Class notes: Advanced Topics in Macroeconomics

Topic: Three Models and Three Near-Linear Methods

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I. Three Models

Three models are compared and can be shown to be observationally equivalent with respect to consumption, investment, hours, and output. (Write out the necessary conditions and the proof is obvious.) The first one is useful because it can be compared easily to national income and product account (NIPA) data. In class, we will match each variable to the analogue in NIPA Tables 1.1.5 and 1.10 at bea.gov. The second model will be useful later when we introduce tax distortions. The third is useful because the equilibrium is easy to compute and generates the same aggregates as the first model.

Model 1. For now, we'll assume that there is no government spending or taxes. These will be brought in later. We have households and corporations. The households choose paths for per capita consumption c_t and leisure ℓ_t to solve:

$$\max_{\{c_t,\ell_t,s_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t,\ell_t) N_t$$

$$\text{subject to} \sum_{t=0}^{\infty} p_t\{c_t + v_t(s_{t+1}-s_t)\} \leq \sum_{t=0}^{\infty} p_t\{d_t s_t + w_t h_t\}$$

where $N_t = (1+\gamma_n)^t$ is the size of the population, v_t is the price of corporate shares, d_t is the amount of per capita distributions paid per share to the shareholders (i.e., households), w_t is the wage rate paid to labor, $h_t = 1-\ell_t$. The price used when summing expenditures and incomes is p_t, which is the Arrow-Debreu price (and equal to the household marginal utility in equilibrium and the rate at which corporations discount future distributions).

Corporations maximize the present value of aggregate distributions D_t to households:

$$E_0 \sum_{t=0}^{\infty} p_t D_t$$

$$\text{s.t.} \quad K_{t+1} = (1-\delta) K_t + X_t$$

$$D_t = F(K_t, Z_t H_t) - w_t H_t - X_t$$

where, again, p_t, is the discount factor for the corporate shareholders, K_t is the capital stock, X_t is gross investment, H_t is the total labor input, and $Z_t = z_t(1+\gamma_z)^t$ is the technology parameter with

$$\log z_{t+1} = \rho \log z_t + \epsilon_{t+1}$$

and $\epsilon \sim N(0, \sigma^2)$.

If we compare this economy to data in the NIPA accounts, we have two product categories: personal consumption expenditures C_t and gross private domestic investment X_t. (In later classes, we will put part of C_t, namely durables, with X_t and that requires some adjustments to the U.S. accounts.) We have three categories of incomes: compensation $w_t H_t$, corporate profits—which are the sum of net dividends D_t and undistributed profits $K_{t+1} - K_t$—and the consumption of fixed capital δK_t. GDP is the sum of products. GDI is the sum of incomes.

We can also compare this economy to data in the Flow of Funds. Specifically, we can compare the total value (or market capitalization) of the corporations, given here by v_t (if the supply of shares is normalized by 1) to the market value of domestic U.S. corporations. We can also compare this to the end of period reproducible cost (or current cost) of capital K_{t+1} found in the BEA's fixed asset tables, because Tobin's Q in this model is equal to 1 here. We will find that the total market capitalization is much more volatile (at every frequency) than the reproducible cost of capital in the corporate sector which hovers around 1 times GDP. (See McGrattan and Prescott (2005), Figure 1.) An open question is why are they so different; what is driving the price of capital?

Model 2. In this case, assume that househoulds invest in capital and rent it to corporations along with their labor. The problem then is

$$\max_{\{c_t, \ell_t, k_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t) N_t$$
$$\text{subject to} c_t + k_{t+1} - (1-\delta) k_t = r_t k_t + w_t h_t$$

where r_t is the rental rate and w_t is the wage rate.

Corporations rent capital and labor and maximize static profits, namely,

$$\max_{K,H} F(K, ZH) - rK - wH.$$

In equilibrium, the factors are paid their marginal product and thus $r = F_k(K, ZH)$ and $w = ZF_h(K, ZH)$. Note that these rental and wage rates are functions of aggregate capital and labor. Thus, households taking prices as given must form expectations about the evolution of K_t and H_t.

Model 3. If we want to compute an equilibrium for Model 1 or 2, it is much easier to work with the following planner's problem:

$$\begin{aligned}
&\max_{\{c_t,\ell_t,x_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t) N_t \\
&\text{subject to } \; c_t + x_t = F\left(k_t, (1+\gamma_z)^t z_t h_t\right) \\
&\qquad N_{t+1}k_{t+1} = [(1-\delta)k_t + x_t] N_t \\
&\qquad \log z_t = \rho \log z_{t-1} + \epsilon_t, \epsilon \sim N\left(0, \sigma^2\right) \\
&\qquad h_t + \ell_t = 1 \\
&\qquad c_t, x_t \geq 0 \quad \text{in all states}
\end{aligned}$$

and then construct whatever data we want to compare to the U.S. accounts. For example, if we want to compare corporate net dividends from Model 2 (which only has them implicitly), we construct

$$D_t = N_t [F(k_t, Z_t h_t) - w_t h_t - x_t].$$

Next class, we'll discuss some alternative methods for solving the equilibrium decision functions that are *much* faster than iterating on Bellman's equation. Homework 1 will be used to implement these alternative methods.

II. Three Near-Linear Methods

In first-year macro, one of the assigned homeworks requires students to compute equilibria of a simple planner's problem by starting with an initial guess for the value function and iterating on Bellman's equation. Let's call this Method 1. Today, we will contrast this method and the implied equilibria with two alternative methods. The first alternative maps the original nonlinear problem to one with linear constraints and a quadratic objective function. I'll refer to this second approach as the linear-quadratic (LQ) method. Finally, the third method uses a cool trick from Vaughan (1970) who showed that the eigenvalues relevant for the dynamical system come in reciprocal pairs. He then exploited this fact to come up with a very fast way to compute the solution to models with linear constraints and a quadratic objective (or at least those models that are well approximated by such an LQ model).

Method 1.

The starting point is Bellman's equation:

$$V\left(\hat{k}_t, z_t\right) = \max_{\hat{c}_t, h_t, \hat{k}_{t+1}} \left\{U\left(\hat{c}_t, 1-h_t\right) + \beta\left(1+\gamma_n\right) E_t V\left(\hat{k}_{t+1}, z_{t+1}\right)\right\}$$

subject to

$$\begin{aligned}
\hat{c}_t + \hat{x}_t &= \hat{k}_t^{\theta}\left(z_t h_t\right)^{1-\theta} \\
\hat{k}_{t+1} &= \left[\left(1-\delta\right)\hat{k}_t + \hat{x}_t\right] / \left[\left(1+\gamma_z\right)\left(1+\gamma_n\right)\right] \\
\log z_{t+1} &= \rho \log z_t + \epsilon_{t+1},
\end{aligned}$$

where small letters are used to indicate that the variable is per capita and the hat further indicates that it has been divided by the growth in technology (e.g, $k_t = K_t/N_t$, $\hat{k}_t = k_t/(1+\gamma_z)^t$). Here, I am assuming that the choice of utility function is consistent with balanced growth so that I can replace c_t by $\hat{c}_t$ without consequence.

Consider two ways to deal with the expectation operator. The first is to treat z_t as a continuous state variable and use a quadrature method to compute the integral related to the expectation of z next period:

$$\begin{aligned}
E\left[V\left(\hat{k}', z'\right)|z\right] &= \int V\left(\hat{k}', \rho z + \epsilon\right) f\left(\epsilon\right) d\epsilon \\
&\approx \sum_i \omega_i V\left(\hat{k}', \rho z + \epsilon_i\right) f\left(\epsilon_i\right)
\end{aligned}$$

where $f(\cdot)$ is the density function of a normally distributed random variable. The second is to treat z_t as a Markov chain and replace the expectation with a sum:

$$E\left[V\left(\hat{k}', z_j\right)|z_i\right] \approx \sum_j \text{prob}\left(z_j|z_i\right) V\left(\hat{k}', z_j\right)$$

where z_i and z_j are the exogenous states today and tomorrow, respectively.

An easy (but extremely tedious) way to compute an approximate solution for the value function and policy functions is to guess an initial function V (say, a piecewise linear or bi-linear function over a grid for the states), solve the right hand side maximization problem for all possible states $(\hat{k}, z)$—say, by checking all possible triplets of $\hat{c}, h, \hat{k}'$ until a maximum value is found—and then updating the guess for V.

We turn next to the LQ approximation.

Method 2.

Suppose the original maximization problem has the following general form:

$$\begin{aligned} \max_{\{u_t\}_{t=0}^{\infty}} \ & \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \ | X_0\right] \\ \text{subject to} \quad & X_{t+1} = g\left(X_t, u_t, \epsilon_{t+1}\right) \\ & X_0 \text{ given} \end{aligned}$$

where X_t is the vector of states, u_t is the vector of controls (e.g., decisions and prices), r is the objective function which is known, g governs the evolution of the state vector and is also known, ϵ are shocks affecting this evolution which we'll assume to be iid.

The first step is to map the original problem into the following related problem:

$$\begin{aligned} \max_{\{u_t\}_{t=0}^{\infty}} \ & \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left(X_t' Q X_t + u_t' R u_t + 2X_t' W u_t\right) \\ \text{subject to} \quad & X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1} \\ & X_0 \text{ given} \end{aligned} \tag{1}$$

where

$$\begin{aligned} r\left(X_t, u_t\right) &\simeq X_t' Q X_t + u_t' R u_t + 2X_t' W u_t \\ g\left(X_t, u_t, \epsilon_{t+1}\right) &\simeq AX_t + Bu_t + C\epsilon_{t+1}, \end{aligned} \tag{2}$$

with Q and R symmetric. That is, we solve a problem with a quadratic objective function and linear constraints. Note that implicit in our formulation of (1) are the assumptions that X_t is contained in the agents' information sets at time t and that the agents know the objective function and transition functions for all variables.

To obtain the functions in (2), we take a second and first-order Taylor expansion of the corresponding nonlinear functions around the steady state of the system. Thus, when evaluated at the stationary point, the original and approximated functions have the same value.

To find the steady state of the system, we first set the disturbance term ϵ_t to its unconditional mean. Without loss of generality, assume the mean is zero. We then find the first order conditions of the resulting nonstochastic version of the model:

$$\begin{aligned} &\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \\ &\text{subject to} \quad X_{t+1} = g\left(X_t, u_t, 0\right) \end{aligned} \tag{3}$$

and X_0 given. Formulating the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{r\left(X_t, u_t\right) - \lambda'_{t+1}\left(X_{t+1} - g\left(X_t, u_t, 0\right)\right)\} \tag{4}$$

and taking derivatives with respect to u_t and X_{t+1}, we obtain the following first-order conditions

$$\begin{aligned} \frac{\partial r\left(X_t, u_t\right)}{\partial u_t} + \frac{\partial g\left(X_t, u_t, 0\right)'}{\partial u_t} \lambda_{t+1} &= 0 \\ \beta \frac{\partial r\left(X_{t+1}, u_{t+1}\right)}{\partial X_{t+1}} - \lambda_{t+1} + \beta \frac{\partial g\left(X_{t+1}, u_{t+1}, 0\right)'}{\partial X_{t+1}} \lambda_{t+2} &= 0 \end{aligned} \tag{5}$$

for $t \geq 0$, where $\{\lambda_t\}$ is a sequence of Lagrange multipliers. Eliminating time subscripts from (5) and the constraint in (3), we then get the following set of nonlinear equations:

$$\begin{aligned} \frac{\partial r\left(X, u\right)}{\partial u} + \frac{\partial g\left(X, u, 0\right)'}{\partial u} \lambda &= 0 \\ \beta \frac{\partial r\left(X, u\right)}{\partial X} - \lambda + \beta \frac{\partial g\left(X, u, 0\right)'}{\partial X} \lambda &= 0 \\ X - g\left(X, u, 0\right) &= 0 \end{aligned} \tag{6}$$

This is a set of $2m+n$ equations with $2m+n$ unknowns, X, u, λ. The fixed point of this system is the steady state, say $\bar{X}, \bar{u}, \bar{\lambda}$, around which we take first and second-order Taylor expansions of g and r. Thus, we have the problem given by (1).

Thus far, we have derived the first order conditions for the original nonlinear problem that imply a set of equations for finding the steady state (or more precisely, the balanced growth path). We take a second order Taylor expansion of the objective function ($r(X,u)$) around the steady state to get matrices Q, R, and W. We take a first-order Taylor expansion of the constraints ($g(X,u,\epsilon)$) around the steady state to get A, B, C.

Next, we need to put some conditions on these matrices to ensure that the optimal solution to our problem yields a stable system (and that we are maximizing, not minimizing). The relevant conditions are usually stated in terms of a problem with $\beta = 1$ and $W = 0$. We can reformulate our problem so that there is no discounting or cross-products as follows. Let

$$\begin{aligned}
\tilde{X}_t &= \beta^{\frac{t}{2}} X_t \\
\tilde{u}_t &= \beta^{\frac{t}{2}} \left(u_t + R^{-1} W' X_t\right) \\
\tilde{A} &= \sqrt{\beta} \left(A - B R^{-1} W'\right) \\
\tilde{B} &= \sqrt{\beta} B \\
\tilde{Q} &= Q - W R^{-1} W'.
\end{aligned}$$

Assume that $\tilde{Q}$ and R are negative definite matrices (which is an assumption that can be weakened) and assume that there exists a matrix $\tilde{F}$ such that $\tilde{A} - \tilde{B}\tilde{F}$ has eigenvalues inside the unit circle. In this case, the system is stable and, in the language of control theorists, $(\tilde{A}, \tilde{B})$ is stabilizable. The matrix $\tilde{F}$ that is relevant for us is the matrix governing the optimal solution, namely, $\tilde{u}_t = -\tilde{F}\tilde{X}_t$.

If the conditions above are satisfied, then the optimal policy function for the original optimization problem is the time-invariant linear rule:

$$\begin{aligned}
u_t = -F X_t, \qquad F &= \left(R + \beta B' P B\right)^{-1} \left(\beta B' P A + W'\right) \\
&= \left(R + \tilde{B}' P \tilde{B}\right)^{-1} \tilde{B}' P \tilde{A} + R^{-1} W' \\
&\tilde{F} + R^{-1} W'.
\end{aligned} \tag{7}$$

The matrix P in (7) is the steady-state solution to the matrix Riccati difference equation

$$\begin{aligned} P_t &= Q + \beta A' P_{t+1} A - \left(\beta A' P_{t+1} B + W\right)\left(R + \beta B' P_{t+1} B\right)^{-1}\left(\beta B' P_{t+1} A + W'\right) \\ &= \tilde{Q} + \tilde{A}' P_{t+1} \tilde{A} - \tilde{A}' P_{t+1} \tilde{B}\left(R + \tilde{B}' P_{t+1} \tilde{B}\right)^{-1} \tilde{B}' P_{t+1} \tilde{A} \end{aligned} \tag{8}$$

as $t \to -\infty$, with terminal condition $P_T \leq 0$.

There have been many algorithms developed for the solution of the discrete-time Riccati equation. In all cases, we take as given the matrices A, B, Q, R, W and scalar β (or equivalently $\tilde{A}$, $\tilde{B}$, $\tilde{Q}$, and R), tolerance criteria γ_1 and γ_2, and a matrix norm $\|\cdot\|$. The simplest method is simply direct iteration. To do this, set an initial symmetric Riccati matrix, $P^0 \leq 0$.

a) At iteration n, we compute P^{n+1} and $\tilde{F}^n$ to be

$$\begin{aligned} P^{n+1} &= \tilde{Q} + \tilde{A}' P^n \tilde{A} - \tilde{A}' P^n \tilde{B}\left(R + \tilde{B}' P^n \tilde{B}\right)^{-1} \tilde{B}' P^n \tilde{A} \\ \tilde{F}^n &= \left(R + \tilde{B}' P^n \tilde{B}\right)^{-1} \tilde{B}' P^n \tilde{A} \end{aligned}$$

b) If $\|P^{n+1} - P^n\| < \gamma_1 \|P^n\|$ and $\|\tilde{F}^{n+1} - \tilde{F}^n\| < \gamma_2 \|\tilde{F}^n\|$, go to (c); otherwise, increase n by one and return to (a).

c) Set $F = \tilde{F}^n + R^{-1} W'$, $P = P^n$.

With a steady-state solution to the Riccati matrix, we can use (7) to compute F and the law of motion for the state variables:

$$X_{t+1} = \left(A - BF\right) X_t + C\epsilon_{t+1} \tag{9}$$

Furthermore, given an initial condition for the states, X_0, and a realization of the shocks, $\epsilon_t,\ t \geq 0$, we can generate time-series for X_t via (9) and u_t via (7).

We're ready to apply the algorithm to the planner's problem. We can write the original problem as follows:

$$\begin{aligned} &\max_{\{\hat{c}_t, \hat{k}_{t+1}, h_t\}} E \sum_{t=0}^{\infty} \hat{\beta}^t \left[\hat{c}_t \left(1 - h_t\right)^{\psi}\right]^{1-\sigma} \\ \text{subj. to}\quad & \hat{c}_t + \left(1 + \gamma_z\right)\left(1 + \gamma_n\right)\hat{k}_{t+1} - \left(1 - \delta\right)\hat{k}_t = \hat{k}_t^{\theta}\left(z_t h_t\right)^{1-\theta} \\ & \log z_{t+1} = \rho \log z_t + \epsilon_{t+1} \\ & \hat{k}_0, z_0 \text{ given} \end{aligned}$$

where notice that all variables have been transformed so that they are stationary.

We first want to substitute out $\hat{c}_t$ in the objective function so that we have all of the nonlinear terms in the function to be approximated by a quadratic. In other words, we have

$$\max_{\{\hat{k}_{t+1},h_t\}} E \sum_{t=0}^{\infty} \hat{\beta}^t \left[\hat{k}_t^\theta \left(e^{\zeta_t} h_t\right)^{1-\theta} - (1+\gamma_z)(1+\gamma_n)\hat{k}_{t+1} + (1-\delta)\hat{k}_t\right]^{1-\sigma} (1-h_t)^{\psi(1-\sigma)}$$

$$\begin{aligned} \text{subject to} \quad & \zeta_{t+1} = \rho\zeta_t + \epsilon_{t+1} \\ & \hat{k}_0, \zeta_0 \text{ given} \end{aligned}$$

If we want a linear approximation, then we can define the state vector as follows: $X_t = [\hat{k}_t, \zeta_t, 1]'$ and the control vector as follows: $u_t = [\hat{k}_{t+1}, h_t]$. The third element of the state vector is added because the steady states are in general nonzero and a constant term is needed in the decision functions. We can use a second-order Taylor expansion of the objective function to derive the Q, R, and W (which are 3×3, 2×2, and 3×2 respectively). The matrices A, B, and C are as follows:

$$\underbrace{\begin{bmatrix} \hat{k}_{t+1} \\ \zeta_{t+1} \\ 1 \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{k}_t \\ \zeta_t \\ 1 \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \hat{k}_{t+1} \\ h_t \end{bmatrix}}_{u_t} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{C} \epsilon_{t+1}$$

If we want a log-linear approximation, then we can define the state vector as follows: $X_t = [\log \hat{k}_t, \zeta_t, 1]'$ and the control vector as follows: $u_t = [\log \hat{k}_{t+1}, h_t]$. In this case, we need to replace $\hat{k}_t$ with $\exp(\log \hat{k}_t)$ in the objective function, but the matrices A, B, and C stay the same.

Method 3.

Next, we use the insights of Vaughan (1970) to To avoid slow iteration on the Riccati equation, Vaughan (1970) exploits certain properties of the first-order conditions of the LQ problem defined above. (See his paper which is posted on the website.) Vaughan assumes no discounting or cross-product terms, so we'll map the variables and coefficients to $\tilde{X}$, $\tilde{u}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{Q}$ as shown earlier. Also note that because the solution does not depend on the variances and covariances of ϵ, we can abstract from the uncertainty for now. Writing out the Lagrangian, we have

$$\mathcal{L} = \sum_{t=0}^{\infty} \{ \tilde{X}_t' \tilde{Q} \tilde{X}_t + \tilde{u}_t' R \tilde{u}_t - \lambda_{t+1}' \left(X_{t+1} - \tilde{A} \tilde{X}_t - \tilde{B} \tilde{u}_t \right) \} \tag{10}$$

Taking derivatives with respect to $\tilde{u}_t$, $\tilde{X}_{t+1}$, and λ_{t+1}, we obtain the following first-order conditions

$$\begin{aligned} 2R\tilde{u}_t + B'\lambda_{t+1} &= 0 \\ \tilde{Q}\tilde{X}_{t+1} - \lambda_{t+1} + \tilde{A}'\lambda_{t+2} &= 0 \\ \tilde{X}_{t+1} - \tilde{A}\tilde{X}_t - \tilde{B}\tilde{u}_t &= 0 \end{aligned} \tag{11}$$

for $t \geq 0$, where $\{\lambda_t\}$ is a sequence of Lagrange multipliers. Eliminating $\tilde{u}_t$ and letting $\tilde{\lambda}_t = 1/2\lambda_t$, we have:

$$\begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' \\ \tilde{Q}\tilde{A}^{-1} & \tilde{Q}\tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' + \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}$ be the coefficient matrix on the right hand side. Vaughan showed that this matrix can be decomposed and used directly to obtain the Riccati matrix P (and hence the solution to the LQ problem); that is, he showed that

$$\mathcal{H} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1},$$

where the eigenvalues of Λ are outside of the unit circle. Notice that the eigenvalues come in reciprocal pairs. This is an important property that implies a unique stable solution, one that satisfies the transversality condition and ensures a bounded return.

Using the fact that the Lagrange multiplier is the derivative of the value function ($\tilde{\lambda}_t = P\tilde{X}_t$), it is easy to figure out how to set P so as to get a stationary dynamical system for X. Let $W = V^{-1}$. In this case, it is easy to show that:

$$\tilde{X}_{t+1} = \{ V_{11}\Lambda^{-1} \left(W_{11} + W_{12}P \right) + V_{12}\Lambda \left(W_{21} + W_{22}P \right) \} \tilde{X}_t.$$

Since Λ has roots outside the unit circle, it must be the case that $P = -W_{22}^{-1}W_{21}$. Note that since $W = V^{-1}$, this is equivalent to setting $P = V_{21}V_{11}^{-1}$.

In the case that $\tilde{A}$ is not invertible, we can modify the method slightly and use generalized eigenvalues with the following alternative system:

$$\begin{bmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} I & \tilde{B}R^{-1}\tilde{B}' \\ 0 & \tilde{A}' \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix}.$$

Let $\mathcal{H}_1$ be the coefficient matrix for the state and costate in $t+1$, and let $\mathcal{H}_2$ be the coefficient matrix for the state and costate in t. Then, instead of taking eigenvalues of $\mathcal{H}$ as above, we take generalized eigenvalues with the pair $(\mathcal{H}_1, \mathcal{H}_2)$.

Once we have a steady-state solution to the Riccati matrix, we can use the earlier formula to compute F and the law of motion for the state variables:

$$X_{t+1} = (A - BF)\, X_t + C\epsilon_{t+1} \tag{12}$$

Furthermore, given an initial condition for the states, X_0, and a realization of the shocks, $\epsilon_t,\ t \geq 0$, we can generate time-series for X_t and u_t.