Lecture XIV

Solving for the Equilibrium in Models with Idiosyncratic and Aggregate Risk

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Quantitative Macroeconomics

Aiyagari model with aggregate productivity shocks

- **Aggregate and Idiosyncratic Risk.** The aggregate productivity shock z only takes two values, $z \in Z \equiv \{z_b, z_g\}$ with $z_b < z_g$.
- We also assume only two values for the individual productivity shock, $\varepsilon \in E \equiv \{\varepsilon_b, \varepsilon_g\}$ with $\varepsilon_b < \varepsilon_g$.
- Let

$$\pi(z', \varepsilon' | z, \varepsilon) = \Pr(z_{t+1} = z', \varepsilon_{t+1} = \varepsilon' | z_t = z, \varepsilon_t = \varepsilon)$$

  be the Markov chain that describes the joint evolution of the exogenous shocks.
- Notation allows transition probabilities for ε to depend on z, z'

Aiyagari model with aggregate productivity shocks

- **Production side.** From the firm's optimization problem we have:

$$
\begin{aligned}
w &= zF_H(K,H) \\
R &= 1 + zF_K(K,H) - \delta
\end{aligned}
$$

- Note: prices depend on the K/H ratio, not just on K, but the dynamics of H can be perfectly forecasted, conditional on (z, z'), through π because labor supply is exogenous.

- Rig π so that Z is a sufficient statistic for H (KS, 1998)

- H is time varying, but we know how to forecast it, e.g.:

$$
H(z) = \sum_{\varepsilon \in E} \varepsilon \Pi_z(\varepsilon)
$$

Aiyagari model with aggregate productivity shocks

- **State variables–** The two individual states are $(a, \varepsilon) \in S$ and the two aggregate states are $(z, \lambda) \in Z \times \Lambda$ where $\lambda(\cdot)$ is the measure of households across states.

- λ represents the beginning-of-period distribution of wealth and employment status (ε), after this period employment status is realized (wealth is inherited from last period saving decision).

- The individual states are directly budget relevant

- The aggregate states are needed to compute prices

- The law of motion Ψ is needed to forecast prices

Aiyagari model with aggregate productivity shocks

- **Household Problem** in recursive form:

$$
\begin{aligned}
v(a,\varepsilon;z,\lambda) &= \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E, z' \in Z} v(a',\varepsilon';z',\lambda')\pi(z',\varepsilon'|z,\varepsilon) \right\} \\
& \quad s.t. \\
c + a' &= w\left(z,\lambda\right)\varepsilon + R\left(z,\lambda\right)a \\
a' &\geq -\underline{a} \\
\lambda' &= \Psi\left(\lambda,z,z'\right)
\end{aligned}
$$

  where $\Psi\left(\lambda,z,z'\right)$ is the law of motion of the endogenous aggregate state

- Dependence on z' is inherited from π, but it does not apply to the marginal with respect to wealth (wealth is predetermined).

- $\Psi\left(z,z',\lambda\right)$ is the new equilibrium object

Equilibrium

- We focus on RCE, where (i) the individual saving decision rule is a time-invariant function $g(a, \varepsilon; z, \lambda)$ and (ii) next period cross-sectional distribution is a time invariant function $\Psi(\lambda, z, z')$ of the current distribution and of current and next period aggregate shocks.

- Key complication: the decision rule depends on λ which is a distribution.

- Where is this dependence coming from? To solve their problem, households need to forecast prices next period. Prices depend on aggregate capital, and aggregate capital this period and next period, K and K', depends on how assets are distributed in the population.

Why is the distribution a state variable?

- Consider the Euler equation associated to the problem above:

$$u_c\left(R\left(z,\lambda\right)a+w\left(z,\lambda\right)\varepsilon-g\left(a,\varepsilon;z,\lambda\right)\right)\geq$$
$$\beta\mathbb{E}\left[R\left(z',\lambda'\right)u_c\left(R\left(z',\lambda'\right)g\left(a,\varepsilon;z,\lambda\right)+w\left(z',\lambda'\right)\varepsilon'-g\left(g\left(a,\varepsilon;z,\lambda\right),\varepsilon';z',\lambda'\right)\right)\right]$$

- To solve for g, households need to forecast prices next period, and next period prices depend on λ'
- Agents need to know the equilibrium law of motion Ψ to forecast λ' given λ
- Ψ is a mapping from distributions into distributions: complicated
- Note: if prices were exogenous (SOE), the wealth distribution would not be a state variable in this example
- What if the government pays a lump sum transfer and balances the budget with a labor (or capital) income tax?

The Krusell and Smith approach

- Krusell and Smith (KS, 1998) propose to approximate the distribution through a finite set of moments

- Let $\bar{m}$ be a $(M \times 1)$ vector of moments of the *wealth distribution*, i.e., the marginal of λ with respect to a

- Our new state vector has law of motion, in vector notation,

$$\bar{m}' = \Psi\left(z, \bar{m}\right)$$

- Note that we lost the dependence on z' since we are only interested in the wealth distribution.

The Krusell and Smith approach

- To make this approach operational, one needs to: (i) choose M and (ii) specify a functional form for Ψ

- KS main finding: one obtains a very precise forecasting rule by simply setting $M = 1$ and by specifying a law of motion of the form:

$$\log K' = b_z^0 + b_z^1 \log K,$$

or similarly in levels instead of logs.

- Near-aggregation because (i) the wealth-rich have close-to-linear saving rules, (ii) they matter a lot more in the determination of aggregate wealth, and (iii) aggregate shocks do not induce significant wealth redistribution across agents

The Krusell and Smith algorithm

1. Specify a functional form for the law of motion, for example, linear or loglinear. In the linear case:

$$\bar{m}' = B_z^0 + B_z^1 \bar{m}$$

   where B_z^0 is $M \times 1$ and B_z^1 is $M \times M$

2. Guess the matrices of coefficients $\{B_z^0, B_z^1\}$

3. Specify how prices depend on $\bar{m}$. Since $\bar{m}_1 = K$:

$$\begin{aligned} R(z, \bar{m}) &= 1 + zF_K\left(\frac{\bar{m}_1}{H(z)}\right) - \delta \\ w(z, \bar{m}) &= zF_H\left(\frac{\bar{m}_1}{H(z)}\right) \end{aligned}$$

The Krusell and Smith algorithm

4. Solve the household problem and obtain the decision rule $g\left(a, \varepsilon; z, \bar{m}\right)$ with standard methods.

   Note you have additional state variables $\bar{m}$ on which you have to define a grid.

   At every point $\left(a, \varepsilon, z, \bar{m}\right)$ on the grid solve for the choice $a' = a^*$ that satisfies the following EE:

$$
\begin{aligned}
& u_c\left(R\left(z, \bar{m}\right) a + w\left(z, \bar{m}\right) \varepsilon - a^*\right) \geq \\
& \beta \sum_{z', \varepsilon'} [R\left(z', \Psi_1\left(z, \bar{m}\right)\right) u_c(R\left(z', \Psi_1\left(z, \bar{m}\right)\right) a^* + w\left(z', \Psi_1\left(z, \bar{m}\right)\right) \varepsilon' \\
& \qquad\qquad - g\left(a^*, \varepsilon'; z', \Psi\left(z, \bar{m}\right)\right))] \pi\left(z', \varepsilon' | z, \varepsilon\right)
\end{aligned}
$$

   where $\Psi_1(\cdot)$ is the forecasting function of the mean (first moment)

5. Simulate the economy for N individuals and T periods. In each period compute $\bar{m}_t$ from the cross-sectional distribution

The Krusell and Smith algorithm

6. Run an OLS regression

$$\bar{m}_{t+1} = \hat{B}_z^0 + \hat{B}_z^1 \bar{m}_t$$

   and estimate the coefficients $\left\{\hat{B}_z^0, \hat{B}_z^1\right\}$.

   Since the law of motion is time-invariant, we can separate the dates t in the sample where the state is z_b from those where the state is z_g and run two distinct regressions.

7. If $\left\{\hat{B}_z^0, \hat{B}_z^1\right\} \neq \{B_z^0, B_z^1\}$, try a new guess and go back to step 1

8. Continue until convergence: fixed point algorithm in $\left(B_z^0, B_z^1\right)$

9. Assess whether the solution is accurate enough. If fit at the solution is not satisfactory add moments of the distribution or try a different functional form for the law of motion.

Stochastic simulation

- This method approximates the continuum of agents with a large but finite number of agents and uses a random number generator to draw both the aggregate and the idiosyncratic shocks.

1. Simulate the economy for N individuals and T periods. For example, $N = 10,000$ and $T = 1,500$.

2. Draw first a random sequence for the aggregate shocks of length T. Next, one for the individual productivity shocks for each $i = 1, ..., N$, conditional on the time-path for the aggregate shocks.

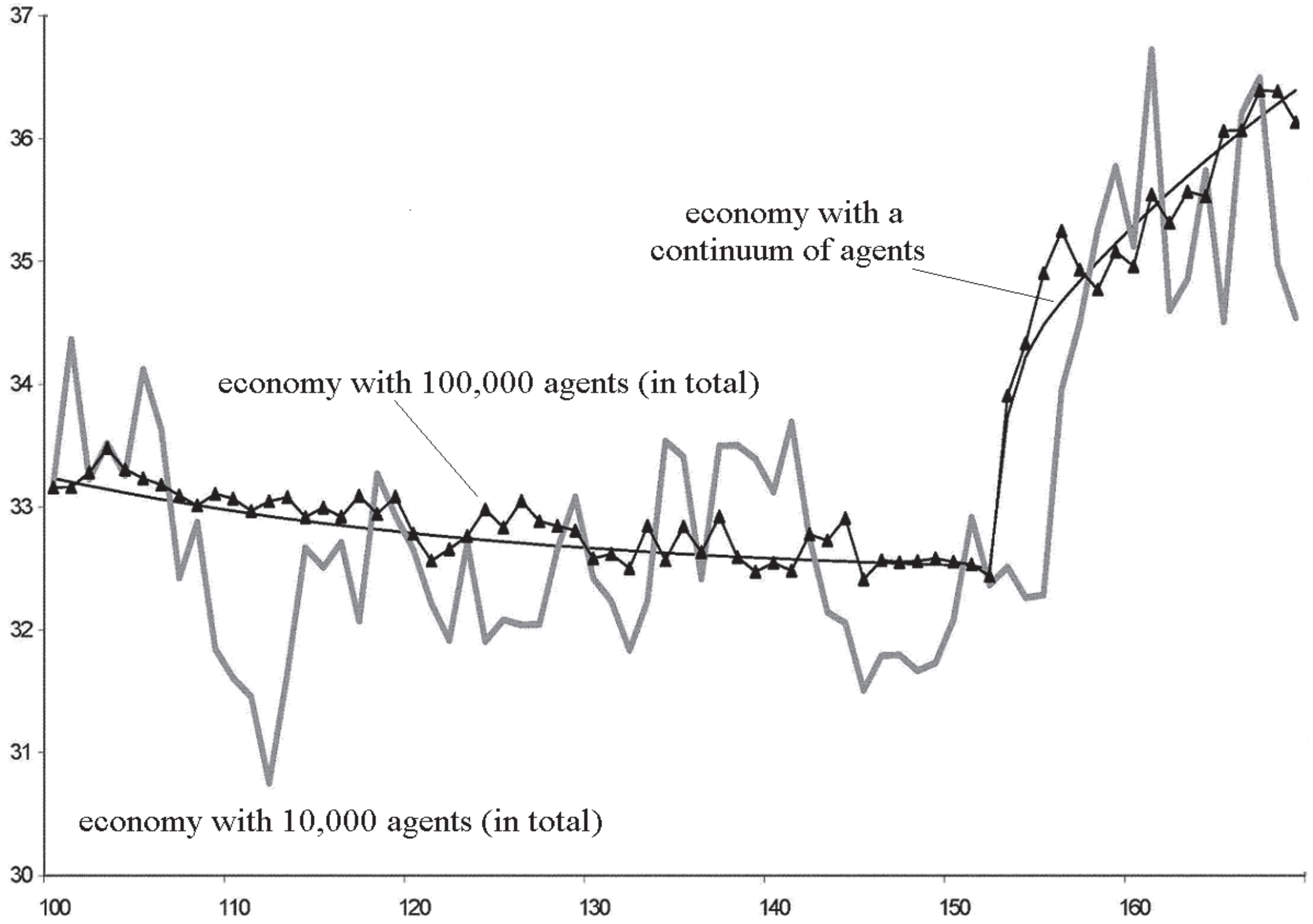
3. Initialize at $t = 0$ from the stationary distribution, for example.

4. Drop the first 500 periods when computing statistics.

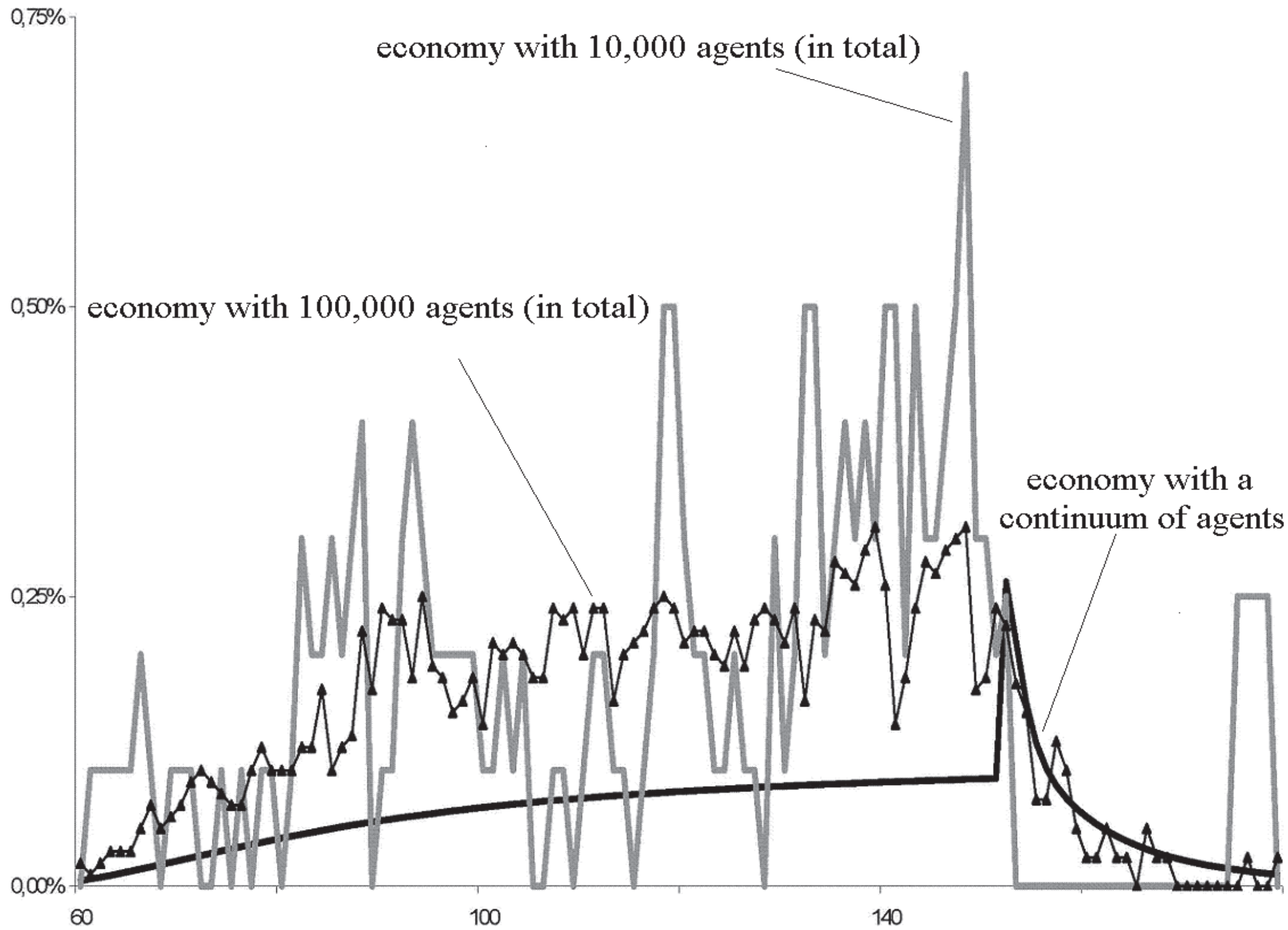
Stochastic simulation

- There will be cross-sectional sampling variation in the simulated cross-sectional data, while— conditional on the aggregate shock— there should be none if the model has a continuum of agents.
- Simulated data tend to cluster and clustering is bad for function approximation.
- Algain-Allais-Den Haan-Rendhal document that moments of asset holdings of the unemployed (which are few) are subject to substantial sampling variation.
- Similarly for agents at the constraint
- It is pretty slow, but it can be parallelized easily across the N dimension

Wealth per capita of the unemployed



Fraction of agents on the borrowing constraint



Non-stochastic simulation

- Same idea as approximation of pdf discussed ealier
- Advantage wrt to approximation of cdf? No calculation of inverse policy function (and hence no monotonicity of the policy function) required.

1. Draw a long series of aggregate shocks of length T.
2. Construct a fine grid over capital $[a_{\min}, a_{\max}]$, say, of 1,000 points, i.e. $J = 1,000$.
3. Initialize the distribution at $t = 0$ from the stationary distribution $\lambda_0 (a, \varepsilon) = \lambda^* (a, \varepsilon)$

Non-stochastic simulation

4. Suppose we are at a given date t of the simulation with aggregate states $(\bar{m}_t, z_t)$ and next period aggregate shock is z_{t+1}.

   Loop over the finer grid and for every ε and a_j on the finer grid for wealth compute $g\left(a_j, \varepsilon; z_t, \bar{m}_t\right)$

   Identify the two adjacent grid points a_k and a_{k+1} that contain $g(a_j, \cdot)$. Then compute:

$$\lambda_{t+1}\left(a_{k+1}, \varepsilon'\right) = \sum_{\varepsilon \in E} \pi\left(\varepsilon', z_{t+1} | \varepsilon, z_t\right) \frac{a_{k+1} - g\left(a_j, \varepsilon; z_t, \bar{m}_t\right)}{a_{k+1} - a_k} \lambda_t\left(a_j, \varepsilon\right)$$

$$\lambda_{t+1}\left(a_k, \varepsilon'\right) = \sum_{\varepsilon \in E} \pi\left(\varepsilon', z_{t+1} | \varepsilon, z_t\right) \frac{g\left(a_j, \varepsilon; z_t, \bar{m}_t\right) - a_k}{a_{k+1} - a_k} \lambda_t\left(a_j, \varepsilon\right)$$

5. Use these discretized distributions to compute the moments $\bar{m}_t$ period by period. This method is called the histogram method.

Explicit aggregation (Den Haan-Rendhal, 2010)

- Idea: derive aggregate laws of motion directly from individual policy rules without simulating cross-sectional distr. of agents

- Decision rules (suppose we discretize wrt to $\bar{m}$) can be written as:

$$g\left(a, \varepsilon; z, \bar{m}\right) = \sum_{j=0}^{J} \phi_j \left(\varepsilon; z, \bar{m}\right) a^j$$

- End of period aggregate wealth (denoted by $\hat{}$), for type ε is:

$$\hat{m}_{\varepsilon}\left(1\right) = \int \sum_{j=0}^{J} \phi_j \left(\varepsilon; z, \bar{m}\right) a^j d\lambda_{\varepsilon} = \sum_{j=0}^{J} \phi_j \left(\varepsilon; z, \bar{m}\right) \int a^j d\lambda_{\varepsilon} = \sum_{j=0}^{J} \phi_j \left(\varepsilon; z, \bar{m}\right) \bar{m}_{\varepsilon}\left(j\right)$$

where the index 1 denotes the first moment, and λ_{ε} denotes the wealth distribution conditional on type ε.

- It depends on all the higher beginning-of-period moments $\bar{m}_{\varepsilon}\left(j\right)$.

Explicit aggregation method

- When $J = 1$ and decision rules are linear, the equation above is sufficient to calculate exactly $\hat{m}_{\varepsilon}(1)$

- And thus, together with the value of z', next period aggregate wealth $\bar{m}'_{\varepsilon'}(1)$ since:

$$\bar{m}'_{\varepsilon'}(1) = \sum_{\varepsilon \in E} \pi(\varepsilon', z'; \varepsilon, z)\, \hat{m}_{\varepsilon}(1)\, \Pi_z(\varepsilon)$$

- The calculation is virtually impossible if there is just a little bit of nonlinearity in the decision rules.

- For simplicity, suppose that $J = 2$. The aggregate states are then $(\bar{m}_{\varepsilon}(1), \bar{m}_{\varepsilon}(2))$ for all ε.

Explicit aggregation method

- How do we compute $\hat{m}_\varepsilon(2)$?

$$
\begin{aligned}
\hat{m}_\varepsilon(2) &= \int g(a,\varepsilon;z,\bar{m})^2\, d\lambda_\varepsilon = \sum_{j=0}^{2} \int \left[\phi_j(\varepsilon;z,\bar{m})\, a^j\right]^2 d\lambda_\varepsilon \\
&= \phi_0(\varepsilon;z,\bar{m})^2 + 2\phi_0(\varepsilon;z,\bar{m})\,\phi_1(\varepsilon;z,\bar{m}) \int a\, d\lambda_\varepsilon \\
&+ \left[2\phi_0(\varepsilon;z,\bar{m})\,\phi_2(\varepsilon;z,\bar{m}) + \phi_1(\varepsilon;z,\bar{m})^2\right] \int a^2 d\lambda_\varepsilon \\
&+ 2\phi_1(\varepsilon;z,\bar{m})\,\phi_2(\varepsilon;z,\bar{m}) \int a^3 d\lambda_\varepsilon + \phi_2(\varepsilon;z,\bar{m})^2 \int a^4 d\lambda_\varepsilon \\
&= F(\bar{m}_\varepsilon(1), \bar{m}_\varepsilon(2), \bar{m}_\varepsilon(3), \bar{m}_\varepsilon(4))
\end{aligned}
$$

We need first 4 moments of the wealth distribution.

- But to predict the third and four moments, we need even more moments...and so on

- Conclusion: whenever $J > 1$ one has to include an infinite set of moments as state variables to get an exact solution

Explicit aggregation method (case $J = 2$)

- Den Haan and Rendhal (2010) suggest the following algorithm:

1. Define, e.g., as aggregate states, $\bar{m} = (\bar{m}_\varepsilon(1), \bar{m}_\varepsilon(2))$

2. Use a quadratic approximation for $(a')^2$ that you can use in the aggregation step:

$$(a'_\varepsilon)^2 = h(a, \varepsilon; z, \bar{m}) \equiv \psi_0(\varepsilon; z, \bar{m}) + \psi_1(\varepsilon; z, \bar{m})\, a + \psi_2(\varepsilon; z, \bar{m})\, a^2$$

then, the explicit aggregation step involves computing

$$\begin{aligned}\hat{m}_\varepsilon(2) &= \int \left[\psi_0(\varepsilon; z, \bar{m}) + \psi_1(\varepsilon; z, \bar{m})\, a + \psi_2(\varepsilon; z, \bar{m})\, a^2\right] d\lambda_\varepsilon \\ &= \psi_0(\varepsilon; z, \bar{m}) + \psi_1(\varepsilon; z, \bar{m})\, \bar{m}_\varepsilon(1) + \psi_2(\varepsilon; z, \bar{m})\, \bar{m}_\varepsilon(2)\end{aligned}$$

- Thus, to predict the second moment, you only need the second moment!

Explicit aggregation method (XPA) in practice

- In practice, Den Haan and Rendahl (2010) approximate individual policy rule with a high-order spline, but...
- ... they obtain the aggregate law of motion by aggregating a simple linear approximation of the individual policy rule, and show that they can get an accurate solution with this approach.
- Alternative implementation of XPA when aggregate K is enough:

$$\begin{aligned} \hat{m}_{\varepsilon}(1) &= \int g(a,\varepsilon;z,\bar{m})d\lambda_{\varepsilon} \simeq g(\bar{m}(1),\varepsilon;z,\bar{m}(1)) \\ \bar{m}(1) &= \sum_{\varepsilon\in E} g(\bar{m}(1),\varepsilon;z,\bar{m}(1))\Pi_z(\varepsilon) \end{aligned}$$

- In general: XPA much faster (10-50 times) because you don't need to do the simulation step

Assessing accuracy of the approximated law of motion

- KS suggest to compute the R^2 to measure the fit of the regression on the simulated data and use it to assess accuracy of the approximation for the law of motion.

- Unfortunately, it is not a good measure of fit: solutions with an R^2 in excess of 0.9999 sometimes can be inaccurate.

- Why? We want to assess the accuracy of the law of motion

$$K_{t+1} = b^0_{z_t} + b^1_{z_t} K_t$$

- Recall that this law of motion is based on the best linear fit of the time series for average capital $\{K^*_t\}$ obtained from a panel simulated via the decision rules, jointly with a sequence for $\{Z_t\}$.

- Thus K^*_{t+1} and K^*_t are only related through the decision rules, not directly through the previous law of motion

Assessing accuracy of the approximated law of motion

- Define: $u_{t+1} = K^*_{t+1} - K_{t+1}$ where K^*_t is the true value of the capital stock obtained from the simulation and K_{t+1} is the predicted one based on the law of motion. Then:

$$u_{t+1} = K^*_{t+1} - \left(b^0_{z_t} + b^1_{z_t} K^*_t\right)$$

  since each period one starts with the true value and evaluates how the approximation performs starting from the truth.

- It is a one-step ahead forecast error of the law of motion

- Suppose the approximating law of motion is bad and would want to push the observations away from the truth each period. The error terms defined this way underestimate the problem, because the true dgp (“ * ”) is used each period to put the approximating law of motion back on track

Meaningless of the R^2

Table 1: Meaninglessness of the R^2

equation	R^2	$\widehat{\sigma}_u$	implied properties mean	stand. dev.
$\alpha_3 = 0.96404$ (fitted regression)	0.99999729	4.1×10^{-5}	3.6723	0.0248
$\alpha_3 = 0.954187$	0.99990000	2.5×10^{-4}	3.6723	0.0217
$\alpha_3 = 0.9324788$	0.99900000	7.9×10^{-4}	3.6723	0.0174
$\alpha_3 = 0.8640985$	0.99000000	2.5×10^{-3}	3.6723	0.0113

Notes: The first row corresponds to the fitted regression equation. The subsequent rows are based on aggregate laws of motion in which the value of α_3 is changed until the indicated level of the R^2 is obtained. α_1 is adjusted to keep the fitted mean capital stock equal.

$$K_{t+1} = \alpha_1 + \alpha_2 z_t + \alpha_3 K_t$$

Assessing accuracy of the approximated law of motion

- Another problem is that R^2 is based on mean (squared) errors, but it is best to define discrepancies as maximal errors.

- Finally, if you express the law of motion in terms of changes ΔK_{t+1} instead of levels K_{t+1}, even though the law of motion is the same, you get:

$$\Delta K_{t+1} = b^0_{z_t} + \left(b^1_{z_t} - 1\right) K_t$$

whose R^2 are a lot lower, of the order of 0.85.

Assessing accuracy of the approximated law of motion

- Alternative procedure. Define the error as

$$\tilde{u}_{t+1} = K^*_{t+1} - \left(b^0_{z_t} + b^1_{z_t} K_t\right),$$

  i.e., use K_t instead of K^*_t to predict the capital stock next period, thus we allow errors to propagate over time

- Then, report the maximal error in the simulation.

- One can even plot the true and approximated series, and by analyzing it one can figure out in which histories deviations are the largest.

- Even better: do it for ΔK_t

- P.S. KS also suggest to use T-step ahead forecast error for u_{t+1}: much better that R^2, and nearly equivalent to the alternative R^2

Trivial market clearing

- When prices are determined by marginal product of the aggregate state variable (e.g., physical or human capital), then iterating over the law of motion for the aggregate state yields a solution where prices are consistent with market clearing

- Given a simulated history of $\{z_t\}$, and an initial condition K_t: (i) prices at t depend only on K_t which is known; (ii) individual saving decisions do not affect prices at t, but they aggregate into the same K_{t+1} predicted by the law of motion, and hence the asset market clears next period; (iii) next period prices equal again the marginal product of K_{t+1}.

- What makes market clearing trivial is that prices (wages and rates of return) only depend on a pre-determined variable K_t

- Suppose now decisions at time t affect prices at time t. Examples: (i) endogenous labor supply; (ii) risk-free bond; (iii) housing market

Endogenous labor supply

- Define first-step decision rules for assets and hours worked $g_a(\varepsilon, a, z, K; H)$ and $g_h(\varepsilon, a, z, K; H)$, a function of aggregate labor input H as well

  Define aggregate laws of motion:

$$\begin{aligned} K' &= b_z^0 + b_z^1 K = \Psi_K(z, K) \\ H &= d_z^0 + d_z^1 K = \Psi_H(z, K) \end{aligned}$$

- Guess initial decision rules and aggregate laws of motion
- At every point on the grid solve for (h^*, a^*) using the EE:

$$u_c(R(z,K,H)a + w(z,K,H)\varepsilon h^* - a^*) \geq \beta \sum_{z',\varepsilon'} [R(z', \Psi_K(z,K), \Psi_H(z', \Psi_K(z,K))) \cdot$$
$$\cdot u_c(R(z', \Psi_K(z,K), \Psi_H(z', \Psi_K(z,K)))a^* +$$
$$w(z', \Psi_K(z,K), \Psi_H(z', \Psi_K(z,K)))\varepsilon' g_h(\varepsilon', a^*, z', \Psi_K(z,K), \Psi_H(z'; \Psi_K(z,K)))$$
$$-g_a(a^*, \varepsilon', z', \Psi_K(z,K); \Psi_H(z', \Psi_K(z,K))))]\pi(z', \varepsilon' | z, \varepsilon)$$

Endogenous labor supply

and the intratemporal first order condition:

$$v_h\left(h^*\right) = w\left(z,K,H\right)\varepsilon \cdot u_c\left(R\left(z,K,H\right)a + w\left(z,K,H\right)\varepsilon - a^*\right)$$

- Once these first-step decision rules are obtained from this system of equations, use the decision rules to simulate an artificial panel, but at every date t one must solve for H_t^* that satisfies the labor market clearing condition at date t:

$$\int g_h\left(\varepsilon,a,z_t,K_t,H_t^*\right)d\lambda_t = F_H\left(\frac{w\left(z_t,K_t,H_t^*\right)}{z_t},K_t\right)^{-1}$$

- This time series $\{z_t,H_t^*,K_t\}$ is used to update the guess of the aggregate laws of motion.

- Once converged is achieved, we can solve for the *second-step (and final) decision rules* $g_a\left(\varepsilon,a,z,K\right)$ and $g_h\left(\varepsilon,a,z,K\right)$ only as a function of K

Asset in zero net (or exogenous) supply

- Examples: household-supplied IOU, government bond, land
- Price of bond q. Cash-in-hand:

$$\omega = R(z, K)\,a + b$$

- Define the decision rule for capital and bonds $g_a(\varepsilon, \omega, z, K; q)$ and $g_b(\varepsilon, \omega, z, K; q)$ and the aggregate laws of motion:

$$\begin{aligned} K' &= b_z^0 + b_z^1 K = \Psi_K(z, K) \\ q &= d_z^0 + d_z^1 K = \Psi_q(z, K) \end{aligned}$$

- Obtained these first-step decision rules, the simulation step requires that, at each date t, one looks for the q_t^* that satisfies the market clearing condition:

$$\int g_b(\varepsilon, \omega, z_t, K_t; q_t^*)\,d\lambda_t = 0$$

Asset in zero net (or exogenous) supply

- Time series $\{z_t, q_t^*, K_t\}$ is used to update the guess of the aggregate laws of motion.
- Upon convergence, we can solve for the second-step decision rules $g_a(\varepsilon, \omega, z, K)$ and $g_h(\varepsilon, \omega, z, K)$ only as a function of K
- Shortcut (no need to add q as a state): instead solve for

$$g_d(\varepsilon, \omega, z, K) = g_b(\varepsilon, \omega, z, K) + q(z, K)$$

- Imposing market clearing at every t, aggregation of $g_d(\cdot)$ gives the bond price which is then used in the simulation step to update the law of motion:

$$q_t = \int g_d(\varepsilon, \omega, z_t, K_t) d\lambda_t$$

- Then, upon convergence, obtain $g_b(\cdot) = g_d(\cdot) - q(z, K)$

Projection-Perturbation approach (Reiter JEDC 09)

- It combines features of projection and perturbation methods
- Idea: compute a solution that is fully nonlinear in the idiosyncratic shocks, but only linear in the aggregate shocks
- The solution method has three steps:
  1. Provide a finite representation of economy at any date t, i.e.:
     (a) representing the saving function g_a by a vector ϕ_t which contains the values of g_a at the grid points over a (if g_a is approximated through a spline) or the polynomial coefficients (if g_a is approximated by a family of orthogonal polynomials).

     (b) representing the distribution as a vector $\boldsymbol{\lambda}_t = \{\boldsymbol{\lambda}_{\varepsilon,t}\}_{\varepsilon \in \mathcal{E}}$ of probability mass of households of each type ε within specified intervals of asset holdings.

Projection-Perturbation approach

2. Compute the steady-state of the economy, i.e., the stationary economy when aggregate shocks are zero. This step yields a finite representation for the stationary saving function and the invariant distribution $\{\phi^*, \boldsymbol{\lambda}^*\}$.

3. Compute a first-order perturbation of all variables $\{\phi_t, \boldsymbol{\lambda}_t, z_t\}$ around the steady-state solution of the model with uninsurable risk $\{\phi^*, \boldsymbol{\lambda}^*, 0\}$.

- Note that we are treating (i) the coefficients of the policy function and (ii) the quantiles of the distribution as variables in the perturbation step

Details on step 1

- Grid over ε, call it $\mathcal{E}$, with n_ε points
- Grid for the consumption policy rule over a, call it $\mathcal{A}^p$ with n_a^p points, the dimension of the vector ϕ_t
- A denser grid for the density, call it $\mathcal{A}^d$ with n_a^d points. The dimension of the vectors $\{\boldsymbol{\lambda}_{\varepsilon,t}\}$ (one for each value of ε) is $n_a^d - 1$.
- Need to define the system of equations representing the economy at date t. It comprises of:
- The law of motion for the exogenous aggregate state:

$$\log z_{t+1} = \rho \log z_t + \sigma \eta_{t+1}$$

Note: it is convenient to treat z_t as a continuous shock. Drawback: we cannot easily handle dependence of π from z

Details on step 1

- Euler equation. We have one equation for each point $(\varepsilon, a_j) \in \mathcal{E} \times \mathcal{A}^p$, i.e., $n_\varepsilon \times n_a^p$ equations:

$$
\begin{aligned}
&u'\left(w\left(z_t, \boldsymbol{\lambda}_t\right)\varepsilon + R\left(z_t, \boldsymbol{\lambda}_t\right) a_j - g_a\left(a_j, \varepsilon; \boldsymbol{\phi}_t\right)\right) \geq \\
&\beta \mathbb{E}_t[R\left(z_{t+1}, \boldsymbol{\lambda}_{t+1}\right) \textstyle\sum_{\varepsilon' \in \mathcal{E}} u'(w\left(z_{t+1}, \boldsymbol{\lambda}_{t+1}\right)\varepsilon' + R\left(z_{t+1}, \boldsymbol{\lambda}_{t+1}\right) g_a\left(a_j, \varepsilon; \boldsymbol{\phi}_t\right) \\
&-g_a\left(g_a\left(a_j, \varepsilon; \boldsymbol{\phi}_t\right), \varepsilon'; \boldsymbol{\phi}_{t+1}\right))\pi\left(\varepsilon', \varepsilon\right)]
\end{aligned}
$$

- Equilibrium prices (2 equations):

$$
\begin{aligned}
w\left(z_t, \boldsymbol{\lambda}_t\right) &= z_t F_H\left(\sum_{\varepsilon \in \mathcal{E}} \sum_{a_k \in \mathcal{A}^d} a_k \lambda_{\varepsilon,t}\left(a_k\right), H\left(z_t\right)\right) \\
R\left(z_t, \boldsymbol{\lambda}_t\right) &= 1 + z_t F_K\left(\sum_{\varepsilon \in \mathcal{E}} \sum_{a_k \in \mathcal{A}^d} a_k \lambda_{\varepsilon,t}\left(a_k\right), H\left(z_t\right)\right) - \delta
\end{aligned}
$$

Details on step 1

- The law of motion for the pdf, $n_\varepsilon \times \left(n_a^d - 1\right) \times 2$ equations. Two equations for each point $(\varepsilon, a_j) \in \mathcal{E} \times \mathcal{A}^d$

$$
\begin{aligned}
\lambda_{\varepsilon',t+1}\left(a_{k+1}\right) &= \sum_{\varepsilon \in \mathcal{E}} \pi\left(\varepsilon', \varepsilon\right) \frac{a_{k+1} - g_a\left(a_j, \varepsilon; \boldsymbol{\phi}_t\right)}{a_{k+1} - a_k} \lambda_{\varepsilon,t}\left(a_j\right) \\
\lambda_{\varepsilon',t+1}\left(a_k\right) &= \sum_{\varepsilon \in \mathcal{E}} \pi\left(\varepsilon', \varepsilon\right) \frac{g_a\left(a_j, \varepsilon; \boldsymbol{\phi}_t\right) - a_k}{a_{k+1} - a_k} \lambda_{\varepsilon,t}\left(a_j\right)
\end{aligned}
$$

- As a result: system of $1 + (n_\varepsilon \times n_a^p) + 2 + n_\varepsilon \times \left(n_a^d - 1\right) \times 2$ equations that can be written as:

$$
\mathbb{E}_t\left[\mathcal{F}\left(y_{t+1}, y_t, x_{t+1}, x_t\right)\right] = 0
$$

where $x_t^1 = z_t, \quad x_t^2 = \boldsymbol{\lambda}_t, \quad y_t = \boldsymbol{\phi}_t$

Details on steps 2 and 3

- Steady state of the system:

$$\mathcal{F}\left(y^{*}, y^{*}, x^{*}, x^{*}\right) = 0$$

  requires computing the policy functions and invariant distribution at the steady-state $(z = 0)$

- We know how to do first-order perturbations of $(\boldsymbol{\phi}_t, \boldsymbol{\lambda}_t, z_t)$ around $(\boldsymbol{\phi}^{*}, \boldsymbol{\lambda}^{*}, 0)$.

- Lots of equations, e.g., if $n^{\varepsilon} = 2$, $n_a^p = 30$ and $n_a^d = 1,000$ we have over 4,000 equations. Most costly is the state space for the distribution

- Remedies: yes, the "smooth density approximation"

Smooth approximation for asset density

- Define a smooth $n - th$ order polynomial approximation of the asset density for type ε at date t as $P\left(a; \boldsymbol{\kappa}_t\left(\varepsilon\right)\right)$ where $\boldsymbol{\kappa}_t\left(\varepsilon\right)$ is a vector of coefficients.

- $P\left(a; \boldsymbol{\kappa}_t\left(\varepsilon\right)\right)$, together with the decision rule at date t, is sufficient to obtain $P\left(a; \boldsymbol{\kappa}_{t+1}\left(\varepsilon\right)\right)$

- Then, two options:

  1. Reiter-style: First-order perturbation of $\boldsymbol{\kappa}_t\left(\varepsilon\right)$ wrt to z: much lower-dimensional than the vector of density quantiles

  2. KS-style: Look for a law of motion of the coefficients $\boldsymbol{\kappa}\left(\varepsilon\right)$, if following a KS-style algorithm

Algorithm

1. Use $P(a; \boldsymbol{\kappa}_t(\varepsilon))$ and $g_a(a, \varepsilon; \boldsymbol{\phi}_t)$ to determine a set of moments for the end-of-period distribution $\{\hat{m}_{\varepsilon,t}(j)\}$ and, through the law of motion of the idiosyncratic shock, determine the beginning of next period moments $\{\bar{m}_{\varepsilon,t+1}(j)\}$

2. Given the vector $\{\bar{m}_{\varepsilon,t+1}(j)\}$, find the values $\boldsymbol{\kappa}_{t+1}(\varepsilon)$ of the coefficients of the approximating density $P(a; \boldsymbol{\kappa}_{t+1}(\varepsilon))$ that ensure that the moments of the approximating density are close enough to $\{\bar{m}_{\varepsilon,t+1}(j)\}$.

- 1. and 2. determine, implicitly, a mapping between $\boldsymbol{\kappa}_t(\varepsilon)$ and $\boldsymbol{\kappa}_{t+1}(\varepsilon)$, and thus between current and next-period distribution as a function of z, that can be included in the system of equations and appropriately perturbated

Details on step 2.

- Algan, Allais, and Den Haan (2008) suggest using:

$$
\begin{aligned}
P\left(a;\boldsymbol{\kappa}_t\left(\varepsilon\right)\right) &= \kappa_t^0\left(\varepsilon\right)\exp[\kappa_t^1\left(\varepsilon\right)\left(a-\bar{m}_{\varepsilon,t}\left(1\right)\right) \\
&+ \kappa_t^2\left(\varepsilon\right)\left(\left(a-\bar{m}_{\varepsilon,t}\left(1\right)\right)^2-\bar{m}_{\varepsilon,t}\left(2\right)\right)+\ldots \\
&+ \kappa_t^n\left(\varepsilon\right)\left(\left(a-\bar{m}_{\varepsilon,t}\left(1\right)\right)^n-\bar{m}_{\varepsilon,t}\left(n\right)\right)]
\end{aligned}
$$

- Step 2 is a root-finding problem: find $\{\boldsymbol{\kappa}_t\left(\varepsilon\right)\}$ that solve a set of equations. With this form for density, the coefficients (except for $\kappa_t^0\left(\varepsilon\right)$), can be found with the following minimization routine:

$$
\min_{\{\kappa_t^1(\varepsilon),\ldots,\kappa_t^n(\varepsilon)\}}\int P\left(a;\boldsymbol{\kappa}_t\left(\varepsilon\right)\right)da
$$

- This minimization leads to the right answer, because the FOCs correspond to the condition that the first n moments of $P\left(a;\boldsymbol{\kappa}_t\left(\varepsilon\right)\right)$ equal to the corresponding moments obtained from decision rule.

Details on step 2.

For example, the FOC wrt to $\kappa_t^2(\varepsilon)$ is :

$$\int \left[\left(a - \bar{m}_{\varepsilon,t}(1)\right)^2 - \bar{m}_{\varepsilon,t}(2) \right] P\left(a; \boldsymbol{\kappa}_t(\varepsilon)\right) da = 0$$

- $\kappa_t^0(\varepsilon)$ is determined residually by the condition that the density integrates to one

- Advantage for projection-perturbation method: just need to keep track of $\{\boldsymbol{\kappa}_t(\varepsilon)\}$, many fewer parameters than size of $\mathcal{A}^d$ grid (say, twenty instead of one thousand): faster perturbation step.

- Disadvantage: slower updating step than with the histogram method because it requires solving a minimization problem each time