1 Problem statement and formulation

Given a nonlinear heat conduction problem, solve for the sensitivity of the solution to different parameters which affect the thermal conductivity. The problem can be written as

$$\frac{d}{dx}\left(k(T)\frac{dT}{dx}\right) = 0 \tag{1}$$

subject to

$$T(0) = T_L \quad \text{and} \quad T(L) = T_R \tag{2}$$

The thermal conductivity is represented by two piecewise linear segments:

$$k(T) = \begin{cases} k_1\left(1 - \dfrac{T - T_1}{T_2 - T_1}\right) + k_2\left(\dfrac{T - T_1}{T_2 - T_1}\right), T_1 \leqslant T \leqslant T_2 \\ k_2\left(1 - \dfrac{T - T_2}{T_3 - T_2}\right) + k_3\left(\dfrac{T - T_2}{T_3 - T_2}\right), T_2 < T \leqslant T_3 \end{cases} \tag{3}$$

Defining the residual as

$$\mathcal{R} = \frac{d}{dx}\left(k(T)\frac{dT}{dx}\right) \tag{4}$$

then multiplying by a weight functon and integrating over an element,

$$\pi = \int_\Omega w\frac{d}{dx}\left(k(T)\frac{dT}{dx}\right)\, d\Omega \tag{5}$$

Next, we apply integration by parts,

$$\pi = \oint_\Gamma k(T)w\frac{dT}{dx}\hat{\mathbf{n}}\, d\Gamma - \int_\Omega k(T)\frac{dw}{dx}\frac{dT}{dx}\, d\Omega \tag{6}$$

Now, the weight function can be written as

$$w(x) = \sum_j V_j N_j \tag{7}$$

where N_j represents the shape functions and V_j represents a vector of unknown coefficients. Using the same shape functions for the approximation function:

$$\widetilde{T}(x) = \sum_i c_i N_i \tag{8}$$

Inserting these into (6) and assuming no Neumann boundary conditions,

$$\pi = -\int_\Omega k(T)\left[\sum_j V_j\frac{dN_j}{dx}\right]\left[\sum_i c_i\frac{dN_i}{dx}\right]\, d\Omega \tag{9}$$

Now, making the functional stationary with respect to V_j and dropping the summations for clarity,

$$\frac{\partial \pi}{\partial V_j} = \left[\int_\Omega k(T) \frac{dN_i}{dx} \frac{dN_j}{dx} \, d\Omega \right] c_i = 0 \tag{10}$$

In this case, since the Lagrange basis is used, the coefficients represent temperatures at the nodes. Integrating over the element in the computational domain,

$$\left[\int k(T(x)) \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} J \, d\xi \right] c_i = 0 \tag{11}$$

This problem can be written as

$$\mathbf{K}(\mathbf{u}) \, \mathbf{u} = \mathbf{f} \tag{12}$$

where $\mathbf{u}$ is a vector which holds all the coefficients and $\mathbf{f}$ is full of zeros because there is no source term. Because the problem is nonlinear, the stiffness matrix is a function of the solution $\mathbf{u}$. Piccard iteration is used to iterative solve in the following manner:

$$\mathbf{K}(\mathbf{u}^n)\mathbf{u}^{n+1} = \mathbf{f} \tag{13}$$

2 Derivation of SACVM

If we want to calculate the sensitivities of the solution, $\mathbf{u}$, with respect to some parameters, $\mathbf{x}$, we can do so using the following

$$\mathbf{K}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{K}}{\partial \mathbf{x}}\mathbf{u} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{14}$$

We may then solve for the sensitivity as

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{K}^{-1} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{K}}{\partial \mathbf{x}}\mathbf{u} \right) \tag{15}$$

Writing the derivatives using the Piccard iteration and dropping the load vector due to zero source terms,

$$\left(\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{x}} \right)^{n+1} = -\mathbf{K}(\bar{\mathbf{u}})^{-1} \left(\frac{\partial \mathbf{K}(\bar{\mathbf{u}})}{\partial \mathbf{x}} \right)^n \bar{\mathbf{u}} \tag{16}$$

where the overbar denotes the solution to the original nonlinear problem. We must now find a way to compute the gradient of the stiffness matrix with respect to the parameters. The complex variable method for approximating derivatives can be written as

$$\frac{df}{dx} = \frac{\mathrm{Im}[f(x+ih)]}{h} + \mathcal{O}[h^2]$$

Now, applying the complex variable method to the derivative of the stiffness matrix,

$$\left(\frac{\partial \mathbf{K}(\bar{\mathbf{u}})}{\partial \mathbf{x}}\right)^n = \underbrace{\frac{\operatorname{Im}\left[\mathbf{K}(\bar{\mathbf{u}} + i\mathbf{h}_{\mathbf{u}})\right]}{\mathbf{h}_{\mathbf{u}}}}_{\frac{\partial \mathbf{K}(\mathbf{u})}{\partial \mathbf{u}}} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^n \tag{17}$$

This $\mathbf{h}_{\mathbf{u}}$ is the perturbation in temperature due to the perturbation in the thermal conductivity. *Given a perturbation in thermal conductivity, we must solve for the corresponding perturbation in the temperature field.*

Another way to look at it:

$$[\Delta \mathbf{u}]^{n+1} = -\mathbf{K}(\bar{\mathbf{u}})^{-1} \left[\Delta \mathbf{K}(\bar{\mathbf{u}})\right]^n \bar{\mathbf{u}}$$

$$[\Delta \mathbf{u}]^{n+1} = -\mathbf{K}(\bar{\mathbf{u}})^{-1} \operatorname{Im}\left[\mathbf{K}(\bar{\mathbf{u}} + i[\Delta \mathbf{u}]^n)\right] \bar{\mathbf{u}}$$