Class notes: Advanced Topics in Macroeconomics

Topic: Finite Element Methods

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In today's class, we first discussed how to expand the analysis of the last class and then worked through test cases for an application to Aiyagari's (1994) model with heterogeneous agents.

First, we discussed higher-order approximations for the basis functions. For example, consider a quadratic approximation of d^n on element e. Let $\bar{x}$ be the local coordinate of the element, which has node 1 located at x_1^e. Then $\bar{x} = x - x_1^e$, where x is the global coordinate. Locally, assume that the nodes are located at $\bar{x} = 0$, $\bar{x} = \alpha\ell_e$, and $\bar{x} = \ell_e$, with $0 < \alpha < 1$. As in the linear case, we can write the approximation on element e as a linear combination of basis functions that have the property that they are equal to 1 at one node and 0 at the other nodes; that is,

$$d_e^n(\bar{x};\theta) = \theta_1^e \psi_1^e(\bar{x}) + \theta_2^e \psi_2^e(\bar{x}) + \theta_3^e \psi_3^e(\bar{x}),$$

where the ψ_i^e's are constructed to satisfy $d_e^n(0) = \theta_1^e$, $d_e^n(\alpha\ell_e) = \theta_2^e$, and $d_e^n(\ell_e) = \theta_3^e$. The only functions that satisfy these conditions are given by

$$\begin{aligned}
\psi_1^e(\bar{x}) &= \left(1 - \frac{\bar{x}}{\ell_e}\right)\left(1 - \frac{\bar{x}}{\alpha\ell_e}\right) \\
\psi_2^e(\bar{x}) &= \frac{\bar{x}}{\alpha(1-\alpha)\ell_e}\left(1 - \frac{\bar{x}}{\ell_e}\right) \\
\psi_3^e(\bar{x}) &= -\frac{\alpha\bar{x}}{(1-\alpha)\ell_e}\left(1 - \frac{\bar{x}}{\alpha\ell_e}\right).
\end{aligned} \tag{1}$$

Notice that there are only three elements of the vector θ used to approximate the function on any particular element.

Second, we considered solving a two-dimensional problem, like the growth model with a continuous stochastic shock. In this case, the representation of the approximate solution is then

$$c^n(k,z;\theta) = \sum_{i=1}^{n} \theta_i \psi_i(k,z).$$

When applying the finite element method, we divide up the domain into smaller nonoverlapping subdomains called elements. If the domain is two-dimensional and rectangular: $\Omega = [0, \bar{k}] \times [-1, 1]$, then a reasonable choice for the element shape, therefore, is also a rectangle. Suppose that we divide the domain into smaller rectangular subdomains which do not overlap. Each element will be a rectangle in Ω, say, $[k_i, k_{i+1}] \times [z_j, z_{j+1}]$, where k_i is the ith grid point for the the capital stock and z_j is the jth grid point for the technology shock.

Consider two types of approximations over the rectangular elements: linear and quadratic. Suppose the representation for consumption on some element e is linear,

$$c_e^n\,(k, z) = a + b\,k + c\,z + d\,kz. \tag{2}$$

Because there are four unknowns, we require an element with four nodes. If we place the four nodes at the corners of the rectange, then we can uniquely define the geometry of the element and use the values of the solution at the four nodes to pin down the constants in equation (2). That is, as in the one-dimensional case, we can rewrite the approximation in (2) so that $c_e^n(k, z; \theta) = \sum_i \theta_i^e \psi_i^e(k, z)$, $i = 1, \ldots, 4$, where the basis functions are such that ψ_i^e is 1 at node i and zero at the other three nodes on the element.

Before giving formulas for the basis functions, it is convenient to first consider a mapping from global coordinates (k, z) to local coordinates (ξ, η) defined on a master element. This is done for convenience, since the master element has a fixed set of coordinates, while each element in Ω has a different set of coordinates. Thus, we can construct basis functions once but use them for each element. Consider functions $\xi(k)$ and $\eta(z)$ that map a typical element $[k_i, k_{i+1}] \times [z_j, z_{j+1}]$ to the square $[-1, 1] \times [-1, 1]$; that is, $\xi(k) = (2k - k_i - k_{i+1})/(k_{i+1} - k_i)$ and $\eta(z) = (2z - z_j - z_{j+1})/(z_{j+1} - z_j)$. Assume that the four nodes of the master element are $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$ using the local coordinates. In this case, the basis functions are constructed so that $c_e^n(\xi, \eta; \theta) = \sum_i \theta_i^e \psi_i^e(\xi, \eta)$ with $\theta_1^e = c_e^n(-1, -1; \theta)$, $\theta_2^e = c_e^n(1, -1; \theta)$, $\theta_3^e = c_e^n(1, 1; \theta)$, and $\theta_4^e = c_e^n(-1, 1; \theta)$. These restrictions imply that

$$c_e^n\,(\xi, \eta; \theta) = \frac{1}{4}\,(1 - \xi)\,(1 - \eta)\;\theta_1^e + \frac{1}{4}\,(1 + \xi)\,(1 - \eta)\;\theta_2^e$$

$$+\frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\,\theta_3^e+\frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\,\theta_4^e. \tag{3}$$

Using these bases, we can proceed as in the one-dimensional case and write out the residuals and solve the nonlinear and sparse system of equations.

Turning next to the Aiyagari (1994) model, we considered two test cases (which were fully described in the technical appendix for Aiyagari and McGrattan (2003) available on my website). In the first test case, we assume that the household solves

$$\max_{\{c_t,a_{t+1}\}}\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)$$
$$\text{subject to}\quad c_t+a_{t+1}=\left(1+r\right)a_t+w.$$

This specification assumes that there is no uncertainty ($e_t = 1$); therefore, wages are constant. The dynamic program for this example involves the following form for Bellman's equation:

$$v\left(x\right)=\max_{0\leq y\leq Rx+w}\{u\left(Rx+w-y\right)+\beta v\left(y\right)\}, \tag{4}$$

where $y_t = x_{t+1}$ is the asset position next period and $R = 1 + r$ is the gross return. A conjectured solution is as follows:

$$\begin{cases} y=0 & \text{if } x\in[0,m_1], \\ y=\frac{-m_1^2}{m_2-m_1}+\frac{m_1}{m_2-m_1}\;x & \text{if } x\in\left(m_1,m_2\right], \\ y=\frac{\left(m_3m_1-m_2^2\right)}{m_3-m_2}+\frac{m_2-m_1}{m_3-m_2}\;x & \text{if } x\in\left(m_2,m_3\right], \\ \vdots \end{cases} \tag{5}$$

where m_j, $j = 1, 2, \ldots$, will be calculated below. Note that the solution assumes that if $x = m_{j+1}$, then $y = m_j$.

The Lagrangian for the maximization in the right-hand side of Eq. (4) is given by

$$L=u\left(xR+w-y\right)+\beta v\left(y\right)+p\left(Rx+w-y\right)+qy,$$

where p and q are multipliers. The first-order conditions for this problem are

$$\begin{aligned} &-u'\left(Rx+w-y\right)+\beta v'\left(y\right)-p+q=0, \\ &p\geq 0,\quad Rx+w-y\geq 0,\quad p\left(Rx+w-y\right)=0, \\ &q\geq 0,\quad y\geq 0,\quad qy=0. \end{aligned} \tag{6}$$

If we assume that the conjecture above is correct, then when $x \in [0, m_1)$ we have the $y \geq 0$ constraint binding. Therefore, if we assume that $Rx + w > 0$, then it must be true that $y < Rx + w$, $p = 0$, and

$$v'(0) = \frac{1}{\beta} u'(Rx + w) - \frac{q}{\beta} \leq \frac{1}{\beta} u'(Rx + w).$$

Furthermore, from Bellman's equation, we get

$$v(x) = u(Rx + w) + \beta v(0), \quad \text{for } x \in (0, m_1) \quad \text{and } v(0) = \frac{\beta}{1-\beta} u(w),$$

and taking derivatives, we get

$$v'(x) = Ru'(Rx + w) < \frac{1}{\beta} u'(Rx + w), \tag{7}$$

since $\beta R < 1$.

Consider next the interval $(m_1, m_2]$. The conjectured solution is such that in this interval, the constraint $y \geq 0$ is not binding. If we assume that $Rx + w > y$, then the first-order conditions imply

$$v'(y) = \frac{1}{\beta} u'(Rx + w - y). \tag{8}$$

If $y = 0$ at $x = m_1$, then

$$v'(0) = \frac{1}{\beta} u'(Rm_1 + w). \tag{9}$$

Using Eq. (7) evaluated at $x = 0$ and Eq. (9), we get

$$u'(Rm_1 + w) = \beta R u'(w),$$

which gives us an equation for m_1. For example, if $u(c) = c^{1-\mu}/(1-\mu)$, then

$$m_1 = \frac{w\left(1 - (\beta R)^{\frac{1}{\mu}}\right)}{(\beta R)^{\frac{1}{\mu}} R},$$

and $y = 0$ in the interval $[0, m_1]$.

Now we want to compute the asset function for the next interval $(m_1, m_2]$. If the conjecture in Eq. (5) is correct, then Eq. (8) holds, as does

$$v'(x) = Ru'(Rx + w - g(x)), \tag{10}$$

which is the derivative of the value function once y is replaced by the optimal policy $y = g(x)$. The conjecture assumes that $y = m_1$ when $x = m_2$, and by Eq. (8) and Eq. (10), we have,

$$u'(Rm_2 + w - m_1) = \beta Ru'(Rm_1 + w).$$

Note that this equation can be solved for m_2. If we follow the same logic for the remaining m's, we find that, in general,

$$u'(Rm_{j+1} + w - m_j) = \beta Ru'(Rm_j + w - m_{j-1}), \quad j = 1, \ldots, \text{ and } m_0 = 0. \tag{11}$$

Thus, given m_1 and m_2, we can compute m_3 and so on.

What we have done is conjectured a solution and derived the functions analytically. It is easy to show that the solution is, in fact, piecewise linear and that the conjecture is correct.

Now we consider the finite element approximation. Let $\beta = 0.95$, $w = 1.0$, $r = 0.02$, $u(c) = c^{1-\mu}/(1-\mu)$, and $\mu = 3$. We can use the formula in Eq. (11) to derive the exact solution. In a series of figures (see the document online), we plot the true solution and the finite element approximations. We first use a grid that can produce an exact match to the true solution. Then we vary the grids and show how the approximation looks when the grids don't match. We also show results with and without the imposition of the boundary condition.

The second test case is relevant for computing the distribution of assets, which has discontinuities throughout because of the Markov chain used for modeling the idiosyncratic shocks. To mimic this, we suppose that the productivities can take on two possible values and the decision functions are given by

$$\alpha(x, i) = \begin{cases} \max(0, -0.25 + x), & \text{if } i = 1 \\ 0.5 + 0.5x, & \text{if } i = 2, \end{cases}$$

with $\pi_{1,1} = \pi_{2,2} = 0.8$. Recall that we want to compute $H(x,i) = Pr(x_t < x \,|\, e_t = e(i))$, which solves:

$$H\left(x,i\right) = \sum_{j=1}^{m} \pi_{j,i} H\left(\alpha^{-1}\left(x,j\right),j\right) I\left(x \geq \alpha\left(0,j\right)\right), \tag{12}$$

where π is the transition matrix for the Markov chain governing earnings and I is an indicator function (i.e., $I(x > y)$ is equal to one if $x > y$ and is equal to zero otherwise).

It is relatively easy to show that the following equations must hold for this simple example:

$$\begin{aligned}
H\left(0,1\right) &= 0.8H\left(0.25,1\right), \\
H\left(0,2\right) &= 0.2H\left(0.25,1\right), \\
H\left(0.25,1\right) &= 0.8H\left(0.5,1\right), \\
H\left(0.25,2\right) &= 0.2H\left(0.5,1\right), \\
H\left(0.5,1\right) &= 0.8H\left(0.75,1\right) + 0.2H\left(0,2\right), \\
H\left(0.5,2\right) &= 0.2H\left(0.75,1\right) + 0.8H\left(0,2\right), \\
H\left(0.75,1\right) &= 0.8H\left(1.0,1\right) + 0.2H\left(0.5,2\right), \\
H\left(0.75,2\right) &= 0.2H\left(1.0,1\right) + 0.8H\left(0.5,2\right), \\
H\left(0.875,1\right) &= 0.8H\left(1.125,1\right) + 0.2H\left(0.75,2\right), \\
H\left(0.875,2\right) &= 0.2H\left(1.125,1\right) + 0.8H\left(0.75,2\right), \\
H\left(1.0,1\right) &= 0.5, \\
H\left(1.0,2\right) &= 0.5.
\end{aligned}$$

If we assume that $H(x,i) = 0.5$ for $x > 1$, then the above expressions can easily be solved. We can first determine $H(0,j)$, $H(0.25,j)$, $H(0.5,j)$, and $H(0.75,j)$ for j =1,2 by solving $Ax = b$, where

$$A = \begin{bmatrix} 1 & -.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -.8 & -.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -.2 & 0 \\ 0 & -.2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -.2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -.2 & -.8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -.8 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .4 \\ 0 \\ 0 \\ 0 \\ .1 \end{bmatrix}.$$

The solution is $H(0,1) = 0.225$, $H(0.25,1) = 0.282$, $H(0.5,1) = 0.352$, $H(0.75,1) = 0.426$, $H(0,2) = 0.056$, $H(0.25,2) = 0.070$, $H(0.5,2) = 0.130$, and $H(0.75,2) = 0.204$. Note that we can back out the other points from these solutions by applying the formula in Eq. (12). Again, the technical appendix online shows comparisons of the exact and approximate solutions for different choices of the grid and the order of the approximation.