ECON 8185 - HW 2

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Incomplete Version

Question 1

We'll consider the following Prototype model from Ellen's Homework 2, which is the same as CKM:

$$\max_{c_t, x_t, l_t} E \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t l_t^{\psi}\right)^{1-\sigma}}{1-\sigma} N_t$$

S.T.

$$\begin{gathered} c_t + (1 + \tau_{xt} x_t = r_t k_t + (1 - \tau_{ht}) w_t h_t + T_t \\ N_{t+1} k_{t+1} = [(1-\delta) k_t + x_t] N_t \\ h_t + l_t = 1 \\ S_t = P S_{t-1} + Q \epsilon_t, \quad S_t = [\ln z_t, \tau_{ht}, \tau_{xt}, \ln g_t] \\ c_t, x_t \geq 0 \end{gathered}$$

Where $N_t = (1+\gamma_n)^t$ and firm tecnhology is $Y_t = K_t^{\theta}(Z_t L_t)^{1-\theta}$. γ_z is the rate of labor-augmenting technical progress. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint is $Y_t = N_t(c_t + x_t + g_t)$. We must work with detrended variables, we should use "hat" variables, but typing this is time consuming so from now on all variables should be understood as detrended unless I say something about them.

The detrended FOC's of this model are:

$$\begin{gathered} c_t + (1+\gamma_z)(1+\gamma_n) k_{t+1} - (1-\delta) k_t + g_t = y_t = k_t^{\theta} (z_t h_t)^{1-\theta} \\ \psi \frac{c_t}{1-h_t} = (1-\tau_{ht})(1-\theta) \left(\frac{k_t}{h_t}\right)^{\theta} z_t^{1-\theta} \\ c_t^{-\sigma} (1-h_t)^{\psi(1-\sigma)} (1+\tau_{xt}) = \beta (1+\gamma_z)^{-\sigma} E_t c_{t+1}^{-\sigma} (1-h_{t+1})^{\psi(1-\sigma)} \left(\theta k_{t+1}^{\theta} (z_{t+1} h_{t+1})^{1-\theta} + (1-\delta)(1+ \right. \end{gathered}$$

Defining some parameters:

In [1]:
```
using Plots, NLsolve, ForwardDiff, DataFrames, LinearAlgebra, QuantEcon, Plots
, Optim, Statistics
#Parameters:
δ = 1   #depreciation rate
θ = 1/3  #capital share of output
β = 0.9  #Discouting
σ = 2  #Elasticity of Intertemporal Substitution
ψ = 1    #Labor parameter
γn= 0.00    #Population growth rate
γz= 0.00   #Productivitu growth rate
gss = 0.1 #average g
τxss = 0.05 #average τx
τhss = 0.02 #average τh
zss = 0.0 #average z (z is in logs)


#Parameters to be estimated
ρg = 0.8
ρx = 0.5
ρh = 0.7
ρz = 0.9

σg= 0.02
σx = 0.01
σz = 0.02
σh = 0.01
```

```
┌ Info: Recompiling stale cache file C:\Users\jgsla\.julia\compiled\v1.0\NLso
lve\KFCNP.ji for NLsolve [2774e3e8-f4cf-5e23-947b-6d7e65073b56]
└ @ Base loading.jl:1184
┌ Info: Recompiling stale cache file C:\Users\jgsla\.julia\compiled\v1.0\Quan
tEcon\V0Mv9.ji for QuantEcon [fcd29c91-0bd7-5a09-975d-7ac3f643a60c]
└ @ Base loading.jl:1184
```

Out[1]: 0.01

Substituting for c, we get an euqation for k, and one for h. Below, I find the Steady State values:

In [2]:
```
#Function with the FOCs
zss = exp(zss)

function SS!(eq, vector::Vector)
    k,h = (vector)
    k1 = k
    h1 = h
    g, τx,τh, z = gss,τxss,τhss, zss
    z1 = z
    τx1 = τx

    c = k * ((z *h)^(1-θ))^(1/θ) - ((1+γz)*(1+γn)*k1-(1-δ)*k+ g )^(1/θ)
    c1 = c
    eq[1] = (ψ *c)^(1/θ)  - (k/h)*((1-h)*(1-τh)*(1-θ)*z^(1-θ))^(1/θ)

    eq[2] = (c^(-σ) *(1-h)^(ψ*(1-σ))*(1+τx)  - (1-δ)*(1+τx1)* β*(1+γz)^(-σ) *
c1^(-σ) * (1-h1)^(ψ*(1-σ)))^(-1/θ) -
     (β*(1+γz)^(-σ) * c1^(-σ) * (1-h1)^(ψ*(1-σ)) * θ*(z1*h1)^(1-θ))^(-1/θ)* k1
    return eq
end

kss = (θ*β)^(1/θ)

SteadyState = nlsolve(SS!, [0.2,0.8],ftol = :1.0e-20, method = :trust_region ,
 autoscale = true)
kss,hss = SteadyState.zero
```

Out[2]:
```
2-element Array{Float64,1}:
 0.06450057178639043
 0.8483596073490938
```

Log-linearizing the equations we get the following system of equations:

$$0 = E_t[a_1\tilde{k}_t + a_2\tilde{k}_{t+1} + a_3\tilde{h}_t + a_4\tilde{z}_t + a_5\tilde{\tau}_{ht} + a_6\tilde{g}_t]$$
$$0 = E_t[b_1\tilde{k}_t + b_2\tilde{k}_{t+1} + b_3\tilde{k}_{t+2} + b_4\tilde{h}_t + b_5\tilde{h}_{t+1}b_6\tilde{z}_t + b_7\tilde{\tau}_{xt} + b_8\tilde{g}_t + b_9\tilde{z}_{t+1} + b_{10}\tau_{xt+1} + b_{11}\tilde{g}_{t+1}]$$

Where tilde variables are log deviations from Steady State. Stacking up the equations we get:

$$0 = E_t\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & b_3 & b_5\end{bmatrix}\begin{bmatrix}\tilde{k}_{t+1}\\ \tilde{k}_{t+2}\\ \tilde{h}_{t+1}\end{bmatrix} + \begin{bmatrix}0 & -1 & 0\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_4\end{bmatrix}\begin{bmatrix}\tilde{k}_t\\ \tilde{k}_{t+1}\\ \tilde{h}_t\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ a_4 & a_5 & 0 & a_6 & 0 & 0 & 0 & 0\\ b_6 & 0 & b_7 & b_8 & b_9 & 0 & b_{10} & b_{11}\end{bmatrix}$$

We call the first matrix A_1, and the second A_2. The code below log-linearizes and find these matrices:

In [3]:
```
function loglineq1(vector::Vector)
    k,k1,h,z,τh,g= vector

    c = k * ((z *h)^(1-θ))^(1/θ) - ((1+γz)*(1+γn)*k1-(1-δ)*k+ g )^(1/θ)
    eq =(ψ *c)^(1/θ)  - (k/h)*((1-h)*(1-τh)*(1-θ)*z^(1-θ))^(1/θ)

    return eq
end
function loglineq2(vector::Vector)
    k,k1,k2,h,h1,z,τx,g,z1,τx1,g1 = (vector)
    c = k * ((z *h)^(1-θ))^(1/θ) - ((1+γz)*(1+γn)*k1-(1-δ)*k+ g )^(1/θ)
    c1 = k * ((z1 *h1)^(1-θ))^(1/θ) - ((1+γz)*(1+γn)*k2-(1-δ)*k1+ g1 )^(1/θ)
    eq =  (c^(-σ) *(1-h)^(ψ*(1-σ))*(1+τx)  - (1-δ)*(1+τx1)* β*(1+γz)^(-σ) * c1
^(-σ) * (1-h1)^(ψ*(1-σ)))^(-1/θ) -
     (β*(1+γz)^(-σ) * c1^(-σ) * (1-h1)^(ψ*(1-σ)) * θ*(z1*h1)^(1-θ))^(-1/θ)* k1
    return eq
end



#Log deviations
T=ForwardDiff.gradient(loglineq1,[kss,kss,hss,zss,τhss,gss])
a =[-kss*T[1]/(kss*T[1]),-kss*T[2]/(kss*T[1]),-hss*T[3]/(kss*T[1]),
-zss*T[4]/(kss*T[1]),-τhss*T[5]/(kss*T[1]),-gss*T[6]/(kss*T[1])]
#if ψ==0
#    a[1],a[2:end]=-1,zeros(5)
#end

T=ForwardDiff.gradient(loglineq2,[kss,kss,kss,hss,hss,zss,τxss,gss,zss,τxss,gs
s])
b = [kss*T[1]/(-kss*T[1]),kss*T[2]/(-kss*T[1]),kss*T[3]/(-kss*T[1]),hss*T[4]/(
-kss*T[1]),
hss*T[5]/(-kss*T[1]),zss*T[6]/(-kss*T[1]),τxss*T[7]/(-kss*T[1]),gss*T[8]/(-kss
*T[1]),
zss*T[9]/(-kss*T[1]),τxss*T[10]/(-kss*T[1]),gss*T[11]/(-kss*T[1])]

A1 = [1 0 0; 0 0 0; 0 b[3] b[5]]
A2 = [0 -1 0; a[1] a[2] a[3]; b[1] b[2] b[4]]
U = [0 0 0 0 0 0 0 0;
a[4] a[5] 0 a[6] 0 0 0 0;
b[6] 0 b[7] b[8] b[9] 0 b[10] b[11]]

A1,A2
```

Out[3]:
```
([1.0 0.0 0.0; 0.0 0.0 0.0; 0.0 0.15248 1.12256], [0.0 -1.0 0.0; -1.0 0.16145
4 -10.5341; -1.0 -0.24338 -0.186122])
```

We look for a solution of the form:

$$\tilde{k}_{t+1} = A\tilde{k}_t + BS_t$$
$$Z_t = CX_t + DS_t$$
$$S_t = PS_{t-1} + Q\epsilon_t$$

Where $Z_t = [\tilde{k}_{t+1}, \tilde{h}_t]'$ and S_t are the stochastic exogenous variables. We compute the generalized eigenvalues and eigenvectors for matrices A_1 and $-A_2$ because A_1 is not invertible. Thus, $A_2V = -A_1V\Pi$ and we can get A and C by:

$$A = V_{11}\Pi_{1,1}V_{1,1}^{-1}$$
$$C = V_{2,1}V_{1,1}^{-1}$$

In [4]:
```
eig = eigen(A1,-A2)
V=eig.vectors
Π = eig.values
#Sorting
for j=1:3
for i=1:2
    if eps(Float64)<abs(Π[i+1])<abs(Π[i])
        Π[i],Π[i+1] = Π[i+1],Π[i]
        V[:,i],V[:,i+1] = V[:,i+1],V[:,i]
    elseif abs(Π[i]) < eps(Float64)
        Π[i],Π[end] =Π[end],Π[i]
        V[:,i],V[:,end]=V[:,end],V[:,i]
    end
end
end
if abs(Π[1])>1
    error("All Eigen Values outside unit circle")
end
Π= Diagonal(Π)
```

Out[4]:
```
3×3 Diagonal{Float64,Array{Float64,1}}:
 0.273179    ⋅          ⋅
  ⋅        -0.632321   ⋅
  ⋅          ⋅         5.12332e-20
```

In [5]:
```
A = V[1,1]*Π[1,1]*inv(V[1,1])
C = V[2:end,1]*(V[1,1])
C = hcat(C,zeros(2,1))
```

Out[5]:
```
2×2 Array{Float64,2}:
  0.273179    0.0
 -0.00289734  0.0
```

In [6]:
```
P = [ρz 0 0 0;
0 ρh 0 0 ;
0 0 ρx 0 ;
0 0 0 ρg]
Q = [σz 0 0 0;
0 σh 0 0 ;
0 0 σx 0 ;
0 0 0 σg]
```

Out[6]:
```
4×4 Array{Float64,2}:
 0.02  0.0   0.0   0.0
 0.0   0.01  0.0   0.0
 0.0   0.0   0.01  0.0
 0.0   0.0   0.0   0.02
```

Finally, to get the matrices B and D, we just need to solve a linear system of equations (see Ellen's notes):

In [7]:
```
function system!(eq,vector::Vector)
    #vector = rand(8)
    #eq= rand(8)
    B=vector[1:4]'
    D2 = vector[5:8]'

    eq[1:4] = a[2].*B .+ a[3].*D2 .+ [a[4] a[5] 0 a[6]]
    eq[5:8] = b[2].*B .+ b[3].*A.*B .+ b[3].*B*P .+ b[4].*D2 .+ b[5].*C[2].*B
.+ b[5].*B*P.+
    [b[6] 0 b[7] b[8]].+[b[9] 0 b[10] b[11] ]*P
 return     eq
end

Sol = nlsolve(system!, ones(8),ftol = :1.0e-20, method = :trust_region , autos
cale = true)
D=ones(2,4)
D[1,:]= Sol.zero[1:4]
D[2,:]= Sol.zero[5:8]
```

Out[7]:
```
4-element Array{Float64,1}:
 -0.16820511698158988
 -0.002517589267184217
  0.0002701330297408085
  0.02143783795673327
```

Now, I will rewrite the model in the form of the exercise, namely:

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$
$$Y_t = CX_t + \omega_t$$

Where, $X_t = [k_t, s_t]$, $s_t = [z_t, \tau_{ht}, \tau_{xt}, g_t]$, $Y_t = [y_t, x_t, h_t]$ and as before:

$$s_{t+1} = Ps_t + Q\varepsilon_{t+1}$$

We need to log linearize y, x since we already done it for labor:

In [8]:
```
#Rewritting to match Anmol's notation
A = hcat(vcat(C[1],zeros(4,1)),vcat(D[1,:]',P))
B = hcat(zeros(5,1),vcat(zeros(1,4),Q))


#We have h as function of states. To find the Matrix B, we need to find y and
 x
#as a function of states

function kt1(vector::Vector)
    k,z,τh,τx,g = vector
    tilde = log.([k,z,τh,τx,g]).-log.([kss,zss,τhss,τxss,gss])
    for i = 1:length(tilde)
        if isnan(tilde[i])
            tilde[i] = 0
        end
    end

    k1= A[1,:]' * tilde
    return k1
end


function ht(vector::Vector)
    k,z,τh,τx,g = vector
    tilde = log.([k,z,τh,τx,g]).-log.([kss,zss,τhss,τxss,gss])
    for i = 1:length(tilde)
        if isnan(tilde[i])
            tilde[i] = 0
        end
    end
    h = C[2,1]*(log(k)-log(kss)) + D[2,:]' * tilde[2:end]
    return h
end
```

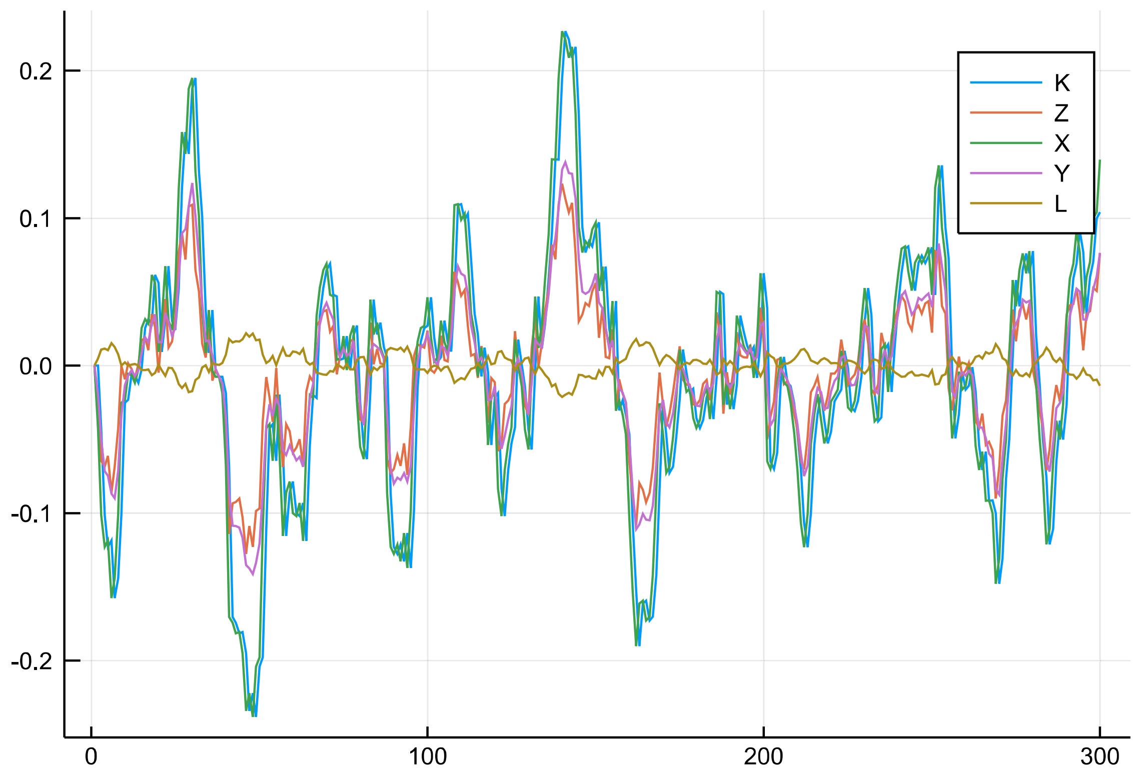
Out[8]:
```
ht (generic function with 1 method)
```

In [9]:
```
#Log-linearizing y as a function of states
function yt(vector::Vector)
    k,z,τh,τx,g = vector
    h = exp(ht(vector)+log(hss))
    y = k^θ * (z*h)^(1-θ)
    return y
end

#GDP
yss = kss^(θ)*(zss*hss)^(1-θ)

T=ForwardDiff.gradient(yt,[kss,zss,τhss,τxss,gss])
ycoefs = [kss*T[1]/yss,zss*T[2]/yss,τhss*T[3]/yss,τxss*T[4]/yss,gss*T[5]/yss]



#log linearizing x as function of states
function xt(vector::Vector)
    k,z,τh,τx,g = vector
    k1 = exp(kt1(vector)+log(kss))
    x= (1+γn)*(1+γz)k1 - (1-δ)k

    return x
end
xss = (1+γz)*(1+γn)*kss-(1-δ)*kss
T=ForwardDiff.gradient(xt,[kss,zss,τhss,τxss,gss])
xcoefs = [kss*T[1]/xss,zss*T[2]/xss,τhss*T[3]/xss,τxss*T[4]/xss,gss*T[5]/xss]

#We have the matrix C!
C = [ycoefs[1] ycoefs[2] ycoefs[3] ycoefs[4] ycoefs[5];
xcoefs[1] xcoefs[2] xcoefs[3] xcoefs[4] xcoefs[5];
C[2,1] D[2,1] D[2,2] D[2,3] D[2,4]]
```

Out[9]:
```
3×5 Array{Float64,2}:
  0.331402     0.55453   -0.00167839   0.000180089   0.0142919
  0.273179     1.41284   -0.000681521  0.0176249    -0.151655
 -0.00289734  -0.168205  -0.00251759   0.000270133   0.0214378
```

In [10]:
```
#defining the vectors
T=300
X= ones(5,T).* [0,0,0,0,0]
Y = ones(3,T).*[0,0,0]


for t=1:T

    if t>1
    X[:,t] = A*X[:,t-1]+ B*randn(5,1)
    end
    Y[:,t] = C*X[:,t]
end
t=2
A*X[:,t-1]
plot([X[1,:],X[2,:],Y[2,:],Y[1,:],Y[3,:]],labels = ["K","Z","X","Y","L"])
```

Out[10]:



The code above is summarized in a function State_Space which returns the matrices A,B,C

In [47]:
```
include("State_Space.jl")
```

Out[47]: State_Space (generic function with 2 methods)

Question 2

Data

We have quarterly data from Q3/1947 - Q4/2017

A brief description of data used is below:

h: Total hours per capita from Prescott, Ueberfeldt, and Cociuba

y: Real GDP from BEA NIPA table 1.1.6

g: Government consumption from BEA NIPA table 3.9.5 deflated by PCE

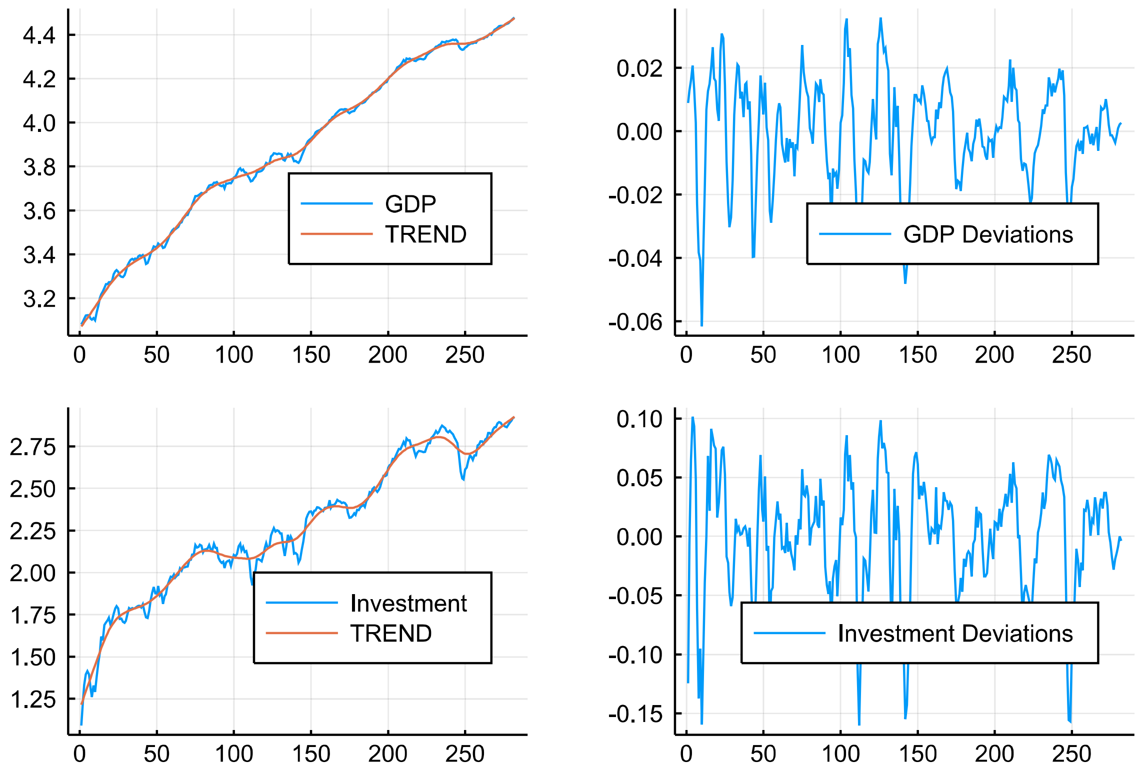
x: Real gross private domestic investment from BEA NIPA table 1.1.6 + government gross investment 3.9.5 (deflated by PCE)

y,g,x are in percapita terms (data was divided by population from Prescott, Ueberfeldt, and Cociuba)

We consider only logs and the trend of y,g and x was removed using HP filter 1600

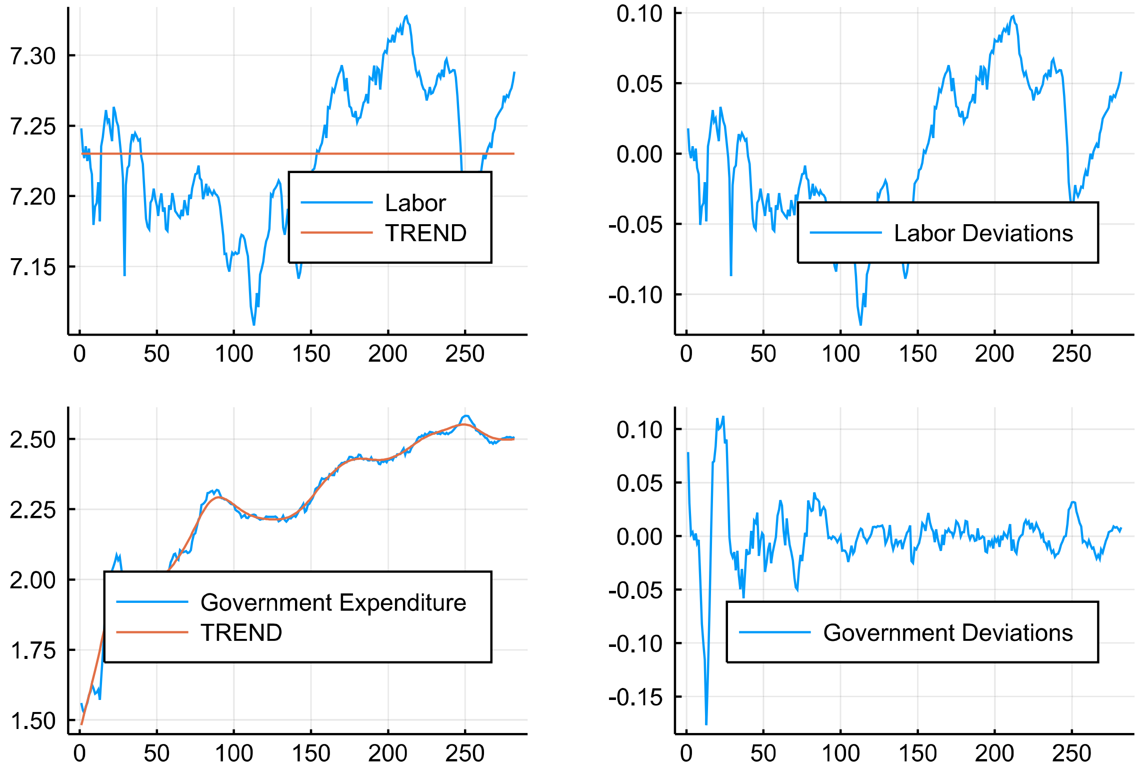
In [12]:
```
include("load_data.jl")
DATA = loaddata()
plot(
    plot([DATA[:GDP],DATA[:GDP_trend]],legend = :bottomright, label = ["GDP", "TREND"]),
    plot([DATA[:GDP_dev]], label = ["GDP Deviations"],legend = :bottomright),
    plot([DATA[:Investment],DATA[:Investment_trend]],legend = :bottomright, label = ["Investment","TREND"]),
    plot([DATA[:Investment_dev]],legend = :bottomright, label = ["Investment Deviations"])

    )
```

Out[12]:



In [13]:
```
plot(      plot([DATA[:Labor],DATA[:Labor_trend]],legend = :bottomright, label
 = ["Labor","TREND"]),
        plot([DATA[:Labor_dev]],legend = :bottomright, label = ["Labor Deviati
ons"]),
        plot([DATA[:GOV],DATA[:GOV_trend]],legend = :bottomright, label = ["Go
vernment Expenditure","TREND"]),
        plot([DATA[:GOV_dev]],legend = :bottomright, label = ["Government Devi
ations"])
    )
```

Out[13]:



Question 3

The Code below implements the Kalman Filter. I follow Ljunqvist and Sargent chapter 2 for that. We will plot the results below with the data above. Note that at this point I did not estimate P and Q.

In [14]:
```
using Statistics
include("KalmanFilter.jl")
function KalmanFilter(Y,A,B,C)
#Y: Observed variables


n = size(A)[1] #number of state variables
m = size(C)[1] #number of measurement variables
T = size(Y)[2] #Sample size

#Initializing the state variables
X = zeros(n,T)
#Variance initial guess
Σ = ones(n,n)*ones(n,n)'
d=10
while d>10^(-15)
    Σ1 = A*Σ*A' + B*B'
    d = maximum(abs.(Σ-Σ1))
    Σ=Σ1
end

#Run the Kalman filter algorithm (see Ljunqvist Sargent)
a=ones(m,T)
Ω = C*Σ*C'
for t = 1:T-1
    a[:,t] = Y[:,t] - C*X[:,t] #Error
    K = A*Σ*C' / (C*Σ*C') #Kalman Gain
    X[:,t+1] = A*X[:,t] +K*a[:,t] #Update the states
    Σ = B*B' + (A-K*C)*Σ*(A-K*C)' #Variance matrix
    Ω = [Ω C*Σ*C'] #store the relevant values
end
a[:,T] = Y[:,T] - C*X[:,T]
Ω = [Ω C*Σ*C']

return X, a, Ω
end
```

Out[14]: KalmanFilter (generic function with 1 method)

With the Kalman Filter, we get the Likelihood function:

In [15]:
```
function likelihood(Y,Ω,a)
    m,T = size(Y)
    L = 0
for t=1:T
    iteration = t
    L = L +(-T*m/2 * log(2*π)-0.5*log(det(Ω[:,m*t-(m-1):(m*t)])) -
    0.5*a[:,t]'*(Ω[:,m*t-(m-1):(m*t)]\a[:,t]))
end

return L
end
```

Out[15]: likelihood (generic function with 1 method)

Now, we need to maximize it to get the estimates for P and Q, we assume that Q is a diagonal matrix. This code was also tested with simulated data and it works!

In [49]:
```
include("KalmanFilter.jl")
initial = [0.5,0.5,0.5,0.5,0.001,0.001,0.001,0.1,0.1,0.1,0.1,0.1] #ρz,ρh,ρx,ρ
g,σz,σh,σx,σg,gss,τxss,τhss,zss
DATA = loaddata()
Y = vcat(DATA[:GDP_dev]',DATA[:Investment_dev]',DATA[:Labor_dev]',DATA[:GOV_de
v]')
d=10
lower = zeros(12)
upper = [1.0,1,1,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5]
while d>10^(-7)
    global d, initial
    bla = optimize(maxloglikelihood,lower, upper, initial)
    d = maximum(abs.(initial - bla.minimizer))
    println(d)
    initial = bla.minimizer
end
ρz,ρh,ρx,ρg,σz,σh,σx,σg,gss,τxss,τhss,zss = initial
P = [ρz  0 0 0;
0 ρh 0  0;
0 0 ρx 0;
0 0 0 ρg]
```

```
0.27845266514081085
0.28239040792151315
0.0
```

Out[49]:
```
4×4 Array{Float64,2}:
 0.401329  0.0       0.0      0.0
 0.0       0.949834  0.0      0.0
 0.0       0.0       0.81503  0.0
 0.0       0.0       0.0      0.86479
```

In [50]:
```
Q = [σz 0 0 0;
0 σh 0  0;
0 0 σx 0;
0 0 0 σg]
```

Out[50]:
```
4×4 Array{Float64,2}:
 0.0324907  0.0       0.0       0.0
 0.0        0.208172  0.0       0.0
 0.0        0.0       0.137513  0.0
 0.0        0.0       0.0       0.013652
```

Above are the estimated P and Q Matrices. I don't know why, the estimated variances for the investment and laboor wedges are very high.

In [54]:
```
steadystates = [gss,τxss,τhss,zss]
```

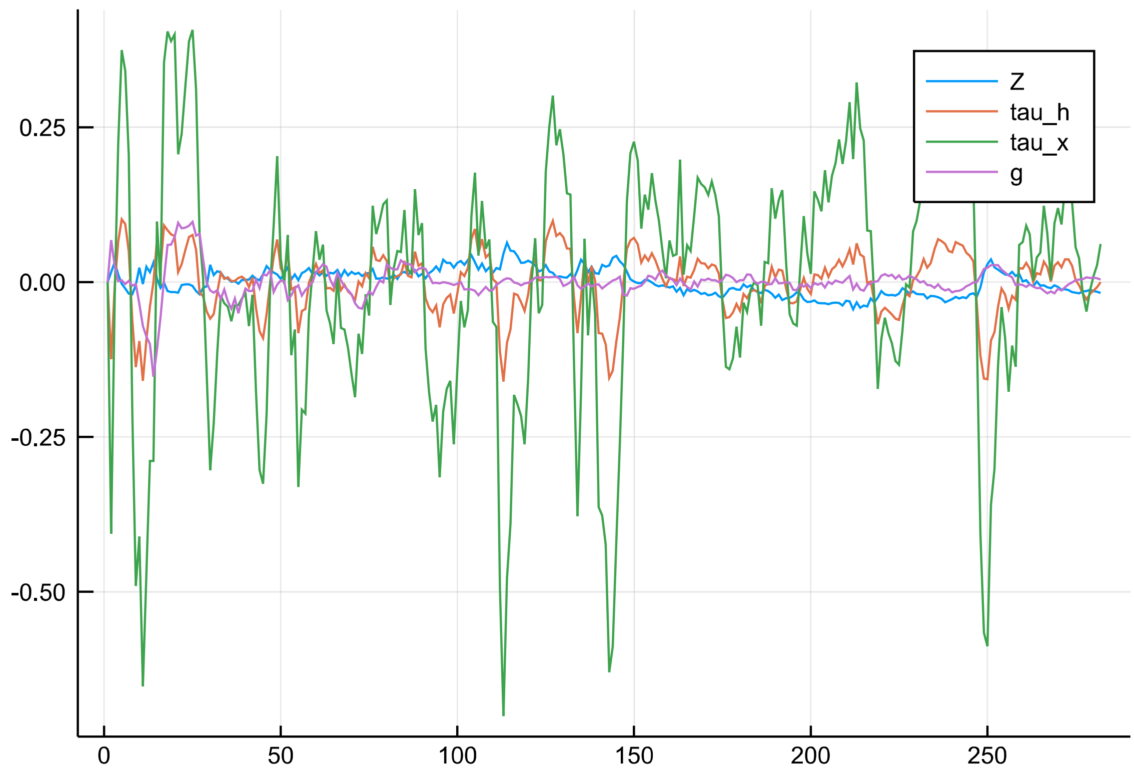
Out[54]:
```
4-element Array{Float64,1}:
 0.08932582477144198
 0.2623856718955942
 0.23590125815590599
 0.23506887642069707
```

Question 4

Below are the simulated wedges

In [53]:
```
using LaTeXStrings
params_calibrated = [δ,θ,β,σ,ψ,γn,γz]
steadystates = [gss,τxss,τhss,zss]
A,B,C = State_Space(params_calibrated,steadystates, P,Q)
X, a, Ω = KalmanFilter(Y,A,B,C)
plot([X[2,:],X[1,:],X[4,:],X[5,:]],labels = ["Z","tau_h","tau_x","g"])
```

Out[53]:



Question 5