Class notes: Advanced Topics in Macroeconomics

Topic: Three Near-Linear Methods (cont)

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In last class, we introduced the third method for solving near-linear methods, but did not complete it. I repeat the notes here and then describe how to implement all for the most simple problem.

Method 3 (repeated)

Next, we use the insights of Vaughan (1970) to To avoid slow iteration on the Riccati equation, Vaughan (1970) exploits certain properties of the first-order conditions of the LQ problem defined above. (See his paper which is posted on the website.) Vaughan assumes no discounting or cross-product terms, so we'll map the variables and coefficients to $\tilde{X}$, $\tilde{u}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{Q}$ as shown earlier. Also note that because the solution does not depend on the variances and covariances of ϵ, we can abstract from the uncertainty for now. Writing out the Lagrangian, we have

$$\mathcal{L} = \sum_{t=0}^{\infty} \{ \tilde{X}_t' \tilde{Q} \tilde{X}_t + \tilde{u}_t' R \tilde{u}_t - \lambda_{t+1}' \left(X_{t+1} - \tilde{A} \tilde{X}_t - \tilde{B} \tilde{u}_t \right) \} \tag{1}$$

Taking derivatives with respect to $\tilde{u}_t$, $\tilde{X}_{t+1}$, and λ_{t+1}, we obtain the following first-order conditions

$$\begin{aligned} 2R\tilde{u}_t + B'\lambda_{t+1} &= 0 \\ \tilde{Q}\tilde{X}_{t+1} - \lambda_{t+1} + \tilde{A}'\lambda_{t+2} &= 0 \\ \tilde{X}_{t+1} - \tilde{A}\tilde{X}_t - \tilde{B}\tilde{u}_t &= 0 \end{aligned} \tag{2}$$

for $t \geq 0$, where $\{\lambda_t\}$ is a sequence of Lagrange multipliers. Eliminating $\tilde{u}_t$ and letting $\tilde{\lambda}_t = 1/2\lambda_t$, we have:

$$\begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' \\ \tilde{Q}\tilde{A}^{-1} & \tilde{Q}\tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' + \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}$ be the coefficient matrix on the right hand side. Vaughan showed that this matrix can be decomposed and used directly to obtain the Riccati matrix P (and hence the solution to the LQ problem); that is, he showed that

$$\mathcal{H} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1},$$

where the eigenvalues of Λ are outside of the unit circle. Notice that the eigenvalues come in reciprocal pairs. This is an important property that implies a unique stable solution, one that satisfies the transversality condition and ensures a bounded return.

Using the fact that the Lagrange multiplier is the derivative of the value function ($\tilde{\lambda}_t = P\tilde{X}_t$), it is easy to figure out how to set P so as to get a stationary dynamical system for X. Let $W = V^{-1}$. In this case, it is easy to show that:

$$\tilde{X}_{t+1} = \{V_{11}\Lambda^{-1}\left(W_{11} + W_{12}P\right) + V_{12}\Lambda\left(W_{21} + W_{22}P\right)\}\tilde{X}_t.$$

Since Λ has roots outside the unit circle, it must be the case that $P = -W_{22}^{-1}W_{21}$. Note that since $W = V^{-1}$, this is equivalent to setting $P = V_{21}V_{11}^{-1}$.

In the case that $\tilde{A}$ is not invertible, we can modify the method slightly and use generalized eigenvalues with the following alternative system:

$$\begin{bmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} I & \tilde{B}R^{-1}\tilde{B}' \\ 0 & \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}_1$ be the coefficient matrix for the state and costate in $t+1$, and let $\mathcal{H}_2$ be the coefficient matrix for the state and costate in t. Then, instead of taking eigenvalues of $\mathcal{H}$ as above, we take generalized eigenvalues with the pair $(\mathcal{H}_1, \mathcal{H}_2)$.

Once we have a steady-state solution to the Riccati matrix, we can use the earlier formula to compute F and the law of motion for the state variables:

$$X_{t+1} = \left(A - BF\right)X_t + C\epsilon_{t+1} \tag{3}$$

Furthermore, given an initial condition for the states, X_0, and a realization of the shocks, $\epsilon_t,\ t \geq 0$, we can generate time-series for X_t and u_t.

A Simple Example

In class, three students are asked to go to the board and solve the following problem:

$$\max_{c_t} E_0 \ln (c_t)$$
$$\text{subject to} c_t + k_{t+1} = A k_t^{\theta}$$

by applying the three different methods without any computer assistance. The student applying dynamic programming makes a guess of $V_0(k)$ and iterates backwards. Things go well if they use $V_0(k) = 0$ as an initial guess. The student applying an LQ method constructs a second-order Taylor expansion of the return function and a first-order Taylor expansion of the constraints. Things go well if they substitute out for c_t and use the fact that k_{t+1} is the control and equal to the state k_t in one period when writing out the linear constraints. The student applying Vaughan's method can see all of the insight of Vaughan by working with the second order difference equation in k_{t+2}, k_{t+1}, and k_t. In particular, it is obvious that roots come in reciprocal (or β-reciprocal) pairs.