0·010	−0·003	0·007
1·1	0·003	−0·008	−0·018	2·7	−0·011	−0·003	0·007
1·2	0·002	−0·010	−0·014	2·8	−0·011	−0·002	0·006
1·3	0·001	−0·011	−0·009	2·9	−0·011	−0·002	0·005
1·4	0·000	−0·011	−0·005	3·0	−0·011	−0·001	0·004
1·5	−0·001	−0·012	−0·001				

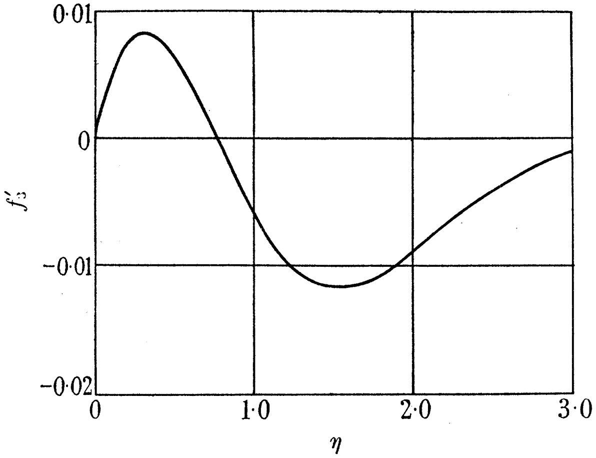

FIGURE 4

The integration of equation (90) has not been carried out but a crude examination indicates that f_5 should have much the same form as f_3 but considerably smaller in magnitude possibly of the order of 20 % of f_3.

It appears therefore that even when $\beta x/a_0$ has reached unity the difference in u at a given value of Y (or η) from its corresponding incompressible value is small and of the order of 1 %. This does not of course imply that the change in u at a given value of the true ordinate y is small. In fact the change in scale is much the most important effect here. Bearing in mind the remarks on order of magnitude made above we can write equation (92) as

$$y = \left(\frac{p_0}{p}\right)^{\frac{1}{2}}\left(\frac{\nu_0}{\beta}\right)^{\frac{1}{2}}\left[\eta - \frac{(\gamma-1)}{2a_0^2}\beta^2x^2\int_0^\eta f_1'^2\,d\eta\right]$$

approximately. Thus

$$y = \left(\frac{p_0}{p}\right)^{\frac{1}{2}}\left(\frac{\nu_0}{\beta}\right)^{\frac{1}{2}}\left[\eta - \frac{(\gamma-1)}{4a_0^2}\beta^2x^2\{\eta + f_1'' + f_1f_1' - f_1''(0)\}\right]. \tag{98}$$

Velocity distributions corresponding to $\beta x/a_0 = 0$, 0·5 and 1·0 (beyond which stage it seems inadvisable to proceed without calculating f_5) are shown in figure 5.

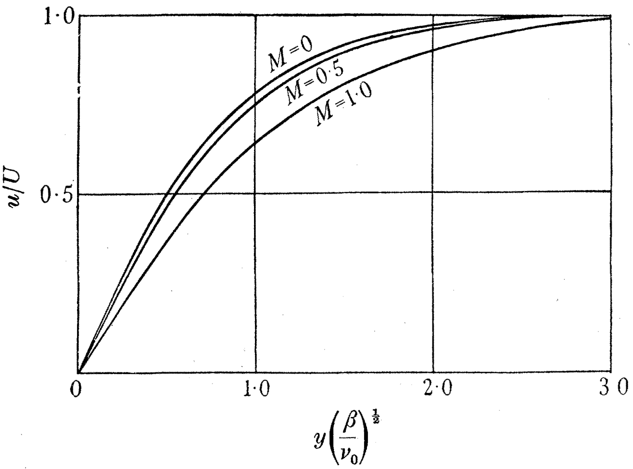

FIGURE 5

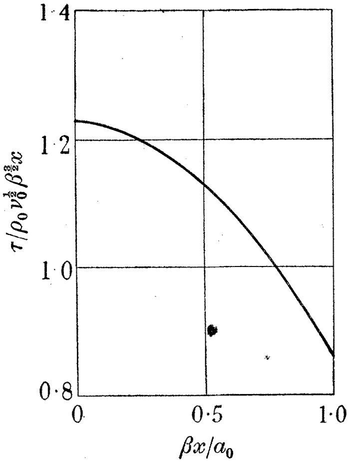

FIGURE 6

The skin friction at the wall τ given by equation (94) is shown graphically in figure 6 for values of $\beta x/a_0$ between 0 and 1. Although $\nu_0^{\frac{1}{2}}\beta^{-\frac{3}{2}}x^{-1}(\partial u/\partial Y)_0$ increases with Mach number from 1·23 at $\beta x/a_0 = 0$ to 1·29 at $\beta x/a_0 = 1$ the actual skin friction coefficient $\tau/\rho_0\nu_0^{\frac{1}{2}}\beta^{\frac{3}{2}}x$ falls from 1·23 to 0·87 as a consequence of the changing scale. As one would indeed expect in the light of the velocity distributions the factor $(p/p_0)^{\frac{1}{2}}$ representing the effect of change of scale predominates and leads to this substantial decrease in skin friction coefficient with Mach number. A good approximation to the skin friction is, in fact, provided by reducing the incompressible value by the factor $(p/p_0)^{\frac{1}{2}}$ in this range of variation of $\beta x/a_0$.

A problem rather similar to the one discussed here is provided by the boundary layer along the wall of a converging channel. Here the effects of compressibility on the main flow are known and to this extent the problem is less artificial than that of the present paragraph.

9. SERIES EXPANSIONS FOR THE GENERAL MAINSTREAM VELOCITY DISTRIBUTION

The considerations of the previous section make it evident that if the velocity distribution in the main stream is given as a series of ascending powers of x

$$U = \beta_1x + \beta_2x^2 + \beta_3x^3 + \dots \tag{99}$$

a solution could be obtained in the form

$$\chi = (\nu_0\beta_1)^{\frac{1}{2}}x[F_1(\eta)+xF_2(\eta)+x^2F_3(\eta)+\dots], \tag{100}$$

where

$$\eta = \left(\frac{\beta_1}{\nu_0}\right)^{\frac{1}{2}} Y. \tag{101}$$

By writing

$$\left.\begin{aligned} F_1 &= f_1, \quad F_2 = \frac{3\beta_2}{\beta_1}f_2, \\ F_3 &= \frac{4\beta_3}{\beta_1}g_3+\frac{4\beta_2^2}{\beta_1^2}h_3+\frac{\beta_1^2}{a_0^2}k_3, \\ F_4 &= \frac{5\beta_4}{\beta_1}g_4+\frac{5\beta_2\beta_3}{\beta_1^2}h_4+\frac{5\beta_2^3}{\beta_1^3}k_4+\frac{\beta_1\beta_2}{a_0^2}q_4, \quad \text{etc.,} \end{aligned}\right\} \tag{102}$$

the equations for the functions F can be split up into equations independent of both Reynolds and Mach numbers, just as in the corresponding problem for incompressible flow.

The functions f_1, f_2, g_3, h_3, g_4, h_4, k_4 are in fact the functions derived for incompressible flow (Howarth 1934). The new functions k_3, q_4 introduced by compressibility are given by the equations

$$4f_1'k_3'-3f_1''k_3-f_1k_3'' = \frac{(\gamma-1)}{2}(1-f_1'^2)-\frac{\gamma}{2}f_1f_1''+k_3''' \tag{103}$$

and

$$5f_1'q_4'-4f_1''q_4-f_1q_4'' = q_4'''-3(5f_2'k_3'-3f_2''k_3-2f_2k_3'')$$
$$+(\gamma-1)[(1-3f_1'f_2')+3(1-f_1'^2)]-\frac{3\gamma}{2}(f_1f_2''+f_1''f_2+f_1f_1''), \tag{104}$$

for which the boundary conditions are

$$k_3 = k_3' = q_4 = q_4' = 0 \text{ at } \eta = 0, \quad k_3' = q_4' = 0 \text{ at } \eta = \infty.$$

As might be expected equation (103) is identical with (87) of which the solution is given in table 2 (where it is called f_3) when $\gamma = 1{\cdot}4$. The functions f_1, f_2, g_3, h_3 and k_4 are tabulated by Howarth (1934).

In view of the generally slow convergence of the series involved and the consequently limited applicability of the results it has not been considered worthwhile to calculate the compressibility effects beyond the function k_3 already calculated.

The skin friction at the wall is

$$\mu_w\left(\frac{\partial u}{\partial y}\right)_w = \mu_0\left(\frac{\partial u}{\partial Y}\right)_w\left(\frac{p}{p_0}\right)^{\frac{1}{2}}$$
$$= \rho_0\nu_0^{\frac{1}{2}}\beta_1^{\frac{3}{2}}\left(\frac{p}{p_0}\right)^{\frac{1}{2}}x\Bigg[f_1''(0)+\frac{3\beta_2x}{\beta_1}f_2''(0)+\left\{\frac{4\beta_3}{\beta_1}g_3''(0)+\frac{4\beta_2^2}{\beta_1^2}h_3''(0)\frac{\beta_1^2}{2a_0^2}k_3''(0)\right\}x^2$$
$$+\left\{\frac{5\beta_4}{\beta_1}g_4''(0)+\frac{5\beta_2\beta_3}{\beta_1^2}h_4''(0)+\frac{5\beta_2^3}{\beta_1^3}k_4''(0)+\frac{\beta_1\beta_2}{2a_0^2}q_4''(0)\right\}x^3+\dots\Bigg]. \tag{105}$$

A similar method applies to velocity distribution

$$U = \beta_1x+\beta_3x^3+\beta_5x^5+\dots \tag{106}$$

corresponding to a symmetrical obstacle.

The stream function χ is given by a series of the form

$$\chi = (\nu_0\beta_1)^{\frac{1}{2}} x[F_1(\eta)+x^2F_3(\eta)+x^4F_5(\eta)+\dots], \tag{107}$$

where

$$\eta = \left(\frac{\beta_1}{\nu_0}\right)^{\frac{1}{2}} Y. \tag{108}$$

$$\left.\begin{aligned} F_1 &= f_1, \\ F_3 &= \frac{4\beta_3}{\beta_1} f_3 + \frac{\beta_1^2}{2a_0^2} g_3, \\ F_5 &= \frac{6\beta_5}{\beta_1} g_5 + \frac{6\beta_3^2}{\beta_1^2} h_5 + \frac{\beta_1\beta_3}{2a_0^2} k_5 + \frac{\beta_1^4}{2a_0^4} q^5, \end{aligned}\right\} \tag{109}$$

Here again the functions f_1, f_3, g_5, h_5 are the ones occurring in incompressible flow (Howarth 1934, where they are tabulated). The equations determining g_3, k_5 and q_5 are

$$4f_1'g_3' - 3f_1''g_3 - f_1g_3'' = g_3''' + \frac{(\gamma-1)}{2}(1-f_1'^2) - \frac{\gamma}{2}f_1f_1'', \tag{110}$$

$$\begin{aligned} 6f_1'k_5' - 5f_1''k_5 - f_1k_5'' = k_5''' &- 12[2f_3'g_3' - f_1''g_3 - f_1g_3''] \\ &+ (\gamma-1)\{1-4f_1'f_3'\} - 2\gamma(f_1f_3'' + f_1''f_3), \end{aligned} \tag{111}$$

$$\begin{aligned} 6f_1'q_5' - 5f_1''q_5 - f_1q_5'' = q_5''' &- 3(g_3'^2 - g_3g_3'') - (\gamma-1)f_1'g_3' \\ &- \frac{\gamma}{2}(f_1'g_3'' + f_1g_3'') + \gamma\frac{(\gamma-1)}{4}(1-f_1'^2) - \gamma\frac{(\gamma-1)}{4}f_1f_1'', \end{aligned} \tag{112}$$

with the boundary conditions

$$g_3 = g_3' = k_5 = k_5' = q_5 = q_5' = 0 \text{ at } \eta = 0, \quad g_3' = k_5' = q_5' = 0 \text{ at } \eta = \infty.$$

Equation (110) is identical with (103) and (87) the solution being given in table 2 (where it is called f_3). Equation (112) is identical with (88).

Beyond this stage the number of new functions introduced by compressibility effects begins to increase rapidly and the value of the method except at small Mach numbers is doubtful. For this reason the integration has not been carried beyond g_3.

10. The flat plate with a retarding linear velocity gradient

A problem which has been thoroughly investigated in incompressible flow is that which occurs when

$$U = U_0 - U_1x. \tag{113}$$

This velocity distribution has been shown to lead to separation of the layer when

$$\frac{U_1x}{U_0} = 0{\cdot}120.$$

We can apply series methods similar to the ones developed above to determine the effects of compressibility. We take conditions at the leading edge as our standard and denote them by a suffix 0. Then we can find a solution of equation (31) in the form

$$\chi = U_0^{\frac{1}{2}}x^{\frac{1}{2}}\nu_0^{\frac{1}{2}}\{F_0(\eta) - 8x^*F_1(\eta) + (8x^*)^2F_2(\eta) - \dots\}, \tag{114}$$

where $$\eta = \frac{1}{2}\frac{YU_0^{\frac{1}{2}}}{\nu_0^{\frac{1}{2}}x^{\frac{1}{2}}} \quad \text{and} \quad x^* = \frac{U_1 x}{U_0}. \tag{115}$$

If we put
$$\left.\begin{aligned} F_0 &= f_0, \\ F_1 &= f_1 + M_0^2 g_1, \\ F_2 &= f_2 + M_0^2 g_2 + M_0^4 h_2, \text{ etc.,} \end{aligned}\right\} \tag{116}$$

where M_0 is the Mach number at the leading edge we find that $f_0, f_1, f_2, \ldots,$ are the functions occurring in the incompressible solution and given by Howarth (1938). The equations to determine g_1, g_2, h_2 are

$$g_1''' + f_0 g_1'' - 2f_0' g_1' + 3f_0'' g_1 = -\frac{(\gamma-1)}{8}(4 - f_0'^2) + \frac{\gamma}{8} f_0 f_0'', \tag{117}$$

$$g_2''' + f_0 g_2'' - 4f_0' g_2' + 5f_0'' g_2 = -\frac{(\gamma-1)}{64}(4 - f_0'^2) + \frac{\gamma}{64} f_0 f_0'' - \frac{(\gamma-1)}{8}(1 - 2f_0' f_1')$$
$$+ \frac{\gamma}{8}(f_0'' f_1 + f_1'' f_0) + 4f_1' g_1' - 3f_1 g_1'' - 3f_1'' g_1, \tag{118}$$

$$h_2''' + f_0 h_2'' - 4f_0' h_2' + 5f_0'' h_2 = -\frac{(\gamma-1)^2}{64}(4 - f_0'^2) + \gamma\frac{(\gamma-1)}{64} f_0 f_0''$$
$$+ \frac{(\gamma-1)}{4} f_0' g_1' + \frac{\gamma}{8}(f_0'' g_1 + f_0 g_1'') + 2g_1'^2 - 3g_1 g_1''. \tag{119}$$

The boundary conditions are

$$g_1 = g_1' = g_2 = g_2' = h_2 = h_2' = 0 \text{ at } \eta = 0, \quad g_1' = g_2' = h_2' = 0 \text{ at } \eta = \infty.$$

The velocity distribution is given by

$$u = \frac{U_0}{2}[f_0'(\eta) - 8x^*\{f_1'(\eta) + M_0^2 g_1'(\eta)\} + 8x^{*2}\{f_2'(\eta) + M_0^2 g_2(\eta) + M_0^4 h_2(\eta)\} + \ldots]. \tag{120}$$

The ordinate y is determined by

$$\begin{aligned} y &= \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \int_0^Y \frac{T}{T_0}\, dY \\ &= \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \left[Y + \frac{(\gamma-1)}{2} M_0^2 \int_0^Y \left(1 - \frac{u^2}{U_0^2}\right) dY \right. \\ &= \left(\frac{p_0}{p}\right)^{\frac{1}{\gamma}} \frac{2\nu_0^{\frac{1}{2}} x^{\frac{1}{2}}}{U_0^{\frac{1}{2}}} \left[\eta + \frac{(\gamma-1)}{2} M_0^2 \int_0^\eta \left(1 - \frac{u^2}{U_0^2}\right) d\eta\right], \end{aligned} \tag{121}$$

and $$\frac{p}{p_0} = \left[1 + \frac{(\gamma-1)}{8} M_0^2 \{8x^* - \tfrac{1}{16}(8x^*)^2\}\right]^{\gamma/(\gamma-1)}. \tag{122}$$

The skin friction is given by

$$\mu_w\left(\frac{\partial u}{\partial Y}\right)_w = \mu_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\left(\frac{\partial u}{\partial Y}\right)_w = \frac{\rho_0 \nu_0^{\frac{1}{2}} U_0^{\frac{3}{2}}}{4}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}} [f_0''(0) - (8x^*)\{f_1''(0) + M_0^2 g_1''(0)\}$$
$$+ (8x^*)^2\{f_2''(0) + M_0^2 g_2''(0) + M_0^4 h_2''(0)\} + \ldots]. \tag{123}$$

3-2

The new functions introduced by compressibility effects again multiply rapidly. It is to be noted that each of the functions F after F_0 contains a term in M_0^2 so that the compressibility correction at small Mach numbers is correspondingly more difficult to assess than in the accelerated flow problem discussed above.

TABLE 3

η	g_1	g_1'	g_1''	η	g_1	g_1'	g_1''
0·0	0·000	0·000	0·088	1·6	0·013	−0·010	−0·010
0·1	0·000	0·008	0·068	1·7	0·012	−0·010	−0·003
0·2	0·002	0·014	0·049	1·8	0·011	−0·010	0·002
0·3	0·003	0·018	0·030	1·9	0·010	−0·010	0·006
0·4	0·005	0·020	0·013	2·0	0·009	−0·009	0·009
0·5	0·007	0·020	−0·002	2·1	0·008	−0·008	0·011
0·6	0·009	0·019	−0·015	2·2	0·007	−0·007	0·012
0·7	0·011	0·017	−0·025	2·3	0·007	−0·006	0·011
0·8	0·012	0·014	−0·033	2·4	0·006	−0·005	0·011
0·9	0·014	0·011	−0·037	2·5	0·006	−0·004	0·009
1·0	0·015	0·007	−0·039	2·6	0·005	−0·003	0·008
1·1	0·015	0·003	−0·038	2·7	0·005	−0·002	0·007
1·2	0·015	−0·001	−0·035	2·8	0·005	−0·002	0·005
1·3	0·015	−0·004	−0·030	2·9	0·005	−0·001	0·004
1·4	0·014	−0·006	−0·023	3·0	0·005	−0·001	0·003
1·5	0·014	−0·008	−0·017				

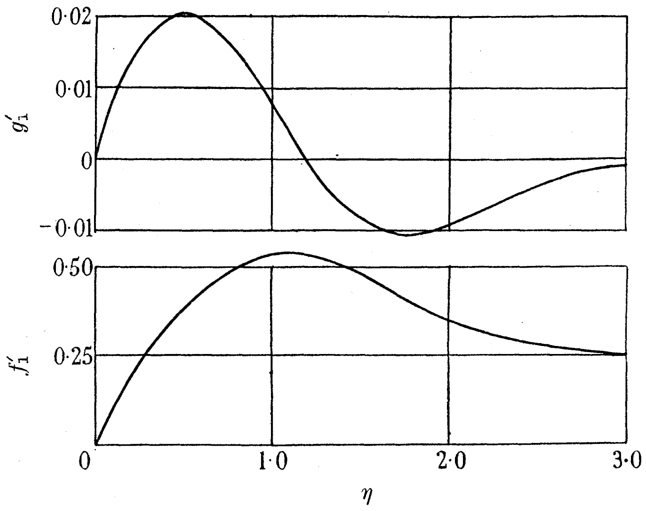

FIGURE 7

The function g_1 has been calculated and is tabulated in table 3; g_1' is shown graphically in figure 7 together with f_1' (on a different scale) for comparison; $g_1''(0)$ is equal to 0·0878. It will be seen that when $M_0 = 1$ the effect of compressibility is to increase numerically the coefficient of $8x^*$ in the velocity series in, say, the inner third of the boundary layer and to decrease it in the remainder. The maximum percentage increase is 6 at $\eta = 0{\cdot}5$ and the maximum percentage decrease is 3 at $\eta = 1{\cdot}8$. This then corresponds to a decrease of total velocity in the inner third and an increase in the remainder. Furthermore when $M_0 = 1$, we have from equation (123)

$$\frac{4\tau}{\rho_0 \nu_0^{\frac{1}{2}} U_0^{\frac{3}{2}}} = \left(\frac{p}{p_0}\right)^{\frac{1}{2}} [1\cdot3284 - (8x^*)(1\cdot1083) + O(8x^*)^2], \tag{124}$$

so that when $8x^* = 0\cdot1$ which is as far as we can safely proceed without further terms

$$\frac{4\tau}{\rho_0 \nu_0^{\frac{1}{2}} U_0^{\frac{3}{2}}} = 1\cdot228. \tag{125}$$

This value is substantially the same as that ($1\cdot226$) obtained at the same point in the incompressible solution. The decrease in skin friction associated with the velocity corrections in the x, Y plane is in fact in this case just offset by the scale correction factor $(p/p_0)^{\frac{1}{2}}$ in passing from the (x, Y) to the (x, y) plane.

The main interest of this problem presumably lies in obtaining the point of separation and this is of course determined completely by conditions in the x, Y plane since $\partial u/\partial Y$ and $\partial u/\partial y$ vanish together. Separation is therefore given by

$$f_0''(0) - 8x^*\{f_1''(0) + M_0^2 g_1''(0)\} + (8x^*)^2\{f_2''(0) + M_0^2 g_2''(0) + M_0^4 h_2''(0)\} + \ldots = 0. \tag{126}$$

The structure of the equations (118) and (119) introduced by compressibility considerations suggests that $g_2''(0)$ and $h_2''(0)$ will be appreciably smaller in magnitude than $g_1''(0)$ and of the same sign as $g_1''(0)$. Thus the effect of compressibility in the coefficient of $8x^*$ tends to move separation forward whilst the corresponding effect in the coefficient of $(8x^*)^2$ is to move it backwards. To see the order of the effects involved let us assume that all the compressibility terms save $g_1''(0)$ may be neglected when $M_0 = 1$. We then find, following out the same procedure as the incompressible work (Howarth 1938), that for separation x^* lies between $0\cdot115$ and $0\cdot123$ as compared with the interval $0\cdot119$ to $0\cdot129$ found for incompressible flow. It would appear then that when $M_0 = 1$ the separation point should not move forward by more than 4 % of the distance from the leading edge to the point of separation in incompressible flow.

It appears, in fact, to be impracticable to use this series method to determine the influence of compressibility on separation in view of the number of functions it would be necessary to calculate. It is, however, open to us to make use of the Kármán-Pohlhausen method which though not very accurate should be adequate to give results at least qualitatively correct.

With $U = U_0 - U_1 x$ equation (46) becomes

$$\frac{d\lambda}{dx} = -\frac{U_1}{U_0 - U_1 x}\left[g(\lambda) + \gamma M^2\{\lambda^2 h(\lambda) + \lambda\} + \frac{(\gamma-1)}{2} M^2 j(\lambda)\right], \tag{127}$$

where, changing the notation slightly from that in § 5, M is the local Mach number corresponding to the velocity $U_0 - U_1 x$. M is therefore a known function of x and equation (127) requires a numerical or graphical integration. However, as a first approximation since the velocity change from leading edge to separation in incompressible flow is known to be relatively small (the series approach gives 12 % and

Pohlhausen's method itself gives $15\frac{1}{2}$ %) it appears plausible to replace M^2 in (127) by a mean value $\overline{M^2}$. If we adopt this approximation then

$$\log\frac{(U_0-U_1x)}{U_0}=\int_0^\lambda\frac{d\lambda}{g(\lambda)+\gamma\overline{M^2}\{\lambda^2h(\lambda)+\lambda\}+\frac{1}{2}(\gamma-1)\,\overline{M^2}j(\lambda)},\qquad(128)$$

since λ is zero at the leading edge. The value x_s of x at separation is given then by

$$\log\left(1-\frac{U_1x_s}{U_0}\right)=-\int_{-12}^{0}\frac{d\lambda}{g(\lambda)+\gamma\overline{M^2}\{\lambda^2h(\lambda)+\lambda\}+\frac{1}{2}(\gamma-1)\,\overline{M^2}j(\lambda)}.\qquad(129)$$

This leads to the following values for U_1x_s/U_0 when $\gamma = 1\cdot4$:

$\overline{M}$	U_1x_s/U
0	0·156
1	0·148
$\sqrt{10}$	0·105
10	0·044

These values of U_1x_s/U_0 imply a change in M^2 from leading edge to separation of 31 % when $\overline{M} = 1$, 43 % when $\overline{M} = \sqrt{10}$ and 66 % when $\overline{M} = 10$. It therefore appears desirable to reject this approximation and to integrate equation (127) as it stands; this has been done graphically. The results obtained are shown in the following table where M_0 refers to the Mach number at the leading edge.

M_0	U_1x_s/U_0	Mach number at separation
0	0·156	0
1	0·148	0·83
$\sqrt{10} = 3\cdot16$	0·107	2·4
10	0·052	5·4

The results do not differ as much as might be expected at first sight from the approximate ones obtained above if $\overline{M}$ is interpreted as the leading edge Mach number. This is due to the fact that $g(\lambda)$ is least and $k(\lambda)$ is greatest at the leading edge so that the value of M^2 plays its most important part there.

The percentage movement of separation between $M_0 = 0$ and 1 is 5 and may be compared with the outside figure of 4 % obtained from the series solution.

†The changes in the position of separation as M_0 increases beyond 1 are quite large and indicate that, for the same mainstream *velocity distribution*, as the Mach number increases the point of separation moves forward quite rapidly in the supersonic region. A somewhat similar problem has been discussed by Illingworth (unpublished) who, assuming viscosity and conductivity constant, has considered the effect of Mach number on separation for a *given linearly increasing pressure* along a flat plate. Although not strictly comparable we can consider the relation between the two sets of results by imagining the incident pressure kept constant and effecting the changes in Mach number by incident density variations. Thus in the problem of the present paper we have, if the suffix zero refers to the leading edge

$$\frac{p}{p_0}=\left[1+\frac{(\gamma-1)}{2}M_0^2\left(1-\frac{U^2}{U_0^2}\right)\right]^{\gamma/(\gamma-1)},$$

† The contents of this paragraph have been amended in consequence of a query raised by Professor Goldstein.

and $$\frac{1}{p_0}\frac{\partial p}{\partial x} = \gamma M_0^2 \alpha(1-\alpha x)\left\{1+\frac{(\gamma-1)}{2}M_0^2(2\alpha x-\alpha^2x^2)\right\}^{1/(\gamma-1)},$$

where $\alpha = U_1/U_0$. The considerations of the present paper can thus be applied crudely to Illingworth's problem if we make α vary with M_0 so that αM_0^2 remains constant. Illingworth's results are calculated at $M_0 = 0{\cdot}1$, $1{\cdot}0$ and $\sqrt{10}$ so that if we take $\alpha = 1$ when $M_0 = 0{\cdot}1$ we must have $\alpha = 0{\cdot}01$ when $M_0 = 1$ and $\alpha = 0{\cdot}001$ when $M_0 = \sqrt{10}$. The results tabulated above then show that separation occurs (since $\alpha = U_1/U_0$) at $x = 0{\cdot}156$ when $M_0 = 0{\cdot}1$, at $x = 14{\cdot}8$ when $M_0 = 1$ and at $x = 107$ when $M_0 = \sqrt{10}$ (assuming as is reasonable that when $M_0 = 0{\cdot}1$ separation occurs at the incompressible value $0{\cdot}156$ for U_1x/U_0). Thus the distances from leading edge to separation at these three Mach numbers are in the ratios $1:95:686$ whereas Illingworth's values are in the ratio $1:118:4216$. The basis of this crude comparison becomes increasingly invalid as M_0 increases but it serves to substantiate qualitatively the large backward movements of separation found by Illingworth when the Mach number is increased and the pressure gradient remains constant. On the basis of this comparison it appears to be more convenient to consider compressibility effects on separation by fixing attention on a given mainstream velocity distribution rather than on a given pressure distribution since the magnitude of such effects is thereby much reduced.

11. General discussion of the theory and its results

The assumptions $\mu \propto T$ and $\sigma = 1$ have been shown to lead to a considerable simplification of the boundary layers equation in compressible flow. This simplification is achieved by a transformation

$$Y = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\int_0^y \frac{T_s}{T}\,dy \tag{130}$$

of the co-ordinate y normal to the boundary. In the (x, Y) plane the equation of motion can be transformed into

$$\frac{\partial^2\chi}{\partial x\,\partial Y}\frac{\partial\chi}{\partial Y}-\frac{\partial^2\chi}{\partial Y^2}\frac{\partial\chi}{\partial x} = U\frac{dU}{dx}\left[1+\frac{(\gamma-1)}{2a_1^2}\left\{U^2-\left(\frac{\partial\chi}{\partial Y}\right)^2\right\}-\frac{\gamma}{2a_1^2}\chi\frac{\partial^2\chi}{\partial Y^2}\right]+\nu_s\frac{\partial^3\chi}{\partial Y^3}, \tag{131}$$

which is identical with the standard incompressible form apart from the term

$$G \equiv 1+\frac{(\gamma-1)}{2a_1^2}\left\{U^2-\left(\frac{\partial\chi}{\partial Y}\right)^2\right\}-\frac{\gamma}{2a_1^2}\chi\frac{\partial^2\chi}{\partial Y^2} \tag{132}$$

in square brackets which is simply unity in that case. The boundary conditions are $\chi = 0$, $\partial\chi/\partial Y = 0$ when $Y = 0$, $\partial\chi/\partial Y \to U$ when $Y \to \infty$ and again are identical with the corresponding incompressible conditions. The relation between velocity and temperature is, since $\sigma = 1$,

$$Jc_pT+\tfrac{1}{2}u^2 = \text{const.} \tag{133}$$

The solution of a problem of compressible flow then falls into two parts. First of all we have to solve the flow equation (131) in the x, Y plane† and then we have to

† This is made possible by the form of the boundary conditions which are applied at 0 and ∞ and so are independent of the (unknown) Y scale.

transform the results into the actual x, y plane by means of the inverse transformation.

$$y = \left(\frac{p_s}{p}\right)^{\frac{1}{2}} \int_0^Y \frac{T}{T_s} dY, \tag{134}$$

and the energy integral (133).

In problems in which dU/dx vanishes such as flow along a flat plate and flow in wakes and jets equation (131) is identical with the incompressible equation and the co-ordinate Y can be interpreted as the distance y_i normal to the layer in the incompressible flow.

The effects of compressibility are then determined entirely by the change of scale given by equation (134). The skin friction along the flat plate is found to be independent of Mach number but the boundary layer thickness increases considerably being double the incompressible thickness at a Mach number of 3·5.

In all other problems the alterations produced by the factor G and the change of scale have both to be obtained. Two examples have been considered in detail. First, for a flow in which the velocity in the mainstream increases linearly with distance from a stagnation point a solution of (131) in series has been obtained. Up to mainstream Mach numbers of unity the solution of (131) differs but slightly from the corresponding incompressible solution. Here again then the most important changes which occur arise from the change of scale which again leads to a thickening of the boundary layer and in this problem to a reduction by a factor $(p/p_0)^{\frac{1}{2}}$ in the skin friction p being the mainstream pressure and p_0 the stagnation pressure.

Secondly, the problem of flow along a flat plate with a linearly retarding pressure gradient has been considered both by the series method and Pohlhausen's. Separation which is one of the features of such a flow is determined entirely by equation (131) since $\partial u/\partial y$ and $\partial u/\partial Y$ vanish together. The change of scale is important in determining the boundary layer thickness and the variation of skin friction along the surface up to the point of separation. It is fairly evident on general grounds that for a given retarding mainstream velocity distribution (131) will lead to a forward movement of separation with increasing Mach number. For, the effect of the pressure gradient term $U(dU/dx)$ is enhanced by a factor $G_0 = 1 + \frac{(\gamma - 1)}{2a_1^2} U^2$ at the boundary so that even though the term G may fall below unity in the outer part of the layer one would expect earlier separation.

This prediction is certainly fulfilled in the particular problem solved. Pohlhausen's method gives a forward movement of separation of 5 % at $M = 1$, 33 % at $M = 10$ and 66 % at $M = 10$ these percentages being percentages of the distance from leading edge to separation in incompressible flow.

We can consider qualitatively the general effects of compressibility in the following way. Let us fix attention on a certain mainstream velocity distribution say one which rises from zero at a stagnation point to a maximum and then falls off again. Let us imagine that this distribution is maintained unaltered and take as our standard conditions those in the mainstream at the stagnation point and denote them by a suffix zero. Then we can examine compressibility effects by considering a sequence of values of a_0; increasing compressibility will of course correspond to decreasing a_0.

Considering a particular value of a_0, the mainstream Mach number U/a_1 is given by $\dfrac{U}{\{a_0^2-\frac{1}{2}(\gamma-1)\,U^2\}^{\frac{1}{2}}}$ and the pressure ratio p/p_0 by

$$\frac{p}{p_0}=\left[1-\frac{(\gamma-1)}{2}\frac{U^2}{a_0^2}\right]^{\gamma/(\gamma-1)}.$$

The true ordinate y is determined from Y by the relation

$$y=\left(\frac{p_0}{p}\right)^{\frac{1}{2}}\int_0^Y\frac{T}{T_0}\,dY=\left(\frac{p_0}{p}\right)^{\frac{1}{2}}\left[Y-\frac{(\gamma-1)}{2a_0^2}\int_0^Y u^2\,dY\right]$$

and the skin friction τ by

$$\tau=\mu_w\left(\frac{\partial u}{\partial y}\right)_w=\mu_0\left(\frac{p}{p_0}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial Y}\right)_0.$$

If we start by considering a_0 very large we have the ordinary incompressible solution (with kinematic viscosity $\nu_0=\mu_0/\rho_0$) which gives in particular $Y=y_i$ (the incompressible ordinate) and a distribution of skin friction which increases from zero to a maximum and then falls off to zero again if the retarding pressure gradient is sufficiently maintained.

Now consider a finite value of a_0. The velocity u in the x, Y plane will differ from the incompressible velocity u_i at the same point of x, y_i plane by an amount depending on the factor G. In the accelerated region one would expect u to increase more rapidly than u_i near the boundary since $G>1$ and then to increase more slowly than u_i in the outer part of the layer. In the retarded layer on the other hand one would expect u to increase less rapidly than u_i near the boundary and more rapidly in the outer part of the layer. Thus we should expect $(\partial u/\partial Y)_0$ to be greater than $(\partial u_i/\partial y)_0$ in the accelerated region and less in the retarded region. The factor $(p/p_0)^{\frac{1}{2}}$ will be less than or equal to unity everywhere and hence we should expect a decreased skin friction coefficient in the retarded region on both counts.† As remarked above separation is determined entirely by $(\partial u/\partial Y)_0$ and would on the whole be expected to occur earlier than in the incompressible flow. For although there is some increase in $(\partial u/\partial Y)_0$ in the accelerated region, if the two examples discussed are typical, this increase will be more than offset by the decrease in the retarded region.

In the accelerated region the two factors affecting the skin friction produce opposite effects and it is not possible to draw any general conclusions. If the accelerated flow solution in § 8 can be taken as a guide the factor $(p/p_0)^{\frac{1}{2}}$ predominates and leads here too to a reduction in skin friction coefficient.

The actual boundary layer ordinate is given by

$$y=\left(\frac{p_0}{p}\right)^{\frac{1}{2}}\left[Y-\frac{(\gamma-1)}{2a_0^2}\int_0^Y u^2\,dY\right]>\left(\frac{p_0}{p}\right)^{\frac{1}{2}}Y\left[1-\frac{(\gamma-1)}{2a_0^2}U^2\right]$$
$$\doteqdot y_i\left[1-\frac{(\gamma-1)}{2a_0^2}U^2\right]^{-(2-\gamma)/2(\gamma-1)}.$$

† This statement is not at variance with the results found for the flat plate with a retarding pressure gradient in § 10 for there standard conditions were defined by values at the leading edge and not stagnation values as here. Thus the incompressible solution taken as standard in the present paragraph corresponds to higher kinematic viscosity and hence to higher skin friction than that referred to in § 10.

Thus the boundary layer ordinate y is always greater than the corresponding ordinate y_i of the incompressible flow and one would therefore expect increased boundary layer thickness throughout the whole layer.

To sum up then, the effects of compressibility on such a boundary layer would appear to be

(i) a reduced skin friction coefficient everywhere,

(ii) an increased boundary layer thickness,

(iii) an earlier separation,

as compared with the corresponding problem in incompressible flow with kinematic viscosity $\nu_0 = \mu_0/\rho_0$.

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Intensity measurements in the very soft X-ray region

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In the grazing incidence grating spectroscopy of the soft X-ray region, it is customary to measure the intensity distribution in the emission bands from solids by methods involving the assumption of the photographic reciprocity law. A procedure has been evolved by which this assumption is avoided. An account is given of the experimental technique and of the essential precautions. Characteristic blackening curves are obtained for the photographic plates in the soft X-ray region and under the actual conditions of spectroscopy.

Intensity distribution curves have been obtained for the K emission bands of graphite, lampblack, diamond and carborundum. The emission band of lampblack is similar to that of graphite but is somewhat wider. The difference between the present curves and those previously obtained appears most clearly in diamond and carborundum. The energy distribution of the p electrons in the valence bands of the solids is calculable from the curves.

Introduction

The discovery, by Compton & Doan, that X-ray spectra of adequate intensity and adequate dispersion could be obtained with a ruled grating in the grazing incidence position, made possible the spectroscopy of X-rays of wave-lengths greater than 20 A. With the method of calibration by the vacuum spark, and with the replacement of plane by concave gratings, the modern soft X-ray spectrographs have