Tensor categories

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Introduction

In many fields of mathematics, one is naturally led to study tensor products of certain objects (e.g. sheaves in algebraic geometry, cobordisms in topology, modules in commutative algebra). All of these notions fit into the framework of *monoidal categories*, which gives an abstract definition of what a tensor product structure on a category should be. The aim of this course is to give an introduction to monoidal categories and tensor categories, the latter being certain monoidal categories endowed with some extra structure. (They are abelian and have a linear structure that is compatible with the tensor product, over some field.)

The first half of the course deals mainly with category-theoretical notions, starting from the definition of a monoidal category and then discussing important additional properties such as rigidity (the existence of duals) and braidings (functorial isomorphisms $X \otimes Y \cong Y \otimes X$). In the second half, we will turn our attention to tensor categories. Our prime example is the category of $\mathbf{Rep}_{\Bbbk}(G)$ finite-dimensional representations of a group G over a field $\Bbbk$, and we will discuss reconstruction theorems that allow us to recover a group (or Hopf algebra) from the corresponding category of representations, together with its monoidal structure and the *forgetful functor* that sends a representation to the underlying vector space. This gives rise to bijections between certain types of groups and Hopf algebras, up to isomorphism, and certain kinds of tensor categories, up to monoidal equivalence, which are broadly referred to as *Tannaka duality.*

Author's note

These notes are the my own synopsis of material that has been collected from many different sources, but most importantly, from [EGNO15]. Further important references include [ML98, DM82, EGNO09] and some websites such as nLab, MathStackExchenge, and Wikipedia. No originality is claimed, except in the presentation of the material, and all mistakes should be considered my responsibility.

These notes are also a work in progress. If you find any mistakes or typos and if you have comments or suggestions, please let me know.

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1 Monoidal categories

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) a class $\mathrm{Ob}(\mathcal{C})$ of *objects* of $\mathcal{C}$;

(2) for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *homomorphisms* from A to B;

(3) for every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition map*

$$\circ \colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

such that the following axioms hold:

(a) for $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon Z \to W$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(b) for $X \in \mathrm{Ob}(\mathcal{C})$, there exists an *identity homomorphism* id_X such that $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f\colon Y \to X$ and $g\colon X \to Y$.

Remark 1.2. (1) The homomorphisms in a category are often simply referred to as *morphisms*.

(2) As in points (a) and (b) of the definition, we often denote a homomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ by an arrow $f\colon X \to Y$.

(3) A morphism $f\colon X \to Y$ is called an *isomorphism* if there exists a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In that case, we write $g = f^{-1}$ and $X \cong Y$.

Example 1.3. We list some important examples of categories:

- **Set**: the category of sets, with maps between sets as homomorphisms;
- **Grp**: the category of groups with group homomorphisms;
- **AbGrp**: the category of abelian groups with group homomorphisms;
- $\mathbf{Vect}_{\Bbbk}$: the category of finite-dimensional $\Bbbk$-vector spaces with $\Bbbk$-linear maps, for a given field $\Bbbk$;
- $A-$**Mod**: the category of A-modules with A-module homomorphisms, for a given algebra A; we write $A-$**mod** for the subcategory of finite-dimensional A-modules.

Definition 1.4. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns

(1) to every object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$;

(2) to every homomorphism $f\colon X \to Y$ in $\mathcal{C}$ a homomorphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$;

in such a way that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \qquad \text{and} \qquad F(f \circ g) = F(f) \circ F(g).$$

Remark 1.5. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the product category $\mathcal{C} \times \mathcal{D}$ whose objects are the pairs (X, Y) of objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, and where

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((X, Y), (Z, W)\big) = \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{D}}(Y, W)$$

for $X, Z \in \mathrm{Ob}(\mathcal{C})$ and $Y, W \in \mathrm{Ob}(\mathcal{D})$. The composition and the identity morphisms are defined component-wise in the obvious way. A functor from $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ to some other category $\mathcal{E}$ is often called a *bifunctor*.

Definition 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta\colon F \to G$ is a family of morphisms $\eta_A\colon F(A) \to G(A)$ in $\mathcal{D}$, for every object A of $\mathcal{C}$, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

For an object A of $\mathcal{C}$, we call η_A the component of η at A. A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

Remark 1.7. The functors between two categories $\mathcal{C}$ and $\mathcal{D}$ form a category $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ whose morphisms are natural transformations between functors. The composition of natural transformations is defined componentwise.

Definition 1.8. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an equivalence if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Definition 1.9. A *monoidal category* is a tuple $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, where

- $\mathcal{C}$ is a category,
- $\mathbf{1}$ is an object of $\mathcal{C}$, called the *unit object*,
- $\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor called the *tensor product*,
- $a\colon\ - \otimes(- \otimes -) \to (- \otimes -) \otimes -$ is a natural isomorphism, called the *associativity constraint*,
- $\lambda\colon \mathbf{1} \otimes - \to \mathrm{id}_{\mathcal{C}}$ and $\rho\colon\ - \otimes \mathbf{1} \to \mathrm{id}_{\mathcal{C}}$ are natural isomorphisms, called the (left and right) *unitors*,

subject to the following axioms:
Pentagon axiom: For all objects A, B, C, D of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[rr, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \arrow[dl, "\alpha_{A,B,C\otimes D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & & (A \otimes (B \otimes C)) \otimes D \arrow[dll, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \\
& & ((A \otimes B) \otimes C) \otimes D & &
\end{tikzcd}$$

In other words, we have

$$\big(\alpha_{A,B,C\otimes D}\big) \circ \big(\alpha_{A,B\otimes C,D}\big) \circ \big(\mathrm{id}_A \otimes \alpha_{B,C,D}\big) = \big(\alpha_{A\otimes B,C,D}\big) \circ \big(\alpha_{A,B,C} \otimes \mathrm{id}_D\big).$$

Unit axiom / triangle axiom: For all objects A, B of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
A \otimes (\mathbf{1} \otimes B) \arrow[rr, "\alpha_{A,\mathbf{1},B}"] \arrow[dr, "\mathrm{id}_A \otimes \lambda_B"'] & & (A \otimes \mathbf{1}) \otimes B \arrow[dl, "\rho_A \otimes \mathrm{id}_B"] \\
& A \otimes B &
\end{tikzcd}$$

In other words, we have

$$(\rho_A \otimes \mathrm{id}_B) \circ \alpha_{A,\mathbf{1},B} = \mathrm{id}_A \otimes \lambda_B.$$

Remark 1.10. (1) When no confusion is possible, we simply write $\mathcal{C}$ instead of $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$. We also say that $(\mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is a *monoidal structure* on the category $\mathcal{C}$.

(2) Being monoidal is **not** a property of a given category, but an *additional structure*. A category can admit more than one monoidal structure. (See the examples below.)

(3) The fact that $\otimes$ is a bifunctor means that for suitable morphisms a, b, c, d in $\mathcal{C}$, we have

$$(a \otimes c) \circ (b \otimes d) = (a \circ b) \otimes (c \circ d).$$

(4) Instead of assuming the existence of $\mathbf{1}$ with natural transformations λ and ρ subject to the unit axiom, one can start from the (seemingly weaker, but actually equivalent) assumptions that there exists an object $\mathbf{1}$ with an isomorphism $\iota\colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ such that the functors $\mathbf{1} \otimes -$ and $- \otimes \mathbf{1}$ are equivalences. (No additional assumptions on the isomorphism ι are necessary.) See Sections 2.1 and 2.2 of [EGNO15] for more details.

Example 1.11. (1) The category **Set** of sets has a monoidal structure where the tensor product is given by the cartesian product and the unit object is a singleton $\{\bullet\}$.

(2) The category **Grp** of groups has a monoidal structure where the tensor product is given by the cartesian product and the unit object is the trivial group $\{1\}$.

(3) The category **AbGrp** of abelian groups has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\mathbb{Z}} -$ and the unit object is the group $\mathbb{Z}$ of integers. It also inherits a different monoidal structure from the category **Grp**; see the previous point.

(4) The category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$ has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\Bbbk} -$ and the unit object is the one-dimensional vector space $\Bbbk$.

(5) The category $\mathbf{Rep}_{\Bbbk}(G)$ of finite dimensional representations of a group G over a field $\Bbbk$ admits a monoidal structure where the tensor product is the usual tensor product of representations and the unit object is the trivial one-dimensional representation $\Bbbk$. More precisely, if we identify $\mathbf{Rep}_{\Bbbk}(G)$ with the category of finite-dimensional modules over the group algebra $\Bbbk[G]$ then the action of $\Bbbk[G]$ on the tensor product $M \otimes N$ of two $\Bbbk[G]$-modules M and N is uniquely determined by $g \cdot (m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$.

In the following, we refer to the objects of $\mathbf{Rep}(G)$ as *G-modules*.

(6) For a category $\mathcal{C}$, the category $\mathbf{End}(\mathcal{C}) = \mathbf{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors of $\mathcal{C}$ has a monoidal structure, where the tensor product is given by the composition of functors and the unit object is the identity functor $\mathrm{id}_{\mathcal{C}}$. The associativity constraints and unitors are identity natural transformations.

The two next examples will seem quite trivial for now, but they will become more interesting later when we add extra structure:

(7) Let G be a monoid and A an abelian group. Then we can define a monoidal category $\mathcal{C}_A^G$ with objects $\mathrm{Ob}(\mathcal{C}_A^G) = \{\delta_g \mid g \in G\}$ indexed by G and homomorphisms

$$\mathrm{Hom}_{\mathcal{C}_A^G}(\delta_g, \delta_h) = \begin{cases} A & \text{if } g = h, \\ \varnothing & \text{otherwise,} \end{cases}$$

for $g, h \in G$. The tensor product is defined by $\delta_g \otimes \delta_h = \delta_{gh}$ and by $a \otimes a' = aa' \in A$ for $g, h \in G$ and $a, a' \in A$, the unit object is δ_e (where $e \in G$ is the unit element) and the associativity constraints and unitors are identity maps.

(8) Let G be a monoid and let $\mathbf{Vect}_{\Bbbk}^G$ be the category of finite-dimensional G-graded $\Bbbk$-vector spaces $V = \bigoplus_{g \in G} V_g$, with homomorphisms given by grading-preserving linear maps. (That is, for $V = \bigoplus_g V_g$ and $W = \bigoplus_g W_g$ two G-graded vector spaces, the homomorphisms from V to W in $\mathbf{Vect}_{\Bbbk}^G$ are the linear maps $f\colon V \to W$ that satisfy $f(V_g) \subseteq W_g$ for all $g \in G$.) Then the

monoidal structure on $\mathbf{Vect}_{\Bbbk}$ induces a monoidal structure $\mathbf{Vect}_{\Bbbk}^{G}$, where the grading on the tensor product of G-graded vector spaces V and W is given by

$$(V \otimes W)_g = \bigoplus_{hh'=g} V_h \otimes W_{h'}$$

for $g \in G$. The tensor product of two homomorphisms in $\mathbf{Vect}_{\Bbbk}^{G}$ is just the usual tensor product of linear maps. The unit object is the one-dimensional vector space $\Bbbk = \Bbbk_e$ whose unique non-zero grading piece is indexed by the unit object $e \in G$. The associativity constraint and the unitors come from the category $\mathbf{Vect}_{\Bbbk}$.

Observe that there is a faithful functor $i_{\Bbbk}^{G} \colon \mathcal{C}_{\Bbbk^\times}^{G} \to \mathbf{Vect}_{\Bbbk}^{G}$ with $i_{\Bbbk}^{G}(\delta_g) = \Bbbk_g$ the one-dimensional vector space with grading concentrated in degree g, for $g \in G$, and with the obvious definition on homomorphisms. This functor is compatible with the tensor product (up to a natural isomorphism); it is an example of a *monoidal functor* (to be defined shortly).

In our final example, we demonstrate that there are monoidal categories with a less obvious choice of associativity constraint.

(9) Let G be a monoid, let A an abelian group and let ω be a 3-cocycle for G with values in A, i.e. a map $\omega \colon G^{\times 3} \to A$ with

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) \tag{1.1}$$

for $g_1, g_2, g_3, g_4 \in G$. Then we can define a monoidal category $\mathcal{C}_A^{G,\omega}$ with underlying category $\mathcal{C}_A^{G}$ and with tensor product and unit object defined as in point (7), but with associativity constraint α^ω defined by

$$\alpha^\omega_{g,h,k} = \omega(g,h,k) \colon \delta_g \otimes (\delta_h \otimes \delta_k) = \delta_{ghk} \longrightarrow \delta_{ghk} = (\delta_g \otimes \delta_h) \otimes \delta_k$$

for $g, h, k \in G$. Observe that the 3-cocycle condition implies that $\mathcal{C}_A^{G,\omega}$ satisfies the pentagon axiom. (In fact, a map $\omega \colon G^{\times 3} \to A$ defines an associativity constraint for $\mathcal{C}_A^{G}$ if and only if ω is a 3-cocycle.) The unitors are defined by $\lambda_g = \omega(e,e,g)$ and $\rho_g = \omega(g,e,e)^{-1}$ for $g \in G$, and the unit axiom becomes the equation $\omega(g,e,h) = \omega(g,e,e) \cdot \omega(e,e,h)$ for $g, h \in G$ (which also follows from (1.1) by setting $g_2 = g_3 = e$).

Given a 3-cocycle $\omega \colon G^{\times 3} \to \Bbbk^\times$ with values in the multiplicative group of a field $\Bbbk$, we can extend the associativity constraint α^ω on $\mathcal{C}_{\Bbbk^\times}^{G}$ to an associativity constraint α^ω on $\mathbf{Vect}_{\Bbbk}^{G}$ via

$$\alpha^\omega_{\Bbbk_g,\Bbbk_h,\Bbbk_k} = \omega(g,h,k) \cdot \alpha_{\Bbbk_g,\Bbbk_h,\Bbbk_k},$$

for $g, h, k \in G$, extended by additivity, where α denotes the 'usual' associativity constraint in $\mathbf{Vect}_{\Bbbk}^{G}$. (Note that every object of $\mathbf{Vect}_{\Bbbk}^{G}$ is a direct sum of objects of the form $\Bbbk_g$ with $g \in G$.)

Remark 1.12. Given a monoidal category $\mathcal{C} = (\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, we define the *opposite monoidal category* $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}, \mathbf{1}, \otimes^{\mathrm{op}}, \alpha^{\mathrm{op}}, \rho, \lambda)$ with the same underlying category, but tensor product defined by $X \otimes^{\mathrm{op}} Y = Y \otimes X$ and $f \otimes^{\mathrm{op}} g = g \otimes f$ for objects X, Y and homomorphisms f, g in $\mathcal{C}$, and with associativity constraint given by $\alpha^{\mathrm{op}}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$ for objects X, Y, Z of $\mathcal{C}$.

This is not to be confused with the *reverse category* $\mathcal{C}^{\mathrm{rev}}$ with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. The latter can also be endowed with a canonical monoidal structure. Note that $\mathcal{C}^{\mathrm{rev}}$ is often also called the opposite category of $\mathcal{C}$; we use non-standard terminology here to avoid confusion with the opposite monoidal category defined above.

Lemma 1.13. *For all objects A, B of $\mathcal{C}$, we have*

$$\rho_{A\otimes B} \circ \alpha_{A,B,\mathbf{1}} = \mathrm{id}_A \otimes \rho_B \qquad \text{and} \qquad \lambda_{A\otimes B} \circ \alpha_{\mathbf{1},A,B} = \lambda_A \otimes \mathrm{id}_B.$$

Proof. Consider the following diagram, where all arrows are isomorphisms:

$$\begin{tikzcd}
A \otimes (B \otimes (\mathbf{1} \otimes D)) \arrow[rrrr, "\mathrm{id}_A \otimes \alpha_{B,\mathbf{1},D}"] \arrow[drr, "\mathrm{id}_A \otimes (\mathrm{id}_B \otimes \lambda_D)"'] \arrow[ddd, "\alpha_{A,B,\mathbf{1}\otimes D}"'] & & & & A \otimes ((B \otimes \mathbf{1}) \otimes D) \arrow[dll, "\mathrm{id}_A \otimes (\rho_B \otimes \mathrm{id}_D)"] \arrow[ddd, "\alpha_{A,B\otimes\mathbf{1},D}"] \\
& & A \otimes (B \otimes D) \arrow[d, "\alpha_{A,B,D}"'] & & \\
& & (A \otimes B) \otimes D & & \\
(A \otimes B) \otimes (\mathbf{1} \otimes D) \arrow[urr, "\mathrm{id}_{A\otimes B} \otimes \lambda_D"] \arrow[drr, "\alpha_{A\otimes B,\mathbf{1},D}"'] & & & & (A \otimes (B \otimes \mathbf{1})) \otimes D \arrow[ull, "(\mathrm{id}_A \otimes \rho_B) \otimes \mathrm{id}_D"'] \arrow[dll, "\alpha_{A,B,\mathbf{1}} \otimes \mathrm{id}_D"] \\
& & ((A \otimes B) \otimes \mathbf{1}) \otimes D \arrow[uu, "\rho_{A\otimes B} \otimes \mathrm{id}_D"'] & &
\end{tikzcd}$$

The external pentagon commutes by the pentagon axiom, the quadrangles commute by naturality of the associativity constraint α, and the top triangle and the bottom left triangle commute by the triangle axiom. Since all arrows are isomorphisms, this implies that the bottom right triangle commutes. Setting $D = \mathbf{1}$ and using the natural isomorphism $\rho\colon - \otimes \mathbf{1} \to \mathrm{id}_{\mathcal{C}}$, it follows that the following diagram commutes:

$$\begin{tikzcd}
A \otimes (B \otimes \mathbf{1}) \arrow[rr, "\alpha_{A,B,\mathbf{1}}"] \arrow[dr, "\mathrm{id}_A \otimes \rho_B"'] & & (A \otimes B) \otimes \mathbf{1} \arrow[dl, "\rho_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

This proves the first claim, the second claim can be proven analogously. ☐

Definition 1.14. Let $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \mathbf{1}', \otimes', \alpha', \lambda', \rho')$ be two monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triple $(F, \varphi, \varepsilon)$, where $F\colon \mathcal{C} \to \mathcal{C}'$ is a functor, $\varphi\colon F(- \otimes -) \to F(-) \otimes' F(-)$ is a natural isomorphism and $\varphi\colon F(\mathbf{1}) \to \mathbf{1}'$ is an isomorphism, such that the following diagrams commute:

$$\begin{tikzcd}
& F\big(A \otimes (B \otimes C)\big) \arrow[rr, "F(\alpha_{A,B,C})"] \arrow[dl, "\varphi_{A,B\otimes C}"'] & & F\big((A \otimes B) \otimes C\big) \arrow[dr, "\varphi_{A\otimes B,C}"] & \\
F(A) \otimes' F(B \otimes C) \arrow[dr, "\mathrm{id}_{F(A)} \otimes \varphi_{B,C}"'] & & & & F(A \otimes B) \otimes' F(C) \arrow[dl, "\varphi_{A,B} \otimes \mathrm{id}_{F(C)}"] \\
& F(A) \otimes' \big(F(B) \otimes' F(C)\big) \arrow[rr, "\alpha'_{F(A),F(B),F(C)}"'] & & \big(F(A) \otimes' F(B)\big) \otimes' F(C) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(A\otimes \mathbf{1}) \arrow[r, "\varphi_{A,\mathbf{1}}"] \arrow[d, "F(\rho_A)"'] & F(A)\otimes' F(\mathbf{1}) \arrow[d, "\mathrm{id}_{F(A)}\otimes' \varepsilon"'] \\
F(A) & F(A)\otimes' \mathbf{1}' \arrow[l, "\rho'_{F(A)}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathbf{1}\otimes A) \arrow[r, "\varphi_{\mathbf{1},A}"] \arrow[d, "F(\lambda_A)"] & F(\mathbf{1})\otimes' F(A) \arrow[d, "\varepsilon\otimes' \mathrm{id}_{F(A)}"] \\
F(A) & \mathbf{1}'\otimes' F(A) \arrow[l, "\lambda'_{F(A)}"]
\end{tikzcd}$$

A *monoidal natural transformation* between monoidal functors (F,φ,ε) and $(F',\varphi',\varepsilon')$ is a natural transformation $\psi\colon F\to F'$ such that

$$\varphi'\circ\psi_{-\otimes-} = \psi\otimes'\psi\circ\varphi \qquad \text{and} \qquad \varepsilon=\varepsilon'\circ\psi_{\mathbf{1}},$$

i.e. the following diagrams commute for all objects A,B of $\mathcal{C}$:

$$\begin{tikzcd}
F(A\otimes B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\psi_{A\otimes B}"'] & F(A)\otimes' F(B) \arrow[d, "\psi_A\otimes'\psi_B"'] \\
F'(A\otimes B) \arrow[r, "\varphi'_{A,B}"'] & F'(A)\otimes' F'(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathbf{1}) \arrow[r, "\psi_{\mathbf{1}}"] \arrow[dr, "\varepsilon"'] & F'(\mathbf{1}) \arrow[d, "\varepsilon'"] \\
& \mathbf{1}'
\end{tikzcd}$$

Remark 1.15. (1) Being monoidal for a functor is an additional structure, and not a property. However, being monoidal for a natural transformation is a property.

(2) The composition of two monoidal functors $(F,\varphi,\varepsilon)\colon \mathcal{C}\to\mathcal{D}$ and $(F',\varphi',\varepsilon')\colon \mathcal{D}\to\mathcal{E}$ can be considered as a monoidal functor with structure maps defined as follows, for objects X,Y of $\mathcal{C}$:

$$F'\big(F(X\otimes Y)\big) \xrightarrow{F'(\varphi_{X,Y})} F'\big(F(X)\otimes F(Y)\big) \xrightarrow{\varphi'_{F(X),F(Y)}} F'\big(F(X)\big)\otimes F'\big(F(Y)\big)$$

$$F'\big(F(\mathbf{1})\big) \xrightarrow{F'(\varepsilon)} F'(\mathbf{1}) \xrightarrow{\varepsilon'} \mathbf{1}.$$

(3) Given a monoidal functor $(F,\varphi,\varepsilon)\colon\mathcal{C}\to\mathcal{D}$ such that F is an equivalence of categories, one can show that it is possible to choose a monoidal functor $(G,\psi,\epsilon)\colon\mathcal{D}\to\mathcal{C}$ such that there are monoidal natural isomorphisms $F\circ G\to \mathrm{id}_{\mathcal{D}}$ and $G\circ F\to\mathrm{id}_{\mathcal{C}}$. In that case, we say that $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent.* For more details, see Proposition 4.4.2 in [SR72].

Example 1.16. (1) For a group G and a field $\Bbbk$, the forgetful functor $F\colon \mathbf{Rep}_{\Bbbk}(G)\to\mathbf{Vect}_{\Bbbk}$ which sends a G-module to the underlying vector space is monoidal. (The structure maps φ and ε are identity maps.) For a G-module V and for $g\in G$, let us write $\varphi(g)_V\in\mathrm{End}_{\Bbbk}(V)$ for the action of g on V. Then $\varphi(g)$ defines a natural transformation from the functor F to itself; we write $\varphi(g)\in\mathrm{End}(F)$. (This follows from the fact that for a homomorphism $f\colon V\to W$ of G-modules, the equality $\varphi(g)_W\circ F(f)=F(f)\circ\varphi(g)_V$ is equivalent to $g\cdot f(v)=f(g\cdot v)$ for $v\in V$.) The natural transformation $\varphi(g)$ is monoidal because $\varphi(g)_{V\otimes W}=\varphi(g)_V\otimes\varphi(g)_W$ and $\varphi_{\Bbbk}(g)=\mathrm{id}_{\Bbbk}$, by definition of the tensor product of G-modules and of the trivial G-module. This example will play an important role later in the course.

Conversely, every vector space can be considered as a G-module with the trivial action of G, and this gives rise to a monoidal functor $e\colon\mathbf{Vect}_{\Bbbk}\to\mathbf{Rep}_{\Bbbk}(G)$.

(2) For a commutative ring R, there is a functor $F\colon\mathbf{Set}^{\mathrm{rev}}\to R-\mathbf{Mod}$ that sends a set X to the free R-module $F(X):=R^X=\mathrm{Map}(X,R)$. At the level of homomorphisms, we define $F(f)$ via $g\mapsto g\circ f$, for maps $f\colon X\to Y$ and $g\colon X\to R$. This functor is monoidal with respect to the monoidal structure on $\mathbf{Set}$ via the Cartesian product and on $R-\mathbf{Mod}$ via the usual tensor product of R-modules.

(3) The total cohomology functor $H^*\colon \mathbf{coch}(\mathbf{Vect}_{\Bbbk}) \to \mathbf{Vect}_{\Bbbk}^{\mathbb{Z}}$ from the category of cochain complexes of $\Bbbk$-vector spaces to the category of graded $\Bbbk$-vector spaces is monoidal with respect to the usual derived tensor product on $\mathbf{coch}(\mathbf{Vect}_{\Bbbk})$ by the Künneth theorem: For two cochain complexes $X_\bullet$ and $Y_\bullet$, we have

$$H^i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{j+k=i} H^j(X_\bullet) \otimes H^k(Y_\bullet),$$

matching the definition of the tensor product of $\mathbb{Z}$-graded vector spaces.

(4) Let G and H be groups, let A be an abelian group and let $\omega\colon G^{\times 3} \to A$ and $\pi\colon H^{\times 3} \to A$ be 3-cocycles. Suppose that there is a monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}_A^{G,\omega}$ to $\mathcal{C}_A^{H,\pi}$, for some $\varphi\colon -\otimes- \to -\otimes-$ and $\varepsilon \in \operatorname{End}_{\mathcal{C}_A^G}(\delta_e) = A$. Then F defines a map $f\colon G \to H$ via $F(\delta_g) = \delta_{f(g)}$ for $g \in G$, and f is a homomorphism because

$$\delta_{f(gh)} = F(\delta_{gh}) = F(\delta_g \otimes \delta_h) \cong F(\delta_g) \otimes F(\delta_h) = \delta_{f(g)} \otimes \delta_{f(h)} = \delta_{f(g)f(h)}$$

for $g, h \in G$. Furthermore, φ defines a map $\varphi\colon G \times G \to A$ via

$$A \ni \varphi(g,h) \coloneqq \varphi_{g,h}\colon F(\delta_{gh}) = F(\delta_g \otimes \delta_h) \to F(\delta_g) \otimes F(\delta_h) = \delta_{f(g)}\delta_{f(h)} = \delta_{f(gh)}$$

for $g, h \in G$, and by the definition of monoidal functors, we have

$$\varphi(g,h)\varphi(gh,k)\omega(g,h,k) = \underbrace{\pi\big(f(g), f(h), f(k)\big)}_{=f^*\pi(g,h,k)}\varphi(h,k)\varphi(g,hk)$$

for all $g, h, k \in G$, that is

$$f^*\pi^{-1}\omega(g,h,k) = \varphi(h,k) \cdot \varphi(g,hk) \cdot \varphi(g,h)^{-1} \cdot \varphi(gh,k)^{-1} = d^2\varphi(g,h,k).$$

In other words, the 3-cocycle $\omega f^*\pi^{-1} = d^2\varphi$ is a 3-*coboundary*, so ω and $f^*\pi$ define the same element of the third cohomology group $H^3(G, A)$. (The latter is defined as the quotient of the group of 3-cocycles by the group of 3-coboundaries.) This (and the discussion in point (9) of Example 1.11) relates the equivalence classes of monoidal structures on $\mathcal{C}_A^G$ to $H^3(G, A)$. Similarly, one can relate the equivalence classes of monoidal structures on $\mathbf{Vect}_{\Bbbk}^G$ to $H^3(G, \Bbbk^\times)$. For more details, see Section 2.6 in [EGNO15].

2 Module categories

Unless otherwise stated, we continue to assume in this section that $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is a monoidal category (which we usually abbreviate by $\mathcal{C}$).

Definition 2.1. A *right $\mathcal{C}$-module category* is a quadruple $(\mathcal{M}, \otimes_a, \beta, \vartheta)$, where

- $\mathcal{M}$ is a category,
- $\otimes_a\colon \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is a bifuctor, called the *action*,
- $\beta\colon -\otimes_a(-\otimes-) \to (-\otimes_a-)\otimes_a-$ is a natural isomorphism, called the *associativity constraint*,
- $\vartheta\colon -\otimes_a\mathbf{1} \to \mathrm{id}_{\mathcal{M}}$ is a natural isomorphism, called the *unitor*,

subject to the following axioms:
Pentagon axiom: For all objects M of $\mathcal{M}$ and A, B, C of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
& M \otimes_a (A \otimes (B \otimes C)) \arrow[r, "\mathrm{id}_M \otimes \alpha_{A,B,C}"] \arrow[dl, "\beta_{M,A,B\otimes C}"'] & M \otimes_a ((A \otimes B) \otimes C) \arrow[dr, "\beta_{M,A\otimes B,C}"] & \\
(M \otimes_a A) \otimes_a (B \otimes C) \arrow[drr, "\beta_{M\otimes_a A,B,C}"'] & & & (M \otimes_a (A \otimes B)) \otimes_a C \arrow[dl, "\beta_{M,A,B} \otimes \mathrm{id}_C"] \\
& & ((M \otimes_a A) \otimes_a B) \otimes_a C &
\end{tikzcd}$$

In other words, we have

$$\big(\beta_{M\otimes_a A,B,C}\big) \circ \big(\beta_{M,A,B\otimes C}\big) = \big(\beta_{M,A,B} \otimes \mathrm{id}_C\big) \circ \big(\beta_{M,A\otimes B,C}\big) \circ \big(\mathrm{id}_M \otimes \alpha_{A,B,C}\big).$$

Unit axiom / triangle axiom: For all objects M of $\mathcal{M}$ and A of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
M \otimes_a (\mathbf{1} \otimes A) \arrow[rr, "\beta_{M,\mathbf{1},A}"] \arrow[dr, "\mathrm{id}_M \otimes_a \lambda_A"'] & & (M \otimes_a \mathbf{1}) \otimes_a A \arrow[dl, "\vartheta_M \otimes_a \mathrm{id}_B"] \\
& M \otimes_a A &
\end{tikzcd}$$

In other words, we have

$$(\vartheta_M \otimes_a \mathrm{id}_A) \circ \beta_{M,\mathbf{1},A} = \mathrm{id}_M \otimes_a \lambda_A.$$

Remark 2.2. (1) We can analogously define a *left $\mathcal{C}$-module category* to be a tuple $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ with an action bifunctor $\otimes_a \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, an associativity constraint

$$\beta \colon (- \otimes -) \otimes_a - \to - \otimes_a (- \otimes_a -)$$

and a unitor $\vartheta \colon \mathbf{1} \otimes_a - \to \mathrm{id}_{\mathcal{M}}$.

(2) Being a module category over $\mathcal{C}$ is not a property of a given category but an additional structure.

Example 2.3. (1) $\mathcal{C}$ is a right $\mathcal{C}$-module category if we set $\otimes_a = \otimes$, $\beta = \alpha$ and $\vartheta = \rho$.

(2) For two categories $\mathcal{C}$ and $\mathcal{D}$, the category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is a right $\mathbf{End}(\mathcal{C})$-module category, where $\otimes_a$ is defined by composition of functors:

$$(F, G) \mapsto F \otimes_a G \coloneqq F \circ G, \qquad (\eta, \nu) \mapsto \eta \otimes_a \nu \coloneqq \eta\nu$$

for functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta \colon F \to F'$ and $\nu \colon G \to G'$.

(3) For a field $\Bbbk$ and a group G, the category $\mathbf{Rep}_{\Bbbk}(G)$ is a right $\mathbf{Vect}_{\Bbbk}(G)$-module category. Indeed, viewing a $\Bbbk$-vector space as a trivial G-module gives rise to a (monoidal) functor $e \colon \mathbf{Vect}_{\Bbbk}(G) \to \mathbf{Rep}_{\Bbbk}(G)$, and we can define $- \otimes_a - = - \otimes e(-)$

Lemma 2.4. *Given a left module category $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ over $\mathcal{C}$, there is a monoidal functor*

$$(F, \varphi, \varepsilon) \colon \mathcal{C} \longrightarrow \mathbf{End}(\mathcal{M})$$

with $F = \mathrm{id}_{\mathcal{C}} \otimes_a -$,

$$\varphi_{A,B} = \beta_{A,B,-} \colon F(A \otimes B) = (A \otimes B) \otimes_a - \xrightarrow{\sim} A \otimes_a (B \otimes_a -) = F(A) \circ F(B)$$

and $\varepsilon = \vartheta \colon \mathbf{1} \otimes_a - \to \mathrm{id}_{\mathcal{M}}$.

Proof. This is straightforward to check using the definitions. □

Lemma 2.5. *A monoidal functor* $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathbf{End}(\mathcal{M})$ *gives rise to a left* $\mathcal{C}$*-module category structure* $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ *via* $- \otimes_a - = F(-)(-)$,

$$\beta_{A,B,M} = (\varphi_{A,B})_M \colon (A \otimes B) \otimes_a M = F(A \otimes B)(M) \xrightarrow{\sim} \big(F(A) \circ F(B)\big)(M) = A \otimes_a (B \otimes_a M)$$

and $\vartheta = \varepsilon\colon \mathbf{1} \otimes_a - = F(\mathbf{1})(-) \xrightarrow{\sim} \mathrm{id}_{\mathcal{M}}$.

Remark 2.6. Combining Lemmas 2.4 and 2.5, we see that there is a one-to-one correspondence between left $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors $\mathcal{C} \to \mathbf{End}(\mathcal{M})$.

Definition 2.7. Let $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ and $(\mathcal{M}', \otimes_a', \beta', \vartheta')$ be two right $\mathcal{C}$-module categories. A *right* $\mathcal{C}$*-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (F, γ), where $F\colon \mathcal{M} \to \mathcal{M}'$ is a functor and

$$\gamma\colon F(- \otimes_a -) \longrightarrow F(-) \otimes_a' -$$

is a natural isomorphism such that the following diagrams commute for all objects X, Y of $\mathcal{C}$ and M of $\mathcal{M}$:

$$\begin{tikzcd}
F\big(M \otimes_a (X \otimes Y)\big) \arrow[rr, "F(\beta_{M,X,Y})"] \arrow[dd, "\gamma_{M,X\otimes Y}"'] & & F\big((M \otimes_a X) \otimes_a Y\big) \arrow[d, "\gamma_{M\otimes_a X,Y}"] \\
& & F(M \otimes_a X) \otimes_a' Y \arrow[d, "\gamma_{M,X} \otimes_a' \mathrm{id}_Y"] \\
F(M) \otimes_a' (X \otimes Y) \arrow[rr, "\beta'_{F(M),X,Y}"'] & & \big(F(M) \otimes_a' X\big) \otimes_a' Y
\end{tikzcd}$$

$$\begin{tikzcd}
F(M \otimes_a \mathbf{1}) \arrow[rr, "\gamma_{M,\mathbf{1}}"] \arrow[dr, "F(\vartheta_M)"'] & & F(M) \otimes_a' \mathbf{1} \arrow[dl, "\vartheta'_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

A *right* $\mathcal{C}$*-module natural transformation* from a right $\mathcal{C}$-module functor $(F, \gamma)\colon \mathcal{M} \to \mathcal{M}'$ to a right $\mathcal{C}$-module functor $(F', \gamma')\colon \mathcal{M} \to \mathcal{M}'$ is a natural transformation $\psi\colon F \to F'$ such that the following diagram commutes for all objects X of $\mathcal{C}$ and M of $\mathcal{M}$:

$$\begin{tikzcd}
F(M \otimes_a X) \arrow[r, "\gamma_{M,X}"] \arrow[d, "\psi_{M\otimes_a X}"'] & F(M) \otimes_a' X \arrow[d, "\psi_M \otimes_a' \mathrm{id}_X"] \\
F'(M \otimes_a X) \arrow[r, "\gamma'_{M,X}"'] & F'(M) \otimes_a' X
\end{tikzcd}$$

We write $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{M})$ for the monoidal category of right $\mathcal{C}$-module endofunctors of a right $\mathcal{C}$-module category $\mathcal{M}$, with homomorphisms given by the right $\mathcal{C}$-module natural transformations. The tensor product is defined as the composition of functors. (There is a canonical choice of structure maps which endows the composition of two right $\mathcal{C}$-module functors with the structure of a right $\mathcal{C}$-module functor, cf. part (2) of Remark 1.15)

3 Strictness and coherence

Definition 3.1. A monoidal category $\mathcal{C}$ is called *strict* if

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z \qquad \text{and} \qquad \mathbf{1} \otimes X = X = X \otimes \mathbf{1}$$

for all objects X, Y, Z of $\mathcal{C}$ (note that we require equalities and not isomorphisms) and if all associativity constraints and unitors are identity maps.

Example 3.2. The monoidal category $\mathbf{End}(\mathcal{D})$ of endofunctors of a category $\mathcal{D}$ is strict, and so is the category $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{M})$ of right $\mathcal{C}$-module endofunctors of a right $\mathcal{C}$-module category $\mathcal{M}$.

Theorem 3.3. *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof (sketch). We show that $\mathcal{C}$ is equivalent to the strict category $\mathcal{C}' := \mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C})$ of right $\mathcal{C}$-module endofunctors of $\mathcal{C}$. We can define a functor $F\colon \mathcal{C} \to \mathcal{C}'$ by

$$F(A) = (A \otimes - \,,\, \alpha_{A,-,-}) \qquad \text{and} \qquad F(f) = f \otimes -$$

for every object A of $\mathcal{C}$ and every morphism $f\colon A \to B$ in $\mathcal{C}$. A functor $G\colon \mathcal{C}' \to \mathcal{C}$ is given by

$$G(X) = X(\mathbf{1}) \qquad \text{and} \qquad G(\eta) = \eta_{\mathbf{1}}$$

for a right $\mathcal{C}$-module endofunctor (X, γ) of $\mathcal{C}$ and a right $\mathcal{C}$-module natural transformation $\eta\colon X \to Y$. Then one can endow F and G with structure maps to make them monoidal functors, and check that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to the identity functor on $\mathcal{C}'$. Indeed, given a right $\mathcal{C}$-module endofunctor (X, γ) of $\mathcal{C}$ and an object A of $\mathcal{C}$, we have an isomorphism

$$F \circ G\big((X, \gamma)\big)(A) = X(\mathbf{1}) \otimes A \xrightarrow{\gamma^{-1}_{\mathbf{1},A}} X(\mathbf{1} \otimes A) \xrightarrow{X(\lambda_A)} X(A)$$

which gives rise to a right $\mathcal{C}$-module natural isomorphism

$$\varphi_{(X,\gamma)}\colon F \circ G\big((X, \gamma)\big) = \big(X(\mathbf{1}) \otimes -, \alpha_{X(\mathbf{1},-,-)}\big) \longrightarrow (X, \gamma)$$

that is natural in (X, γ). This yields the desired (monoidal) natural isomorphism $F \circ G \to \mathrm{id}_{\mathrm{End}_{\mathrm{end}-\mathcal{C}}(\mathcal{C})}$. The details of the proof are left to the reader. $\square$

Example 3.4. In this example, we construct a strict monoidal category that is monoidally equivalent to $\mathbf{Vect}_{\Bbbk}$ for a field $\Bbbk$. Let $\mathbf{Mat}_{\Bbbk}$ be the category whose objects are the natural numbers

$$\mathrm{Ob}(\mathbf{Mat}_{\Bbbk}) := \mathbb{N} = \{0, 1, 2, \ldots\},$$

with homomorphisms from $m \in \mathbb{N}$ to $n \in \mathbb{N}$ given by the set

$$\mathrm{Hom}_{\mathbf{Mat}_{\Bbbk}}(m, n) := \mathrm{Mat}_{n \times m}(\Bbbk)$$

of $n \times m$-matrices over $\Bbbk$ (i.e. matrices with m columns and n rows). Composition is given by matrix multiplication. We can define a monoidal structure on $\mathbf{Mat}$ by $m \otimes n = m \cdot n$ for $m, n \in \mathbb{N}$ at the level of objects. The tensor product of two matrices $A = (a_{ij}) \in \mathrm{Mat}_{n \times m}(\Bbbk)$ and $B = (b_{ij}) \in \mathrm{Mat}_{n' \times m'}(\Bbbk)$ is the *Kronecker product*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix} \in \mathrm{Mat}_{(nn') \times (mm')}(\Bbbk).$$

and the unit object is $1 \in \mathbb{N}$. (It is straightforward to verify that the Kronecker product is associative.)

We can define a functor $F\colon \mathbf{Mat}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ by $F(m) = \Bbbk^{\oplus m}$ for $m \in \mathbb{N}$, at the level of objects. For $A \in \mathrm{Mat}_{n \times m}(\Bbbk)$, we define $F(A) \in \mathrm{Hom}_{\Bbbk}(\Bbbk^{\oplus m}, \Bbbk^{\oplus n})$ to be left multiplication by A on the column vector space $\Bbbk^{\oplus m}$. It is straightforward to see that F is fully faithful and essentially surjective, hence an equivalence of categories, and that F is monoidal.

Theorem 3.5. *Let $A_1, \dots, A_n$ be objects of $\mathcal{C}$ and let P and Q be two paranthesized tensor products of $A_1, \dots, A_n$ (in this order, but not necessarily with the same parenthesization), possibly with copies of the unit object $\mathbf{1}$ inserted in different places. Let $f\colon P \to Q$ and $g\colon P \to Q$ be two isomorphisms that are obtained by composing tensor products of identity morphisms, associativity constraints, unitors and their respective inverses. Then $f = g$.*

Proof. This follows from the strictness theorem. More specifically, let $F\colon \mathcal{C} \to \mathcal{C}_0$ be a monoidal equivalence from $\mathcal{C}$ to a strict monoidal category $\mathcal{C}_0$. Then we have $F(P) = F(Q)$ and $F(f) = F(g)$. Since F induces a bijection between $\mathrm{Hom}_{\mathcal{C}}(P, Q)$ and $\mathrm{Hom}_{\mathcal{C}_0}\big(F(P), F(Q)\big)$, it follows that $f = g$. $\square$

Remark 3.6. An example of a setting in which we can use the coherence theorem 3.5 is Lemma 1.13: For objects A and B of $\mathcal{C}$, the homomorphisms

$$\rho_{A\otimes B} \circ \alpha_{A,B,\mathbf{1}}\colon A \otimes (B \otimes \mathbf{1}) \longrightarrow A \otimes B \qquad \text{and} \qquad \mathrm{id}_A \otimes \rho_B\colon A \otimes (B \otimes \mathbf{1}) \longrightarrow A \otimes B$$

coincide. This does not yield an alternative proof of Lemma 1.13 because the lemma is used in the proof of the strictness theorem 3.3, which is used in turn in the proof of the coherence theorem 3.5.

Another example is that for objects A, B, C, D, E of $\mathcal{C}$, all of the isomorphism between

$$A \otimes (B \otimes (C \otimes (D \otimes E))) \qquad \text{and} \qquad (((A \otimes B) \otimes C) \otimes D) \otimes E$$

that are obtained as compositions of tensor products of identity homomorphisms and associativity constraints coincide.

4 Duals and rigidity

We continue to assume that $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is a monoidal category. In order to simplify the notation, we will often ignore associativity constraints and unitors in the following.

In this section, we want to discuss the notion of dual objects in monoidal categories. We start with a motivating example.

Example 4.1. Let $\Bbbk$ be a field, let V be a finite-dimensional $\Bbbk$-vector space and let $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ be the dual space of V. Then there is a canonical evaluation map

$$\mathrm{ev}_V\colon V^* \otimes V \longrightarrow \Bbbk \qquad \text{with} \quad \xi \otimes v \longmapsto \xi(v).$$

Furthermore, for a basis $\{v_1, \dots, v_n\}$ of V with dual basis $\{v_1^*, \dots, v_n^*\}$ of V^* (defined via $v_i^*(v_j) = \delta_{ij}$), there is a coevaluation map

$$\mathrm{coev}_V\colon \Bbbk \longrightarrow V \otimes V^*, \qquad \lambda \longmapsto \lambda \cdot \sum_{i=1}^{n} v_i \otimes v_i^*,$$

and this map is independent of the choice of basis. Using the definitions, one easily checks that the composition

$$V \cong \Bbbk \otimes V \xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V \otimes \Bbbk \cong V$$

coincides with the identity map on V, that is $(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.

Now suppose that we are given a vector space W with linear maps

$$e\colon W \otimes V \longrightarrow \Bbbk \qquad \text{and} \qquad c\colon \Bbbk \longrightarrow V \otimes W.$$

Then we can define linear maps $f\colon W \to V^*$ and $g\colon V^* \to W$ via

$$f(w) = e(w \otimes -) \qquad \text{and} \qquad g(\xi) = (\xi \otimes \mathrm{id}_W) \circ c(1),$$

for $w \in W$ and $\xi \in V^*$. Now one can further show that f and g are mutually inverse if and only if

$$(\mathrm{id}_V \otimes e) \circ (c \otimes \mathrm{id}_V) = \mathrm{id}_V \qquad \text{and} \qquad (e \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes c) = \mathrm{id}_W.$$

This motivates the definition of duals below.

Definition 4.2. A *left dual* of an object X of $\mathcal{C}$ is an object X^* of $\mathcal{C}$ together with homomorphisms

$$\mathrm{ev}_X \colon X^* \otimes X \longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}_X \colon \mathbf{1} \longrightarrow X \otimes X^*,$$

called *evaluation* and *coevaluation*, such that the compositions

$$X \cong \mathbf{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathbf{1} \cong X$$

and

$$X^* \cong X^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} \mathbf{1} \otimes X^* \cong X^*$$

afford the identity homomorphisms on X and X^*, respectively. The equalities

$$\mathrm{id}_X = (\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \qquad \text{and} \qquad \mathrm{id}_{X^*} = (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X)$$

are called the *zig-zag relations*.

A *right dual* of X is an object ${}^*X$ of $\mathcal{C}$ together with homomorphisms

$$\mathrm{ev}'_X \colon X \otimes X^* \longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}'_X \colon \mathbf{1} \longrightarrow X^* \otimes X,$$

called *evaluation* and *coevaluation*, such that the compositions

$$X \cong X \otimes \mathbf{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} \mathbf{1} \otimes X \cong X$$

and

$$X^* \cong \mathbf{1} \otimes X^* \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{X^*}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{ev}'_X} X^* \otimes \mathbf{1} \cong X^*$$

afford the identity homomorphisms on X and X^*, respectively.

Example 4.3. (1) The dual space V^* of a finite-dimensional vector space V is a left (and right) dual of V in $\mathbf{Vect}_{\Bbbk}$.

(2) For a G-module M in $\mathbf{Rep}_{\Bbbk}(G)$, the dual space M^* becomes a G-module via $(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$. The linear maps $\mathrm{ev}_M \colon M^* \otimes M \to \Bbbk$ and $\mathrm{coev}_M \colon \Bbbk \to M \otimes M^*$ are homomorphisms of G-modules, so M^* is a left (and right) dual of M in $\mathbf{Rep}_{\Bbbk}(G)$.

(3) Let G be a monoid. If for $g \in G$, the one-dimensional graded vector space $\Bbbk_g$ has a left dual then g has an inverse in G, since a tensor product $\big(\bigoplus_h V_h\big) \otimes \Bbbk_g$ admits a non-zero homomorphism to $\Bbbk_e$ only if $hg = e$ for some $h \in G$. If G is a group then every G-graded vector space $V = \bigoplus_g V_g$ has a left (and right) dual, given by the dual space V^* with grading defined by $V^*_g = (V_{g^{-1}})^*$, for $g \in G$.

Lemma 4.4. *If an object X of $\mathcal{C}$ admits a left (or right) dual then the latter is unique up to isomorphism.*

Proof. Let X_1^* and X_2^* be two left dual objects of X and denote by e_1, e_2, c_1, c_2 the corresponding evaluation and coevaluation homomorphisms. We define two homomorphisms

$$f \colon X_1^* \cong X_1^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X_1^*} \otimes c_2} X_1^* \otimes X \otimes X_2^* \xrightarrow{e_1 \otimes \mathrm{id}_{X_2^*}} \mathbf{1} \otimes X_2^* \cong X_2^*$$

and

$$g \colon X_2^* \cong X_2^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X_2^*} \otimes c_1} X_2^* \otimes X \otimes X_1^* \xrightarrow{e_2 \otimes \mathrm{id}_{X^*}} \mathbf{1} \otimes X_1^* \cong X_1^*$$

and consider the following commutative diagram:

$$\begin{tikzcd}
X_1^* \arrow[r, "\mathrm{id}_{X_1^*} \otimes c_1"] \arrow[d, "\mathrm{id}_{X_1^*} \otimes c_2"'] & X_1^* \otimes X \otimes X_1^* \arrow[d, "\mathrm{id}_{X_1^*} \otimes c_2 \otimes \mathrm{id}_{X \otimes X_1^*}"'] \arrow[dr, "\mathrm{id}_{X_1^* \otimes X \otimes X_1^*}"] & \\
X_1^* \otimes X \otimes X_2^* \arrow[r, "\mathrm{id}_{X_1^* \otimes X \otimes X_2^*} \otimes c_1"'] \arrow[d, "e_1 \otimes \mathrm{id}_{X_2^*}"'] & X_1^* \otimes X \otimes X_2^* \otimes X \otimes X_1^* \arrow[r, "\mathrm{id}_{X_1^* \otimes X} \otimes e_2 \otimes \mathrm{id}_{X_1^*}"'] \arrow[d, "e_1 \otimes \mathrm{id}_{X_2^* \otimes X \otimes X_1^*}"'] & X_1^* \otimes X \otimes X_1^* \arrow[d, "e_1 \otimes \mathrm{id}_{X_1^*}"'] \\
X_2^* \arrow[r, "\mathrm{id}_{X_2^*} \otimes c_1"'] & X_2^* \otimes X \otimes X_1^* \arrow[r, "e_2 \otimes \mathrm{id}_{X_1^*}"'] & X_1^*
\end{tikzcd}$$

The squares commute by the bifunctoriality of the tensor product and the triangle commutes because X_2^* is a left dual of X. Hence the composition along the top right boundary of the diagram coincides with the composition along the bottom left boundary. The former is the identity on X_1^* because X_1^* is a left dual of X, and the latter equals $g \circ f$. Analogously, one sees that $\mathrm{id}^*_{X_2} = f \circ g$, and the claim follows. □

Remark 4.5. More specifically, the left dual of an object of $\mathcal{C}$ is unique up to a unique isomorphism in the following sense: In the notation of Lemma 4.4, assume that we have an homomorphism $h\colon X_1^* \to X_2^*$ such that $e_1 = e_2 \circ (h \otimes \mathrm{id}_X)$ and $c_2 = (\mathrm{id}_X \otimes h) \circ c_1$. Then one can show that h coincides with the isomorphism $f = (e_1 \otimes \mathrm{id}_{X_2^*}) \circ (\mathrm{id}_{X_1^*} \otimes c_2)$ from the proof of Lemma 4.4.

Lemma 4.6. *If an object X of $\mathcal{C}$ has a left dual X^* then X is a right dual of X^*, that is ${}^*(X^*) = X$. Analogously, if X has a right dual then $({}^*X)^* = X$.*

Proof. For the first claim, set $\mathrm{ev}'_{X^*} = \mathrm{ev}_X$ and $\mathrm{coev}'_{X^*} = \mathrm{coev}_X$. The second claim is analogous. □

Lemma 4.7. *Let X and Y be objects of $\mathcal{C}$ with left duals. Then $Y^* \otimes X^*$ is a left dual of $X \otimes Y$. Similarly, if X and Y have right duals then ${}^*Y \otimes {}^*X$ is a right dual of $X \otimes Y$.*

Proof. We define

$$\mathrm{ev}_{X \otimes Y}\colon Y^* \otimes X^* \otimes X \otimes Y \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^*}} Y^* \otimes \mathbf{1} \otimes Y \cong Y^* \otimes Y \xrightarrow{\mathrm{ev}_Y} \mathbf{1}$$

and

$$\mathrm{coev}_{X \otimes Y}\colon \mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \cong X \otimes \mathbf{1} \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*}} X \otimes Y \otimes Y^* \otimes X^*,$$

omitting associativity constraints and unitors. It is straightforward to check that these homomorphisms satisfy the zig-zag relations. □

Lemma 4.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor. If an object X of $\mathcal{C}$ has a left dual X^* then $F(X^*)$ is a left dual of $F(X)$.*

Proof. Consider the homomorphisms

$$\mathrm{ev}_{F(X)}\colon F(X^*) \otimes F(X) \xrightarrow{\varphi^{-1}_{X^*,X}} F(X \otimes X^*) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\varepsilon} \mathbf{1}_{\mathcal{D}}$$

and

$$\mathrm{coev}_{F(X)}\colon \mathbf{1}_{\mathcal{D}} \xrightarrow{\varepsilon^{-1}} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes X^*) \xrightarrow{\varphi_{X,X^*}} F(X) \otimes F(X^*).$$

It is straightforward to verify that $\mathrm{ev}_{F(X)}$ and $\mathrm{coev}_{F(X)}$ satisfy the zig-zag relations, using the zig-zag relations for ev_X and coev_X. □

Definition 4.9. A functor $G\colon \mathcal{D} \to \mathcal{C}$ is called *right adjoint* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \cong \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big)$$

of functors from $\mathcal{C} \times \mathcal{D}$ to **Set**. In that case, we also say that F is *left adjoint* to G and write $F \dashv G$.

Remark 4.10. We have $F \dashv G$ if and only if there exist natural transformations

$$\varepsilon : FG \longrightarrow \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad \eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow GF,$$

called the *unit* and the *counit* of the adjunction $F \dashv G$, such that the compositions

$$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F\,\eta} FGF \xrightarrow{\varepsilon\,\mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F \qquad \text{and} \qquad G = \mathrm{id}_{\mathcal{C}} \circ G \xrightarrow{\eta\,\mathrm{id}_G} GFG \xrightarrow{\mathrm{id}_G\,\varepsilon} G \circ \mathrm{id}_{\mathcal{D}} = G$$

are equal to id_F and id_G, respectively. The equalities $\mathrm{id}_F = (\varepsilon\,\mathrm{id}_F) \circ (\mathrm{id}_F\,\eta)$ and $\mathrm{id}_G = (\mathrm{id}_G\,\varepsilon) \circ (\eta\,\mathrm{id}_G)$ are called the *zig-zag relations.*

Given a natural isomorphism

$$\psi\colon \mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big),$$

we can define

$$\varepsilon_X = \psi^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})\colon X \longrightarrow GF(X) \qquad \text{and} \qquad \eta_Y = \psi_{G(Y),Y}(\mathrm{id}_{G(Y)})\colon FG(Y) \longrightarrow Y$$

for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$. Conversely, given a unit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and a counit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$, we obtain a natural isomorphism

$$\psi\colon \mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big)$$

via

$$\psi_{X,Y}(f) = \varepsilon_X \circ F(f)\colon F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_X} Y$$

for objects X of $\mathcal{C}$ and Y of $\mathcal{D}$ and a homomorphism $f\colon X \to G(Y)$ in $\mathcal{D}$.

Example 4.11. In the monoidal category $\mathbf{End}(\mathcal{D})$ of endofunctors of a category $\mathcal{D}$, a left dual of a functor $F\colon \mathcal{D} \to \mathcal{D}$ is the same as a left adjoint of F, and a right dual is the same as a right adjoint.

Lemma 4.12. *If an object X of $\mathcal{C}$ has a left dual X^* then there are adjunctions*

$$- \otimes X \dashv - \otimes X^* \qquad \textit{and} \qquad X^* \otimes - \dashv X \otimes -.$$

In particular, for objects Y and Z of $\mathcal{C}$, there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(Y \otimes X, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z \otimes X^*) \qquad \textit{and} \qquad \mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes Z)$$

which are natural in Y and Z.

Proof. The unit and the counit of the adjunction $- \otimes X \dashv - \otimes X^*$ are given by

$$\varepsilon_Y\colon (Y \otimes X^*) \otimes X \xrightarrow{\alpha_{Y,X^*,X}} Y \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_X} Y$$

and

$$\eta_Y\colon Y \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_Y} (X \otimes X^*) \otimes Y \xrightarrow{\alpha^{-1}_{X,X^*,Y}} X \otimes (X^* \otimes Y).$$

The zig-zag relations for ε and η follow from the zig-zag-relations for the evaluation and coevaluation, and the first isomorphism between Hom-sets is immediate from Remark 4.10. The second adjunction and isomorphism of Hom-sets are obtained analogously. $\square$

Example 4.13. Consider the category **AbGrp** $= \mathbb{Z} - $ **Mod** of abelian groups (or $\mathbb{Z}$-modules), with the usual tensor product $\otimes = \otimes_{\mathbb{Z}}$ over $\mathbb{Z}$ and the unit object $\mathbb{Z}$. Observe that for any two abelian groups A and B, the set $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ can be considered as an abelian group via pointwise addition, and that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \cong A$ (by evaluation at $1 \in \mathbb{Z}$). If the group $A = \mathbb{Z}/3\mathbb{Z}$ has a left dual A^* then

$$A^* \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A^*) \cong \mathrm{Hom}(A, \mathbb{Z}) = 0$$

by Lemma 4.12, contradicting the zig zag relations. Hence $\mathbb{Z}/3\mathbb{Z}$ does not admit a left dual in **AbGrp**.

Remark 4.14. Before we continue discussing duals, some reminders about category theory are in order. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *faithful* (or *full*) if the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{f \mapsto F(f)} \mathrm{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big)$$

is injective (respectively surjective). The functor F is called *fully faithful* if it is full and faithful, and it is called *essentially surjective* if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ such that $F(X) \cong Y$. One can show that a functor is fully faithful and essentially surjective if and only if it is an equivalence.

Now for every object X of $\mathcal{C}$, we have a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \longrightarrow \mathbf{Set}$$

and for a homomorphism $f\colon X \to Y$ in $\mathcal{C}$, there is a natural transformation

$$\mathrm{Hom}(f, -)\colon \mathrm{Hom}_{\mathcal{C}}(Y, -) \to \mathrm{Hom}_{\mathcal{C}}(Y, -)$$

with components

$$\mathrm{Hom}(f, Z)\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \qquad g \mapsto g \circ f.$$

These data give rise to a functor

$$\mathcal{C}^{\mathrm{rev}} \longrightarrow \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$$

which is fully faithful by Yoneda's lemma. In particular, given two objects X and Y of $\mathcal{C}$ such that the functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorphic, there exists an isomorphism $X \cong Y$ in $\mathcal{C}$.

Remark 4.15. Using Lemma 4.12, we can give an alternative proof of Lemma 4.4: For an element X of $\mathcal{C}$ with two left duals X_1^* and X_2^*, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X_1^*, -) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) \cong \mathrm{Hom}_{\mathcal{C}}(X_2^*, -),$$

and Yoneda's lemma implies that $X_1^* \cong X_2^*$.

Definition 4.16. Assume that X and Y are objects of $\mathcal{C}$ that have left duals and let $f\colon X \to Y$ be a homomorphism. The left dual of f is the homomorphism

$$f^*\colon Y^* \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{Y^*, X, X^*}} (Y^* \otimes X) \otimes X^* \xrightarrow{\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}} (Y^* \otimes Y) \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} X^*$$

If X and Y have right duals then one similarly defines the right dual ${}^*f\colon {}^*Y \to {}^*X$.

Remark 4.17. Assume that X and Y are objects of $\mathcal{C}$ with left duals, and let $f\colon X \to Y$ be a homomorphism. Then there is a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X^*,-) \arrow[r, "g \mapsto g \circ f^*"] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathcal{C}}(Y^*,-) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) \arrow[r, "g \mapsto (f \otimes \mathrm{id}) \circ g"'] & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -)
\end{tikzcd}$$

where the vertical arrows are given by Remark 4.15. By Yoneda's lemma, f^* is the unique homomorphism from Y^* to X^* that makes this diagram commute.

Lemma 4.18. *Let X, Y and Z be objects of $\mathcal{C}$ that have right duals. For homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$, we have $(g \circ f)^* = f^* \circ g^*$.*

Proof. This follows from Remark 4.17 and the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X^*,-) \arrow[r, "h \mapsto h \circ f^*"] \arrow[d, "\wr"'] \arrow[rr, bend left, "h \mapsto h \circ (f^* \circ g^*)"] & \mathrm{Hom}_{\mathcal{C}}(Y^*,-) \arrow[r, "h \mapsto h \circ g^*"] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathcal{C}}(Z^*,-) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) \arrow[r, "h \mapsto (f \otimes \mathrm{id}) \circ h"'] \arrow[rr, bend right, "h \mapsto \big((g \circ f) \otimes \mathrm{id}\big) \circ h"'] & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -) \arrow[r, "h \mapsto (g \otimes \mathrm{id}) \circ h"'] & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -)
\end{tikzcd}$$

where the vertical arrows are given by Remark 4.15. ☐

Corollary 4.19. *If every object X in $\mathcal{C}$ has a left dual X^* then there is a contravariant left duality functor $(-)^*\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$ with $X \mapsto X^*$ and $f \mapsto f^*$ for all objects X and all homomorphisms f in $\mathcal{C}$. Analogously, if every object X in $\mathcal{C}$ has a right dual ${}^*X$ then there is a right duality functor ${}^*(-)\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$.*

Proof. This follows from Lemmas 4.4 and 4.18. ☐

Remark 4.20. Suppose that all objects in $\mathcal{C}$ have right duals. Then the right duality functor canonically defines a monoidal functor $\big((-)^*, \varphi, \varepsilon\big)\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}^{\mathrm{op}}$. This follows from Lemmas 4.4 and 4.7 and the fact that $(f \otimes g)^* = g^* \otimes f^*$ for homomorphisms f and g in $\mathcal{C}$. (This fact can be proven by arguing as in Lemma 4.18.)

Definition 4.21. A monoidal category $\mathcal{C}$ is called *rigid* if every object X has a right dual X^* and a left dual ${}^*X$.

Remark 4.22. In a rigid monoidal category, the left duality functor is an equivalence and its quasi-inverse is the right duality functor by Lemma 4.6.

The name *rigid* is justified by the following lemma:

Lemma 4.23. *Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories and let $(F, \varphi, \varepsilon)$ and $(G, \varphi', \varepsilon')$ be monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. Further let $u\colon F \to G$ be a monoidal natural transformation. Then u is a natural isomorphism.*

Proof. The proof will be given in the exercises. ☐

5 Braided monoidal categories

Definition 5.1. A *braiding* on $\mathcal{C}$ is a natural isomorphism $\beta\colon -\otimes- \to -\otimes^{\mathrm{op}}-$ (i.e. with components $\beta_{A,B}\colon A\otimes B \to B\otimes A$) that satisfies the *hexagon axiom*, that is, such that the following diagrams commute for all objects A,B,C of $\mathcal{C}$:

$$\begin{tikzcd}
 & A\otimes(C\otimes B) \arrow[r, "\alpha_{A,C,B}"] & (A\otimes C)\otimes B \arrow[dr, "\beta_{A,C}\otimes \mathrm{id}_B"] & \\
A\otimes(B\otimes C) \arrow[ur, "\mathrm{id}_A\otimes\beta_{B,C}"] \arrow[dr, "\alpha_{A,B,C}"'] & & & (C\otimes A)\otimes B \\
 & (A\otimes B)\otimes C \arrow[r, "\beta_{A\otimes B,C}"'] & C\otimes(A\otimes B) \arrow[ur, "\alpha_{C,A,B}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (B\otimes A)\otimes C \arrow[r, "\alpha^{-1}_{B,A,C}"] & B\otimes(A\otimes C) \arrow[dr, "\mathrm{id}_B\otimes\beta_{A,C}"] & \\
(A\otimes B)\otimes C \arrow[ur, "\beta_{A,B}\otimes\mathrm{id}_C"] \arrow[dr, "\alpha^{-1}_{A,B,C}"'] & & & B\otimes(C\otimes A) \\
 & A\otimes(B\otimes C) \arrow[r, "\beta_{A,B\otimes C}"'] & (B\otimes C)\otimes A \arrow[ur, "\alpha^{-1}_{B,C,A}"'] &
\end{tikzcd}$$

A *braided monoidal category* is a pair $(\mathcal{C},\beta)$, where $\mathcal{C}$ is a monoidal category and β is a braiding.

Example 5.2. (1) The category **Set** with the monoidal structure by Cartesian product admits a braiding β with $\beta_{X,Y}\colon X\times Y \to Y\otimes X$ given by $(x,y)\mapsto(y,x)$ for sets X and Y.

(2) The category $\Bbbk-\mathbf{Vect}$ of vector spaces over a field $\Bbbk$ with the ordinary tensor product has a braiding β with $\beta_{X,Y}\colon X\otimes Y\to Y\otimes X$ determined by $x\otimes y\mapsto y\otimes x$ for $\Bbbk$-vector spaces X and Y and $x\in X$, $y\in Y$.

Lemma 5.3. *In a braided monoidal category $(\mathcal{C},\beta)$, we have*

$$\lambda_X = \rho_X\circ\beta_{\mathbf{1},X},\qquad \rho_X=\lambda_X\circ\beta_{X,\mathbf{1}}\qquad \text{and}\qquad \beta_{\mathbf{1},X}=\beta^{-1}_{X,\mathbf{1}}$$

for all objects X of $\mathcal{C}$. In other words, the following diagrams commute:

$$\begin{tikzcd}
X\otimes\mathbf{1} \arrow[rr, "\beta_{X,\mathbf{1}}"] \arrow[dr, "\rho_X"'] & & \mathbf{1}\otimes X \arrow[dl, "\lambda_X"] \\
 & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}\otimes X \arrow[rr, "\beta_{\mathbf{1},X}"] \arrow[dr, "\lambda_X"'] & & X\otimes\mathbf{1} \arrow[dl, "\rho_X"] \\
 & X &
\end{tikzcd}$$

Proof. Using the hexagon axiom, the naturality of β and the unitors, and the coherence theorem (see Theorem 3.5), we get the following commutative diagram:

$$
\begin{tikzcd}
 & & & X & & & \\
X \otimes \mathbf{1} \arrow[urrr, "\rho_X"] & & \mathbf{1} \otimes (X \otimes \mathbf{1}) \arrow[ll, "\lambda_{X \otimes \mathbf{1}}"'] \arrow[rr, "\alpha_{\mathbf{1},X,\mathbf{1}}"] & & (\mathbf{1} \otimes X) \otimes \mathbf{1} \arrow[rr, "\rho_{\mathbf{1} \otimes X}"] \arrow[dr, "\beta_{\mathbf{1},X} \otimes \mathrm{id}_{\mathbf{1}}"] & & \mathbf{1} \otimes X \arrow[ulll, "\lambda_X"'] \\
 & \mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[ur, "\mathrm{id}_{\mathbf{1}} \otimes \beta_{\mathbf{1},X}"] \arrow[dr, "\alpha_{\mathbf{1},\mathbf{1},X}"] \arrow[ddl, "\lambda_{\mathbf{1} \otimes X}"] & & & & (X \otimes \mathbf{1}) \otimes \mathbf{1} \arrow[ddr, "\rho_{X \otimes \mathbf{1}}"] & \\
 & & (\mathbf{1} \otimes \mathbf{1}) \otimes X \arrow[rr, "\beta_{\mathbf{1} \otimes \mathbf{1},X}"'] \arrow[dll, "\lambda_{\mathbf{1}} \otimes \mathrm{id}_X"] & & X \otimes (\mathbf{1} \otimes \mathbf{1}) \arrow[ur, "\alpha_{X,\mathbf{1},\mathbf{1}}"] \arrow[drr, "\mathrm{id}_X \otimes \lambda_{\mathbf{1}}"] & & \\
\mathbf{1} \otimes X \arrow[uuu, "\beta_{\mathbf{1},X}"] \arrow[rrrrrr, "\beta_{\mathbf{1},X}"'] & & & & & & X \otimes \mathbf{1} \arrow[uuu, "\beta_{X,\mathbf{1}}"']
\end{tikzcd}
$$

This implies that we have

$$\beta_{1,X} = \beta_{X,1} \circ \lambda_X^{-1} \circ \rho_X \circ \beta_{1,X}$$

and therefore $\lambda_X \circ \beta_{X,\mathbf{1}}^{-1} = \rho_X$. The equations $\rho_X \circ \beta_{X,\mathbf{1}} = \lambda_X$ and $\beta_{\mathbf{1},X} = \beta_{X,\mathbf{1}}^{-1}$ can be proven analogously. ☐

Lemma 5.4. *Let $\mathcal{C}$ be a strict monoidal category and let β be a braiding on $\mathcal{C}$. Then $\mathcal{C}$ satisfies the* braid relations*, that is, for objects X, Y, Z of $\mathcal{C}$, we have*

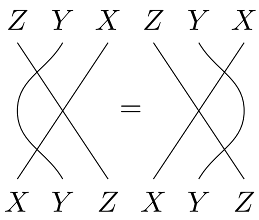
$$(\beta_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes \beta_{Y,X}) \circ (\beta_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \beta_{Y,Z}).$$

Proof. Since $\mathcal{C}$ is strict, the diagrams in the hexagon axiom become triangles, and by naturality of the braiding, we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & X \otimes Y \otimes Z \arrow[r, "\beta_{X,Y} \otimes \mathrm{id}_Z"] \arrow[dl, "\mathrm{id}_X \otimes \beta_{Y,Z}"'] \arrow[dd, "\beta_{X \otimes Y,Z}"] & Y \otimes X \otimes Z \arrow[dr, "\mathrm{id}_Y \otimes \beta_{X,Z}"] \arrow[dd, "\beta_{Y \otimes X,Z}"'] & \\
X \otimes Z \otimes Y \arrow[dr, "\beta_{X,Z} \otimes \mathrm{id}_Y"'] & & & Y \otimes Z \otimes X \arrow[dl, "\beta_{Y,Z} \otimes \mathrm{id}_X"] \\
 & Z \otimes X \otimes Y \arrow[r, "\mathrm{id}_Z \otimes \beta_{X,Y}"'] & Z \otimes Y \otimes X &
\end{tikzcd}
$$

The braid relation can be read off along the perimeter of the diagram. ☐

Remark 5.5. The braid relations can be depicted by the following diagram:



Remark 5.6. Recall from Theorem 3.3 that every monoidal category is equivalent to a strict monoidal category. This (or a more elaborate version of the argument in the proof of Lemma 5.4) can be used to prove a version of the braid relations for non-strict monoidal categories.

Definition 5.7. A braiding β on $\mathcal{C}$ is called *symmetric* if it satisfies $\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all objects X, Y of $\mathcal{C}$. A *symmetric monoidal category* is a braided monoidal category with a symmetric braiding β.

Example 5.8. Let G be a group and A an abelian group, and suppose that $\mathcal{C}_A^G$ admits a braiding. Then we have

$$\delta_{gh} = \delta_g \otimes \delta_h \cong \delta_h \otimes \delta_g \cong \delta_{hg}$$

for all $g, h \in G$, so $gh = hg$ and G is abelian. A braiding $\beta\colon - \otimes - \to - \otimes^{\mathrm{op}} -$ on $\mathcal{C}_A^G$ is the same as a collection

$$A \quad \ni \quad \beta_{g,h}\colon \delta_{gh} = \delta_g \otimes \delta_h \longrightarrow \delta_h \otimes \delta_g \cong \delta_{hg}$$

satisfying the hexagon axiom, i.e. a map $\beta\colon G \times G \to A$ such that

$$\beta(g,h) \cdot \beta(h,k) = \beta(gh,k) \qquad \text{and} \qquad \beta(g,h) \cdot \beta(g,k) = \beta(g,hk)$$

for $g, h, k \in G$. In other words, a braiding on $\mathcal{C}_A^G$ is the same as a $\mathbb{Z}$-bilinear map $\beta\colon G \times G \to A$. The braiding is symmetric if and only if $\beta(g,h) \cdot \beta(h,g) = e$ for all $g, h \in G$.

For a field $\Bbbk$ one similarly finds that $\mathbf{Vect}_{\Bbbk}^G$ admits a braiding if and only if G is abelian, and that every braiding corresponds to a choice of $\mathbb{Z}$-bilinear map $G \times G \longrightarrow \Bbbk^\times$.

Now suppose that $G = \mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$ is the cyclic group of order n. For an n-th root of unity $\zeta \in \Bbbk^\times$, we can define a biliear map $\beta\colon G \times G \longrightarrow \Bbbk^\times$ via

$$\beta(\overline{a}, \overline{b}) = \zeta^{\overline{a \cdot b}}.$$

For $n = 2$ and $\zeta = -1$, the resulting braided monoidal category

$$\mathbf{SVect}_{\Bbbk} = \big(\mathbf{Vect}_{\Bbbk}^{\mathbb{Z}/2\mathbb{Z}}, \beta\big)$$

is called the category of *super $\Bbbk$-vector spaces*. It is straightforward to see that $\mathbf{SVect}_{\Bbbk}$ is a symmetric monoidal category.

Definition 5.9. A *braided monoidal functor* between braided monoidal categories $(\mathcal{C}, \beta)$ and $(\mathcal{D}, \beta')$ is a monoidal functor $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ such that for all objects X, Y of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "\gamma_{X,Y}"] \arrow[d, "F(\beta_{X,Y})"'] & F(X) \otimes F(Y) \arrow[d, "\beta'_{F(X),F(Y)}"] \\
F(Y \otimes F(X)) \arrow[r, "\gamma_{Y,X}"'] & F(Y) \otimes F(X)
\end{tikzcd}$$

If $(\mathcal{C}, \beta)$ and $(\mathcal{D}, \beta')$ are symmetric then we call $(F, \varphi, \varepsilon)$ a *symmetric monoidal functor*.

6 Enriched categories

Fix a symmetric monoidal category $\mathcal{S}$.

Definition 6.1. An $\mathcal{S}$-*enriched category* $\mathcal{C}$ consists of the following data:

(1) a class $\mathrm{Ob}(\mathcal{C})$ of *objects* of $\mathcal{C}$;

(2) for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an object $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of $\mathcal{S}$;

(3) for every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition homomorphism* in $\mathcal{S}$:

$$c_{X,Y,Z} \colon \mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes_{\mathcal{S}} \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

such that the following axioms hold:

(a) for $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, we have

$$c_{X,Y,W} \circ (c_{Y,Z,W} \otimes \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}) \circ \alpha_{\mathrm{Hom}_{\mathcal{C}}(Z,W),\mathrm{Hom}_{\mathcal{C}}(Y,Z),\mathrm{Hom}_{\mathcal{C}}(X,Y)} = c_{X,Z,W} \circ (\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(Z,W)} \otimes c_{X,Y,Z}).$$

(b) for $X \in \mathrm{Ob}(\mathcal{C})$, there exists an *identity homomorphism* $\mathrm{id}_X \colon \mathbf{1}_{\mathcal{S}} \to \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that

$$c_{X,X,Y} \circ (\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \otimes \mathrm{id}_X) = \rho_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \qquad \text{and} \qquad c_{X,Y,Y} \circ (\mathrm{id}_Y \otimes \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}) = \lambda_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}.$$

Example 6.2. (1) A **Set**-enriched category is just an ordinary category.

(2) A **AbGrp**-enriched category is also called a *pre-additive* category. The category of **AbGrp** abelian groups can be considered as a pre-additive category.

(3) For a field $\Bbbk$, a $\mathbf{Vect}_{\Bbbk}$-enriched category is also called a $\Bbbk$-*linear category*. The category $\mathbf{Rep}_{\Bbbk}(G)$ of representations of a group G can be considered as a $\Bbbk$-linear category.

Definition 6.3. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{S}$-enriched categories. An $\mathcal{S}$-enriched functor $F \colon \mathcal{C} \to \mathcal{D}$ consists of a map $F \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and for every pair of objects X, Y in $\mathcal{C}$, a homomorphism $F_{X,Y} \colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}\big(F(X), F(Y)\big)$ in $\mathcal{S}$ such that the following diagrams commute, for all objects X, Y, Z of $\mathcal{C}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes_{\mathcal{S}} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "c^{\mathcal{C}}_{X,Y,Z}"] \arrow[d, "F_{Y,Z} \otimes F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{C}}\big(F(Y),F(Z)\big) \otimes_{\mathcal{S}} \mathrm{Hom}_{\mathcal{D}}\big(F(X),F(Y)\big) \arrow[r, "c^{\mathcal{D}}_{X,Y,Z}"'] & \mathrm{Hom}_{\mathcal{D}}\big(F(X),F(Z)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbf{1}_{\mathcal{S}} \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_{F(X)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[rr, "F_{X,X}"'] & & \mathrm{Hom}_{\mathcal{D}}\big(F(X),F(X)\big)
\end{tikzcd}$$

Remark 6.4. Let $\mathcal{C}$ be a category and let $(X_i)_{i \in I}$ be a collection of objects of X. A *product* of $X_1, \dots, X_n$, if it exists, is an object $X = \prod_i X_i$ of $\mathcal{C}$ with homomorphisms $\pi_i \colon X \to X_i$ (called *projections*) such that for every object Y of $\mathcal{C}$ with homomorphisms $f_i \colon Y \to X_i$, there exists a unique homomorphism $f \colon Y \to X$ such that $\pi_i \circ f = f_i$.

$$\begin{tikzcd}
Y \arrow[d, dashed, "f"'] \arrow[dr, "f_i"] & \\
X \arrow[r, "\pi_i"'] & X_i
\end{tikzcd}$$

If it exists, the product (with the projections) is uniquely determined by the universal property, up to a (unique) isomorphism.

Definition 6.5. An *additive category* is a preadditive category $\mathcal{C}$ such that for every finite collection $(X_i)_{i \in I}$ of objects of $\mathcal{C}$, the product $\prod_i X_i$ exists.

Remark 6.6. (1) In the definition of an additive category, one can equivalently require that finite coproducts $\coprod_i X_i$ exist. (Coproducts are defined by the universal property that is obtained by reversing all arrows in Remark 6.4.) One can show that finite coproducts in additive categories coincide with finite products, that is $\prod_i X_i \cong \coprod_i X_i$.

(2) In every additive category, we can form the empty product, which coincides with the empty coproduct and is called the *zero object* 0 of $\mathcal{C}$. By the universal properties, every object X of $\mathcal{C}$ admits a unique homomorphism $0 \to X$ and a unique homomorphism $X \to 0$ (i.e. 0 is an *initial* and *terminal* object of $\mathcal{C}$). This also implies that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{0\}$, the trivial group.

Remark 6.7. In an additive category $\mathcal{C}$, we can define the notions of *kernel* and *cokernel* for a homomorphism $f : X \to Y$: The kernel of f, if it exists, is an object $K = \ker(f)$ of $\mathcal{C}$ with a homomorphism $k : K \to X$ such that $f \circ k = 0$ (with respect to the abelian group structure on $\mathrm{Hom}_{\mathcal{C}}(K, Y)$) and such that for every object K' with a homomorphism $k' : K' \to Y$ satisfying $f \circ k' = 0$, there is a unique homomorphism $f' : K' \to K$ such that $k' = k \circ f'$.

$$\begin{tikzcd}
K \arrow[r, "k"'] \arrow[rr, bend left, "0"] & X \arrow[r, "f"'] & Y \\
K' \arrow[u, dashed, "f'"] \arrow[ur, "k'"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Dually, the cokernel $c : Y \to C = \mathrm{cok}(f)$ of f, if it exists, is defined by the universal property displayed in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & Y \arrow[r, "c"'] \arrow[dr, "c'"'] & C \arrow[d, dashed, "f'"] \\
& & C'
\end{tikzcd}$$

Definition 6.8. An additive category is called *pre-abelian* if every homomorphism has a kernel and a cokernel.

Remark 6.9. In a pre-abelian category $\mathcal{C}$, every homomorphism $f : X \to Y$ with kernel $k : K \to X$ and cokernel $c : Y \to C$ admits a canonical factorization

$$\begin{tikzcd}
X \arrow[r, "p"'] \arrow[rrr, bend left, "f"] & \mathrm{cok}(k) \arrow[r, dashed, "\overline{f}"'] & \ker(c) \arrow[r, "i"'] & Y
\end{tikzcd}$$

where $p\colon X \to \mathrm{cok}(k)$ is the cokernel of k and $i\colon \ker(c) \to Y$ is the kernel of c. This follows from the universal properties in Remark 6.7; the proof is left to the reader.

Definition 6.10. An *abelian category* is a pre-abelian category such that for every homomorphism $f\colon X \to Y$, the homomorphism $\overline{f}$ in the factorization from Remark 6.9 is an isomorphism.

Remark 6.11. In an abelian category, every homomorphism $f\colon X \to X$ has a factorization

$$X \xrightarrow{p} I \xrightarrow{i} Y,$$

where $p\colon X \to I$ is the cokernel of the kernel $k\colon \ker(f) \to X$ of f and $i\colon I \to Y$ is the kernel of the cokernel $c\colon Y \to \mathrm{cok}(f)$ of f. The object $I = \mathrm{im}(f)$ (with the homomorphisms p and i) is the *image* of f.

Definition 6.12. Let $\mathcal{C}$ be an abelian category. A *complex* in $\mathcal{C}$ is a sequence

$$\cdots \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \xrightarrow{d_{i+1}} \cdots$$

of objects X_i of $\mathcal{C}$ and homomorphisms $d_i\colon X_i \to X_{i+1}$, for $i \in \mathbb{Z}$, such that $d_{i+1} \circ d_i = 0$. The complex is called *exact* if d_i induces an isomorphism $\mathrm{im}(d_i) \to \ker(d_{i+1})$ for all $i \in \mathbb{Z}$.

Definition 6.13. An $\mathcal{S}$-enriched monoidal category is a monoidal category $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ such that $X \otimes -$ and $- \otimes X$ are $\mathcal{S}$-enriched functors for every object X of $\mathcal{C}$.

7 Hopf algebras

For the remainder of this Section, let us fix a field $\Bbbk$.

Definition 7.1. (1) An *algebra* over $\Bbbk$ is a $\Bbbk$-vector space A with linear maps $\mu\colon C \otimes C \to C$ and $\eta\colon \Bbbk \to C$, called *multiplication* and *unit*, respectively, such that

$$\mu \circ (\mu \otimes \mathrm{id}_C) = \mu \circ (\mathrm{id}_C \otimes \mu) \qquad \text{and} \qquad \mu \circ (\mathrm{id}_C \otimes \eta) = \mathrm{id}_C = \mu \circ (\eta \otimes \mathrm{id}_C).$$

(2) A *coalgebra* over $\Bbbk$ is a $\Bbbk$-vector space C with linear maps $\delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to \Bbbk$, called *comultiplication* and *counit*, respectively, such that

$$(\delta \otimes \mathrm{id}_C) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta \qquad \text{and} \qquad (\mathrm{id}_C \otimes \varepsilon) \circ \delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C) \circ \delta.$$

Definition 7.2. (1) A homomorphism between $\Bbbk$-algebras (A, μ, η) and (A', μ', η') is a $\Bbbk$-linear map $\varphi\colon A \to A'$ such that

$$\mu' \circ (\varphi \otimes \varphi) = \varphi \circ \mu \qquad \text{and} \qquad \varphi \circ \eta = \eta'.$$

(2) A homomorphism between $\Bbbk$-coalgebras (C, δ, ε) and $(C', \delta', \varepsilon')$ is a $\Bbbk$-linear map $\varphi\colon C \to C'$ such that

$$(\varphi \otimes \varphi) \circ \delta = \delta' \circ \varphi \qquad \text{and} \qquad \varepsilon' \circ \varphi = \varepsilon.$$

Definition 7.3. (1) A *left module* over a $\Bbbk$-algebra (A, μ, η) is a $\Bbbk$-vector space M with a linear map $\mu_M\colon C \otimes C \to C$ and $\eta\colon \Bbbk \to C$, called *action* of A, such that

$$\mu_M \circ (\mu \otimes \mathrm{id}_M) = \mu_M \circ (\mathrm{id}_A \otimes \mu) \qquad \text{and} \qquad \mu_M \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M.$$

A homomorphism between left A-modules (M, μ_M) and $(M', \mu_{M'})$ is a $\Bbbk$-linear map $f\colon M \to M'$ such that $f \circ \mu_M = \mu_{M'} \circ (\mathrm{id}_A \otimes f)$.

(2) A *left comodule* over a $\Bbbk$-coalgebra (C, δ, ε) is a $\Bbbk$-vector space N with a linear map $\delta_N \colon N \to C \otimes N$, called *coaction* of C, such that

$$(\delta \otimes \mathrm{id}_N) \circ \delta_N = (\mathrm{id}_C \otimes \delta_N) \circ \delta_N \qquad \text{and} \qquad (\varepsilon \otimes \mathrm{id}_N) \circ \delta_N = \mathrm{id}_N.$$

A homomorphism between left C-comodules (N, δ_N) and $(N', \delta_{N'})$ is a $\Bbbk$-linear map $g \colon N \to N'$ such that $\delta_{N'} \circ g = (\mathrm{id}_A \otimes g) \circ \delta_N$.

Remark 7.4. Given two $\Bbbk$-algebras A and A', the tensor product $A \otimes A'$ has a canonical $\Bbbk$-algebra structure with multiplication $(\mu \otimes \mu') \circ s_{2,3}$ and unit $\eta \otimes \eta'$, where

$$s_{2,3} \colon (A \otimes A') \otimes (A \otimes A') \to (A \otimes A) \otimes (A' \otimes A')$$

is the $\Bbbk$-linear map that swaps the second and the third component of the tensor product. Similarly, the tensor product of two $\Bbbk$-coalgebras has a canonical $\Bbbk$-coalgebra structure with comultiplication $s_{2,3} \circ (\delta \otimes \delta')$ and counit $\varepsilon \otimes \varepsilon'$.

Definition 7.5. A *bialgebra* over $\Bbbk$ is a quintuple $(B, \mu, \eta, \delta, \varepsilon)$ such that (B, μ, η) is a $\Bbbk$-algebra, (B, δ, ε) is a $\Bbbk$-coalgebra and δ and ε are algebra homomorphisms (or equivalently, μ and η are coalgebra homomorphisms).

Remark 7.6. Let $(B, \mu, \eta, \delta, \varepsilon)$ be a $\Bbbk$-bialgebra and let M and M' be B-modules. Then $M \otimes M'$ has a canonical B-module structure with action map

$$(\mu_M \otimes \mu_{M'}) \circ s_{2,3} \circ (\delta \otimes \mathrm{id}_{M \otimes M'}) \colon B \otimes M \otimes N \longrightarrow M \otimes N,$$

where

$$s_{2,3} \colon (B \otimes B) \otimes (M \otimes M') \to (B \otimes M) \otimes (B \otimes M')$$

is the $\Bbbk$-linear map that swaps the second and the third component of the tensor product. Similarly, the tensor product of two B-comodules N and N' has a canonical B-comodule structure with coaction map $(\mu \otimes \mathrm{id}_{N \otimes N'}) \circ s_{2,3} \circ (\delta_N \otimes \delta_{N'})$.

Definition 7.7. An *antipode* on a bialgebra algebra B is a $\Bbbk$-linear map $S \colon B \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes \mathrm{id}_B"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\delta"] \arrow[dr, "\delta"'] \arrow[rr, "\varepsilon"] & & \Bbbk \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}_B \otimes S"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

A pair (B, S) of a bialgebra and an antipode is called a *Hopf algebra.*

Lemma 7.8. *Let $\mathcal{C}$ be a rigid abelian monoidal category. Then the functors $X \otimes -$ and $- \otimes X$ are exact for all objects X of $\mathcal{C}$.*

Proof. By Lemma 4.12, the functor $X \otimes -$ is right adjoint to $X^* \otimes -$ and left adjoint to ${}^*X \otimes -$. Now the claim about $X \otimes -$ follows because every right adjoint functor is left exact and every left adjoint functor is right exact, and the claim about $- \otimes X$ is obtained analogously. $\square$

8 Reconstruction theory - the finite case

Definition 8.1. A *finite abelian* $\Bbbk$*-linear category* is an abelian category $\mathcal{A}$ that is equivalent to $A - \mathbf{mod}$ for some finite-dimensional $\Bbbk$-algebra A.

Remark 8.2. A $\Bbbk$-linear abelian category $\mathcal{A}$ is finite if and only if the following conditions are satisfied:

(1) all Hom-spaces in $\mathcal{A}$ are finite-dimensional;

(2) $\mathcal{A}$ has finitely many simple objects;

(3) $\mathcal{A}$ has enough projectives (i.e. every object is a quotient of a projective object);

(4) every object has a finite composition series.

In that case we can choose a *projective generator* of $\mathcal{A}$ (i.e. a projective object P such that every simple object in $\mathcal{A}$ is a quotient of P) and set $A = \mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}}$, and the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ induces an equivalence between $\mathcal{A}$ and $A - \mathbf{mod}$.

Observe that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ defines a functor from $\mathcal{A}$ to $\mathbf{Vect}_{\Bbbk}$ which is exact becuase P is projective and faithful because P is a generator. Conversely, if $G\colon \mathcal{A} \to \mathbf{Vect}_{\Bbbk}$ is an exact and faithful functor then one can define an A-module structure on $P \coloneqq G(A^*)^*$ such that G is naturally isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P, -)$. (See the discussion below.) Since G is exact and faithful, it follows that P is a projective generator of $\mathcal{A} = A - \mathbf{mod}$. By Yoneda's lemma, we have

$$\mathrm{End}(G) \cong \mathrm{End}\big(\mathrm{Hom}_{\mathcal{A}}(P, -)\big) \cong \mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}}$$

In order to construct the A-module structure on $P = G(A^*)^*$, note that since A is an A-bimodule, so is the dual space A^*; hence the right action $r_a\colon A^* \to A^*,\ \xi \to \xi \cdot a$ of an element $a \in A$ is a left A-module homomorphism. Now we can define a right action of A on $G(A)$ via $\vartheta \cdot a = G(r_a)(\vartheta)$ for $\vartheta \in G(A^*)$ and $a \in A$, and so $G(A)^*$ is indeed a left A-module. For the injective left A-module A^*, we have

$$\mathrm{Hom}_{\mathcal{A}}(P, A^*) \cong \mathrm{Hom}_{A}(A, G(A^*)) \cong G(A^*),$$

where the second Hom-space is a space of right A-module homomorphisms. For an arbitrary A-module M, we can choose an injective presentation $0 \to M \to A^{*\oplus a} \to A^{*\oplus b}$, and the latter induces by exactness an isomorphism between $G(M)$ and $\mathrm{Hom}(P, M)$. This homomorphism is natural in M because of the lifting property for injective resolutions and functoriality of cokernels.

Remark 8.3. Let $\mathcal{C}$ be a small category and let $F\colon \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ be a functor into the category of finite-dimensional vector spaces, for some field $\Bbbk$. Then the natural endomorphisms of F can be considered as a subspace of the product $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\Bbbk}\big(F(X)\big)$, consisting of the elements $(\vartheta_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ such that for every homomorphism $f\colon X \to Y$ in $\mathcal{C}$, we have $\vartheta_Y\colon F(f) = F(f) \circ \vartheta_Y$. In other words, $\mathrm{End}(F)$ is the equalizer in the diagram

$$\mathrm{End}(F) \longrightarrow \prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\Bbbk}\big(F(X)\big) \rightrightarrows \prod_{X \to Y} \mathrm{Hom}_{\Bbbk}\big(F(X), F(Y)\big),$$

where the second product runs over all homomorphisms $f\colon X \to Y$ in $\mathcal{C}$ and the two arrows on the right are given by $x \mapsto F(f) \circ x$ for $x \in \mathrm{End}_{\Bbbk}\big(F(X)\big)$ and $y \mapsto y \circ F(f)$ for $y \in \mathrm{End}_{\Bbbk}\big(F(Y)\big)$, respectively. This construction is known as the **end** of the functor $\mathrm{Hom}_{\Bbbk}\big(F(-), F(-)\big)\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ and denoted by

$$\mathrm{End}(F) = \int_{X \in \mathcal{C}} \mathrm{End}_{\Bbbk}\big(F(X)\big).$$

The integral sign hints on the fact that there is a 'Fubini theorem' for the **end**s; see Section IX.8 in [ML98]. Also note that **end**s can be considered as a particular kind of limit; see Section IX.5 in [ML98].

Remark 8.4. For a $\Bbbk$-linear abelian category $\mathcal{C}$ and two functors $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ and $G\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$, we can define a bifunctor $F \otimes G = F(-) \otimes_\Bbbk G(-)\colon \mathcal{C} \times \mathcal{C} \to \mathbf{Vect}_\Bbbk$.

Proposition 8.5. *Let $\mathcal{C}$ be a $\Bbbk$-linear category and let $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ and $G\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ be two $\Bbbk$-linear functors such that $\mathrm{End}(F)$ and $\mathrm{End}(G)$ are finite-dimensional. Then there is an isomorphism of $\Bbbk$-algebras*

$$\alpha_{F,G}\colon \mathrm{End}(F) \otimes \mathrm{End}(G) \longrightarrow \mathrm{End}(F \otimes G)$$

such that

$$\alpha_{F,G}(\eta \otimes \vartheta)_{X,Y} = \eta_X \otimes_\Bbbk \vartheta_Y$$

for $\eta \in \mathrm{End}(F)$ and $\vartheta \in \mathrm{End}(G)$.

Proof. One can check that $\alpha_{F,G}$ coincides with the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{End}(F) \otimes \mathrm{End}(G) &= \Big(\int_{X\in\mathcal{C}} \mathrm{End}_\Bbbk\big(F(X)\big) \Big) \otimes \mathrm{End}(G) \\
&\overset{(1)}{\cong} \int_{X\in\mathcal{C}} \Big(\mathrm{End}_\Bbbk\big(F(X)\big) \otimes \mathrm{End}(G) \Big) \\
&\cong \int_{X\in\mathcal{C}} \Big(\mathrm{End}_\Bbbk\big(F(X)\big) \otimes \int_{Y\in\mathcal{C}} \mathrm{End}_\Bbbk\big(G(Y)\big) \Big) \\
&\overset{(1)}{\cong} \int_{X\in\mathcal{C}} \int_{Y\in\mathcal{C}} \Big(\mathrm{End}_\Bbbk\big(F(X)\big) \otimes \mathrm{End}_\Bbbk\big(G(Y)\big) \Big) \\
&\overset{(2)}{\cong} \int_{(X,Y)\in\mathcal{C}\times\mathcal{C}} \Big(\mathrm{End}_\Bbbk\big(F(X)\big) \otimes \mathrm{End}_\Bbbk\big(G(Y)\big) \Big) \\
&\cong \int_{(X,Y)\in\mathcal{C}\times\mathcal{C}} \mathrm{End}_\Bbbk\big(F(X) \otimes G(Y)\big) \\
&= \mathrm{End}(F \otimes G).
\end{aligned}$$

Here, we use the facts that

(1) tensoring with a finite dimensional vector space has a left and right adjoint functor (given by tensoring with the dual space); hence it is a *continuous funcor*. This means that tensoring with a finite dimensional vector space commutes with limits in $\mathbf{Vect}_\Bbbk$, and in particular with **end**s.

(2) there is a Fubini theorem for **end**s; see Section IX.8 in [ML98]. $\square$

Remark 8.6. If (F, φ, ϵ) is an exact faithful monoidal functor then we can define linear maps

$$\varepsilon\colon \mathrm{End}(F) \longrightarrow \Bbbk \qquad \text{and} \qquad \delta\colon \mathrm{End}(F) \longrightarrow \mathrm{End}(F) \otimes \mathrm{End}(F)$$

via $\varepsilon(\eta) = \eta_{\mathbf{1}} \in \mathrm{End}_\Bbbk\big(F(\mathbf{1})\big) \cong \mathrm{End}_\Bbbk(\Bbbk) \cong \Bbbk$ and $\delta(\eta) = \alpha_{F,F}^{-1}(\tilde{\eta})$ for $\eta \in \mathrm{End}(F)$, where $\tilde{\eta} \in \mathrm{End}(F\otimes F)$ is defined by $\tilde{\eta}_{X,Y} = \varphi_{X,Y} \circ \eta_{X\otimes Y} \circ \varphi_{X,Y}^{-1}$.

Proposition 8.7. *Let $\mathcal{C}$ be a finite $\Bbbk$-linear abelian monoidal category and let $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ be a faithful monoidal functor. The algebra $H = \mathrm{End}(F)$ is a bialgebra with counit $\delta\colon H \to H \otimes H$ and counit $\varepsilon\colon H \to \Bbbk$ as defined in Remark 8.6*

Proof. We will show that δ is a coassociative algebra homomorphism. The properties relating to the counit are left to the reader. Note that $(\delta \otimes \mathrm{id}_H) \circ \delta$ induces via Proposition 8.5 an algebra homomorphism

$$\hat{\delta}\colon H = \mathrm{End}(F) \longrightarrow \mathrm{End}\big((F \otimes F) \otimes F\big), \qquad \eta \longmapsto \hat{\eta}$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & & \mathrm{End}(F) \otimes \mathrm{End}(F) \otimes \mathrm{End}(F) \arrow[dd, "\alpha_{F,F} \otimes \mathrm{id}_H"] \\
 & \mathrm{End}(F) \otimes \mathrm{End}(F) \arrow[ur, "\delta \otimes \mathrm{id}_H"] \arrow[dr, "\tilde{\delta} \otimes \mathrm{id}_H"'] \arrow[dd, "\alpha_{F,F}"'] & \\
\mathrm{End}(F) \arrow[ur, "\delta"] \arrow[dr, "\tilde{\delta}"'] \arrow[ddrr, bend right, "\hat{\delta}"'] & & \mathrm{End}(F \otimes F) \otimes \mathrm{End}(F) \arrow[dd, "\alpha_{F\otimes F,F}"'] \\
 & \mathrm{End}(F \otimes F) \arrow[dr, "\psi"] & \\
 & & \mathrm{End}\big((F \otimes F) \otimes F\big)
\end{tikzcd}$$

The homomorphism $\psi\colon \mathrm{End}(F \otimes F) \to \mathrm{End}\big((F \otimes F) \otimes F\big)$ is defined by $\mu \mapsto \mu'$, where

$$\mu'_{X,Y,Z} = (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \mu_{X \otimes Y} \circ (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)})^{-1}$$

for objects X, Y, Z of $\mathcal{C}$, and it follows that

$$\begin{aligned}
\hat{\eta}_{X,Y,Z} &= (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \tilde{\eta}_{X \otimes Y, Z} \circ (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)})^{-1} \\
&= (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \varphi_{X \otimes Y, Z} \circ \eta_{(X \otimes Y) \otimes Z} \circ \varphi^{-1}_{X \otimes Y, Z} \circ (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)})^{-1},
\end{aligned}$$

for $\eta \in \mathrm{End}(F)$. Analogously, $(\mathrm{id}_H \otimes \delta) \circ \delta$ induces an algebra homomorphism $H \to \mathrm{End}\big(F \otimes (F \otimes F)\big)$ with $\eta \mapsto \check{\eta}$ such that

$$\check{\eta}_{X,Y,Z} = (\mathrm{id}_{F(X)} \otimes \varphi_{Y,Z}) \circ \varphi_{X, Y \otimes Z} \circ \eta_{X \otimes (Y \otimes Z)} \circ \varphi^{-1}_{X, Y \otimes Z} \circ (\mathrm{id}_{F(X)} \otimes \varphi_{Y,Z})^{-1}.$$

In order to show that δ is coassociative, it suffices to show that $\hat{\eta}_{X,Y,Z} = \check{\eta}_{X,Y,Z}$ for all objects X, Y, Z of $\mathcal{C}$ (disregarding associativity constraints in $\mathbf{Vect}_{\Bbbk}$). Since (F, φ, ϵ) is a monoidal functor, we have

$$(\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \varphi_{X \otimes Y, Z} \circ F(\alpha_{X,Y,Z}) = (\mathrm{id}_{F(X)} \otimes \varphi_{Y,Z}) \circ \varphi_{X, Y \otimes Z},$$

and it follows that

$$\begin{aligned}
\check{\eta}_{X,Y,Z} &= (\mathrm{id}_{F(X)} \otimes \varphi_{Y,Z}) \circ \varphi_{X, Y \otimes Z} \circ \eta_{X \otimes (Y \otimes Z)} \circ \varphi^{-1}_{X, Y \otimes Z} \circ (\mathrm{id}_{F(X)} \otimes \varphi_{Y,Z})^{-1} \\
&= (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \varphi_{X \otimes Y, Z} \circ F(\alpha_{X,Y,Z}) \circ \eta_{X \otimes (Y \otimes Z)} \circ F(\alpha^{-1}_{X,Y,Z}) \circ \varphi^{-1}_{X \otimes Y, Z} \circ (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)})^{-1} \\
&= (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)}) \circ \varphi_{X \otimes Y, Z} \circ \eta_{(X \otimes Y) \otimes Z} \circ \varphi^{-1}_{X \otimes Y, Z} \circ (\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)})^{-1} \\
&= \hat{\eta}_{X,Y,Z},
\end{aligned}$$

where in the third step, we use the fact that $\eta \in \mathrm{End}(F)$.

In order to show that δ is a homomorphism, it suffices to show that the homomorphism

$$\tilde{\delta}\colon \mathrm{End}(F) \longrightarrow \mathrm{End}(F \otimes F), \qquad \eta \longmapsto \tilde{\eta}$$

is an algebra homomorphism, where $\tilde{\eta}_{X,Y} = \varphi_{X,Y} \circ \eta_{X \otimes Y} \circ \varphi^{-1}_{X,Y}$ for objects X, Y of $\mathcal{C}$. (Recall from Remark 8.6 that $\tilde{\delta} = \alpha_{F,F} \circ \delta$, where $\alpha_{F,F}$ is an algebra isomorphism by Proposition 8.5.) For $\eta, \nu \in \mathrm{End}(F)$, we have

$$\tilde{\delta}(\eta \circ \nu)_{X,Y} = \varphi_{X,Y} \circ (\eta \circ \nu)_{X \otimes Y} \circ \varphi^{-1}_{X,Y} = (\varphi_{X,Y} \circ \eta_{X \otimes Y} \circ \varphi^{-1}_{X,Y}) \circ (\varphi_{X,Y} \circ \eta_{X \otimes Y} \circ \varphi^{-1}_{X,Y}) = \tilde{\delta}(\eta)_{X,Y} \circ \tilde{\delta}(\nu)_{X,Y},$$

whence $\tilde{\delta}(\eta \circ \nu) = \tilde{\delta}(\eta) \circ \tilde{\delta}(\nu)$ and $\tilde{\delta}$ is an algebra homomorphism. $\square$

Remark 8.8. Let $\mathcal{C}$ be a $\Bbbk$-linear category with a $\Bbbk$-linear functor $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$. Then for every object X of $\mathcal{C}$, we can endow $F(X)$ with an $\mathrm{End}(F)$-module structure via the representation

$$c_X\colon \mathrm{End}(F) \longrightarrow \mathrm{End}_\Bbbk\big(F(X)\big), \qquad \eta \longmapsto \eta_X.$$

The definition of natural transformations implies that for a homomorphism $f\colon X \to Y$ in $\mathcal{C}$, the $\Bbbk$-linear map $F(f)$ is a homomorphism of $\mathrm{End}(F)$-modules.

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & F(X) \arrow[d, "F(f)"] \\
F(Y) \arrow[r, "\eta_Y"'] & F(Y)
\end{tikzcd}$$

Thus, the functor F induces a functor $\hat{F}\colon \mathcal{C} \longrightarrow \mathrm{End}(F) - \mathbf{mod}$.

Now suppose that $\mathcal{C}$ is a monoidal category and that $F = (F, \varphi, \varepsilon)$ is a monoidal functor, so that $\mathrm{End}(F)$ is a bialgebra. We claim that $(\hat{F}, \varphi, \varepsilon)$ is a monoidal functor from $\mathcal{C}$ to the monoidal category $\mathrm{End}(F) - \mathbf{mod}$. Indeed, by the definition of the comultiplication $\delta\colon \mathrm{End}(F) \to \mathrm{End}(F) \otimes \mathrm{End}(F)$, the action of $\eta \in \mathrm{End}(F)$ on $F(X) \otimes F(Y)$ is given by $\varphi_{X,Y} \circ \eta_{X\otimes Y} \circ \varphi^{-1}_{X,Y}$ for objects X, Y of $\mathcal{C}$. (See the commutative diagram below, where $c_{X,Y}(\mu) = \mu_{X,Y}$ and the rightmost vertical arrow is induced by the tensor product.)

$$\begin{tikzcd}
 & \mathrm{End}(F) \otimes \mathrm{End}(F) \arrow[r, "c_X \otimes c_Y"] \arrow[d, "\alpha_{F,F}"'] & \mathrm{End}_\Bbbk\big(F(X)\big) \otimes \mathrm{End}_\Bbbk\big(F(Y)\big) \arrow[d, "\wr"] \\
\mathrm{End}(F) \arrow[ur, "\delta"] \arrow[dr, "\tilde{\delta}"'] & & \\
 & \mathrm{End}(F \otimes F) \arrow[r, "c_{X,Y}"'] & \mathrm{End}_\Bbbk\big(F(X) \otimes F(Y)\big)
\end{tikzcd}$$

Since η acts on $F(X \otimes Y)$ via $\eta_{X\otimes Y}$, the $\Bbbk$-linear map $\varphi_{X,Y}\colon F(X \otimes Y) \to F(X) \otimes F(Y)$ is a homomorphism of $\mathrm{End}(F)$-modules. Similarly, one sees that $\varepsilon\colon F(\mathbf{1}) \to \Bbbk$ is a homomorphism of $\mathrm{End}(F)$-modules, and the axioms for a monoidal functor follow from the fact that $(F, \varphi, \varepsilon)$ is a monoidal functor from $\mathcal{C}$ to $\mathbf{Vect}_\Bbbk$.

Remark 8.9. Suppose that every object X in $\mathcal{C}$ has a right dual X^*, and recall from Lemma 4.8 that for a monoidal functor $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$, we have canonically $F(X^*) = F(X)^*$. Since right and left duals coincide in $\mathbf{Vect}_\Bbbk$, we also have canonically $F(X^*)^* \cong F(X)$, and we can define an anti-homomorphism

$$S\colon \mathrm{End}(F) \longrightarrow \mathrm{End}(F)$$

such that $S(a)_X = a^*_{X^*}$. If $\mathcal{C}$ is rigid (i.e. if every object X of $\mathcal{C}$ also has a left-dual ${}^*X$) then S has an inverse such that $S^{-1}(a) = a^*_{{}^*X}$.

Proposition 8.10. *Let $\mathcal{C}$ be a $\Bbbk$-linear monoidal category with right duals and let $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ be a monoidal functor. Then the map $S\colon \mathrm{End}(F) \to \mathrm{End}(F)$ from Remark 8.9 is an antipode for the bialgebra* $\mathrm{End}(F)$.

Proof. For $a, b \in \mathrm{End}(F) = B$, we have

$$\begin{aligned}
\mu \circ (S \otimes \mathrm{id}_B)(a \otimes b)_X &= S(a)_X \circ b_X \\
&= \big(\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}\big) \circ \big(\mathrm{id}_{F(X)} \otimes a_{X^*} \otimes \mathrm{id}_{F(X)}\big) \circ \big(\mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}\big) \circ b_X \\
&= \big(\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}\big) \circ \big(\mathrm{id}_{F(X)} \otimes a_{X^*} \otimes b_X\big) \circ \big(\mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}\big),
\end{aligned}$$

and thus for $c \in \mathrm{End}(F \otimes F)$, we obtain

$$\mu \circ (S \otimes \mathrm{id}_B)\big(\alpha_{F,F}^{-1}(c)\big)_X = \big(\mathrm{id}_{F(X)} \otimes F(\mathrm{ev}_X)\big) \circ (\mathrm{id}_{F(X)} \otimes c_{X^*,X}) \circ \big(F(\mathrm{coev}_X) \otimes \mathrm{id}_{F(X)}\big)$$

(Recall that $\alpha_{F,F}(a \otimes b)_{X,Y} = a_X \otimes b_Y$.) By the definition of the comultiplication δ (see Remark 8.6), this implies that

$$\mu \circ (S \otimes \mathrm{id}) \circ \delta(a) = \big(\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}\big) \circ \big(\mathrm{id}_{F(X)} \otimes (\varphi_{X^*,X} \circ a_{X^* \otimes X} \circ \varphi_{X^*,X}^{-1})\big) \circ \big(\mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}\big)$$

for $a \in \mathrm{End}(F)$. Now the equality $\mu \circ (S \otimes \mathrm{id}_B) \otimes \delta = \eta \circ \varepsilon$ follows from the commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)}"'] & F(X) \otimes F(X)^* \otimes F(X) \arrow[d, "\mathrm{id}_{F(X)} \otimes \varphi_{X^*,X}^{-1}"] \\
F(X) \arrow[d, "\mathrm{id}_{F(X)} \otimes a_{\mathbf{1}}"'] & F(X) \otimes F(X^* \otimes X) \arrow[l, "\mathrm{id}_{F(X)} \otimes F(\mathrm{ev}_X)"'] \arrow[d, "\mathrm{id}_{F(X)} \otimes a_{X^* \otimes X}"] \\
F(X) \arrow[d, "\mathrm{id}_{F(X)}"'] & F(X) \otimes F(X^* \otimes X) \arrow[l, "\mathrm{id}_{F(X)} \otimes F(\mathrm{ev}_X)"'] \arrow[d, "\mathrm{id}_{F(X)} \otimes \varphi_{X^*,X}"] \\
F(X) & F(X) \otimes F(X)^* \otimes F(X) \arrow[l, "\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}"]
\end{tikzcd}$$

where the top square and the bottom square commute by the definitions of evaluation and coevaluation for $F(X)$ in the proof of Lemma 4.8 (and in the case of the top square, the zig-zag relations), and the middle square commutes by the naturality of a. (The composition along the top, right and bottom is $\mu \circ (S \otimes \mathrm{id}) \circ \delta(a)_X$ by the above, and the composition along the left-hand side is $\eta \circ \varepsilon(a)_X$.) The equality $\mu \circ (S \otimes \mathrm{id}_B) \otimes \delta = \eta \circ \varepsilon$ can be proven analogously. $\square$

Theorem 8.11. *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F) \qquad \textit{and} \qquad H \mapsto (H - \mathbf{mod}, F_H)$$

are mutually inverse bijections between

(1) $\Bbbk$-linear abelian monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and bialgebras over $\Bbbk$, up to isomorphism.

(2) tensor categories $\mathcal{C}$ over $\Bbbk$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors and Hopf algebras over $\Bbbk$, up to isomorphism.

Potential topics for talks

The topics that appear in gray have already been chosen.

(1) pivotal categories, traces and dimension (?);

(2) Frobenius-Perron dimension;

(3) monoidal categories by generators and relations (via diagrams);

(4) the Temperley-Lieb category (and knot invariants);

(5) tensor triangular geometry;

(6) the geometric Satake equivalence;

(7) negligible morphisms and semisimplification;

(8) the Drinfel'd double and the Drinfel'd center;

(9) $\mathfrak{sl}_n$-webs;

(10) interpolation categories;

(11) $\cdots$

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