Tensor categories

Jonathan Gruber

January 12, 2023

Introduction

In many fields of mathematics, one is naturally led to study tensor products of certain objects (e.g. sheaves in algebraic geometry, cobordisms in topology, modules in commutative algebra). All of these notions fit into the framework of *monoidal categories*, which gives an abstract definition of what a tensor product structure on a category should be. The aim of this course is to give an introduction to monoidal categories and tensor categories, the latter being certain monoidal categories endowed with some extra structure. (They are abelian and have a linear structure that is compatible with the tensor product, over some field.)

The first half of the course deals mainly with category-theoretical notions, starting from the definition of a monoidal category and then discussing important additional properties such as rigidity (the existence of duals) and braidings (functorial isomorphisms $X \otimes Y \cong Y \otimes Y$). In the second half, we will turn our attention to tensor categories. Our prime example is the category of $\mathbf{Rep}_{\Bbbk}(G)$ finite-dimensional representations of a group G over a field $\Bbbk$, and we will discuss reconstruction theorems that allow us to recover a group (or Hopf algebra) from the corresponding category of representations, together with its monoidal structure and the *forgetful functor* that sends a representation to the underlying vector space. This gives rise to bijections between certain types of group and Hopf algebras, up to isomorphism, and certain kinds of tensor categories, up to monoidal equivalence, that is broadly referred to as *Tannaka duality.*

Author's note

These notes are the my own synopsis of material that has been collected from many different sources, but most importantly, from [EGNO15]. Further important references include [ML98, DM82, EGNO09] and some websites such as nLab, MathStackExchenge, and Wikipedia. No originality is claimed, except in the presentation of the material, and all mistakes should be considered my responsibility.

These notes are also a work in progress. If you find any mistakes or typos and if you have comments or suggestions, please let me know.

I would like to thank Johannes Flake for helpful discussions and encouragement, and for making his own notes available.

1 Monoidal categories

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) a class $\mathrm{Ob}(\mathcal{C})$ of *objects* of $\mathcal{C}$;

(2) for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *homomorphisms* from A to B;

(3) for every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition map*

$$\circ \colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

such that the following axioms hold:

(a) for $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon Z \to W$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(b) for $X \in \mathrm{Ob}(\mathcal{C})$, there exists an *identity homomorphism* id_X such that $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f\colon Y \to X$ and $g\colon X \to Y$.

Remark 1.2. (1) The homomorphisms in a category are often simply referred to as *morphisms*.

(2) As in points (a) and (b) of the definition, we often denote a homomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ by an arrow $f\colon X \to Y$.

(3) A morphism $f\colon X \to Y$ is called an *isomorphism* if there exists a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In that case, we write $g = f^{-1}$ and $X \cong Y$.

Example 1.3. We list some important examples of categories:

- **Set**: the category of sets, with maps between sets as homomorphisms;
- **Grp**: the category of groups with group homomorphisms;
- **AbGrp**: the category of abelian groups with group homomorphisms;
- $\mathbf{Vect}_{\Bbbk}$: the category of finite-dimensional $\Bbbk$-vector spaces with $\Bbbk$-linear maps, for a given field $\Bbbk$;
- $A-\mathbf{Mod}$: the category of A-modules with A-module homomorphisms, for a given algebra A; we write $A-\mathbf{mod}$ for the subcategory of finite-dimensional A-modules.

Definition 1.4. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns

(1) to every object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$;

(2) to every homomorphism $f\colon X \to Y$ in $\mathcal{C}$ a homomorphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$;

in such a was that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \qquad \text{and} \qquad F(f \circ g) = F(f) \circ F(g).$$

Remark 1.5. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the product category $\mathcal{C} \times \mathcal{D}$ whose objects are the pairs (X, Y) of objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, and where

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((X, Y), (Z, W)\big) = \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{D}}(Y, W)$$

for $X, Z \in \mathrm{Ob}(\mathcal{C})$ and $Y, W \in \mathrm{Ob}(\mathcal{D})$. The composition and the identity morphisms are defined component-wise in the obvious way. A functor from $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ to some other category $\mathcal{E}$ is often called a *bijunctor*.

Definition 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta\colon F \to G$ is a family of morphisms $\eta_A\colon F(A) \to G(A)$ in $\mathcal{D}$, for every object A of $\mathcal{C}$, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

For an object A of $\mathcal{C}$, we call η_A the component of η at A. A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

Remark 1.7. The functors between two categories $\mathcal{C}$ and $\mathcal{D}$ form a category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ whose morphisms are natural transformations between functors. The composition of natural transformations is defined componentwise.

Definition 1.8. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an equivalence if there exists a functor $g\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $g \circ f$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Definition 1.9. A *monoidal category* is a tuple $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, where

- $\mathcal{C}$ is a category,
- $\mathbf{1}$ is an object of $\mathcal{C}$, called the *unit object*,
- $\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor called the *tensor product*,
- $a\colon\ - \otimes(- \otimes -) \to (- \otimes -) \otimes -$ is a natural isomorphism, called the *associativity constraint*,
- $\lambda\colon \mathbf{1} \otimes - \to \mathrm{id}_{\mathcal{C}}$ and $\rho\colon\ - \otimes \mathbf{1} \to \mathrm{id}_{\mathcal{C}}$ are natural isomorphisms, called the (left and right) *unitors*,

subject to the following axioms:
Pentagon axiom: For all objects A, B, C, D of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[rr, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \arrow[dl, "\alpha_{A,B,C\otimes D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & & (A \otimes (B \otimes C)) \otimes D \arrow[dll, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \\
& & ((A \otimes B) \otimes C) \otimes D & &
\end{tikzcd}$$

In other words, we have

$$\big(\mathrm{id}_A \otimes \alpha_{B,C,D}\big) \circ \big(\alpha_{A,B\otimes C,D}\big) \circ \big(\alpha_{A,B,C\otimes D}\big) = \big(\alpha_{A,B,C} \otimes \mathrm{id}_D\big) \circ \big(\alpha_{A\otimes B,C,D}\big).$$

Unit axiom / triangle axiom: For all objects A, B of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
A \otimes (\mathbf{1} \otimes B) \arrow[rr, "\alpha_{A,\mathbf{1},B}"] \arrow[dr, "\mathrm{id}_A \otimes \lambda_B"'] & & (A \otimes \mathbf{1}) \otimes B \arrow[dl, "\rho_A \otimes \mathrm{id}_B"] \\
& A \otimes B &
\end{tikzcd}$$

In other words, we have

$$\alpha_{A,\mathbf{1},B} \circ (\rho_A \otimes \mathrm{id}_B) = \mathrm{id}_A \otimes \lambda_B.$$

Remark 1.10. (1) When no confusion is possible, we simply write $\mathcal{C}$ instead of $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$. We also say that $(\mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is a *monoidal structure* on the category $\mathcal{C}$.

(2) Being monoidal is **not** a property of a given category, but an *additional structure*. A category can admit more than one monoidal structure. (See the examples below.)

(3) The fact that $\otimes$ is a bifunctor means that for suitable morphisms a, b, c, d in $\mathcal{C}$, we have

$$(a \otimes c) \circ (b \otimes d) = (a \circ b) \otimes (c \circ d).$$

(4) Instead of assuming the existence of $\mathbf{1}$ with natural transformations λ and rho subject to the unit axiom, one can start from the (seemingly weaker, but actually equivalent) assumptions that there exists an object $\mathbf{1}$ with an isomorphism $\iota \colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ such that the functors $\mathbf{1} \otimes -$ and $- \otimes \mathbf{1}$ are equivalences. (No additional assumptions on the isomorphism ι are necessary.) See Sections 2.1 and 2.2 of [EGNO15] for more details.

Example 1.11. (1) The category **Set** of sets has a monoidal structure where the tensor product is given by the cartesian product and the unit object is a singleton $\{\bullet\}$.

(2) The category **Grp** of groups has a monoidal structure where the tensor product is given by the cartesian product and the unit object is the trivial group $\{1\}$.

(3) The category **AbGrp** of abelian groups has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\mathbb{Z}} -$ and the unit object is the group $\mathbb{Z}$ of integers. It also inherits a different monoidal structure from the category **Grp**; see the previous point.

(4) The category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$ has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\Bbbk} -$ and the unit object is the one-dimensional vector space $\Bbbk$.

(5) The category $\mathbf{Rep}_{\Bbbk}(G)$ of finite dimensional representations of a group G over a field $\Bbbk$ admits a monoidal structure where the tensor product is the usual tensor product of representations and the unit object is the trivial one-dimensional representation $\Bbbk$. More precisely, if we identify $\mathbf{Rep}_{\Bbbk}(G)$ with the category of finite-dimensional modules over the group algebra $\Bbbk[G]$ then the action of $\Bbbk[G]$ on the tensor product $M \otimes N$ of two $\Bbbk[G]$-modules M and N is uniquely determined by $g \cdot (m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$.

In the following, we refer to the objects of $\mathbf{Rep}(G)$ as *G-modules*.

(6) For a category $\mathcal{C}$, the category $\mathbf{End}(\mathcal{C}) = \mathbf{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors of $\mathcal{C}$ has a monoidal structure, where the tensor product is given by the composition of functors and the unit object is the identity functor $\mathrm{id}_{\mathcal{C}}$. The associativity constraints and unitors are identity natural transformations.

The two next examples will seem quite trivial for now, but they will become more interesting later when we add extra structure:

(7) Let G be a monoid and A an abelian group. Then we can define a monoidal category $\mathcal{C}_A^G$ with objects $\mathrm{Ob}(\mathcal{C}_A^G) = \{\delta_g \mid g \in G\}$ indexed by G and homomorphisms

$$\mathrm{Hom}_{\mathcal{C}_A^G}(\delta_g, \delta_h) = \begin{cases} A & \text{if } g = h, \\ \varnothing & \text{otherwise,} \end{cases}$$

for $g, h \in G$. The tensor product is defined by $\delta_g \otimes \delta_h = gh$ and by $a \otimes a' = aa' \in A$ for $g, h \in G$ and $a, a' \in A$, the unit object is δ_e (where $e \in G$ is the unit element) and the associativity constraints and unitors are identity maps.

(8) Let G be a monoid and let $\mathbf{Vect}_{\Bbbk}^G$ be the category of finite-dimensional G-graded $\Bbbk$-vector spaces $V = \bigoplus_{g \in G} V_g$, with homomorphisms given by grading-preserving linear maps. (That is, for $V = \bigoplus_g V_g$ and $W = \bigoplus_g W_g$ two G-graded vector spaces, the homomorphisms from V to W in $\mathbf{Vect}_{\Bbbk}^G$ are the linear maps $f \colon V \to W$ that satisfy $f(V_g) \subseteq W_g$ for all $g \in G$.) Then the

monoidal structure on $\mathbf{Vect}_{\Bbbk}$ induces a monoidal structure $\mathbf{Vect}_{\Bbbk}^{G}$, where the grading on the tensor product of G-graded vector spaces V and W is given by

$$(V \otimes W)_g = \bigoplus_{hh'=g} V_h \otimes W_{h'}$$

for $g \in G$. The tensor product of two homomorphisms in $\mathbf{Vect}_{\Bbbk}^{G}$ is just the usual tensor product of linear maps. The unit object is the one-dimensional vector space $\Bbbk = \Bbbk_e$ whose unique non-zero grading piece is indexed by the unit object $e \in G$. The associativity constraint and the unitors come from the category $\mathbf{Vect}_{\Bbbk}$.

Observe that there is a faithful functor $i_{\Bbbk}^{G} \colon \mathcal{C}_{\Bbbk^\times}^{G} \to \mathbf{Vect}_{\Bbbk}^{G}$ with $i_{\Bbbk}^{G}(\delta_g) = \Bbbk_g$ the one-dimensional vector space with grading concentrated in degree g, for $g \in G$, and with the obvious definition on homomorphisms. This functor is compatible with the tensor product (up to a natural isomorphism); it is an example of a *monoidal functor* (to be defined shortly).

In our final example, we demonstrate that there are monoidal categories with a less obvious choice of associativity constraint.

(9) Let G be a monoid, let A an abelian group and let ω be a 3-cocycle for G with values in A, i.e. a map $\omega \colon G^{\times 3} \to A$ with

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) \tag{1.1}$$

for $g_1, g_2, g_3, g_4 \in G$. Then we can define a monoidal category $\mathcal{C}_A^{G,\omega}$ with underlying category $\mathcal{C}_A^G$ and with tensor product and unit object defined as in point (7), but with associativity constraint α^ω defined by

$$\alpha^\omega_{g,h,k} = \omega_{g,h,k} \colon \delta_g \otimes (\delta_h \otimes \delta_k) = \delta_{ghk} \longrightarrow \delta_{ghk} = (\delta_g \otimes \delta_h) \otimes \delta_k$$

for $g, h, k \in G$. Observe that the 3-cocycle condition implies that $\mathcal{C}_A^{G,\omega}$ satisfies the pentagon axiom. (In fact, a map $\omega \colon G^{\times 3} \to A$ defines an associativity constraint for $\mathcal{C}_A^G$ if and only if ω is a 3-cocycle.) The unitors are defined by $\lambda_g = \omega(e, e, g)$ and $\rho_g = \omega(g, e, e)^{-1}$ for $g \in G$, and the unit axiom becomes the equation $\omega(g, e, h) = \omega(g, e, e) \cdot \omega(e, e, h)$ for $g, h \in G$ (which also follows from (1.1) by setting $g_2 = g_3 = e$).

Given a 3-cocycle $\omega \colon G^{\times 3} \to \Bbbk^\times$ with values in the multiplicative group of a field $\Bbbk$, we can extend the associativity constraint α^ω on $\mathcal{C}_{\Bbbk^\times}^{G}$ to an associativity constraint α^ω on $\mathbf{Vect}_{\Bbbk}^{G}$ via

$$\alpha^\omega_{\Bbbk_g, \Bbbk_h, \Bbbk_k} = \omega(g, h, k) \cdot \alpha_{\Bbbk_g, \Bbbk_h, \Bbbk_k},$$

for $g, h, k \in G$, extended by additivity, where α denotes the 'usual' associativity constraint in $\mathbf{Vect}_{\Bbbk}^{G}$. (Note that every object of $\mathbf{Vect}_{\Bbbk}^{G}$ is a direct sum of objects of the form $\Bbbk_g$ with $g \in G$.)

Remark 1.12. Given a monoidal category $\mathcal{C} = (\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, we define the *opposite monoidal category* $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}, \mathbf{1}, \otimes^{\mathrm{op}}, \alpha^{\mathrm{op}}, \rho, \lambda)$ with the same underlying category, but tensor product defined by $X \otimes^{\mathrm{op}} Y = Y \otimes X$ and $f \otimes^{\mathrm{op}} g = g \otimes f$ for objects X, Y and homomorphisms f, g in $\mathcal{C}$, and with associativity constraint given by $\alpha^{\mathrm{op}}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$ for objects X, Y, Z of $\mathcal{C}$.

This is not to be confused with the *reverse category* $\mathcal{C}^{\mathrm{rev}}$ with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The latter can also be endowed with a canonical monoidal structure. Note that $\mathcal{C}^{\mathrm{rev}}$ is often also called the opposite category of $\mathcal{C}$; we use non-standard terminology here to avoid confusion with the opposite monoidal category defined above.

Lemma 1.13. *For all objects A, B of $\mathcal{C}$, we have*

$$\rho_{A \otimes B} \circ \alpha_{A,B,\mathbf{1}} = \mathrm{id}_A \otimes \rho_B \qquad \text{and} \qquad \lambda_{A \otimes B} \circ \alpha_{\mathbf{1},A,B} = \lambda_A \otimes \mathrm{id}_B.$$

Proof. Consider the following diagram, where all arrows are isomorphisms:

$$
\begin{tikzcd}
A \otimes (B \otimes (\mathbf{1} \otimes D)) \arrow[rrrr, "\mathrm{id}_A \otimes \alpha_{B,\mathbf{1},D}"] \arrow[drr, "\mathrm{id}_A \otimes (\mathrm{id}_B \otimes \lambda_D)"] \arrow[ddd, "\alpha_{A,B,\mathbf{1}\otimes D}"'] & & & & A \otimes ((B \otimes \mathbf{1}) \otimes D) \arrow[dll, "\mathrm{id}_A \otimes (\rho_B \otimes \mathrm{id}_D)"'] \arrow[ddd, "\alpha_{A,B\otimes\mathbf{1},D}"] \\
& & A \otimes (B \otimes D) \arrow[d, "\alpha_{A,B,D}"'] & & \\
& & (A \otimes B) \otimes D & & \\
(A \otimes B) \otimes (\mathbf{1} \otimes D) \arrow[urr, "\mathrm{id}_{A\otimes B} \otimes \lambda_D"] \arrow[drr, "\alpha_{A\otimes B,\mathbf{1},D}"] & & & & (A \otimes (B \otimes \mathbf{1})) \otimes D \arrow[ull, "(\mathrm{id}_A \otimes \rho_B) \otimes \mathrm{id}_D"'] \arrow[dll, "\alpha_{A,B,\mathbf{1}} \otimes \mathrm{id}_D"] \\
& & ((A \otimes B) \otimes \mathbf{1}) \otimes D \arrow[uu, "\rho_{A\otimes B} \otimes \mathrm{id}_D"'] & &
\end{tikzcd}
$$

The external pentagon commutes by the pentagon axiom, the quadrangles commute by naturality of the associativity constraint α, and the top triangle and the bottom left triangle commute by the triangle axiom. Since all arrows are isomorphisms, this implies that the bottom right triangle commutes. Setting $D = \mathbf{1}$ and using the natural isomorphism $\rho\colon - \otimes \mathbf{1} \to \mathrm{id}_{\mathcal{C}}$, it follows that the following diagram commutes:

$$
\begin{tikzcd}
A \otimes (B \otimes \mathbf{1}) \arrow[rr, "\alpha_{A,B,\mathbf{1}}"] \arrow[dr, "\mathrm{id}_A \otimes \rho_B"'] & & (A \otimes B) \otimes \mathbf{1} \arrow[dl, "\rho_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}
$$

This proves the first claim, the second claim can be proven analogously. $\square$

Definition 1.14. Let $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \mathbf{1}', \otimes', \alpha', \lambda', \rho')$ be two monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triple $(F, \varphi, \varepsilon)$, where $F\colon \mathcal{C} \to \mathcal{C}'$ is a functor, $\varphi\colon F(- \otimes -) \to F(-) \otimes' F(-)$ is a natural isomorphism and $\varphi\colon F(\mathbf{1}) \to \mathbf{1}'$ is an isomorphism, such that the following diagrams commute:

$$
\begin{tikzcd}
& F\big(A \otimes (B \otimes C)\big) \arrow[rr, "F(\alpha_{A,B,C})"] \arrow[dl, "\varphi_{A,B\otimes C}"'] & & F\big((A \otimes B) \otimes C\big) \arrow[dr, "\varphi_{A\otimes B,C}"] & \\
F(A) \otimes' F(B \otimes C) \arrow[dr, "\mathrm{id}_{F(A)} \otimes \varphi_{B,C}"'] & & & & F(A \otimes B) \otimes' F(C) \arrow[dl, "\varphi_{A,B} \otimes \mathrm{id}_{F(C)}"] \\
& F(A) \otimes' \big(F(B) \otimes' F(C)\big) \arrow[rr, "\alpha'_{F(A),F(B),F(C)}"'] & & \big(F(A) \otimes' F(B)\big) \otimes' F(C) &
\end{tikzcd}
$$

$$\begin{tikzcd} F(A\otimes \mathbf{1}) \arrow[r, "\varphi_{A,\mathbf{1}}"] \arrow[d, "F(\rho_A)"'] & F(A)\otimes' F(\mathbf{1}) \arrow[d, "\mathrm{id}_{F(A)}\otimes'\varepsilon"'] \\ F(A) & F(A)\otimes' \mathbf{1}' \arrow[l, "\rho'_{F(A)}"] \end{tikzcd} \qquad \begin{tikzcd} F(\mathbf{1}\otimes A) \arrow[r, "\varphi_{\mathbf{1},A}"] \arrow[d, "F(\lambda_A)"] & F(\mathbf{1})\otimes' F(A) \arrow[d, "\varepsilon\otimes'\mathrm{id}_{F(A)}"] \\ F(A) & \mathbf{1}'\otimes' F(A) \arrow[l, "\lambda'_{F(A)}"] \end{tikzcd}$$

A *monoidal natural transformation* between monoidal functors $(F, \varphi, \varepsilon)$ and $(F', \varphi', \varepsilon')$ is a natural transformation $\psi\colon F \to F'$ such that

$$\varphi' \circ \psi_{-\otimes -} = \psi \otimes' \psi \circ \varphi \qquad \text{and} \qquad \varepsilon = \varepsilon' \circ \psi_{\mathbf{1}},$$

i.e. the following diagrams commute for all objects A, B of $\mathcal{C}$:

$$\begin{tikzcd} F(A\otimes B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "\psi_{A\otimes B}"'] & F(A)\otimes' F(B) \arrow[d, "\psi_A\otimes'\psi_B"'] \\ F'(A\otimes B) \arrow[r, "\varphi'_{A,B}"'] & F'(A)\otimes' F'(B) \end{tikzcd} \qquad \begin{tikzcd} F(\mathbf{1}) \arrow[r, "\psi_{\mathbf{1}}"] \arrow[dr, "\varepsilon"'] & F'(\mathbf{1}) \arrow[d, "\varepsilon'"] \\ & \mathbf{1}' \end{tikzcd}$$

Remark 1.15. Being monoidal for a functor is an additional structure, and not a property. However, being monoidal for a natural transformation is a property.

Example 1.16. (1) For a group G and a field $\Bbbk$, the forgetful functor $F\colon \mathbf{Rep}_\Bbbk(G) \to \mathbf{Vect}_\Bbbk$ which sends a G-module to the underlying vector space is monoidal. (The structure maps φ and ε are identity maps.) This example will play an important role later in the course.

Conversely, every vector space can be considered as a G-module with the trivial action of G, and this gives rise to a monoidal functor $e\colon \mathbf{Vect}_\Bbbk \to \mathbf{Rep}_\Bbbk(G)$.

(2) For a commutative ring R, there is a functor $F\colon \mathbf{Set}^{\mathrm{rev}} \to R-\mathbf{Mod}$ that sends a set X to the free R-module $F(X) := R^X = \mathrm{Map}(X, R)$. At the level of homomorphisms, we define $F(f)$ via $g \mapsto g \circ f$, for maps $f\colon X \to Y$ and $g\colon X \to R$. This functor is monoidal with respect to the monoidal structure on **Set** via the Cartesian product and on $R-\mathbf{Mod}$ via the usual tensor product of R-modules.

(3) The total cohomology functor $H^*\colon \mathbf{coch}(\mathbf{Vect}_\Bbbk) \to \mathbf{Vect}_\Bbbk^{\mathbb{Z}}$ from the category of cochain complexes of $\Bbbk$-vector spaces to the category of graded $\Bbbk$-vector spaces is monoidal with respect to the usual derived tensor product on $\mathbf{coch}(\mathbf{Vect}_\Bbbk)$ by the Künneth theorem: For two cochain complexes $X_\bullet$ and $Y_\bullet$, we have

$$H^i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{j+k=i} H^j(X_\bullet) \otimes H^k(Y_\bullet),$$

matching the definition of the tensor product of $\mathbb{Z}$-graded vector spaces.

(4) Let G be a group, let A be an abelian group and let ω and π be 3-cocycles with values in A. Suppose that $(\mathrm{id}_{\mathcal{C}_A^G}, \varphi, \varepsilon)$ defines a monoidal functor from $\mathcal{C}_A^{G,\omega}$ to $\mathcal{C}_A^{G,\pi}$, for some $\varphi\colon -\otimes- \to -\otimes-$ and $\varepsilon \in \mathrm{End}_{\mathcal{C}_A^G}(\delta_e) = A$. Then φ defines a map $\varphi\colon G \times G \to A$ via

$$A \ni \varphi(g, h) := \varphi_{g,h}\colon \delta_{gh} = \delta_g \otimes \delta_h \to \delta_g \otimes \delta_h = \delta_{gh}$$

for $g, h \in G$, and by the definition of monoidal functors, we have

$$\varphi(g,h)\varphi(gh,k)\omega(g,h,k) = \pi(g,h,k)\varphi(h,k)\varphi(g,hk)$$

for all $g, h, k \in G$, that is

$$\omega\pi^{-1}(g,h,k) = \varphi(h,k) \cdot \varphi(g,hk) \cdot \varphi(g,h)^{-1} \cdot \varphi(gh,k)^{-1} = d^2\varphi(g,h,k).$$

In other words, the 3-cocycle $\omega\pi^{-1} = d^2\varphi$ is a 3-*coboundary*, so ω and π define the same element of the third cohomology group $H^3(G, A)$. (The latter is defined as the quotient of the group of 3-cocycles by the group of 3-coboundaries.) This (and the discussion in point (9) of Example 1.11) relates the equivalence classes of monoidal structures on $\mathcal{C}_A^G$ to $H^3(G, A)$. Similarly, one can relate the equivalence classes of monoidal structures on $\mathbf{Vect}_{\Bbbk}^G$ to $H^3(G, \Bbbk^\times)$. For more details, see Section 2.6 in [EGNO15].

Potential topics for talks:

(1) pivotal categories, traces and dimension (?);

(2) Frobenius-Perron dimension;

(3) monoidal categories by generators and relations (via diagrams);

(4) the Temperley-Lieb category (and knot invariants);

(5) tensor triangular geometry;

(6) the geometric Satake equivalence;

(7) negligible morphisms and semisimplification;

(8) the Drinfel'd double and the Drinfel'd center;

(9) $\mathfrak{sl}_n$-webs;

(10) interpolation categories;

(11) $\cdots$

References

[DM82] Pierre Deligne and J. S. Milne. Tannakian categories. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982)., 1982.

[EGNO09] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories. Lecture notes for a course taught by P. Etingof at MIT, available at `http://mtm.ufsc.br/~ebatista/2016-2/tensor_categories.pdf`, 2009.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.