ECON 8185 - Homework 2

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Question 1

The model is:

$$\max_{c_t, x_t, l_t} E \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t l_t^{\psi}\right)^{1-\sigma}}{1-\sigma} N_t$$

S.T.

$$c_t + (1 + \tau_{xt})x_t = r_t k_t + (1 - \tau_{ht})w_t h_t + T_t$$

$$N_{t+1}k_{t+1} = [(1-\delta)k_t + x_t]N_t$$

$$h_t + l_t = 1$$

$$S_t = LS_{t-1} + \Sigma\epsilon_t, \quad S_t = [\ln z_t, \tau_{ht}, \tau_{xt}, \ln g_t]$$

$$c_t, x_t \geq 0$$

Where $N_t = (1 + \gamma_n)^t$ and firm tecnhology is $Y_t = K_t^{\theta}(Z_t L_t)^{1-\theta}$. γ_z is the rate of labor-augmenting technical progress. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint is $Y_t = N_t(c_t + x_t + g_t)$. We must work with detrended variables, we should use "hat" variables, but typing this is time consuming so from now on all variables should be understood as detrended unless I say something about them.

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The detrended FOC's of this model are:

$$c_t + (1+\gamma_z)(1+\gamma_n)k_{t+1} - (1-\delta)k_t + g_t = y_t = k_t^\theta (z_t h_t)^{1-\theta}$$

$$\psi \frac{c_t}{1-h_t} = (1-\tau_{ht})(1-\theta)\left(\frac{k_t}{h_t}\right)^\theta z_t^{1-\theta}$$

$$c_t^{-\sigma}(1-h_t)^{\psi(1-\sigma)}(1+\tau_{xt}) = \beta(1+\gamma_z)^{-\sigma} E_t c_{t+1}^{-\sigma}(1-h_{t+1})^{\psi(1-\sigma)}\left(\theta k_{t+1}^\theta (z_{t+1}h_{t+1})^{1-\theta} + (1-\delta)(1+\theta_{xt+1})\right)$$

Substituting for c, we get an equation for k, and one for h. Below, I can find the Steady State values.

Question 1

1.a) Iterate on Bellmans equation

Due to the curse of dimensionality, I won't pursue this path. The memory requirements for the 5 state variables of this problem make it not feasible to implement the same algorithm I did for HW1.

1.b) Map it to a linear quadratic problem

Recall from lecture notes, we have to map the original problem into the following LQ problem:

$$\max_{\{u_t\}_{t=0}^\infty} E_0 \sum_{t=0}^\infty \beta^t (X_t' Q X_t + u_t' R u_t + 2X_t' W u_t)$$

s.t.

$$X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1}$$

X_0 given.

In this case, we have: $u_t = [k_{t+1}, h_t]'$

Note that in this problem X_t may be decomposed as:

$$X_t = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_t$$

Where X_1 are the individual states, X_2 are the exogenous states with known laws of motion ,, and X_3 are the aggregate states with laws of motion that are unknown and need to be computed in equilibrium $[K, H]$. We have that:

$X_1 = [k_t]'$, $X_2 = [z, \tau_{xt}, \tau_{ht}, g_t]$, $X_3 = [K_t, H_t]$, and:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & L & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

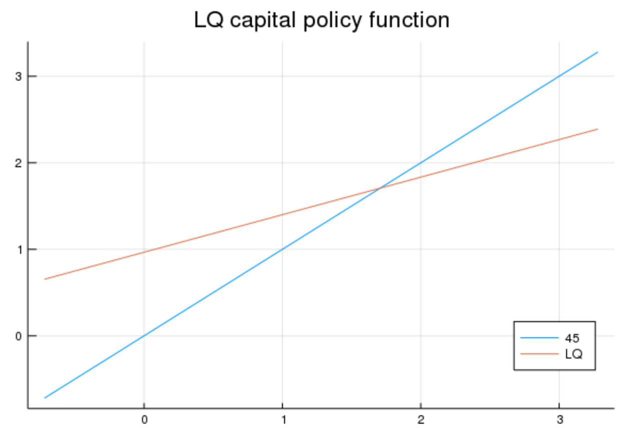
$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Sigma & & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewrite $y_t = [\tilde{X}_1, \tilde{X}_2]'$ and the problem constraint becomes:

$$y_{t+1} = \tilde{A}_y y_t + \tilde{B}_y \tilde{u}_t + A_z \tilde{X}_{3t}$$

Where tilde variables are the undiscounted counterpart of each variable. Matrices A_y, B_y, Q, R and W may be found by second and first order Taylor expansions of the utility function and constraints. Matrix A_z is for now unknown. Following the methods in the lecture notes using Big K, little k trick with $\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ this problem may be solved.

We mapped it to a LQ problem , below I plot the capital policy function by this method for the calibration: $\delta = 0.0464$, $\theta = 1/3$, $\beta = 0.9$, $= 1$, $\phi = 1$, $\gamma_n = 0$, $\gamma_z = 0$, shocks will be assumed to be independent and the AR coefficients will be: $\rho_z = 0.8$, $\rho_g = 0.2$, $\rho_h = 0.6$, $\rho_x = 0.4$ and variances:$\sigma_z = 0.1$,$\sigma_g = 0.05$, $\sigma_h = 0.07$, $\sigma_x = 0.02$. This calibration will be kept constant throughout this homework.



To be honest, I am not super confident about my implementation of this method. Although it works fairly, it is frequent the case that matrix R is not invertible and I simply cannot do anything with this. For this reason, all the following figures will be based on the modified Vaughan method which work smoothly.

1.c) Apply Vaughan's method

I use the modified Vaughan method using the log linearized FOC's as in the lecture notes. Where tilde variables are log deviations from Steady State. Stacking up the equations we get:

$$0 = E_t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & b_5 \end{bmatrix} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{k}_{t+2} \\ \tilde{h}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_4 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{k}_{t+1} \\ \tilde{h}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & 0 & 0 \\ b_6 & 0 & b_7 & b_8 & b_9 & 0 & b_{10} & b_{11} \end{bmatrix} \begin{bmatrix} S_t \\ S_{t+1} \end{bmatrix}$$

We call the first matrix A_1, and the second A_2. We look for a solution of the form:

$$\tilde{k}_{t+1} = A\tilde{k}_t + BS_t$$

$$Z_t = CX_t + DS_t$$

$$S_t = PS_{t-1} + Q\epsilon_t$$

Where $Z_t = [\tilde{k}_{t+1}, \tilde{h}_t]'$ and S_t are the stochastic exogenous variables. We compute the generalized eigenvalues and eigenvectors for matrices A_1 and

A_2 because A_1 is not invertible. Thus, $A_2 V = -A_1 V \Pi$ and we can get A and C by:

$$A = V_{11}\Pi_{1,1}V_{1,1}^{-1}$$

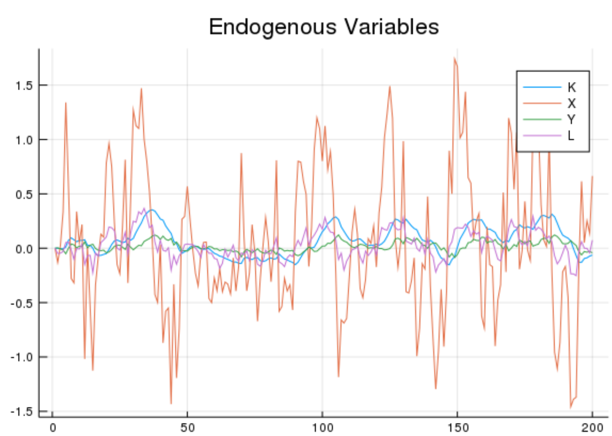
$$C = V_{2,1}V_{1,1}^{-1}$$

Finally, to get the matrices B and D, we just need to solve a linear system of equations. In the section below, I simulate all the relevant variables:

Question 2

First, I simulate the variables with known law of motion:

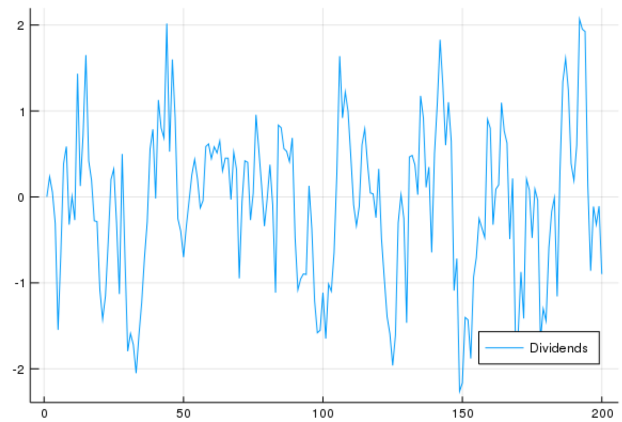
And the endogenous variables implied by the model:



Note that dividends can be retrieved by the following expression:

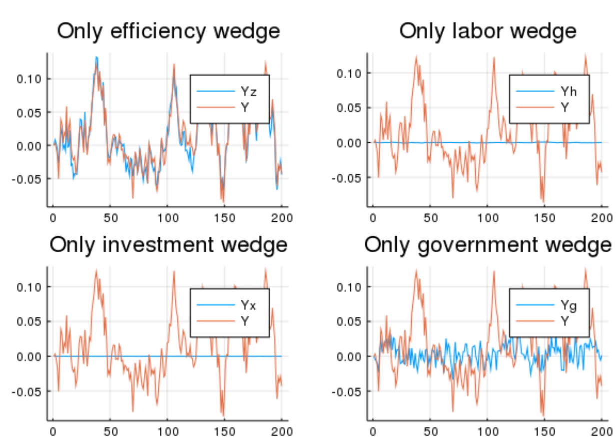
$$D_t = Y_t - w_t H_t - X_t$$

The log linearized dividends series implied by the model is plotted below:



Some interesting uses of the model are illustrated below. If we want to use the model to analyze data, we could estimate the parameters (see HW3) and then perform the following exercise. Using the estimated model, we could shut down all but one shock to asses how important is that shock to explain the data. The graphs below illustrate this using simulated data:

The log linearized dividends series implied by the model is plotted below:



The blue lines represent the Output series of a model with only one shock. As we may see, a model with only the efficiency shock would be able to explain really well our simulated data. This is not surprising since we assumed high persistence and volatility to this process. Another way of doing the same type of analysis is to shut one shock only and see how the model with the three remaiing shocks would behave: