Exercises 8 · Probability

Solution to decision-tree problem

The problem itself, restated. You are the owner of a small electronics company. In 6 months, a proposal is due for an electronic timing system for downhill skiing events in the 2018 Winter Olympic Games. For several years, your company has been developing a new form of "finish beam" that records the time at which a ski racer crosses the finish line. A finish beam is a critical component in any ski timing system. They must be accurate to the millisecond and immune to poor weather. They must also be able to distinguish between a skiier and a puff of snow crossing the beam, and they must maintain a wireless data link with the starting gate in conditions which make radio transmission difficult (e.g. high mountains). When completed, yours will be superior to anything currently on the market. But your progress in research and development has been slower than expected, and you are unsure about whether your staff can perfect the technology and produce the components in time to submit a proposal. You must make a decision about whether to press ahead with an accelerated schedule of research and development, taking into account your uncertainty about future events. Consider these facts:

- The Olympic contract pays $1 million to the winner.
- If your R&D effort succeeds in developing the new beam (which will happen with probability p_1), there is a good chance (probability p_2) that your company will win the contract.
- If you decide to accelerate the R&D schedule for the purposes of developing the new beam, you will need to invest $200,000 in extra funds. With probability $1 - p_1$, this effort will fail to develop the new beam in time.
- If you fail to develop the new beam, there is a small chance (probability p_3) that you will still be able to win the same contract with your older model of timing system that has already been developed. This involves no extra cost for research and development.
- Making a proposal requires developing a prototype timing system at an additional production cost, above and beyond R&D costs. This additional cost would be $50,000 for the new model of timing system, and $40,000 for the old model.
- Finally, if you win the contract, the finished product—whether the new or the old model—will cost an additional $250,000 to produce, above and beyond the cost of building a prototype.

Part A: Each of the 11 terminal arrows of the tree, corresponds to a particular pattern of events that may or may not happen. For each node, fill in the amount of net profit you would make if those events actually happen. Make sure to account for all costs along each path of the tree.

Solution: See the last page.

Part B: For the right-most four stochastic nodes of the tree (F, G, H, and I), fill in the probabilities of the possible outcomes (two for each node), and then compute the expected value of the node in terms of p_1, p_2, and p_3. Remember, the outcome at that node is a random variable, and we know how to calculate expected values of random variables.

Solution: Consider node F first. There are two possible outcomes here: either the proposal succeeds (with probability p_2 and net profit 500K), or it fails (with probability $1 - p_2$ and net profit $-250K$). Therefore the expected value of node F is

$$E(\text{node F}) = 500000p_2 - 250000(1 - p_2) = 750000p_2 - 250000$$

Similarly,

$$\begin{aligned} E(G) &= 750000p_3 - 240000 \\ E(H) &= 750000p_3 - 240000 \\ E(I) &= 710000p_3 - 40000(1 - p_3) = 750000p_3 - 40000 \end{aligned}$$

Part C: For decision nodes D, E, and C, write down a *decision rule*—that is, a criterion expressed in terms of the relevant probabilities (p_1, p_2, and p_3) that tells you which decision you should make to maximize utility in that set of circumstances. Derive these rules by hand, and show your work. You may use software to help with routine calculations, but not to derive the rules.

Solution: I'll work out D for you; you can work out nodes E and C with the same strategy. Imagine standing at node D. Here there are three choices, and we want to take the choice with the maximum expected value. Thus we would propose with the new model if

$$E(F) > E(G) \quad \text{and} \quad E(F) > -200000\,.$$

Thus we take option F if

$$750000p_2 - 250000 > 750000p_3 - 240000 \quad \text{and} \quad 750000p_2 - 250000 > -200000\,.$$

With some algebra we can simplify these further: we take F if

$$p_2 > p_3 + 1/75 \quad \text{and} \quad p_2 > 5/75 = 1/15\,.$$

Similarly, we take G if

$$750000p_3 - 240000 > 750000p_2 - 250000 > \quad \text{and} \quad 750000p_3 - 240000 > -200000\,.$$

These conditions simplify further: We take G if

$$p_3 + 1/75 > p_2 \quad \text{and} \quad p_3 > 4/75\,.$$

If neither of these is true, then we take the sure \$200,000 loss.

The decision rules for the other two decision nodes (E and C) can be worked out using exactly the same process.

Part D: Suppose $p_2 = 0.8$ and $p_3 = 0.1$. Write down an expression for the expected value of Node B in terms of p_1. What value of p_1 makes you indifferent between proceeding with the accelerated R&D effort, or not?

Solution: Running these numbers through your decision rules from the previous step, we find that the optimal choice at node D is to propose the new model. Therefore the expected value of node D is equal to the expected value of the optimal choice, which is

$$E(D) = E(F) = 750000p_2 - 250000 = 0.8 \cdot 750000 - 250000 = 350000\,.$$

Similarly, the optimal choice at node E is to propose with the old model, since $E(H) = -165000 > -200000$. Therefore the expected value of node E is precisely this value of -165000.

Thus in terms of p_1, the expected value of node B is

$$E(B) = p_1 E(\text{node D}) + (1 - p_1)E(\text{node E}) = 350000p_1 - (1 - p_1)165000 = 515000p_1 - 165000$$

Now at node A we have to compare decision B with decision C. Node C has the same expected value of node I, because $E(I) = 35000$ when $p_1 = 0.1$, and therefore node I dominates the "no proposal" option (terminal node 11). Therefore, the expected value of node C is 35000.

Thus the choice at node A boils down to whether

$$E(B) = 515000p_1 - 165000 > 35000$$

If it is, we choose node B (proceed). If it isn't, we choose node C (don't proceed). Notice that the two expected values are exactly equal when

$$515000p_1 - 165000 = 35000\,,$$

or when $p_1 \approx 0.388$. This is our "indifference point." As long as p_1 is larger than this threshold, proceeding with the R&D has a higher expected value.