Lecture VI

Numerical Integration

Gianluca Violante

New York University

Quantitative Macroeconomics

Reservation wage equation

$$w^* \quad = \quad b + \frac{\lambda}{r+\delta} \int_{w^*}^{\bar{w}} (w - w^*)\, dG\,(w)$$

- How do we solve for w^*? We need a method that computes numerically the integral inside a root finding algorithm

- Three approaches to computing integral:

  1. Newton-Cotes methods: employ piecewise polynomial approximations to the integrand with evenly spaced nodes

  2. Gaussian Quadrature: choose nodes and weights efficiently, i.e. they satisfy some moment-matching conditions

  3. Monte-Carlo methods: use equally weighted random nodes

Newton-Cotes: Trapezoid rule

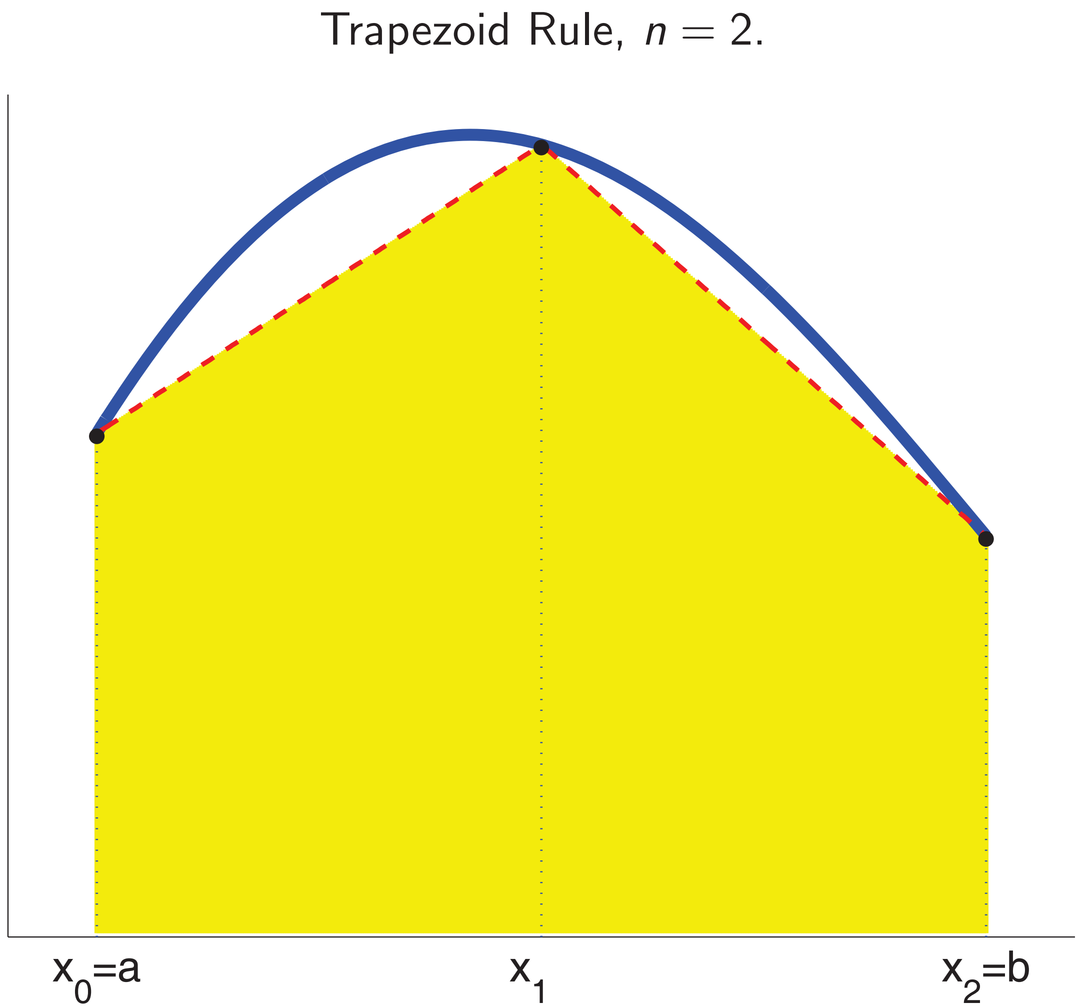
- Approximate f with a piecewise linear polynomial $\tilde{f}$ whose integral is easy to compute:

$$\int_a^b f(x)\,dx \simeq \int_a^b \tilde{f}(x)\,dx$$

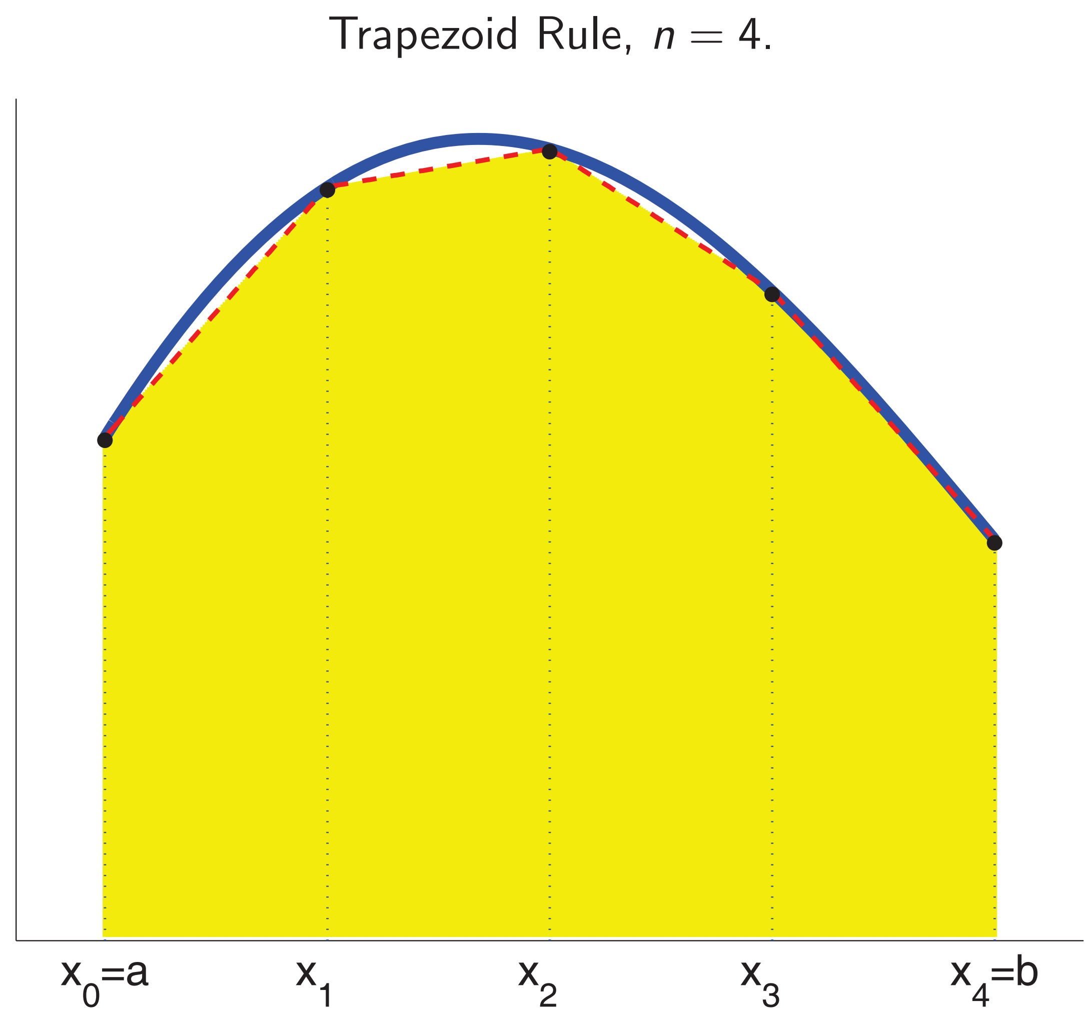
- Partition the integration interval $[a, b]$ into n subintervals of equal length $h = (b - a)/n$ and endpoint nodes $x_i = a + ih$
- Compute the function values $y_i = f(x_i)$ at nodes i.
- Form a piecewise linear approximation of the function between successive points (x_i, x_{i+1})

$$f(x) \simeq f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$$

Example: 2 nodes



Example: 4 nodes



Newton-Cotes: Trapezoid rule

- The area under the piecewise linear appx. for subinterval i is:

$$\int_{x_i}^{x_{i+1}} \tilde{f}\,(x)\,dx \simeq \left[\frac{f\,(x_{i+1}) + f\,(x_i)}{2}\right] \cdot h$$

and hence

$$\int_{x_i}^{x_{i+1}} f\,(x)\,dx \simeq \sum_{i=0}^{N} w_i f\,(x_i)$$

where $w_i = h$ for all i, unless $\{i = 0, N\}$ where $w_i = 1/2$

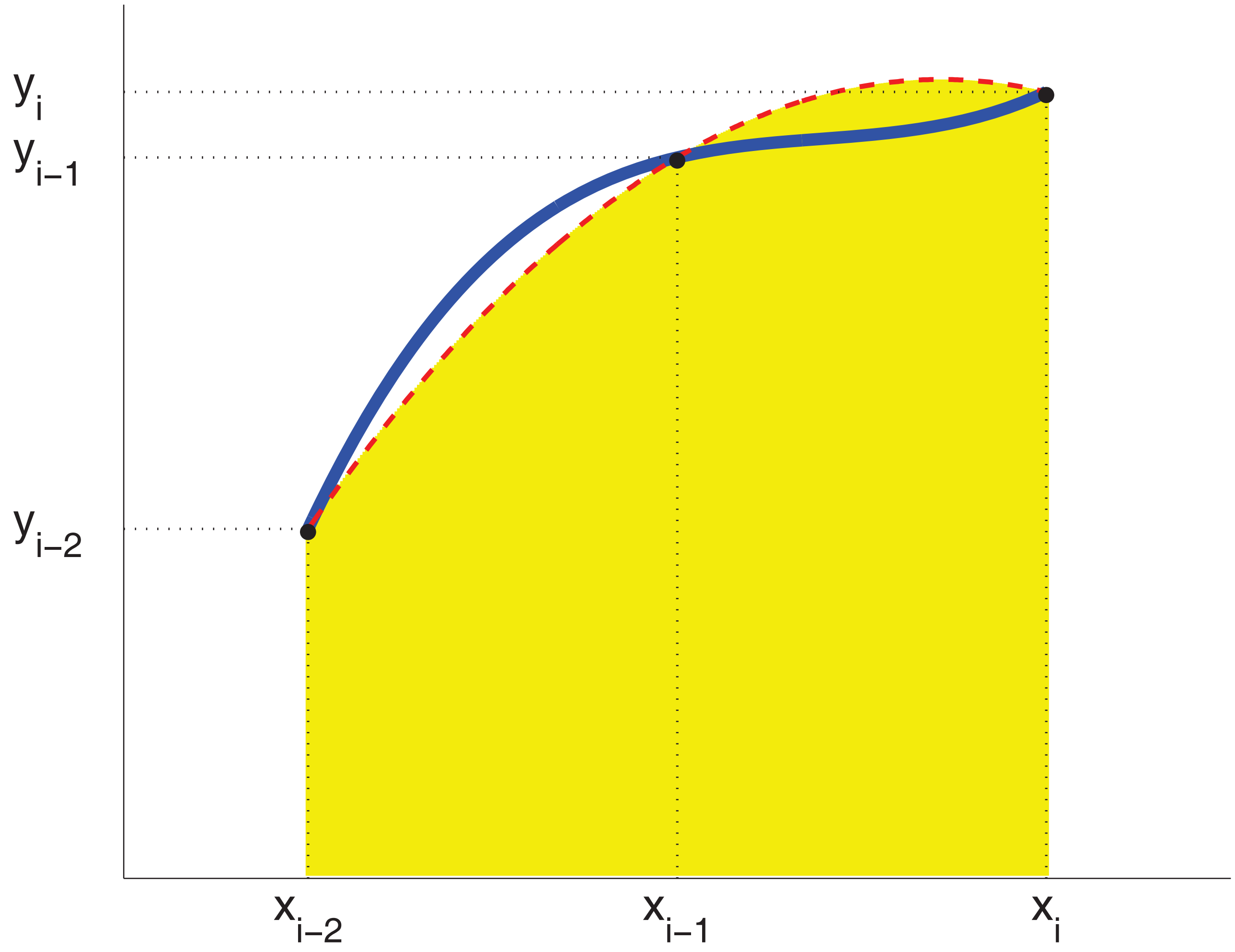
- Simple and robust

- Increasing nodes from N to $M * N$ reduces error by factor of M^2

Newton-Cotes: Simpson rule

- Simpson rule based on quadratic approximation of the function
- Form a piecewise quadratic approximation $\tilde{f}$ that interpolates f at successive triplets of (x_{i-1}, x_i, x_{i+1}) with quadratic functions
- Similar expression as above. Figure out what w_i need to be
- If f is smooth, Simpson rule is preferred because approximation error is square of Trapezoid rule error, hence more accurate
- But if f is nondifferentiable at some points, then trapezoid rule may be better

Example: 2 nodes



Gaussian quadrature

- It builds on idea that (i) a function can be approx. by a polynomial and (ii) weights and quadrature points are chosen efficiently

- Let $\Phi = \{\varphi_k(x)\}_{k=0}^{\infty}$ be a family of polynomials defined on $[a,b]$ with typical element

$$\varphi_k(x) = \sum_{i=0}^{k} \alpha_{k,i} x^i, \text{with} \quad \alpha_{k,k} = 1 \quad \text{(normalization)}$$

- Definition: Φ is orthogonal w.r.t. the weight function $w(x)$ if

$$\langle \varphi_k, \varphi_j \rangle = \int_a^b \varphi_k(x)\, \varphi_j(x)\, w(x)\, dx = 0 \text{ for } k \neq j$$

- Moreover, if $\langle \varphi_k, \varphi_k \rangle = 1$, for all k, then Φ is also said to be orthonormal with respect to $w(x)$

Gaussian quadrature

- Theorem: Suppose that Φ is orthonormal w.r.t. $w(x)$ on $[a, b]$. Let $\{x_i\}_{i=1}^n$ be the zeros of $\varphi_n(x)$. Then $x_i \in [a, b]$. If $f \in C^{(2n)}[a, b]$:

$$\int_a^b f(x)\, w(x)\, dx = \sum_{i=1}^{n} \omega_i f(x_i) + \frac{f(\xi)^{2n}}{(2n)!}$$

  for some $\xi \in [a, b]$. The residual is "small". And the weights are:

$$\omega_i = -\frac{1}{\varphi_n'(x_i)\, \varphi_{n+1}(x_i)} > 0$$

- Corollary: if f is a polynomial of degree $(2n-1)$, then the integration is exact (residual term is zero)
- Theorem gives us zeros of orthogonal polynomials as quadrature nodes and weights (a.k.a. abscissae).
- In practice, nodes/weights tabulated for known polynomial families

Example of exact approximation

- We want to evaluate the definite integral

$$I = \int_{-1}^{1} f\left(x\right) dx, \quad \text{where} \quad f\left(x\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

- Set $n = 2$ since $(2n - 1 = 3)$

- The Gaussian approximation of the integral is:

$$\hat{I} = \sum_{i=1}^{2} \omega_i \left(c_0 + c_1 x_i + c_2 x_i^2 + c_3 x_i^3\right)$$

- Set $I = \hat{I}$

Example of exact approximation

$$
\begin{aligned}
\int_{-1}^{1}\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right) dx &= \sum_{i=1}^{2} \omega_i \left(c_0 + c_1 x_i + c_2 x_i^2 + c_3 x_i^3\right) \\
2c_0 + 0c_1 + \frac{2}{3} c_2 + 0c_3 &= c_0 \left(\omega_1 + \omega_2\right) + c_1 \left(\omega_1 x_1 + \omega_2 x_2\right) + \\
& \quad c_2 \left(\omega_1 x_1^2 + \omega_2 x_2^2\right) + c_3 \left(\omega_1 x_1^3 + \omega_2 x_2^3\right)
\end{aligned}
$$

- The last equation is satisfied if and only if:

$$
\begin{aligned}
2 &= \omega_1 + \omega_2 \\
0 &= \omega_1 x_1 + \omega_2 x_2 \\
\frac{2}{3} &= \omega_1 x_1^2 + \omega_2 x_2^2 \\
0 &= \omega_1 x_1^3 + \omega_2 x_2^3
\end{aligned}
$$

- System of 4 equations into 4 unknowns $(\omega_1, \omega_2, x_1, x_2)$ that gives weights and quadrature points.

Gauss-Chebyshev quadrature on [-1,1]

- Consider an integral of the form:

$$\int_{-1}^{1} f\left(x\right)\left(1-x^{2}\right)^{-1/2} dx$$

- The weighting function is the one that defines Chebyshev polynomials as orthogonal family. Thus, we know that:

$$\int_{-1}^{1} f\left(x\right)\left(1-x^{2}\right)^{-1/2} dx \simeq \sum_{i=1}^{n} \omega_i f\left(x_i\right)$$

$$\omega_i = \frac{\pi}{n}, \quad x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$$

- Convenient because the weight is constant and the abscissas have simple formula

Gauss-Chebyshev quadrature on [a,b]

$$I = \int_a^b f(x)\,dx$$

- Linear change of variable from $x \in (a,b)$ to $y \in (-1,1)$:

$$x = a + \frac{(1+y)(b-a)}{2}$$

- Multiply and divide by $\left(1-y^2\right)^{-1/2}$:

$$\begin{aligned}
I &= \int_{-1}^{1} f\left(a + \frac{(1+y)(b-a)}{2}\right) \frac{\left(1-y^2\right)^{-1/2}}{\left(1-y^2\right)^{-1/2}} \left(\frac{b-a}{2} dy\right) \\
&= \frac{b-a}{2} \int_{-1}^{1} \left[f\left(a + \frac{(1+y)(b-a)}{2}\right) \left(1-y^2\right)^{1/2}\right] \left(1-y^2\right)^{-1/2} dy \\
&\simeq \frac{\pi(b-a)}{2n} \sum_{i=1}^{n} \left[f\left(a + \frac{(1+y_i)(b-a)}{2}\right) \left(1-y_i^2\right)^{1/2}\right]
\end{aligned}$$

Other quadrature methods

- In general, suppose you want to compute an integral of the type

$$\int_a^b f(x)\,\omega(x)\,dx \simeq \sum_{i=1}^{n} \omega_i f(x_i)$$

Range	$\omega(x)$	Polynomial family	Quadrature method
$[-1,1]$	1	Legendre	Gauss-Legendre
$(-1,1)$	$\left(1-x^2\right)^{1/2}$	Chebyshev	Gauss-Chebyshev
$[0,\infty)$	e^{-x}	Laguerre	Gauss-Laguerre
$(-\infty,\infty)$	e^{-x^2}	Hermite	Gauss-Hermite

- ... after an appropriate change of variable

- Nodes and weights already tabulated

Monte-Carlo Integration

- Gaussian approach to multi-dimensional integrals OK up to 3 dimensions: then, it requires too many functional evaluations
- Monte-Carlo useful for multi-dimensional integration
- LLN: if $\{x_i\}$ are iid realizations of a random variable $x \in [0,1]$ and f is continuous, then with probability one:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)\,dx$$

- Draw n realizations of x from $U[0,1]$ and compute the sum:

$$I_n = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

- Integral is a random var. and value depends on realizations

Variance-reduction techniques

- Stratified sampling: divide $[0, 1]$ into two subintervals $[0, \alpha]$ and $[\alpha, 1]$ in order to avoid concentration of draws in one subregion by bad luck and use:

  $$I_n = \frac{\alpha}{n}\sum_{i=1}^{n} f\left(x_{1i}\right) + \frac{(1-\alpha)}{n}\sum_{i=1}^{n} f\left(x_{2i}\right)$$

  so you weight more the larger section

- Antithetic variance: if f is weakly increasing then $f\left(x\right)$ and $f\left(1-x\right)$ are negatively correlated.

  $$I_n = \frac{1}{2n}\sum_{i=1}^{n}\left[f\left(x_i\right) + f\left(1-x_i\right)\right]$$

  produces smaller variance because covar. btw. terms negative