# *Exercises 9 · Utility and decisions; back to regression*

**Due Monday, April 25, 2016**

*Problem 1: utility optimization*

Return to the scenario from the course notes: you, as an investor, must decide how much of your wealth to allocate between a risky asset and a riskless asset. The riskless asset carries no risk, but no return. The risky asset will return at rate $r_g$ with probability $p$ (a good outcome), and rate $r_b$ with probability $1 - p$ (a bad outcome). If your beginning wealth is $W_0$ and if you invest some fraction $c$ of your wealth in the risky asset, where $c$ could be anything between 0 and 1, your possible outcomes $W_0 \cdot (1 + cr_g)$ or $W_0 \cdot (1 + cr_g)$. Your utility of wealth is given by the logarithmic utility function: $u(w) = \log(w)$.

Imagine now that the government levies a tax $t$ on returns from all risky assets. Here $t$ is expressed as a fraction: e.g. if $t = 0.25$ and your asset appreciates by \$100, you will owe \$25 in tax. Your after-tax wealth will therefore either $W_0 \cdot \{1 + (1 - t) \cdot r_g c\}$ or $W_0 \cdot \{1 + (1 - t) \cdot r_b c\}$, with probabilities $p$ and $1 - p$, respectively. (A tax on a negative return is like a subsidy—just like, for example, the subsidy the IRS gives by allowing people to deduct investment losses from their taxable income.)

Using your calculus and algebra skills, derive an equation that characterizes your optimal $c$ (fraction of wealth allocated to the risky asset) as a function of $p$, $r_g$, $r_b$, $W_0$, and the tax rate $t$. By "optimal" I mean "the allocation that maximizes expected utility." Use this equation to answer the following questions.[1]

[1] The algebra here can get messy. I offer two hints that will keep things relatively clean. First, if $f(x)$ obtains a maximum at some point $x^\star$, then $\log f(x)$ also obtains a maximum at $x^\star$. Thus to maximize $f(x)$, we can maximize $\log f(x)$ instead, which is often easier. Second, the derivative of $\log g(x)$ with respect to $x$ is $g'(x)/g(x)$.

(A) Suppose that the tax rate is $t = 0.1$, and that the numbers are otherwise the same as those considered in the course notes: $W_0 = 100$; that $r_g = 0.1$ and $r_b = -0.1$; and that $p = 0.52$. If you wish to maximize expected utility, what fraction of your money $c$ will you allocate to the risky asset now? What is the expected utility of this optimal allocation? Is it better or worse than the expected utility of the optimal allocation in the "no-tax" regime?

(B) Now suppose that the tax rate is even higher: $t = 0.25$, and that the numbers are otherwise the same from Part B. What fraction of your money $c$ will you allocate to the risky asset now? What is the expected utility of this optimal allocation? Is it better or worse than the expected utility when $t = 0.1$?

(C) As a general rule, if investors are risk averse, how does the optimal

level of investment in a risky asset change when you tax its return? How does the tax rate affect the expected utility that investors will enjoy, assuming that they optimize their expected utility for whatever tax rate the government sets?

*Problem 2: bootstrap resampling in portfolio modeling*

Suppose you have $10000 to invest, and are contemplating allocating your wealth among five major asset classes, as represented by five exchange-traded funds:

- SPY, large-cap stocks
- TLT, long-term government bonds
- LQD, long-term corporate bonds
- DBC, commodities (oil, gold, metals, etc)
- VNQ, real estate

You will use Monte Carlo simulation combined with bootstrap resampling (using data from Yahoo finance over the last 10 years) to compare the short-term risk profiles for the following three portfolio allocations:

1. 50% of your wealth in stocks (SPY), 50% in real estate (VNC)
2. 50% in commodities (DBC), 50% in government bonds (TLT)
3. 20% in each of the five asset classes.

Daily data for all five funds is available from Yahoo Finance (remember the `getYahooSeries` command from our R script in class).

Suppose you plan to take a buy-and-hold strategy for four weeks (20 trading days). That is, you do not rebalance your portfolio along the way to compensate for differing gains and losses in each of the assets. Briefly acquaint yourself (using outside sources) with the concept of *value at risk* (VaR) for a financial portfolio. The Wikipedia entry on VaR (clickable link here) should be sufficient for this purpose, but there are plenty of other web resources as well.

Your task: use Monte Carlo simulation coupled with boostrap resampling to estimate the 5% Value at Risk (VaR) for each of these portfolios over this horizon. Which portfolio looks riskiest? Which looks least risky? Why do you think it shook out this way?

To get enough Monte Carlo draws, you may need to leave your computer running awhile in the background. For the purpose of debugging,

I would recommend checking your scripts first with relatively small Monte Carlo sample sizes (e.g. 50), just to make sure they work as intended. Then increase the Monte Carlo sample size and take a break while your computer churns through the computations necessary to get your final answer.

*Problem 3: the assumptions of the normal linear regression model*

The dataset in "consumerexp.csv" contains quarterly data from 1952 to 1956 on consumer expenditure ($y$) and the stock of money in the U.S. economy ($x$), both measured in billions of dollars. A simplified version of the quantitative theory of money supply suggests a model given by

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \,.$$

In general, economists are interested in estimating $\beta_1$ and interpreting it as the "multiplier," thereby relating general levels of consumer expenditure with the amount of money available in the economy. A good understanding of $\beta_1$ is of crucial importance in predicting the effects of monetary and fiscal policy decisions.

(A) Plot consumer expenditures versus time. Is there trend over time in how consumer expenditures are changing?

(B) Regress consumer expenditures on stock of money, and report an approximate 95% confidence interval for the multiplier. You may use either bootstrapping or the simple linear regression model, but be sure to say which one you are using.

(C) Plot the model residuals versus time. Does it appear that consumer expenditures are growing over time, *after adjusting for changes in the money supply*?

(D) Do you see anything in the plot of residuals versus time that makes you worry about a potential violation of the normal linear regression model's key assumptions?

*Practice essay question for those interested (not to turn in)*

We have now encountered two broad philosophies for quantifying uncertainty in statistical models. The first is bootstrapping. The second is making probabilistic assumptions (for example, normality) about the random or unexplained component of the model. Compare these two philosophies. What assumptions do they require? What are their relative strengths and weaknesses?