Exercises 9 · Decision analysis and Monte Carlo

Due Monday, April 27, 2015

Problem 1

Suppose that you are risk averse, and that your utility of money is given by the log utility function:

$$u(w) = \log(w)\,,$$

where w is total wealth. Remember, log means the natural logarithm.

Consider a simple decision problem where you, as an investor, must decide how much of your wealth to allocate between a risky asset and a riskless asset. The riskless asset carries no risk, but no return—for example, stuffing cash under your mattress. The risky asset will return at rate r_g with probability p (a good outcome), and rate r_b with probability $1-p$ (a bad outcome). Therefore if your beginning wealth is w and if you invest x in the risky asset, where x could be anything between 0 and w, your two possible outcomes are to end up with either $w + r_g x$ or $w + r_b x$.

Part A: Suppose that $p = 0.5$, $r_g = 0.1$ (gain 10%), and $r_b = -0.1$ (lose 10%). Regardless of your current wealth w, how much money will you allocate to the risky asset if you are trying to maximize expected utility?

Part B: Suppose that your current wealth is $w = 100$; that $r_g = 0.1$ and $r_b = -0.1$; and that $p = 0.52$ (so that the risky investment carries a positive expected return, but just barely). How much money x will you allocate to the risky asset if you wish to maximize expected utility?

Part C: Imagine now that the government levies a tax t on returns from all risky assets. Your after-tax wealth will therefore either $w + (1-t) \cdot r_g x$ or $w + (1-t) \cdot r_b x$, with probabilities p and $1-p$, respectively. (A tax on a negative return is like a subsidy—just like, for example, the subsidy the IRS gives by allowing people to deduct investment losses from their taxable income.)

Using your calculus and algebra skills, derive an equation that characterizes your optimal x (allocation to the risky asset) as a function of p, r_g, r_b, w, and the tax rate t. By "optimal" I mean "the allocation that maximizes expected utility." Use this equation to answer the following questions.[1]

[1] The algebra here can get messy. I offer two hints that will keep things relatively clean. First, if $f(x)$ obtains a maximum at some point $x^\star$, then $\log f(x)$ also obtains a maximum at $x^\star$. Thus to maximize $f(x)$, we can maximize $\log f(x)$ instead, which is often easier. Second, the derivative of $\log g(x)$ with respect to x is $g'(x)/g(x)$.

1. Suppose that the tax rate is $t = 0.1$, and that the numbers are otherwise the same from Part B. If you wish to maximize expected utility, how much of your money x will you allocate to the risky asset now? What is the expected utility of this optimal allocation? Is it better or worse than the expected utility of the optimal allocation in the "no-tax" regime?
2. Now suppose that the tax rate is even higher: $t = 0.25$, and that the numbers are otherwise the same from Part B. How much of your money x will you allocate to the risky asset now? What is the expected utility of this optimal allocation? Is it better or worse than the expected utility of the optimal allocation from Parts B and C1?
3. As a general rule, if investors are risk averse, how does the optimal level of investment in a risky asset change when you tax its return? How does the tax rate affect the expected utility that investors will enjoy, assuming that they optimize their expected utility for whatever tax rate the government sets?

Problem 2

Suppose you are offered the following bet, and you get to choose how much to wager:

- With probability 0.52, you will win whatever you have wagered.
- With probability 0.48, you will lose whatever you have wagered.

Part A: (Warm-up.) Suppose that your current wealth is $w = \$1000$, and you decide to risk 10% of your wealth on this bet. What is your expected final wealth after one round of the bet?

Part B: Suppose now that you get to repeat the bet as many times as you want. After every single round of betting, you decide to risk 10% of your current wealth on the next bet. In other words, if you have x dollars after the last round of betting, you place $0.1x$ dollars on the next round's wager. If you win, you will wealth will be $1.1x$. If you lose, it will be $0.9x$.

Suppose you start with $1000. Simulate 10,000 rounds of this bet.[2] (I recommend that you build on our R scripts from class. But you can also make a spreadsheet with 10,000 rows.) Make two plots: (1) your simulated trajectory of wealth over every round from 1 to 10,000, and (2) the absolute size of the bet you are making in each round. Make sure that the betting round is on the x-axis of both plots. What happens after 10,000 rounds of betting? Are you

[2] Why 10,000? Because that's roughly the number of trading days over a 40-year period of investment.

rich or broke? In light of the answer from Part A, do you find this surprising?

Part C: Now repeat the simulation—except this time, only risk 0.5% of your current wealth (that is, 1 part in 200) at every round of betting. As before, make two plots: (1) your simulated value of current wealth at every step from 1 to 10,000, and (2) the absolute size of the bet you are making in each round. This time, what happens after 10,000 rounds? Are you rich or broke?

Part D: Now try the following strategy: at every round, risk the fraction of your current wealth that maximizes the expected utility for that single round of betting. Use log utility, $u(w) = \log(w)$. To implement this, you will need to derive a decision rule similar to the kind you derived for Problem 1: that is, the optimal bet size, as a function of current wealth (and implicitly, as a function of the success probability, which in this case is 0.52.)

As in B and C, simulate 10,000 betting rounds. Keep track of three quantities over time: (1) running wealth, (2) absolute bet size round-by-round, and (3) relative bet size, as a fraction of your running wealth. What happens to your wealth and bet sizes over time under this strategy, versus the strategies in Parts B and C?

Part E: Comment on the wisdom of the following statement: "If your goal is to ensure the long-term growth of your capital, you should make bets that carry the highest possible expected return."