Lecture IX

Linear-Quadratic Approximations

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LQ approximation methods

- The LQ method locally approximates the period utility function around the steady state using a quadratic function.
- If u is (approximated as) quadratic, the value function is quadratic, too. The decision rules are linear in the state variables.
- LQ approximation and the method of linearizing FOCs around the steady state are equivalent. In both methods, the optimal decision rules are going to be linear in state variables.
- Since both methods rely on the local approximation, both methods are valid only locally around the steady state of the model.
- The method is easy to implement where the welfare theorems hold and can solve the Social Planner's problem. For solving equilibrium of wider class of economies with distortions, linearization methods are more intuitive.

Stochastic growth model with leisure

$$V(z,k) = \max_{c,k',h} u(c, 1-h) + \beta \sum_{z' \in Z} \pi(z, z') V(k', z')$$

where:

$$\begin{aligned} u(c, 1-h) &= \frac{\left[c^{\theta}(1-h)^{1-\theta}\right]^{1-\gamma}}{1-\gamma} \\ y &= e^{z} k^{\alpha} h^{1-\alpha} \\ y &= c + k' - (1-\delta) k \end{aligned}$$

z follows an AR(1) with mean zero

Steps for LQ Approximation

1. Solve for the steady state
2. Identify the endogenous and exogenous states, and the choice variables
3. Redefine the utility as a function of (endogenous and exogenous) state variables and choice variables.
4. Then, approximate the utility function around the steady state, using a 2nd order Taylor approximation.
5. Use value function iteration to find the optimal value function.

Step 1: Solve for SS $\left(\bar{k}, \bar{c}, \bar{h}\right)$

1. Euler equation:

$$1 = \beta\left(1 + \alpha \bar{k}^{\alpha-1}\bar{h}^{1-\alpha} - \delta\right)$$

2. Intratemporal FOC:

$$\theta\left(1-\bar{h}\right)\cdot(1-\alpha)\,\bar{k}^{\alpha}\bar{h}^{-\alpha} \quad = \quad (1-\theta)\,\bar{c}$$

3. From the resource constraint:

$$\bar{c} + \delta\bar{k} = \bar{k}^{\alpha}\bar{h}^{1-\alpha}$$

substituting $\bar{c}$ out using the last two equations:

$$\theta\left(1-\bar{h}\right)\cdot(1-\alpha)\,\bar{k}^{\alpha}\bar{h}^{-\alpha} = (1-\theta)\left(\bar{k}^{\alpha}\bar{h}^{1-\alpha} - \delta\bar{k}\right)$$

2 equations in 2 unknowns $(\bar{k}, \bar{h})$. Use root finding method to find SS

Steps 2/3: Identify variables and redefine u

- Exogenous state variables: z
- Endogenous state variables: k
- (2×1) vector of control variables: $d = (k', h)$.
- Rewrite the utility function as $u(z, k, d)$ by substituting constraints and define $\bar{u} \equiv u\left(\bar{z}, \bar{k}, \bar{d}\right)$
- Define also the (4×1) vector:

$$w = \begin{bmatrix} z \\ k \\ d \end{bmatrix} = \begin{bmatrix} z \\ k \\ k' \\ h \end{bmatrix}$$

Steps 4: Approximate u with quadratic function

- Using a second order approximation of u around the SS:

$$u\left(z,k,d\right)\simeq\bar{u}+\left(w-\bar{w}\right)^{T}\bar{J}+\frac{1}{2}\left(w-\bar{w}\right)^{T}\bar{H}\left(w-\bar{w}\right)$$

where $\bar{J}$ and $\bar{H}$ are Jacobian and Hessian evaluated at $\left(\bar{z},\bar{k},\bar{d}\right)$.

$$\begin{aligned}
u\left(z,k,d\right) &\simeq \bar{u}+\left(w-\bar{w}\right)^{T}\bar{J}+\frac{1}{2}\left(w-\bar{w}\right)^{T}\bar{H}\left(w-\bar{w}\right)\\
&= \bar{u}-\bar{w}^{T}\bar{J}+\frac{1}{2}\bar{w}^{T}\bar{H}\bar{w}+w^{T}\left(\bar{J}-\bar{H}\bar{w}\right)+\frac{1}{2}w\bar{H}w\\
&= \left[\begin{array}{cc}1 & w^{T}\end{array}\right]\left[\begin{array}{cc}\bar{u}-\bar{w}^{T}\bar{J}+\frac{1}{2}\bar{w}^{T}\bar{H}\bar{w} & \frac{1}{2}\left(\bar{J}-\bar{H}\bar{w}\right)^{T}\\ \frac{1}{2}\left(\bar{J}-\bar{H}\bar{w}\right) & \frac{1}{2}\bar{H}\end{array}\right]\left[\begin{array}{c}1\\ w\end{array}\right]\\
&= \left[\begin{array}{cc}1 & w^{T}\end{array}\right]Q\left[\begin{array}{c}1\\ w\end{array}\right]
\end{aligned}$$

A quadratic form, where the matrix of coefficients Q is $\left(5\times5\right)$.

Step 5: VFI

- We can write the VFI at iteration t as:

$$V_{t+1}(z,k) = \max_{d} \left\{ \begin{bmatrix} 1 & w^T \end{bmatrix} Q \begin{bmatrix} 1 \\ w \end{bmatrix} + \beta \sum_{z'} \pi(z,z') V_t(z',k') \right\}$$

- We know the value function has the same quadratic form as u. Define the vector of states:

$$s = \begin{bmatrix} 1 \\ z \\ k \end{bmatrix}$$

- We postulate a quadratic form:

$$V_t(z,k) = s^T P_t s$$

where P is negative semi-definite and symmetric. Need to find P!

Step 5: VFI

- The approximated Bellman equation looks like

$$s^T P_{t+1} s = \max_d \left\{ \begin{bmatrix} 1 & w^T \end{bmatrix} Q \begin{bmatrix} 1 \\ w \end{bmatrix} + \beta \mathbb{E} \left[(s')^T P_t s' \right] \right\}$$

where the law of motion for the state can be written as a function of the vector w as:

$$s' = \begin{bmatrix} 1 \\ z' \\ k' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ k \\ k' \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon' \\ 0 \end{bmatrix}$$

$$s' = B \begin{bmatrix} 1 \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon' \\ 0 \end{bmatrix}$$

Step 5: VFI

- This is useful because we can rewrite:

$$
\begin{aligned}
s^T P_{t+1} s &= \max_d \begin{bmatrix} 1 & w^T \end{bmatrix} Q \begin{bmatrix} 1 \\ w \end{bmatrix} + \beta \begin{bmatrix} 1 & w^T \end{bmatrix} B^T P_t B \begin{bmatrix} 1 \\ w \end{bmatrix} \\
&\quad + \beta \mathbb{E} \begin{bmatrix} 0 & \varepsilon' & 0 \end{bmatrix} P_t \begin{bmatrix} 0 \\ \varepsilon' \\ 0 \end{bmatrix} \\
&= \max_d \left\{ \begin{bmatrix} 1 & w^T \end{bmatrix} Q \begin{bmatrix} 1 \\ w \end{bmatrix} + \beta \begin{bmatrix} 1 & w^T \end{bmatrix} B^T P_t B \begin{bmatrix} 1 \\ w \end{bmatrix} + \beta \sigma_\varepsilon^2 P_{t,22} \right\}
\end{aligned}
$$

- You can see the certainty equivalence property here: the FOC wrt d (a component of w) will not depend on σ_ε

Step 5: VFI

- Collecting terms

$$s^T P_{t+1} s = \max_d \left[\begin{array}{cc} 1 & w^T \end{array}\right] [Q + M_t] \left[\begin{array}{c} 1 \\ w \end{array}\right]$$

where

$$M_t = \beta B^T P_t B + \left[\begin{array}{cccc} \beta\sigma_\varepsilon^2 P_{t,22} & 0 & ... & 0 \\ 0 & ... & & ... \\ ... & & & \\ & & & \\ 0 & ... & & 0 \end{array}\right]$$

Step 5: VFI

- Let's rewrite the Bellman equation as follows:

$$s^T P_{t+1} s \quad = \quad \max_{d} \left[\begin{array}{cc} s^T & d^T \end{array} \right] \left\{ \left[\begin{array}{cc} Q_{ss} & Q_{sd}^T \\ Q_{sd} & Q_{dd} \end{array} \right] + \left[\begin{array}{cc} M_{t,ss} & M_{t,sd}^T \\ M_{t,sd} & M_{t,dd} \end{array} \right] \right\} \left[\begin{array}{c} s \\ d \end{array} \right]$$

- Recall that s is a (3×1) vector of constant plus states and d a (2×1) vector of decisions

- The Q and M_t matrices are (5×5). $Q_{s\cdot}$ and $M_{t,s\cdot}$ are (2×3) and Q_{dd} and $M_{t,dd}$ are (2×2)

- Thus, multiplying through:

$$s^T P_{t+1} s = \max_{d} \; s^T \left(Q_{ss} + M_{t,ss}\right) s + 2d^T \left(Q_{sd} + M_{t,sd}\right) s$$
$$+ \quad d^T \left(Q_{dd} + M_{t,dd}\right) d$$

Step 5: VFI

- It is a concave program, so the FOC is sufficient.

- The FOC wrt d yields:

$$\begin{aligned} 0 &= 2\left(Q_{sd} + M_{t,sd}\right)s + 2\left(Q_{dd} + M_{t,dd}\right)d \\ d &= -\left(Q_{dd} + M_{t,dd}\right)^{-1}\left(Q_{sd} + M_{t,sd}\right)s \\ d &= \Omega_t^T s \end{aligned}$$

where Ω is (3×2) given by:

$$\Omega_t = -\left(Q_{sd}^T + M_{t,sd}^T\right)\left(Q_{dd} + M_{t,dd}\right)^{-1}$$

Step 5: VFI

- If we use the FOC to substitute out d in the Bellman equation:

$$s^T P_{t+1} s = s^T \left(Q_{ss} + M_{t,ss}\right) s + 2 s^T \Omega_t \left(Q_{sd} + M_{t,sd}\right) s + s^T \Omega_t \left(Q_{dd} + M_{t,dd}\right) \Omega_t^T s$$

- Using the expression for Ω_t:

$$s^T P_{t+1} s = s^T \left[Q_{ss} + M_{t,ss} - \left(Q_{sd}^T + M_{t,sd}^T\right) \left(Q_{dd} + M_{t,dd}\right)^{-1} \left(Q_{\tilde{w}d} + M_{t,sd}\right)\right] s$$

- This suggests a recursion for V:

1. Given P_t (needed to construct M_t):

$$P_{t+1} \quad = \quad Q_{ss} + M_{t,ss} - \left(Q_{sd}^T + M_{t,sd}^T\right) \left(Q_{dd} + M_{t,dd}\right)^{-1} \left(Q_{sd} + M_{t,sd}\right)$$

2. Use P_{t+1} to construct M_{t+1} and go back to step 1 til convergence

Summary of LQ-VFI procedure

1. Guess P_0. Since the value function is concave, we guess a negative semidefinite matrix, for example $P_0 = -I$.

2. Given P_t, update the value function using recursion above and obtain P_{t+1}. The Q matrix is defined by approximating the return function and the M_t matrix by the formula we obtained above

3. Compare P_t and P_{t+1}. If the distance (measured in sup norm) is smaller than the predetermined tolerance level, stop. Otherwise go back to the updating step (step 2) with P_{t+1}.

4. With the optimal P^*, we can compute the decision rules. Check solution is correct by evaluating decision rules at SS.

5. Use the decision rules to simulate the economy.