Class notes: Advanced Topics in Macroeconomics

Topic: Adding Distortions and Near-Linear Methods

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The near-linear methods discussed so far are relevant for problems that map into concave programming problems (like the social planner's problem). Next, we consider modifications in the case that we have distortions such as time-varying distortionary taxes.

In class, we started the discussion of these modified methods by first considering how we could solve the problem in the second homework:

$$\max_{\{c_t, x_t, \ell_t\}} E \sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \psi \log(\ell_t)\} N_t$$

$$\begin{aligned}
\text{subj. to}\quad & c_t + (1+\tau_{xt})\, x_t = r_t k_t + (1-\tau_{ht})\, w_t h_t + \psi_t \\
& N_{t+1} k_{t+1} = [(1-\delta)\, k_t + x_t]\, N_t \\
& h_t + \ell_t = 1 \\
& S_t = P S_{t-1} + Q\epsilon_t, \quad S_t = [\log z_t, \tau_{ht}, \tau_{xt}, \log g_t] \\
& c_t, x_t \geq 0 \quad \text{in all states,}
\end{aligned}$$

where $N_t = (1+\gamma_n)^t$ and firm technology is $Y_t = K_t^{\theta}(Z_t L_t)^{1-\theta}$. Factors are paid their marginal products r and w, and revenues in excess of government purchases of goods and services, $N_t g_t$, are lump-sum transferred to households in amount ψ_t. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint in this economy is $Y_t = N_t(c_t + x_t + g_t)$.

LQ Method

In class, we discussed the possibility of mapping this problem into the LQ framework used earlier (in the first homework). Specifically, how does one deal with the prices? These prices are functions of the aggregate capital K_t and aggregate labor supply H_t. One proposal was to guess a mapping from the pair (K_t, H_t) to (K_{t+1}, H_{t+1}), solve the problem as before, update the guess, and solve the problem again until the candidate solution converges. This idea was quickly dismissed because it involves solving the maximization problem many times.

The alternative is to write out the first order conditions, impose market clearing (e.g., $K_t = N_t k_t$) and construct Vaughan's Hamiltonian as before—although this time, the Q, R, W, A, B matrices will be modified to take into account the distortionary taxes and prices. This is the method laid out in McGrattan (JEDC 1996).

Suppose the original maximization problem has the following general form:

$$\begin{aligned}
&\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t r\left(X_t, u_t\right) \mid X_0\right] \\
&\text{subject to} \quad X_{t+1} = g\left(X_t, u_t, \epsilon_{t+1}\right) \\
&\qquad\qquad\quad X_0 \text{ given}
\end{aligned}$$

where X_t is the vector of states, u_t is the vector of controls (e.g., decisions and prices), r is the objective function which is known, g governs the evolution of the state vector and is also known, ϵ are shocks affecting this evolution which we'll assume to be iid.

As before, the first step is to map the original problem into the following related problem:

$$\begin{aligned}
&\max_{\{u_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left(X_t' Q X_t + u_t' R u_t + 2 X_t' W u_t\right) \\
&\text{subject to} \quad X_{t+1} = A X_t + B u_t + C \epsilon_{t+1} \\
&\qquad\qquad\quad X_0 \text{ given}
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
r\left(X_t, u_t\right) &\simeq X_t' Q X_t + u_t' R u_t + 2 X_t' W u_t \\
g\left(X_t, u_t, \epsilon_{t+1}\right) &\simeq A X_t + B u_t + C \epsilon_{t+1},
\end{aligned} \tag{2}$$

with Q and R symmetric. The second step is to map this into an undiscounted problem without cross-products in the objective function as follows. Let:

$$\begin{aligned}
\tilde{X}_t &= \beta^{t/2} X_t \\
\tilde{u}_t &= \beta^{t/2} \left(u_t + R^{-1} W' X_t \right) \\
\tilde{Q} &= Q - W R^{-1} W' \\
\tilde{A} &= \sqrt{\beta} \left(A - B R^{-1} W' \right) \\
\tilde{B} &= \sqrt{\beta} B.
\end{aligned}$$

When we have distortions, we have to distinguish between aggregate variables (that are arguments of the prices) and individual variables that are choices of consumers or firms (which I'll sometimes refer to as the "big K-little k" problem). When we write out the LQ problem this time, we'll distinguish three types of state variables: individual states (the little k's), the exogenous variables (the taxes and TFP), and the aggregate states (the big K's). In other words, the problem that we'll solve is:

$$\max_{\{\tilde{u}_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \left(\tilde{X}_t' \tilde{Q} \tilde{X}_t + \tilde{u}_t' R \tilde{u}_t \right)$$

$$\text{subject to} \quad \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix}_{t+1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix}_t + \begin{bmatrix} \tilde{B}_1 \\ 0 \\ 0 \end{bmatrix} \tilde{u}_t + C \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}_{t+1} . \quad (3)$$

where $\tilde{X}_1$ are the individual states, $\tilde{X}_2$ are the aggregate or exogenous states with known laws of motion, and $\tilde{X}_3$ are the aggregate states with laws of motion that are unknown and need to be computed in equilibrium. Thus, all matrices in (3) are known with the exception of A_{32} and A_{33}.

For convenience, we can stack the states with known laws of motion into a new vector $y_t = [\tilde{X}_{1t}, \tilde{X}_{2t}]'$ and work with the upper partition of (3):

$$y_{t+1} = \tilde{A}_y y_t + \tilde{B}_y \tilde{u}_t + A_z \tilde{X}_{3t}$$

where $\tilde{A}_y$ is the upper left partition of $\tilde{A}$, $\tilde{B}_y$ is the upper partition of $\tilde{B}$, and A_z is the upper right partition of $\tilde{A}$. These matrices are related to the original problem as follows:

$$\begin{aligned}
\tilde{A}_y &= \sqrt{\beta} \left(A_y - B_y R^{-1} W_y' \right) \\
\tilde{A}_z &= \sqrt{\beta} \left(A_z - B_y R^{-1} W_z' \right) \\
\tilde{B}_y &= \sqrt{\beta} B_y.
\end{aligned}$$

where A_y, B_y, and B_z are the analogous partitions of A and B.

If we form the Lagrangian and take first order conditions with respect to $\tilde{u}_t$ and $\tilde{X}_{t+1}$, we get:

$$\tilde{u}_t = -R^{-1}\tilde{B}_y'\lambda_{t+1}$$

$$\lambda_{t+1} = \tilde{A}_y'\lambda_{t+2} + \tilde{Q}_y y_{t+1} + \tilde{Q}_z\tilde{X}_{3t+1}$$

where λ_t is the multiplier on the constraints and $\tilde{Q}_y$ and $\tilde{Q}_z$ are the upper right and left partitions of $\tilde{Q}$, respectively. These first order conditions are similar to what we had before except that now we have the aggregate variables X_{3t+1} appearing.

At this point, we impose market clearing conditions: which are given by:

$$X_{3t} = \Theta\left[X_{1t}, X_{2t}\right]' + \Psi\tilde{u}_t$$

in the original problem. For example, equating big K_t to little k_t (where both are per capita). Note that when we map our original problem to the undiscounted problem without cross products, we also have to map Θ and Ψ:

$$\tilde{\Theta} = \left(I + \Psi R^{-1}W_z'\right)^{-1}\left(\Theta - \Psi R^{-1}W_y'\right)$$

$$\tilde{\Psi} = \left(I + \Psi R^{-1}W_z'\right)^{-1}\Psi$$

and

$$\tilde{X}_{3t} = \tilde{\Theta}y_t + \tilde{\Psi}\tilde{u}_t.$$

Plugging the expression for $\tilde{X}_{3t}$ equation into the first order conditions and doing some simple algebra yields a system just like before:

$$\begin{bmatrix} y \\ \lambda \end{bmatrix}_t = \begin{bmatrix} \hat{A}^{-1} & \hat{A}^{-1}\hat{B}R^{-1}\tilde{B}_y' \\ \hat{Q}\hat{A}^{-1} & \hat{Q}\hat{A}^{-1}\hat{B}R^{-1}\tilde{B}_y' + \bar{A}' \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix}_{t+1} = H\begin{bmatrix} y \\ \lambda \end{bmatrix}_{t+1}$$

where $\hat{A} = \tilde{A}_y + \tilde{A}_z\tilde{\Theta}$, $\hat{Q} = \tilde{Q}_y + \tilde{Q}_z\tilde{\Theta}$, $\hat{B} = \tilde{B}_y + \tilde{A}_z\tilde{\Psi}$, and $\bar{A} = \tilde{A}_y - \tilde{B}_y R^{-1}\tilde{\Psi}'\tilde{Q}_z'$. The matrix H is analogous to Vaughan's Hamiltonian matrix. Note that if $\tilde{\Theta} = 0$ and $\tilde{\Psi} = 0$, H is Vaughan's matrix with eigenvalues that come in reciprocal pairs. When there are distortions, we will not find this pattern but the numerics are just as simple.

Once we have computed the eigenvalues and eigenvectors of H, we can construct the solution:

$$\tilde{u}_t = -\left(R + \tilde{B}_y'P\hat{B}\right)^{-1}\tilde{B}_y'P\hat{A}y_t$$

where $P = V_{21}V_{11}^{-1}$ as before. Recall that V is the eigenvector matrix of the Hamiltonian matrix, $H = V\Lambda V^{-1}$.

The matrix P can also be found by iteratively solving the modified Riccati equation:

$$P_n = \hat{Q} + \bar{A}' \left(P_{n+1}^{-1} + \hat{B} R^{-1} \tilde{B}_y' \right)^{-1} \hat{A}$$

starting from a negative definite matrix. Where does this come from? Using the fact that the multiplier λ_t is the derivative of the value function with respect to the states y_t, we can use either the first-order conditions or, better yet, the elements of the stacked Hamiltonian matrix H to equate coefficients of y_t after filling in $P_n y_t$ for λ_t and $(\hat{A}^{-1} + \hat{A}^{-1}\hat{B}R^{-1}\tilde{B}_y' P_{n+1})^{-1} y_t$ for y_{t+1}. It turns out that the modified Riccati equation can also be written as follows:

$$P_n = \hat{Q} + \bar{A}' P_{n+1} \hat{A} - \bar{A}' P_{n+1} \hat{B} \left(R + \tilde{B}_y' P_{n+1} \hat{B} \right)^{-1} \tilde{B}_y' P_{n+1} \hat{A}$$

thanks to the following identity:

$$\left(A - BD^{-1}C \right)^{-1} = A^{-1} + A^{-1}B \left(D - CA^{-1}B \right)^{-1} CA^{-1}$$

which is easily verified.

Vaughan's Method

Last week, we applied Vaughan's method to a distorted LQ problem. Today, we'll work with a variant of Vaughan's method that starts with the log-linearized first-order conditions and ends with a solution of the form:

$$\begin{aligned} X_{t+1} &= AX_t + BS_t \\ Z_t &= CX_t + DS_t \\ S_t &= PS_{t-1} + Q\epsilon_t \end{aligned}$$

where X_t are the endogenous state variables, S_t are the exogenous state variables, and Z_t are endogenous decisions or prices that appear in the first-order equations.

Consider the case of Homework 2. Recall that problem is to compute equilibria of the following growth model:

$$\max_{\{c_t,x_t,\ell_t\}} E\sum_{t=0}^{\infty}\beta^t\left\{\left(c_t\ell_t^{\psi}\right)^{1-\sigma}/(1-\sigma)\right\}N_t$$

$$\begin{aligned}
\text{subj. to}\quad & c_t+(1+\tau_{xt})\,x_t=r_tk_t+(1-\tau_{ht})\,w_th_t+\kappa_t\\
& N_{t+1}k_{t+1}=\left[(1-\delta)\,k_t+x_t\right]N_t\\
& h_t+\ell_t=1\\
& S_t=PS_{t-1}+Q\epsilon_t,\quad S_t=[\log z_t,\tau_{ht},\tau_{xt},\log g_t]\\
& c_t,x_t\geq 0\quad\text{in all states,}
\end{aligned}$$

where $N_t=(1+\gamma_n)^t$ and firm technology is $Y_t=K_t^{\theta}(Z_tL_t)^{1-\theta}$. Factors are paid their marginal products r and w, and revenues in excess of government purchases of goods and services, N_tg_t, are lump-sum transferred to households in amount κ_t. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint in this economy is $Y_t=N_t(c_t+x_t+g_t)$.

In this case, the (detrended) first order conditions are:

$$\begin{aligned}
&\hat{c}_t+(1+\gamma_z)(1+\gamma_n)\,\hat{k}_{t+1}-(1-\delta)\,\hat{k}_t+\hat{g}_t=\hat{y}_t=\hat{k}_t^{\theta}\,(z_th_t)^{1-\theta}\\
&\psi\hat{c}_t/(1-h_t)=(1-\tau_{ht})(1-\theta)\left(\hat{k}_t/h_t\right)^{\theta}z_t^{1-\theta}\\
&\hat{c}_t^{-\sigma}(1-h_t)^{\psi(1-\sigma)}(1+\tau_{xt})\\
&\qquad=\beta(1+\gamma_z)^{-\sigma}E_t\hat{c}_{t+1}^{-\sigma}(1-h_{t+1})^{\psi(1-\sigma)}\left(\theta\hat{k}_{t+1}^{-\theta}(z_{t+1}h_{t+1})^{1-\theta}+(1-\delta)(1+\tau_{xt+1})\right)
\end{aligned}$$

If we substitute for $\hat{c}_t$ using the resource constraint and then log-linearize the conditions around the steady state, we get:

$$\begin{aligned}
0&=E_t\{a_1\tilde{k}_t+a_2\tilde{k}_{t+1}+a_3\tilde{h}_t+a_4\tilde{z}_t+a_5\tilde{\tau}_{ht}+a_6\tilde{g}_t\}\\
0&=E_t\{b_1\tilde{k}_t+b_2\tilde{k}_{t+1}+b_3\tilde{k}_{t+2}+b_4\tilde{h}_t+b_5\tilde{h}_{t+1}+b_6\tilde{z}_t+b_7\tilde{\tau}_{xt}\\
&\qquad+b_8\tilde{g}_t+b_9\tilde{z}_{t+1}+b_{10}\tilde{\tau}_{xt+1}+b_{11}\tilde{g}_{t+1}\},
\end{aligned}$$

where $\tilde{k}_t = \log \hat{k}_t / \log \hat{k}_{ss}$, $\tilde{h}_t = \log h_t / \log h_{ss}$, $\tilde{z}_t = \log z_t / \log z_{ss}$, $\tilde{\tau}_{ht} = \tau_{ht}/\tau_{hss}$, $\tilde{\tau}_{xt} = \tau_{ht}/\tau_{hss}$, and $\tilde{g}_t = \log \hat{g}_t / \log \hat{g}_{ss}$. These equations can be stacked up as follows:

$$0 = E_t \left\{ \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & b_5 \end{bmatrix}}_{=A_1} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{k}_{t+2} \\ \tilde{h}_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_4 \end{bmatrix}}_{=A_2} \begin{bmatrix} \tilde{k}_t \\ \tilde{k}_{t+1} \\ \tilde{h}_t \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & 0 & 0 \\ b_6 & 0 & b_7 & b_8 & b_9 & 0 & b_{10} & b_{11} \end{bmatrix} \begin{bmatrix} S_t \\ S_{t+1} \end{bmatrix} \right\}$$

For this problem, we are looking for a solution of the form:

$$\tilde{k}_{t+1} = A\tilde{k}_t + BS_t$$
$$Z_t = CX_t + DS_t$$
$$S_t = PS_{t-1} + Q\epsilon_t$$

where $Z_t = [\tilde{k}_{t+1}, \tilde{h}_t]'$ and S_t are the stochastic exogenous variables.

Applying Vaughan, we compute the *generalized* eigenvalues and eigenvectors for matrices A_1 and $-A_2$ because A_1 is not invertible. (Note that if A_1 were invertible, then we would compute eigenvalues and eigenvectors of $-A_1^{-1}A_2$ as we did in the LQ problem.) Let V be the eigenvectors and Λ is a diagonal matrix with eigenvalues, so that

$$A_2 V = -A_1 V \Lambda.$$

Sort the eigenvalues in Λ and the associated columns in the matrix of eigenvectors V so that the eigenvalue inside the unit circle is in the (1,1) position of Λ. Then,

$$A = V_{11} \Lambda(1,1) V_{11}^{-1}$$
$$C = V_{21} V_{11}^{-1}$$

Note that A is 1×1 and C is 2×1.

Once we have A and C, we plug them into the system above—along with $S_{t+1} = PS_t + Q\epsilon_{t+1}$—and we can form a linear system in B and D. Since the policy function for the first element of Z_t is the same as A, B, we'll just consider the policy rule for $\tilde{h}_t$, namely:

$$\tilde{h}_t = C_2\tilde{k}_t + D_2 S_t$$

Plugging the policy rules into the first-order equations yields the following:

$$\begin{aligned}
0 &= (a_1 + a_2 A + a_3 C_2)\,\tilde{k}_t + (a_2 B + a_3 D_2 + [a_4, a_5, 0, a_6])\,S_t \\
0 &= \left(b_1 + b_2 A + b_3 A^2 + b_4 C_2 + b_5 C_2 A\right)\tilde{k}_t \\
&\quad + (b_2 B + b_3 AB + b_3 BP + b_4 D_2 + b_5 C_2 B + b_5 BP + [b_6, 0, b_7, b_8] + [b_9, 0, b_{10}, b_{11}]\,P)\,S_t
\end{aligned}$$

We can check that the coefficients on $\tilde{k}_t$ are zero at the solution. (Notice that they do not depend on B or D_2.) That leaves eight unknowns, namely four elements of B and four elements of D_2 and eight linear equations, namely the coefficients on S_t, which must be set equal to zero at the solution:

$$\begin{aligned}
0 &= a_2 B + a_3 D_2 + [a_4, a_5, 0, a_6] \\
0 &= b_2 B + b_3 AB + b_3 BP + b_4 D_2 + b_5 C_2 B + b_5 BP + [b_6, 0, b_7, b_8] + [b_9, 0, b_{10}, b_{11}]\,P.
\end{aligned}$$

We just have to stack these eight equations and solve the linear system. To do this, we'll need to use the fact that $vec(FGH) = (H' \otimes F)vec(G)$ for any matrices F, G, and H.

One last thing to note. If we set $\tau_{ht} = 0$, $\tau_{xt} = 0$, and $g_t = 0$, we are back to the simple case of Homework 1. Thus, we have a test case for the codes.