Journal of Monetary Economics 33 (1994) 573–601. North-Holland

The macroeconomic effects of distortionary taxation*

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Received February 1992, final version received November 1992

Prescott (1986) estimates that technology shocks account for 75% of the fluctuations in the postwar U.S. economy. This paper reestimates the contribution of technological change for a standard business cycle model that includes a public sector and fiscal disturbances. I find that a significant fraction of the variance of aggregate consumption, investment, output, capital stock, and hours of work can be explained by disturbances in labor and capital tax rates and government consumption. I also use the model to quantify the welfare costs of capital and labor taxation. For both the time series and welfare calculations, maximum likelihood estimates of taste, technology, and policy parameters are used.

Key words: Business cycle fluctuations; Taxation; Dynamic general equilibrium; Maximum likelihood

JEL classification: E32; E62; C51

1. Introduction

Real business cycle models in the tradition of Kydland and Prescott (1982)[1] assume that technology shocks are the main source of cyclical variation in aggregate time series. Prescott (1986) estimates that technology shocks account

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*This paper is based on my doctoral thesis. I would like to thank the members of my committee, Rody Manuelli, Richard Rogerson, John Taylor, and, in particular, Tom Sargent. I would also like to thank Sumru Altug, Toni Braun, Graham Candler, John Heaton, Ken Judd, Randy Wright, the editors, and an anonymous referee for helpful comments and criticisms. Martie Starr provided invaluable editorial assistance. Any views expressed here are mine and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

[1]See, for example, Benhabib, Rogerson, and Wright (1991), Hansen (1985), Long and Plosser (1983), and Prescott (1986),

for 75% of the fluctuations in the postwar U.S. economy. Such a claim leads naturally to several questions. First, if you accept Prescott's estimate, what accounts for the remaining 25% of fluctuations? Second, if variables or sources of disturbances other than technology are taken into account, will technology still account for 75% of the total fluctuations? Third, will taking omitted variables or sources of disturbances into account resolve the failings of the one-shock model, or will the results be that, relative to the data, the variability of consumption, hours of work, and output is still too low, and the variability of investment and the correlation of real wages and hours are still too high?

I attempt to answer these questions for the U.S. economy by considering two alternative sources of fluctuations: changes in government consumption and tax rates. To quantify the contribution of fiscal disturbances to economic fluctuations, variances are decomposed with fractions explained by innovations in technology, government expenditures, labor tax rates, and capital tax rates. Assuming that fiscal policies are like those in the United States, I find that a significant portion of the variance of the aggregate consumption, output, hours worked, capital stock, and investment can be explained by the government expenditures and factor tax processes. Far less than 75% is attributable to technology for most of the variables.

The context of this study is a standard business cycle model that includes a public sector. The government is described as a sequence of tax rates on capital and labor and a sequence of government consumption. The fiscal policies are set exogenously. Competitive firms produce goods with labor and capital inputs. And households make consumption, labor, and investment decisions, taking as given factor prices for their inputs and fiscal policy. These decisions are used to determine the model's implications for the time series and the welfare costs of taxation.

Fiscal disturbances are likely to have a significant impact on the time series for several reasons. Since they are shocks to labor supply, changes in fiscal variables imply a negative correlation between wages and hours, all other things equal. With only technology shocks driving fluctuations, the correlation is high and positive. If changes in both technology and fiscal variables occur, the model predicts a correlation close to zero, which is approximately what is observed in the data. Fiscal policy can also potentially increase the variation in hours of work and consumption. If tax rates are fluctuating, then substitution between market and nonmarket activities will increase.

I also use the model to quantify the welfare costs of capital and labor taxation. A wide range of estimates of the costs of taxes on factors of production exists. For a version of the model studied here, McGrattan (1989) finds that the estimates are sensitive to parameterizations of preferences and tax policies. Judd (1987) also finds that his estimates of welfare costs are sensitive to parameter choices. In the case of a permanent increase in the tax on labor income, Judd's cost figures range from 2 cents per dollar revenue to over 1 dollar. For the capital income, the range is from 15 cents to over 20 dollars.

Because the time series and welfare implications of the model depend on the choice of parameters, maximum likelihood estimates are computed. To construct the likelihood function, it is necessary to first compute an equilibrium. As do Kydland and Prescott (1982), I apply a method that relies on replacing the nonlinear objective function of the household with a quadratic function. Quadratic costs and linear constraints make an analytical derivation possible even though distortions have been introduced by the tax policies. The results are linear decision functions that make construction of the likelihood very easy.

With parameters of preferences, technologies, and policies given by the maximum likelihood estimates, two measures of the costs of factor taxation are computed. The first measure is the marginal deadweight loss of permanent changes in the tax rates as calculated by Judd (1987). The calculations reported here are not directly comparable to Judd's since Judd assumes that tax rates are constant and I assume that the tax rates, government consumption, and technology shocks are correlated stochastic processes. A change in the labor tax rate induces changes in the capital tax rate, government consumption, and technology. Therefore, the first set of tax experiments should be thought of as changes in the tax processes. The second measure of welfare cost is the compensating variation in consumption required to leave households indifferent between two economies with different levels of taxation. To isolate the effects of a change in one of the tax rates, the computation is done for the system in the steady state. The calculations show that a permanent increase in the capital tax rate is more costly than an increase in the labor tax rate. Elimination of all tax distortions implies benefits of the order of Lucas' (1987) calculations for a 1% increase in the economy's growth rate.

The model is described in section 2. In section 3, a method for computing the optimal investment and labor functions for the households is described. The estimation strategy is outlined in section 4. The empirical results for the United States are reported in section 5. Section 5 describes the data, the parameter estimates, the implications of the model for time series properties, and the implications of the model for welfare of the households. Concluding remarks are given in section 6. An appendix with sources of data is also provided.[2]

2. The model economy

The model economy studied in this paper comprises the government, a large number of identical firms, and a large number of identical households, all of whom are infinitely-lived. Government policy in this economy is characterized

[2]Some recent work that also considers the macroeconomic effects of fiscal policy includes Braun (1994), Cassou (1990), Chang (1992), Greenwood and Huffman (1991).

by sequences of tax rates on capital and labor and a sequence of government consumption. The tax and spending processes are assumed to be exogenous. The firms in the economy are assumed to behave competitively. Therefore, their decision-making can be summarized by the equation of factor prices and marginal products. Household decision-making is somewhat more complicated. This section considers the consumption, labor, and investment decisions of a representative household facing a given fiscal policy and given prices of productive inputs. The description of the household's problem serves as the background to the discussion of the effects of fiscal variables on aggregate fluctuations. In later sections, the equilibrium behavior is characterized and the household's decision functions are used to make precise statements about the effect of government taxation and spending on the business cycle.

The preferences of the household are given by the expected utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^{*t} u(c_{p,t}^* + \pi g_t^*, a(L)l_t), \qquad 0 < \beta^* < 1, \tag{1}$$

where β^* is a discount factor, expectations are conditioned on the information set of the household at time 0, and u is concave in both of its arguments. Current period utility depends on the number of goods consumed, both privately, $c_{p,t}^*$, and publicly, g_t^*, and on current and past hours of leisure, $l_t, l_{t-1}, \ldots$. The specification of the first argument of (1) assumes that government expenditures can influence the household's utility if $\pi \neq 0$. If $\pi > 0$, the marginal utility of consumption decreases with an increase in g_t^*. If $\pi < 0$, the opposite is true. The specification of the second argument assumes that past leisure decisions affect current leisure services. If $a(L) = \sum_{j=0}^{\infty} a_j L^j$, $a(1) = 1$, where L is the lag operator, then one hour of leisure at t gives a_j hours of leisure services at $t+j$. Assuming that services decline geometrically over time implies that $a_j = (1-\eta)^{j-1} a_1$, $0 < \eta < 1$. For $\eta = 1$, only current and lagged hours of leisure affect current utility. If the function a is restricted to have $a_0 = 1$, then utility is time-separable. Hours of work at time t are given by

$$n_t = H - l_t,$$

assuming that the total time allotment, H, is used. Therefore,

$$a(L)l_t = H - a_0 n_t - \eta(1-a_0)h_t,$$

$$h_{t+1} = (1-\eta)h_t + n_t,$$

where $h_t = \sum_{j=1}^{\infty} (1-\eta)^{j-1} n_{t-j}$ is a weighted sum of past hours of work.

The budget constraints of the household are given by

$$c_{p,t}^* + i_t^* \leq (1-\tau_t)r_t k_t^* + (1-\varphi_t)w_t^* n_t + \delta\tau_t k_t^* + \xi_t^*, \qquad t \geq 0. \tag{2}$$

Each period the household purchases consumption, $c_{p,t}^*$, and investment, i_t^*, goods with after-tax income obtained from renting its capital and labor to a firm. The capital income in time t is $r_t k_t^*$, where r is the price of renting capital and k^* is the capital stock of the household. Labor income in t is $w_t^* n_t$, where w^* is the wage rate. If the household behaves competitively, then it is a price-taker in the capital and labor markets. In (2), the prices of consumption and investment goods are normalized to 1. Capital and labor income are taxed at rates τ_t and φ_t, respectively. The tax rates are assumed to be stochastic, exogenous processes. ξ_t^* are lump-sum transfer payments made by the government in period t. The final source of income is depreciation allowances $\delta\tau_t k_t^*$, where $0 \leq \delta \leq 1$ is the constant rate of capital depreciation.

The household owns the technology to convert investment and the current capital stock to next period capital. As in Kydland and Prescott (1982), capital takes time to build. Assuming that N periods are required for building capital, total investment is the sum

$$i_t^* = \sum_{j=0}^{N-1} \phi_{N-j} s_{t-j}^*, \qquad \phi_j \geq 0, \qquad \sum_{j=1}^{N} \phi_j = 1, \tag{3}$$

where s_t is investment starts at time t. The parameters ϕ_j denote the fraction of resources allocated to projects that are j periods from completion. Thus, current investment consists of the value put in place during the first year of projects started in the current period, $\phi_N s_t^*$, the value put in place during the second year of projects started in the previous period, $\phi_{N-1} s_{t-1}^*$, and so on. In this case, the investment projects that add to the end-of-period capital stock at t are those started at $t - N + 1$,

$$k_{t+1}^* = (1-\delta)k_t^* + s_{t-N+1}^*. \tag{4}$$

The decisions of the household depend on factor prices and government transfers. The values of these variables depend in part on the assumed behavior of firms and the government. At each date t, the firm is assumed to choose levels of output y_t^*, capital k_t^*, and labor n_t so as to maximize profits:

$$y_t^* - r_t k_t^* - w_t^* n_t, \tag{5}$$

subject to the technology

$$y_t^* = \lambda_t f(k_t^*, \mu^t n_t), \qquad \mu \geq 1, \tag{6}$$

where f is a constant returns to scale production function, λ_t is a stochastic technological shock, and r_t, w_t^* are input prices taken as given by the firm. Revenues are obtained by selling goods to the households and to the government. Costs are incurred from renting the households' capital and labor. If the rental markets are competitive, as is assumed here, then equilibrium rental and wage rates are equal to the marginal products of capital and labor, respectively. Furthermore, profits of the firm are zero.

The government levies taxes on factors of production to finance a stochastic stream of expenditures. Any revenue that is not used to finance current purchases is transferred to households in a lump-sum payment. Thus, real transfers to households at time t are given by

$$\xi_t^* = \tau_t r_t k_t^* + \varphi_t w_t^* n_t - \tau_t \delta k_t^* - g_t^*. \tag{7}$$

This specification assumes that the government balances its budget each period and never issues debt. Therefore, the transfers act as a residual in eq. (7).

There are four sources of variation in output: the shock to productivity, government consumption, and the two tax rates. The technology shocks and tax rates are assumed to be stationary processes, but government consumption grows over time. An assumption made here is that output (y^*) and its components (c^*, i^*, g^*), capital stock (k^*), investment starts (s^*), real wages (w^*), and government transfers (ξ^*) all grow at the same rate μ. Using the notation $z_t = z_t^*/\mu^t$ for all z on the balanced growth path, eqs. (2) and (5) can be written with elements of z^* replaced by their analogues in z. If the objective function (1) and constraints (3), (4), and (6) are rewritten in terms of stationary variables, then the new objective function and constraints are given by

$$\sum_t \beta^t u(c_{p,t} + \pi g_t, a(L)l_t), \tag{1'}$$

$$i_t = \sum_{j=0}^{N-1} \frac{\phi_{N-j}}{\mu^j} s_{t-j}, \tag{3'}$$

$$k_{t+1} = \frac{1-\delta}{\mu} k_t + \frac{1}{\mu^N} s_{t-N+1}, \tag{4'}$$

$$y_t = \lambda_t f(k_t, n_t), \tag{6'}$$

where β is a function of β^*, μ, and parameters of the utility function. In section 5, the β function is defined for a particular choice of utility. With the above transformations, the process g is stationary. It is assumed that technology shocks (λ), government consumption (g), and tax rates (τ,φ) are governed by the following stationary vector autoregression:

$$v_t = b_0 + b(L)v_{t-1} + b_\varepsilon \varepsilon_t, \tag{8}$$

where $v_t = [\lambda_t, g_t, \tau_t, \varphi_t]'$, $b(L) = b_1 L + \cdots + b_q L^q$, $b(L)$ is invertible, L is a lag operator, ε_t is a vector white noise with $\mathrm{E}\varepsilon_t \varepsilon_t' = I$. The stochastic process, v, like factor inputs and government policies, is taken as given by the households.

The problems of the household and firm are summarized in the following definition of equilibrium. An equilibrium in this economy consists of functions $i(x)$, $n(x)$, and $v(k,h,\underset{\sim}{s},x)$, for $\underset{\sim}{s} = (s_{-1}, \ldots, s_{-N+1})$, $x = (k,h,\underset{\sim}{s},v,\underset{\sim}{v})$, and $\underset{\sim}{v} = (v_{-1}, \ldots, v_{-q})$ such that:

(i) if households choose $\hat{i}$ and $\hat{n}$ that solve

$$v(\hat{k},\hat{h},\hat{\underset{\sim}{s}},x) = \max_{\substack{0 \le \hat{i} \\ 0 \le \hat{n} \le H}} \left\{ \mathscr{U}(\hat{k},\hat{h},\hat{\underset{\sim}{s}},x,i(x),n(x),\hat{i},\hat{n}) + \beta \int v(\hat{k}',\hat{h}',\hat{\underset{\sim}{s}}',x')\,\mathrm{d}G(\varepsilon) \right\},$$

subject to

$$\hat{k}' = \frac{1-\delta}{\mu}\hat{k} + \frac{1}{\mu^N}\hat{s}_{-N+1},$$

$$\hat{h}' = (1-\eta)\hat{h} + \hat{n},$$

$$\hat{s} = \frac{1}{\mu^N \phi_N}(\mu^N \hat{i} - \mu^{N-1}\phi_{N-1}\hat{s}_{-1} + \cdots + \mu\phi_1 \hat{s}_{-N+1}),$$

$$k' = \frac{1-\delta}{\mu}k + \frac{1}{\mu^N}s_{-N+1}, \tag{9}$$

$$h' = (1-\eta)h + n(x), \tag{10}$$

$$s = \frac{1}{\mu^N \phi_N}(\mu^N i(x) - \mu^{N-1}\phi_{N-1}s_{-1} + \cdots + \mu\phi_1 s_{-N+1}), \tag{11}$$

$$v' = b_0 + b_1 v + b_2 v_{-1} + \cdots + b_q v_{-q+1} + b_\varepsilon \varepsilon, \tag{12}$$

$$\begin{aligned} &\mathscr{U}(\hat{k}, \hat{h}, \hat{\underline{s}}, x, i(x), n(x), \hat{i}, \hat{n}) \\ &= u(\lambda f_1(k, n(x))(\hat{k} - \tau\hat{k} + \tau k) + \lambda f_2(k, n(x))(\hat{n} - \varphi\hat{n} + \varphi n(x)) \\ &\quad - \hat{i} + (\pi - 1)g + \delta\tau(\hat{k} - k), H - a_0\hat{n} - \eta(1 - a_0)\hat{h}), \end{aligned} \tag{13}$$

and the distribution, G, for the error process, then $\hat{i} = i(x)$, $\hat{n} = n(x)$;

(ii) if firms choose $\hat{k}$, $\hat{n}$ to maximize the profit function $\lambda f(\hat{k}, \hat{n}) - r(x)\hat{k} - w(x)\hat{n}$, where $r(x) = \lambda f_1(k, n(x))$ and $w(x) = \lambda f_2(k, n(x))$, then $\hat{k} = k$, $\hat{n} = n(x)$.

The first argument of the utility function u in eq. (13) is a weighted function of private and government consumption. Private consumption is equal to income less investment from the household budget constraint, eq. (2), where rental and wage rates have been replaced by marginal products, i.e., $r(x) = \lambda f_1(k, n(x))$, $w(x) = \lambda f_2(k, n(x))$. Some of the variables in the definition are marked with a '$\hat{\ }$' symbol. These variables are specific to the individual. Those without '$\hat{\ }$' are per capita levels. Thus, $\hat{k}$ is the capital stock owned by the individual, whereas k is the per capita level. In equilibrium, the individual and per capita levels are equated. However, if the households are assumed to behave competitively, then it is necessary to distinguish between the individual and per capita levels in (13).

If one can compute the equilibrium investment and labor decision functions, it is possible to use these functions to compare statistical properties of the model's time series with their counterparts in the data. One way to do this is to choose an initial value for the state vector (x_0), use the equilibrium decision functions [$i(x_0)$ and $n(x_0)$] to determine investment and labor, and then calculate the next period's state, x_1, from eqs. (9)–(12). Repeating this procedure for $x_2, x_3, \ldots$ gives time series for the model. In the next section, a method for computing $i(x)$ and $n(x)$ is described.

3. The linear-quadratic model

For general utility and production specifications, it is not possible to analytically derive the decision functions for investment and hours. In this section, a method of approximation is considered that allows an analytical derivation. The method is applied to solving the household's optimization problem rather than that of a fictitious planner [as in Kydland and Prescott (1982)]. Because taxes are distortionary, there no longer is a connection between the Pareto optima and a competitive equilibrium as defined in section 2.

The first step in deriving decision functions follows the procedure used by Kydland and Prescott (1982). The nonlinear objective function $\mathscr{U}$ in (13) is replaced by a quadratic expansion around the steady state (i.e., the limiting values of the states when $\varepsilon_t = 0, t \geq 0$). The steady state values for the exogenous state vector follow directly from eq. (8), i.e., $\bar{v} = (I - b(1))^{-1} b_0$, where $\bar{v} = [\bar{\lambda}, \bar{g}, \bar{\tau}, \bar{\varphi}]'$ and '$\bar{\ }$' indicates the steady state level. Let $\bar{k}$ and $\bar{n}$ be the steady state values of capital and hours worked. For the economy of section 2, $\bar{k}$ and $\bar{n}$ satisfy

$$\left\{a_0 + \frac{\beta\eta(1 - a_0)}{1 - \beta(1 - \eta)}\right\} u_2(\bar{c}, H - \bar{n}) - (1 - \bar{\varphi}) u_1(\bar{c}, H - \bar{n}) \bar{\lambda} f_2(\bar{k}, \bar{n}) = 0, \tag{14}$$

$$\beta(1 - \bar{\tau})\bar{\lambda} f_1(\bar{k}, \bar{n}) + \beta\delta\bar{\tau} - (\mu - \beta + \beta\delta) \sum_{j=1}^{N} \left(\frac{\mu}{\beta}\right)^{j-1} \phi_j = 0, \tag{15}$$

where

$$\bar{c} = \bar{\lambda} f(\bar{k}, \bar{n}) - \mu^{-1}(\mu - 1 + \delta)\left(\sum_{j=1}^{N} \mu^j \phi_j\right)\bar{k} + (\pi - 1)\bar{g}.$$

The steady state values for investment, starts, and weighted past hours are functions of $\bar{k}$ and $\bar{n}$ and follow directly from the household's constraints. The approximate objective function follows directly:

$$\begin{aligned} \mathscr{U}(\hat{k}_t, \hat{h}_t, \hat{s}_t, x_t, i_t, n_t, \hat{i}_t, \hat{n}_t) &\equiv U(X_t), \qquad X_t = [\hat{k}_t, \hat{h}_t, \hat{s}_t, x_t, i_t, n_t, \hat{i}_t, \hat{n}_t]', \\ &\simeq U(\bar{X}) + \left.\frac{\partial U}{\partial X_t}\right|_{\bar{X}}' (X_t - \bar{X}) \\ &\quad + \tfrac{1}{2}(X_t - \bar{X})' \left.\frac{\partial^2 U}{\partial X_t^2}\right|_{\bar{X}} (X_t - \bar{X}), \end{aligned} \tag{16}$$

where the elements of $\bar{X}$ are the steady state values of the elements of X_t. The definition of equilibrium given in section 2 remains the same, with the exception that the utility function in (13) is replaced by the quadratic function (16). Therefore, to determine the equilibrium investment and labor decisions, it is necessary to maximize the discounted sum of the approximation to $U(X)$ in (16) subject to constraints (9)–(12). If per capita investment and labor decision functions $[i(x)$ and $n(x)]$ are known, then the optimization problem is well-specified. Furthermore, if per capita investment and labor decisions are linear functions of the aggregate state vector, x, then the problem with (16) replacing (13) is a standard one: the costs are quadratic and the constraints are linear and methods such as Vaughan's (1970) can be applied.

Since per capita investment and labor decisions are known only after the individual decisions have been calculated, one has to either start with a guess for the per capita functions and iterate or compute an equilibrium using the first-order conditions of the household's problem. If one starts with some guess of the per capita investment and labor decisions that are linear functions of the state vector, then the household's optimization problem is linear-quadratic and standard methods apply. However, the resulting investment and labor decisions of the representative household are functions of the guess of per capita functions. To find the functions for per capita investment and labor that result in precisely the same functions for the individual, one can take the individual household's optimal investment and labor decisions as the next guess for per capita functions and continue as before.[3]

The equilibrium decision functions can also be found without iterating. McGrattan (1994) describes a noniterative algorithm that uses the linear-quadratic optimization problem [e.g., the household's problem with the quadratic function (16) replacing (13)] and the side constraints to be imposed in equilibrium as inputs. The side constraints for equilibrium in the model described here are the equation of individual and per capita levels.[4] The method of King, Plosser, and Rebelo (1988) can also be applied. In their case, the first-order conditions of the optimization problem are linearized, with the resulting difference equations solved in a noniterative way.

If the optimal decision functions are linear in the state variables, then the equilibrium law of motion is given by

$$x_{t+1} = A_o x_t + B\epsilon_{t+1}, \qquad \mathrm{E}\epsilon_t\epsilon_t' = I, \qquad BB' = \Sigma, \tag{17}$$

where $x_t = (k_t, h_t, s_{t-1}, \ldots, s_{t-N+1}, v_t, \ldots, v_{t-q})$. The matrix A_o is derived from the constraints of the household [e.g., eqs. (9)–(12)] with the optimal decision functions for investment and hours replacing $i(x)$ and $n(x)$. The vector ϵ_t has four nonzero elements associated with the error terms in the equation describing the exogenous state variables, v_t. Therefore, the nonzero elements of ϵ_t and rows

[3]The algorithm can be summarized as follows. Choose linear functions $i^0(x)$ and $n^0(x)$ as an initial guess of the equilibrium decision functions. Solve the household's optimization problem with (16) as the quadratic return function. The resulting decision functions for the household, $\hat{i}$ and $\hat{n}$, are linear functions of ($\hat{k}$, $\hat{h}$, $\hat{s}$, and x). Since $\hat{k} = k$, $\hat{h} = h$, $\hat{s} = \underline{s}$ in equilibrium, the individual decisions can be summarized as functions of the aggregate state vector, x. And they provide the next guess of the equilibrium per capita decision functions, $i^1(x)$ and $n^1(x)$.

[4]A technical appendix that describes the method and its application to the model of sections 2 and 3 is available from the author upon request.

of B are $N + 2, \ldots, N + 5$, where N is the number of gestation lags in building capital.

One step remains before comparing the predictions of the model with U.S. time series. To generate simulations of the state vector in eq. (17), it is necessary to first choose a parameterization of utility, production, and policy.

4. Estimation strategy[5]

Previous analyses in the business cycle literature have relied on micro-economic econometric studies to guide their choice of parameters. For example, Prescott (1986, p.14) advocates using information from individual panel data to restrict parameter estimates since he assumes that 'measures obtained from aggregate series and those from individual panel data must be consistent'. If the results of the studies are robust to the parameter choices, then this calibration exercise is valid.

Two sets of results are reported in the next section for postwar data in the United States. The first set is the time series predictions of the model; these results depend crucially on the specification of tax rate processes. The second set is calculations of welfare costs which depend crucially on the choice of utility parameters. Since there is little consensus on *effective* tax rate measures and preference parameters for the representative agent of the economy, there are potential problems with calibrating the model. However, there are econometric procedures that can be applied that circumvent these problems. One such procedure is described in this section.

Suppose that the econometrician has observations on capital stock, hours of leisure or work, investment starts, technology shocks, government consumption, and tax rates. Suppose also that the distribution of ϵ in eq. (17) [or equivalently ε in eq. (8)] is normal. Then the parameters of preferences, technologies, and government policy that are arguments of A_o and Σ can be found by maximizing the Gaussian log-likelihood function.

The entire state vector is not observed for the United States. For example, without parameters for the production function, the technology shock cannot be determined from observations on outputs and inputs of production. Therefore, in addition to (17), it is necessary to specify some relationship between the data that are available and the underlying state vector, x. To (17), add a measurement equation

$$Z_t = Cx_t + e_t, \tag{18}$$

[5]The estimation strategy used here is described in greater detail in Harvey (1981) and Sargent (1989).

which relates observables in period t, Z_t, to the states. The elements of the vector e_t are measurement errors which may be serially correlated, e.g.,

$$e_{t+1} = De_t + \omega_t, \qquad \mathrm{E}\omega_t\omega_t' = \Omega, \qquad \mathrm{E}e_t\omega_t' = 0.$$

The measurement equation can thus be written as follows:

$$z_t = \bar{C}x_t + \bar{e}_t,$$

where $z_t = Z_{t+1} - DZ_t$, $\bar{C} = CA_o - DC$, and $\bar{e}_t = CB\epsilon_{t+1} + \omega_t$.

The matrices A_o, C, and B are nonlinear functions of the parameters of preferences, technology, and fiscal policy. The matrices D and Ω are functions of the measurement error parameters. Define Γ as the vector of parameters to be estimated. If $[\epsilon_{t+1}, \bar{e}_t]$ is a normally distributed white noise process,[6] then the maximum likelihood estimates are obtained by minimizing the following function with respect to the model parameters:[7]

$$\mathscr{L}(\Gamma) = T\{\ln|\Sigma_z| + \mathrm{trace}(\Sigma_z^{-1}S_z)\}, \tag{19}$$

where

$$S_z = \frac{1}{T}\sum_{t=0}^{T-1}(z_t - \hat{z}_t)(z_t - \hat{z}_t)', \qquad \Sigma_z = \mathrm{E}(z_t - \hat{z}_t)(z_t - \hat{z}_t)',$$

$$\hat{z}_t = \mathrm{E}[z_t | z_{t-1}, z_{t-2}, \ldots, z_0, \hat{x}_0],$$

[6]In the definition of equilibrium, investment and hours choices are assumed to be positive. With $[\epsilon_{t+1}, \bar{e}_t]$ normally distributed and linear decision rules for investment and hours as specified in section 3, there is no guarantee that these choices are positive. However, in practice, this is not a problem since means of consumption, investment, output, government purchases, capital stock, and hours are large relative to variances.

[7]Computation of the likelihood function requires the prediction of z given past values ($\hat{z}$) and the covariance matrix, Σ_z. Both are generated from the Kalman filter equations:

$$\hat{x}_{t+1} = A_o\hat{x}_t + Ku_t, \qquad u_t = z_t - \bar{C}\hat{x}_t = z_t - \hat{z}_t,$$

given an initial estimate of the state, $\hat{x}_0$, where

$$\hat{x}_t = \mathrm{E}[x_t | z_{t-1}, z_{t-2}, \ldots, z_0, \hat{x}_0], \quad K = (A_oS\bar{C}' + \Sigma C')\Sigma_z^{-1},$$

$$S = A_oSA_o' + \Sigma - K\Sigma_zK', \quad \Sigma_z = \bar{C}S\bar{C}' + \Omega + C\Sigma C', \quad \Sigma = BB'.$$

The matrices K and S are the Kalman gain and $\mathrm{E}(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$, respectively. A technical appendix with analytical derivatives for the likelihood function is available from the author upon request.

for the sample $\{Z_t, t = 0, \ldots, T]$. The likelihood function is conditioned on an estimate of the initial state vector, $\hat{x}_0$. For the results reported in section 5, the elements of $\hat{x}_0$ are taken to be the steady state values of the corresponding states in x_t.

5. Results

In this section, the methods of sections 3 and 4 are used to study the effects of fiscal disturbances in the postwar United States. The subsections describe the data that are used in the study, the maximum likelihood parameter estimates, the predictions of the model for first and second moments, and the calculations of the welfare costs of factor taxation. The tax policies used in calculating these costs are similar to those used by the United States in the postwar period.

5.1. Data

The data used to estimate the model parameters are per capita aggregate output (i.e., consumption plus investment plus government purchases), investment, government purchases, hours of work, capital stock, tax rate on labor, and tax rate on capital for the United States over the sample 1947:1–1987:4. Definitions and sources of these series are given in the data appendix. Private consumption and transfer payments are not included in this list of observables, since they are redundant. Starts and weighted past hours are assumed to be latent. Therefore, the vector of observables is given by $Z_t = [y_t^*/\mu^t, i_t^*/\mu^t, g_t^*/\mu^t, k_t^*/\mu^t, n_t, \tau_t, \varphi_t]'$, where μ is the geometric rate of growth to be estimated.[8] The data are detrended by μ to make the elements of Z stationary. As shown by Dhrymes (1970), the likelihood function of eq. (19), $\mathscr{L}(\Gamma)$, must be adjusted by adding $\sum_{t=1}^{T} \ln(\mu^{-4t})$ if the data are detrended by μ.

The series of tax rates are constructed using the method of Joines (1981).[9] What is needed are measures of the effective economy-wide marginal tax rates on capital and labor. By Joines' definition, the marginal tax rate for a factor is the sum of a proportional tax on all income, a proportional tax on the factor-specific income, and a nonproportional tax on factor-specific income. The proportional tax on all income is total receipts from proportional taxes which are neither capital- nor labor-specific expressed as a ratio to total income. Taxes used to construct the proportional tax rate on all income include federal, state,

[8]Because output is a nonlinear function of the state x_t, a linear expansion of this function around the steady state is used for the first row of C in eq. (18).

[9]Barro and Sahasakul (1986), Seater (1985), and others have suggested alternative measures of φ_t. Judd (1989) and others have constructed alternative measures of τ_t. The definition of Joines is chosen because it treats labor and capital income in a comparable way.

and local personal and indirect business taxes. The proportional factor-specific tax is the ratio of total receipts from factor-specific income to total income for that factor. Capital-specific proportional taxes include corporate profits taxes, both federal and state, and state personal and indirect business taxes. Capital income includes dividends, interest, rents, royalties, estates, and trusts. Labor income is primarily wages and salaries. The nonproportional tax is the weighted sum of nonproportional tax rates across income groups. The nonproportional tax rate for income group i is assumed to be the difference between the per capita tax liability from bracket $i-1$ to i divided by the difference between the per capita total income from bracket $i-1$ to i. The federal personal income tax is used for the nonproportional tax on both capital and labor. The tax rate on labor differs from capital because Social Security taxes are included as well.

As many have argued [including Joines (1981)], these tax rate measurements are highly suspect. Unresolved questions about the correct treatment of inflation and capital gains, about allocating income to labor or to capital, about the handling of deductions, exemptions, or credits, and about the timing of taxes and income remain. Given the estimation procedure outlined in section 4, however, measurement errors can be handled. For example, if the true tax rate on capital is that constructed by Judd (1989) and if all other series are functions of this process, it will be possible to recover the rate by applying the methods described in section 4.

5.2. Parameter estimates

In table 1, the functional forms for production and utility are displayed in the second column. As in Kydland and Prescott (1982), the number of gestation lags is assumed to be four ($N=4$). Parameter estimates of the production function (θ), depreciation (δ), growth (μ), time-to-build (ϕ_j, $j=1,2,3$), and the utility function (a_0, η, β, γ, α, π) are reported in the third column of table 1. The discount factor, β, is equal to $\beta^* \mu^{\gamma\alpha}$, where β^* is the discount factor of the original problem with growth described in section 2. Total hours of work per quarter, H, is set at 1304.

The estimate of capital's share in the production function, θ, is 0.397 with a standard error of 0.136. This point estimate is higher than estimates typically found for models with taxes absent. But if the tax and no-tax models are to match the same observed capital–output ratios, the no-tax model will over-predict labor's share. To see this, consider constructing a crude estimate of θ as follows. First, assume that the values of β, δ, μ, and ϕ_j, all j, are set equal to the estimates reported in table 1. With the exception of the time-to-build parameters, these parameter estimates are not likely to be affected by including a public sector in the model. For the nonstochastic version of the model, the household's first-order conditions imply that $(1-\bar{\tau})\theta\bar{y}/\bar{k}+\delta\bar{\tau}$ is approximately equal to 0.036. This follows from eq. (15). Using approximate sample averages of

Table 1
Functions and parameter estimates for preferences and technologies.[a]

Description	Function	Parameter estimates
Production function	$f(k_t, n_t) = k_t^{\theta} n_t^{1-\theta}$	$\theta = 0.397$ (0.136)
Capital accumulation	$k_t = \dfrac{1-\delta}{\mu} k_{t-1} + \dfrac{1}{\mu^4} s_{t-4}$	$\delta = 0.0226$, $\mu = 1.0053$ (0.00166) (0.00035)
Investment function	$i_t = \sum_{j=0}^{3} \dfrac{\phi_{4-j}}{\mu^j} s_{t-j}$	$\phi_1 = 0.475$, $\phi_2 = 0.0435$, $\phi_3 = 0.069$ (0.0117) (0.0130) (0.001)
Utility function	$u(c_t, l_t) = (c_t^{\gamma} l_t^{1-\gamma})^{\alpha}/\alpha$	$\gamma = 0.253$, $\alpha = -0.241$ (0.050) (1.23)
Consumption	$c_t = c_{p,t} + \pi g_t$	$\pi = -0.026$ (0.126)
Leisure	$l_t = a(L) l_t$, $a_j = (1-\eta)^{j-1} a_1$	$a_0 = 1$, η arbitrary
Discount factor	β	$\beta = 0.9927$ (0.0075)

[a]c_p is private consumption, g is government consumption, l is hours of leisure, k is the capital stock, n is hours of work, s is investment starts, and i is investment. Standard errors are in parentheses.

output and capital equal to 2600 and 21000, respectively, θ is approximately equal to 0.4 if $\bar{\tau} = 0.5$ and 0.29 if $\bar{\tau} = 0$. Altug (1989), who estimates a version of Kydland and Prescott's (1982) model, found an estimate of 0.3 for this parameter. The estimate obtained here is closer to that of Braun (1994) who estimates a similar model via generalized method of moments and finds capital's share to be 0.45.

The value of depreciation, δ, is 0.0226 with a small standard error. In a model without the time-to-build assumption on capital, this value is pinned down by observations on capital and investment. In the present model, the investment starts enter the capital equation but the steady state of starts is approximately equal to the steady state of investment, the difference being attributable to positive growth rates. Thus, there should be consensus across models for the rate of capital depreciation in the range of the estimate found here.

The estimates for the time-to-build parameters (ϕ_j, $j = 1, 2, 3$) suggest that Kydland and Prescott's choice of equal weights ($\phi_j = 1/N$, $\forall j$) can be rejected. The estimates of table 1 suggest that almost half of the resources (47.5%) are expended in the beginning period of the project and very little is done in the second and third stage. Using data from the U.S. Census on nonresidential construction completions, Taylor (1982) also finds little evidence for equally weighted fractions.

The growth rate, μ, is estimated to be 1.0053 with a standard error of 0.00035. The quarterly growth rate of 0.53% per quarter falls in the range of mean growth rates for output, investment, capital, and government expenditures. The assumption of balanced growth imposes this growth rate on all of the observables except hours of work and tax rates, both of which are assumed to be stationary.

The parameters of the utility function (a_0, η, β, γ, α, π) are also given in table 1. A constrained maximum is attained with a_0 set equal to its upper bound of 1. A value of 1 for a_0 implies that utility is time-separable. With a_0 equal to 1, the value of η can be set arbitrarily. These estimates differ from Altug (1989) and Eichenbaum, Hansen, and Singleton (1988) who find evidence for both n_t and n_{t-1} in the period t utility function.

The value of π which governs the effect of government on the household's utility is -0.026. For $\pi < 0$, the marginal utility of the household is increased with an increase in government expenditures. However, given a standard error of 0.126, the specification of utility that includes only private goods cannot be rejected.

The point estimate for γ implies that the share of consumption in utility is 0.253 with a standard error of 0.05. To interpret this estimate, we follow Eichenbaum, Hansen, and Singleton (1988) and construct a crude estimate of γ as follows. First, note that the first-order conditions of the household's nonlinear problem [eqs. (14) and (15)] imply the following marginal conditions: $u_1(c_t, l_t) = u_2(c_t, l_t)w_t(1 - \varphi_t)$. Using the utility function of table 1, solving for γ, and substituting in steady state values gives $\gamma \approx c/(lw(1-\varphi)+c)$. Using approximate sample averages of consumption, hours, output, and labor tax rates equal to 1400, 300, 2600, and 0.2, respectively, the estimate for γ is 0.26. With $\varphi = 0$, this crude estimate becomes 0.22, which is less than Kydland and Prescott's estimate of 1/3.

The value of α is -0.241 with a standard error of 1.23. Thus, logarithmic preferences ($\alpha = 0$) cannot be rejected. When Eichenbaum, Hansen, and Singleton (1988) use tax-adjusted wage series, they also find little evidence against the hypothesis that preferences are logarithmically separable.

While the standard error on the discount factor, β, is 0.0075, the point estimate of 0.9927 is economically meaningful. In many studies, an estimate for β greater than 1 has been found. Altug (1989) fixes β during estimation to avoid such problems.

The second set of estimates required for simulation of the model is (b_0, $b(L)$, b_ε) for the autoregressive process of eq. (8). Like the parameters in table 1 which govern the household's willingness to substitute between consumption and leisure and capital and labor, choices of the autoregressive parameters have a large impact on the model's prediction of cyclical variation. And the parameters of the autoregressive process, more than those of utility and production, play a key role in determining how much of the variation in output and employment is explained by innovations in technology and fiscal variables. For example, if the tax rates are constant, then all of the variation in the system would be explained by technology shocks and government consumption and the impact of technological shocks would likely be reduced.

In table 2, estimates of the autoregressive process on exogenous states with $q = 2$ are reported. Note that the predicted tax rate on capital is not a white noise process as argued by Judd (1989). In fact, because of the persistence of the observed series, some of the eigenvalues for the autoregressive process are near 1. For stationarity, the eigenvalues of A_o are required to be less than 1 in absolute value. Since the labor tax (table A.1 in the data appendix) has a positive trend over the sample, a constrained maximum is attained with one of the eigenvalues of A_o set equal to its upper bound (which is set at 0.995).

As a check on some of the results, the estimation procedure was repeated with a new labor tax rate series. The new series is obtained by detrending the labor tax series of table A.1. The detrending method used is described in Prescott (1986). Significant differences exist between the estimate of the growth rate, μ, and the vector autoregressive estimates, $(b_0, b(L), b_\varepsilon)$, for the two sets of data.

Table 2

Estimated vector autoregression for exogenous state variables.[a]

v_{t+1} = (technology shock, govt. consumption, capital tax rate, labor tax rate)$'_{t+1}$

$$= \begin{pmatrix} \begin{matrix} 1.09 \\ (0.0816) \end{matrix} & \begin{matrix} 0.00012 \\ (0.000118) \end{matrix} & \begin{matrix} -0.107 \\ (0.163) \end{matrix} & \begin{matrix} 0.893 \\ (0.722) \end{matrix} \\ \begin{matrix} -204 \\ (164) \end{matrix} & \begin{matrix} 1.79 \\ (0.093) \end{matrix} & \begin{matrix} -354 \\ (238) \end{matrix} & \begin{matrix} 1305 \\ (567) \end{matrix} \\ \begin{matrix} -0.118 \\ (0.0616) \end{matrix} & \begin{matrix} 3.75\times10^{-5} \\ (4.49\times10^{-5}) \end{matrix} & \begin{matrix} 1.57 \\ (0.0897) \end{matrix} & \begin{matrix} 0.253 \\ (0.206) \end{matrix} \\ \begin{matrix} -0.00231 \\ (0.0127) \end{matrix} & \begin{matrix} -3.1\times10^{-5} \\ (1.25\times10^{-5}) \end{matrix} & \begin{matrix} 0.0521 \\ (0.025) \end{matrix} & \begin{matrix} 1.76 \\ (0.1) \end{matrix} \end{pmatrix} v_t + \begin{pmatrix} \begin{matrix} -0.143 \\ (0.0927) \end{matrix} & \begin{matrix} -8.74\times10^{-5} \\ (0.000111) \end{matrix} & \begin{matrix} 0.102 \\ (0.16) \end{matrix} & \begin{matrix} -0.938 \\ (0.751) \end{matrix} \\ \begin{matrix} 186 \\ (158) \end{matrix} & \begin{matrix} -0.804 \\ (0.0898) \end{matrix} & \begin{matrix} 322 \\ (211) \end{matrix} & \begin{matrix} -1386 \\ (591) \end{matrix} \\ \begin{matrix} 0.139 \\ (0.0726) \end{matrix} & \begin{matrix} -3.61\times10^{-5} \\ (4.41\times10^{-5}) \end{matrix} & \begin{matrix} -0.678 \\ (0.0838) \end{matrix} & \begin{matrix} -0.317 \\ (0.219) \end{matrix} \\ \begin{matrix} 0.002 \\ (0.0135) \end{matrix} & \begin{matrix} 2.3\times10^{-5} \\ (1.23\times10^{-5}) \end{matrix} & \begin{matrix} -0.0472 \\ (0.0245) \end{matrix} & \begin{matrix} -0.775 \\ (0.0999) \end{matrix} \end{pmatrix} v_{t-1}$$

$$+ \begin{pmatrix} \begin{matrix} 0.0133 \\ (0.00679) \end{matrix} & 0 & 0 & 0 \\ \begin{matrix} 7.84 \\ (1.22) \end{matrix} & \begin{matrix} 5.56 \\ (0.903) \end{matrix} & 0 & 0 \\ \begin{matrix} 0.000842 \\ (0.000475) \end{matrix} & \begin{matrix} -0.000147 \\ (0.000845) \end{matrix} & \begin{matrix} 0.00411 \\ (0.000482) \end{matrix} & 0 \\ \begin{matrix} 0.000286 \\ (0.000208) \end{matrix} & \begin{matrix} 0.000175 \\ (0.000347) \end{matrix} & \begin{matrix} 0.000209 \\ (0.000169) \end{matrix} & \begin{matrix} 0.00141 \\ (0.000253) \end{matrix} \end{pmatrix} \varepsilon_t + \begin{pmatrix} \begin{matrix} 0.0805 \\ (0.0394) \end{matrix} \\ \begin{matrix} 70.99 \\ (33) \end{matrix} \\ \begin{matrix} 0.0337 \\ (0.0192) \end{matrix} \\ \begin{matrix} 0.00617 \\ (0.00776) \end{matrix} \end{pmatrix}, \quad \mathrm{E}\varepsilon_t = 0, \quad \mathrm{E}\varepsilon_t\varepsilon_t' = I$$

[a]Standard errors are in parentheses.

For example, with the filtered data, the maximum likelihood estimate for μ is 1.0046 with standard error 0.00033. With the unfiltered data, the estimate is 1.0053 with standard error 0.00035. However, the other parameters listed in table 1 are not changed significantly when applying the data set with filtered labor tax rates.

The estimates of tables 1 and 2 can be used to compute the equilibrium decision functions for investment and hours of work. In this case, they are

$$\begin{aligned} i_t = & -904 + 0.00093\,k_t + 0.010\,s_{t-1} - 0.0318\,s_{t-2} + 0.119\,s_{t-3} \\ & + 1190\,\lambda_t + 21.4\,\lambda_{t-1} - 1.06\,g_t + 0.366\,g_{t-1} + 13.1\,\tau_t \\ & - 206\,\tau_{t-1} + 1093\,\varphi_t - 766\,\varphi_{t-1}, \\ n_t = & \; 296 - 0.00279\,k_t + 0.00124\,s_{t-1} - 0.00394\,s_{t-2} + 0.0147\,s_{t-3} \\ & + 53.5\,\lambda_t + 2.65\,\lambda_{t-1} - 0.0451\,g_t + 0.0453\,g_{t-1} + 1.62\,\tau_t \\ & - 25.5\,\tau_{t-1} - 76.6\,\varphi_t - 12.8\,\varphi_{t-1}. \end{aligned}$$

The variable h, which is a weighted sum of past hours, does not appear because utility is a function of current hours of work only ($a_0 = 1$). The effect of technology on labor and investment is, not surprisingly, positive. The labor tax has a negative effect on labor but initially a positive effect on investment. From the size of the coefficients, it is clear that the labor tax has a greater impact on the decisions than the capital tax.

The estimates for the measurement error processes are given in table 3. The measurement errors are assumed to be independent. Therefore, D and Ω are diagonal matrices. It was assumed at first that only the tax rates and hours series are measured with error. The inclusion of measurement error on hours of work

Table 3

Measurement error parameter estimates.[a]

$Z_t = Cx_t + e_t,\quad Z_t = [y_t, i_t, g_t, k_t, n_t, \tau_t, \varphi_t]',\quad e_t = De_{t-1} + \omega_t,\quad E\omega_t\omega_t' = \Omega$

$D_{ii} = (0,$	0.969,	0.953,	0.930,	0.994,	0.866,	0.980),	$D_{ij} = 0,$	$i \neq j$
	(0.0251)	(0.0264)	(0.0321)	(0.0149)	(0.0767)	(0.0308)		
$\Omega_{ii} = (0,$	109,	91.8,	1551,	6.44,	$1e-9$,	$9.9e-8$),	$\Omega_{ij} = 0,$	$i \neq j$
	(27.3)	(17.3)	(22.6)	(0.603)	($2.3e-6$)	($4.5e-7$)		

[a] y is output, i is investment, g is government consumption, k is the capital stock, n is hours of work, τ is the capital tax rate, φ is the labor tax rate, x is the vector of states, Z is the vector of observables, e is the vector of measurement errors, and the subscript t indicates the date. Standard errors are in parentheses.

is motivated by the work of Christiano and Eichenbaum (1992) who find a significant difference in their model's empirical performance if they assume that the measures of output and hours are misaligned. However, if the measurement errors on the remaining parameters are set equal to zero, then the predicted innovations in capital and investment are highly serially correlated. Furthermore, including measurement errors on capital, investment, and government consumption significantly improves the likelihood value. Therefore, all variables except output are assumed to be measured with error.

To understand the predictions of the model without measurement errors, it is useful to compare moments for the data to moments of the predicted series. Recall from section 4 that the data are $\{Z_t\}_{t=0}^{T}$. The model's prediction of these series is $CE[x_t | Z_T, \ldots, Z_0, \hat{x}_0]$, when conditioning on the entire sample. In table 4, means and standard deviations for the observed and the predicted series are reported for the sample 1947:1–1987:4. Because the measurement error on output is zero, the model comes very close to predicting correctly its mean and standard deviation. In fact, the first moments predicted by the model match up very well to their data counterparts. The main differences are in the second moments. The variances of both investment and hours are overpredicted, while the variance of government spending is underpredicted. Kydland and Prescott (1982), who compare data and model series after removing low frequencies, find that the model's prediction of the variability of hours is low relative to the data. Here, it is high in part because the estimation procedure attempts to fit all frequencies in the data including the low frequencies. The predicted standard deviation for the tax on capital is close to that of the data. For the tax on labor, the differences are due primarily to the positive trend in this series.

Table 4

Means and standard deviations of predicted and U.S. time series.

	Predicted[a]		United States[b]	
Series	Mean	Std. dev.	Mean	Std. dev.
Output, y	2600	208	2590	210
Investment, i	603	67.2	591	42.1
Govt. consumption, g	609	87.1	598	117
Capital stock, k	21500	813	21400	831
Hours of work, n	297	10.2	301	9.28
Capital tax rate, τ	0.505	0.0382	0.507	0.0388
Labor tax rate, φ	0.240	0.0200	0.229	0.0277

[a]The prediction at date t is $CE[x_t | Z_T, \ldots, Z_0, \hat{x}_0]$ where x_t is the state vector, $(Z_t, t = 0, \ldots, T)$ is the sample of observations, $Z_t = Cx_t + e_t$, and e_t is a vector of measurement errors.

[b]U.S. output, investment, government consumption, and capital in period t have been divided by $\hat{\mu}^t$, where $\hat{\mu} = 1.0053$.

In table 5, specification tests for the innovations are reported. In the first column, the Shapiro and Wilk (1965) test statistic (W) is reported. The null hypothesis being tested is that the innovation (corresponding to the series listed on the left) is normally distributed. Small values of W ($0 < W \leq 1$) lead to rejection of the null. The probability value for the test statistic is given in parentheses. Note that for a critical value of 0.1, the null hypothesis is rejected for innovations in g, τ, and φ. In the third column of table 5, the least-squares estimate and t-statistic are reported for regressions of each innovation (or element of u) on its own lag. Because there is evidence for serial correlation (given a high value of the t-statistic), the hypothesis that the innovations are white noise can be rejected for all but maybe output and the labor tax.

The result that some of the innovations are nonnormal or serially correlated may be because the autoregressive process cannot be estimated in an unconstrained way without assuming nonstationarities. When the estimation procedure is repeated with the filtered labor tax rate series, there is an improvement in the specification test statistics, with the Shapiro–Wilk statistic larger for most series and the serial correlation of the innovations reduced. However, the improvements do not lead me to reject the hypothesis that innovations in the fiscal variables are nonnormal or that innovations in investment and capital are serially correlated. These results indicate that there may be omitted variables which have richer dynamics than is assumed by the first-order serially correlated measurement errors.

5.3. Time series implications

Following Sims (1980), I can use the vector autoregressive representation in (17) and the relationship between states and observables to determine how much

Table 5

Specification tests for innovations.

	Normality		Serial correlation[a]	
Innovations in	Shapiro–Wilk statistic, W	p-value	Least-squares coefficient	t-statistic
Output, y	0.974	(0.119)	0.103	(1.31)
Investment, i	0.982	(0.496)	0.169	(2.17)
Govt. consumption, g	0.936	(0.0001)	0.258	(3.38)
Capital stock, k	0.985	(0.734)	0.708	(12.72)
Hours of work, n	0.978	(0.254)	0.347	(4.71)
Capital tax rate, τ	0.947	(0.0001)	0.255	(3.38)
Labor tax rate, φ	0.913	(0.0001)	0.113	(1.47)

[a]The coefficients are computed by projecting each innovation on its own lag.

of the variance in the aggregate time series can be explained by innovations in technology, government consumption, and tax rates.

First consider the total variance in state variables and observables. The total variance of the state vector x follows directly from (17) and satisfies $\mathrm{var}(x) = A_o \mathrm{var}(x) A_o' + \Sigma$. If measurement errors are not introduced, then the variance in the observed series is simply $\mathrm{var}(Z) = C\mathrm{var}(x)C'$. Since there are four sources of variability of Z, we can ask how much of the variation in x or in Z is due to each source. For example, how much of the variance in output is due to innovations in technology?

Before computing the decomposition of variances, one must decide how to treat the covariation in any two innovations. For the current study, the choice is an easy one. To make the case that fiscal disturbances have an important effect on real aggregate activity, it is best to attribute to technology shocks as much variation as possible. The results reported here assume that technology shocks are ordered first in the vector x, followed by government consumption, the tax on labor income, and lastly the tax on capital. This ordering and a Cholesky factorization of Σ into BB', where B is lower triangular, implies that an impulse to innovations in technology affect not only technology but also government consumption and tax rates. To see this, consider the system in eq. (17) where B has been found by factoring the variance–covariance matrix, Σ. If B is lower triangular, a change in the ith element of ϵ_t affects all equations below and including the ith.

The fraction of variance attributable to the ith disturbance is readily computed from eq. (17). It is found by first solving the equation $V_i = A_o V_i A_o' + B e_{ii} B'$ for V_i, where e_{ii} is a matrix of zeros with a 1 in the (i,i) element. If the elements of V_i are divided by corresponding elements of $\mathrm{var}(x)$, then the (j,k) element of the resulting matrix is the fraction of the covariance between the jth and kth state variables that is due to the ith disturbance in ϵ. To derive the fractions for the vector Z, divide CV_iC' by $C\mathrm{var}(x)C'$.

In table 6, the variances of output, investment, government consumption, capital, hours of work, consumption, and wages are decomposed into the fractions that are explained by technology shocks, government consumption, the tax rate on labor, and the tax rate on capital. The first line of the table can be read as follows: 41% of the variance of output (y) is explained by innovations in technology (λ), 28% of the variance of y is explained by innovations in government consumption (g), 27% of the variance of y is explained by innovations in the labor tax rate (φ), and 4% of the variance of y is explained by innovations in the capital tax rate (τ). The standard errors are computed by assuming a normal approximation to the nonlinear function mapping the parameters in section 5.2 to the variance decomposition. As Runkle (1987) has noted, the confidence intervals of the variance decomposition can be large. The results still suggest, however, that introducing fiscal variables greatly decreases the percentage of variance accounted for by technical change. Note, for example, the large and

Table 6

Variance decomposition.[a]

Series	% variance explained by innovations in			
	Technology shocks	Government consumption	Labor tax rates	Capital tax rates
Output, y	41 (46)	28 (25)	27 (33)	4 (7)
Consumption, c	39 (47)	29 (27)	28 (35)	4 (7)
Investment, i	57 (21)	27 (11)	14 (15)	2 (3)
Govt. consumption, g	19 (21)	44 (15)	32 (23)	5 (7)
Capital stock, k	60 (42)	21 (21)	17 (27)	2 (4)
Hours of work, n	20 (26)	42 (21)	32 (33)	6 (8)
Productivity, y/n	57 (47)	20 (24)	20 (29)	3 (5)
Capital tax rate, τ	16 (16)	30 (22)	30 (29)	24 (18)
Labor tax rate, φ	3 (6)	41 (28)	49 (32)	7 (10)

[a]The ordering of variables in the state vector of eq. (17) puts technology shocks first, government consumption second, labor tax rates third, and capital tax rates fourth. A Cholesky factorization is used to factor the variance–covariance matrix. Standard errors are in parentheses.

significant effect that fiscal variables have on fluctuations in hours of work. Only 20% of the variation is due to technical change. The results of table 6 seems to show that technical change has a large effect on investment and capital, but even in this case the percentage is only 60%.

In fig. 1, percentages of forecast variances of output that are explained by the four innovations are shown for intermediate forecast horizons. Specifically, the graph shows the fractions of $\mathrm{E}\{y_t - \mathrm{E}[y_t|x_0]\}\{y_t - \mathrm{E}[y_t|x_0]\}'$ explained by each innovation, where y_t is output at horizon t and x_0 is the initial state vector. Note that in the limit the decomposition of variance is that displayed in the first row of table 6. However, this figure shows that, for short horizons, innovations in the technology shock have a large effect on output. This result is due in part to ordering technology first in the system of eq. (17). If government consumption is first, then technology has a much smaller role. But for most permutations, innovations in the labor tax rates do not have an immediate effect even though they explain much of the total variation. Fig. 2 displays percentages of forecast variances of hours worked that are explained by the four innovations. Although the technology shock has a smaller effect in the case of hours, it still takes a long time for the fiscal shocks to have an effect. Thus, while fluctuations in tax rates are important when calculating total variation in postwar samples, they are less important when considering predicted variation over shorter horizons.

5.4. *Welfare implications*

In a version of the stochastic growth model in which agents have perfect foresight, Judd (1987) calculates the marginal efficiency cost of capital and labor taxes for a wide range of taste parameters. He finds that the results are very

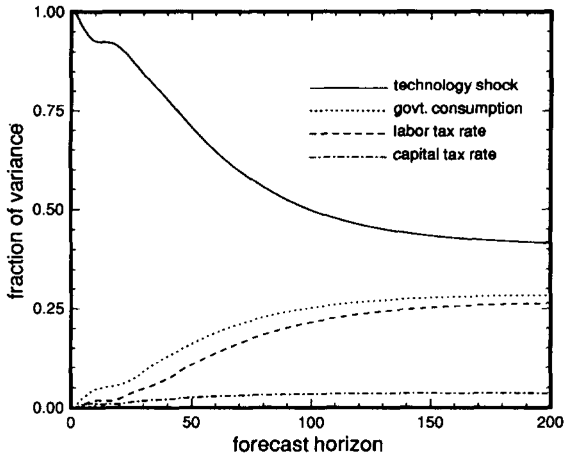

Fig. 1. Fractions of forecast variance of output explained by innovations in technology, government consumption, and tax rates.

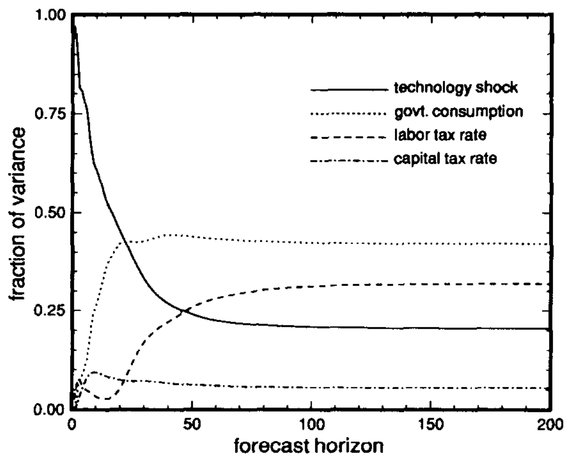

Fig. 2. Fractions of forecast variance of hours explained by innovations in technology, government consumption, and tax rates.

sensitive to the choice of parameterization. In this section, the maximum likelihood estimates of section 5.2 are used to reassess the welfare cost of factor taxation. Two types of calculations are made. First, I calculate the ratio of the change in expected utility to the change in expected revenues, where changes in utility and revenues result from a permanent increase in one of the tax rates. This measure is what Judd (1987) calls *marginal deadweight loss.* The main difference between Judd's exercise and that which is conducted here is our assumptions about the tax rate processes. In Judd's case, the tax rates are constant. Here, the tax rates are modeled as stochastic processes which are not independent. A shock to either tax rate implies changes in the other rate, in government consumption, and in technology. The second welfare measure is the compensating variation in consumption required to leave households indifferent between two economies with different levels of taxation. Compensating variation is only computed for the system in the steady state so that I can consider the effects of changing one tax rate and leaving all other things equal.

To calculate the marginal deadweight loss of a tax change, I need to add a tax shock, τ^s, to one of the tax rate equations; that is, eq. (8) is replaced by

$$\begin{bmatrix} v_t \\ \tau_t^s \end{bmatrix} = \begin{bmatrix} b_0 \\ 0 \end{bmatrix} + \begin{bmatrix} b(L) & e_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{t-1} \\ \tau_{t-1}^s \end{bmatrix} + \begin{bmatrix} b_\varepsilon \varepsilon_t \\ 0 \end{bmatrix}, \qquad v_t = [\lambda_t, g_t, \tau_t, \varphi_t]', \tag{19}$$

where e_i is a vector of zeros with 1 in the ith element and i is either 3 or 4 depending on which tax rate (or, more precisely, which equation) is being changed. The tax shock must also be added to the state vector, x_t. Given these changes, the equilibrium can be computed as before. (See section 3.)

Consider changing the labor tax rate process. Then, the tax shock, τ_t^s, appears in the labor tax equation and e_i in (19) is a vector of zeros with a 1 in the fourth element. If the equilibrium is calculated as described in section 3, one by-product of the calculation is the value function, $v(\hat{k}, \hat{h}, \underline{s}, x)$, where the vector x now includes the tax shock. Assuming that the approximation method of section 3 is applied, the value function is quadratic and easily computed. The computation involves substituting the linear decision functions for investment and labor ($\hat{i}$ and $\hat{n}$) into (16), discounting, and summing over time. The derivative of the (quadratic) value function with respect to τ_0^s, assuming that the value function is evaluated at the initial states, is the estimate of the change in expected utility with a change in the labor tax rate. To express the change in utility in terms of the consumption good at date 0, the derivative of the value function should be divided by p_0^0.

The second calculation needed for the marginal deadweight loss is the change in revenues. The revenues are given by

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t [p_t^0(\tau_t r(x_t) k_t + \varphi_t w(x_t) n(x_t) - \delta \tau_t k_t)] = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathscr{R}(x_t),$$

where $p_t^0 = u_c(c_t + \pi g_t, H - a_0 n_t - \eta(1 - a_0) h_t)$ and $\mathrm{E}_0[\beta^t p_t^0 c_t]$ is the date 0 value of date t consumption.[10] As in the case of the utility function, some

[10] The pricing kernal p_t^0 is related to Arrow–Debreu prices as follows. Let ρ_t^0 be the time 0 state-contingent price of consumption at t contingent on the history $\epsilon^t = (\epsilon_1, \epsilon_2, \ldots, \epsilon_t)$ and x_0 and let $f_t(\epsilon^t)$ be the probability of observing history ϵ^t. Then $p_t^0(\epsilon^t, x_0) = \rho_t^0(\epsilon^t, x_0)/(\beta^t f_t(\epsilon^t))$.

approximation is necessary to compute the present value of revenues. Assume that $\mathscr{R}(x_t) \simeq x_t' M x_t$ for some matrix M. If $R(x_0) = \mathrm{E}_0 \sum_t \beta^t x_t' M x_t$ is the present value of revenues, then

$$R(x_0) = x_0' S x_0 + \beta \operatorname{trace}(S\Sigma)/(1-\beta),$$

$$\frac{\partial R(x_0)}{\partial x_{i0}} = eSx_0, \quad S = \beta A_o' S A_o + M,$$

where e is a vector of zeros with a 1 in the ith element and x_{i0} is the ith element of the initial state vector, x_0. The derivative of $R(x_0)$ with respect to the tax shock, τ_0^s, is the estimate of the change in expected revenues with a change in the tax rate. If revenues are divided by p_0^0, then the present value is in terms of the date 0 consumption good.

The ratio of the change in expected utility to the change in expected revenues is Judd's (1987) measure of deadweight loss. Using the parameter estimates of section 5.2, the marginal deadweight losses due to capital and labor are equal to -1.34 and -1.09, respectively.[11] Thus, a permanent increase in the capital tax rate leads to higher welfare costs than a permanent increase in the labor tax rate. A welfare cost (loss) of 1.34 for the capital tax rate easily falls in the range of estimates of Judd (1987), who finds his estimate to be very sensitive to taste parameters. For $\tau_t = 0.3$, $\varphi_t = 0.3$, Judd's estimates of the loss are in the range of -0.15 to -1.31. For $\tau_t = 0.5$, $\varphi_t = 0.4$, the estimates are between -0.38 and -22.13 (with one estimate positive because revenues fall). The welfare cost of 1.09 for the labor tax is high relative to Judd's estimates.[12] Judd finds estimates of marginal deadweight loss above -0.5 for cases with $\tau_t = 0.3$, $\varphi_t = 0.3$, $t \geq 0$ (and alternative taste parameters) and at or above -1.19 for $\tau_t = 0.5$, $\varphi_t = 0.4$, $t \geq 0$. Note though that a change in the labor tax rate can lead to changes in the remaining elements of v_t if $b(L)$ has nonzero off-diagonal elements. In the case of the parameter estimates of table 2, $b(L)$ does have significant off-diagonal terms. Thus, the costs reported here are the result not of changing one tax rate but rather of changing the tax rate process.

To treat changes in tax rates in isolation, consider the following alternative measure of welfare cost. Define δ_c as the solution to

$$u(\bar{c}(1+\delta_c), \bar{l}) = u(\tilde{c}, \tilde{l}),$$

[11]The present value of government consumption is computed to be 84900, while the present value of distortionary revenues is 83000. The difference is financed by lump-sum taxes.

[12]The welfare costs found here are also high relative to other estimates in the literature. See Judd (1987) for a survey of the results of alternative studies.

where the utility function u is defined in table 1, and $(\bar{c}, \bar{l})$ and $(\tilde{c}, \tilde{l})$ are the steady state levels of consumption and leisure for two different parameterizations of the model. The difference is the choice of the steady state levels of tax rates. For $(\bar{c}, \bar{l})$, I use the parameters in tables 1 and 2 to determine the steady state values. The steady state levels of capital and labor satisfy eqs. (14) and (15), and consumption and leisure are functions of capital and labor. I use exactly the same parameterization for $(\tilde{c}, \tilde{l})$, with the exception of the tax rates. That is, for $(\tilde{c}, \tilde{l})$, I use the parameters of table 1, the values of $\bar{\lambda}$ and $\bar{g}$ used to derive $(\bar{c}, \bar{l})$, and alternative tax rates $(\tilde{\tau}, \tilde{\varphi})$.

The parameterization of tables 1 and 2 imply that the steady state capital and labor tax rates are equal to 0.50 and 0.22. Consider lowering the capital tax rate by 10%, from 0.5 to 0.4. The compensating variation in consumption, δ_c, required to leave households indifferent between the economy with a 50% capital tax rate and a 40% tax rate is 5.5%. The lower tax rate implies a reduction of 5% in distortionary revenues. If the tax rate on labor is lowered from 0.22 to 0.12, then households in the high tax rate economy would require a 3.7% increase in consumption. In this case, the lower tax rate results in a 19% reduction of distortionary revenues. Suppose we lower the labor tax rate so that we find the same distortionary revenues as in the exercise with $(\tilde{\tau}, \tilde{\varphi}) = (0.4, 0.22)$. To do this, the labor tax rate must be reduced by 3%, from 0.22 to 0.19. Households with tax rates (0.5, 0.22) would have to be compensated by a 1.3% increase in consumption to be indifferent. In both cases, the costs of capital taxes exceed those of labor taxes.

If factor taxes are eliminated and replaced by lump-sum taxes (i.e., $\tilde{\tau} = \tilde{\varphi} = 0$), then $\delta_c = 23\%$. This estimate is high relative to those of Cooley and Hansen (1989) who compute the compensating variations in δ_c to evaluate economies with different levels of inflation. They find, for example, that $\delta_c = 0.52\%$ when comparing allocations for an economy with a 10% inflation to the Pareto optimum. The estimate of 23% comes closer to that of Lucas (1987) who finds that households would have to be compensated by an increase in 20% of consumption to be indifferent between a 3% and 2% growth rate.

6. Conclusions

The effects of fiscal variables on aggregate activity have been explored here. Specifically, the effects of government consumption and distorting factor taxes have been considered and contrasted with the effects of technological change. The starting point of the study was the work of Kydland and Prescott (1982) and others who argue that technology shocks are the major determinants of business cycles. While calculations for the model with only technological change do imply a significant role for technology, it is not clear how the results are affected by omitting disturbances such as monetary and fiscal shocks.

The results found here seem to dispute Prescott's (1986) conclusion that technology shocks account for 75% of business cycle fluctuations. I computed the fraction of variance of aggregate time series that results from variation in technological change, government consumption, the tax rate on labor, and the tax rate on capital. Even when the covariation between technology shocks and the fiscal disturbances is attributed to technology, technology shocks account for only 41% of fluctuations in output and only 20% of fluctuations in hours of work. Technology shocks are important, however, for conditional variances of output and investment over short horizons.

A second set of results are calculations of welfare costs due to permanent changes in the tax rates on capital and labor. I found, as many others have found, that the tax on capital is more costly than the tax on labor. And elimination of distortionary taxation would yield benefits comparable to increasing the growth rate of the economy by 1%.

The purpose of this paper has been to quantitatively assess the effect that fiscal variables have on cyclical variation and welfare. In a broader sense, the methods described in this paper can be applied to formulate and estimate a variety of recursive equilibrium models with externalities and distortions, and the present paper serves partly to illustrate the feasibility of using these methods to study artificial economies with relatively large state spaces.

Data appendix

The data used in this study are real aggregate data of the United States for the sample 1947:1–1987:4. All annual series (i.e., capital and tax rates) are log-linearly interpolated to obtain quarterly observations. The final numbers were obtained by dividing the series listed by the population series given below.

(i) $c^*_{p,t}$: personal consumption expenditures of nondurable goods and services (source: *National Income and Product Accounts*, table 1.2, or Citibase variables GCN82, GCS82).

(ii) i^*_t: private fixed investment plus personal consumption expenditures of durable goods (source: *National Income and Product Accounts*, table 1.2, or Citibase variables GIF82, GCD82).

(iii) g^*_t: government purchases of goods and services (source: *National Income and Product Accounts*, table 1.2, or Citibase variable GGE82).

(iv) y^*_t: $c^*_{p,t} + i^*_t + g^*_t$.

(v) n_t: total man-hours employed per week (source: U.S. Department of Labor, BLS, *The Employment Situation–Household Survey*, or Citibase variable LHOURS).

Table A.1

Effective marginal tax rates on capital (τ) and labor (φ), 1947–1987.

t	τ_t	φ_t	t	τ_t	φ_t	t	τ_t	φ_t
1947	0.572	0.199	1961	0.533	0.219	1975	0.508	0.248
1948	0.506	0.185	1962	0.504	0.219	1976	0.518	0.248
1949	0.483	0.171	1963	0.507	0.223	1977	0.502	0.255
1950	0.582	0.182	1964	0.495	0.211	1978	0.486	0.254
1951	0.603	0.216	1965	0.485	0.204	1979	0.478	0.261
1952	0.567	0.207	1966	0.485	0.209	1980	0.480	0.277
1953	0.564	0.209	1967	0.494	0.212	1981	0.450	0.283
1954	0.528	0.193	1968	0.540	0.229	1982	0.440	0.273
1955	0.527	0.195	1969	0.544	0.240	1983	0.428	0.266
1956	0.538	0.201	1970	0.505	0.245	1984	0.422	0.260
1957	0.527	0.208	1971	0.524	0.232	1985	0.429	0.260
1958	0.523	0.206	1972	0.523	0.235	1986	0.464	0.258
1959	0.527	0.212	1973	0.520	0.239	1987	0.467	0.256
1960	0.525	0.215	1974	0.532	0.250			

(vi) k_t^*: constant-dollar net stock of fixed private capital plus net stock of durable goods (source: John Musgrave, 'Fixed Reproducible Tangible Wealth in the U.S.: Revised Estimates', *Survey of Current Business*, January 1986, table 4, row 1, and table 20, column 1 (with updates in *Survey of Current Business*, August 1987 and October 1988).

(vii) *Population measure*: civilian noninstitutional population, 16 years and older (source: U.S. Department of Labor, BLS, *The Employment Situation*, or Citibase variable P16).

(viii) τ_t, φ_t: *effective marginal tax rates on capital and labor* (annual) in table A.1 (sources: *Statistics of Income, Individual Income Tax Returns* (sources of income and taxable income, all returns), *Social Security Bulletin* (tables 2a, 4b); rates constructed using definitions of Joines (1981), series MTRK1, MTRL1; see section 5.1 for a description of these measures).

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