ECON 8185 - Homework 3

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Question 1

The problem is:

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

s.t.

$$c_t + k_{t+1} = Ak_t^{\theta} + (1-\delta)k_t$$

The FOCs imply that:

$$\frac{c_{t+1}}{c_t} = \beta(A\theta k_{t+1}^{\theta-1} + 1 - \delta)$$

Hence,

$$k_{ss} = \left(\frac{1-\beta(1-\delta)}{\beta A\theta}\right)^{\frac{1}{\theta-1}}$$

The functional equation is:

$$F(c)(k) = 1 - \beta\frac{c(k)}{c(Ak^{\theta} + (1-\delta)k - c(k))}(A\theta(A\theta k^{\alpha} + (1-\delta)k - c(k))^{\theta-1} + 1 - \delta) = 0$$

And, in the test case $\delta = 1$, we know the closed form solution:

$$c(k) = (1-\beta\theta)Ak^{\theta}$$

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Let us now define some parameters: $\theta = 0.25$, $\beta = 0.9$ and A is set to make the SS capital sotck equals to the unity.

We will use the finite element method with the piecewise linear base function φ_i as defined in the lecture notes. The elements node $k_i \in [0, 2]$ with their distance increasing exponentially since it is known that the consuption function is less linear close to 0. We have 15 nodes. The Residual equation is:

$$R(k;\alpha) = F(c^n(k;\alpha))$$

$$= 1-\beta \frac{\sum_{i=1}^n \alpha_i \varphi_i(k)}{\sum_{i=1}^n \alpha_i \varphi_i(Ak^\theta + (1-\delta)k - \sum_{i=1}^n \alpha_i \varphi_i(k))} (A\theta(A\theta k^\alpha + (1-\delta)k - \sum_{i=1}^n \alpha_i \varphi_i(k))^{\theta-1} + 1 - \delta) = 0$$

Since the finite element method is a Galerkin method, the base function is the weight we use to minimize the weighted residual:

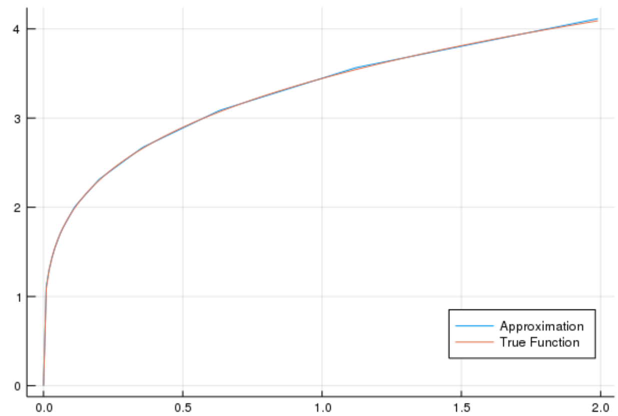
$$\min_\alpha \int_0^{\bar{k}\varphi_i(k)R(k;\alpha)dk}$$

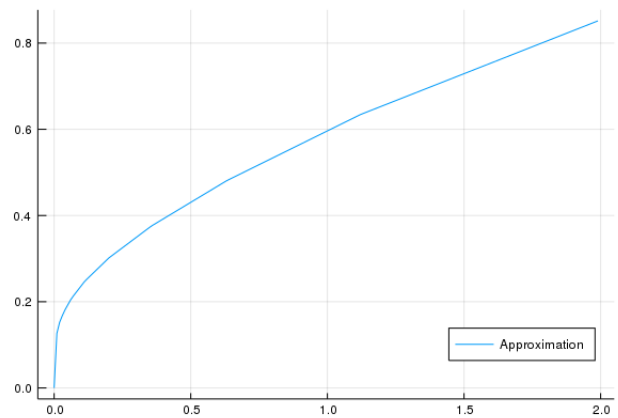
To calculate the integral, we use Gauss-Legendre method with 3 nodes per element. Also note, that we impose the boundary condition $\alpha_1 = 0$ since this imply $c(0) = 0$.

Let's start the minimization procedure. We use the BFGS method which is a Quasi-Newton method. It does not calculates the Hessian as a Newton procedure would do, but it approximates the Hessian by a positive definite approximation which is easier to compute and invert (see Judd's book). Note that this minimization step is extremely sensitive to initial conditions. A grid search or another procedure might be needed to determine the initial parameters.

Below, we plot the approximate consumption function and the true one. As we can see, the approximation is barely indistinguishable from the true function.

Now, we do the same procedure for a case with $\delta = 0.05$. Note that now we can't compare to a known formula, but the shape should be similar to the one above.





Question 3

The Problem is:

$$\max \int_0^\infty e^{-\rho t} \log(c)\, dt$$

s.t.

$$dk = (Ak^\theta - \delta k)dt - c$$

The Bellman's equation for this problem is:

$$\rho V(k) = \max \log(c) + V'(k)(Ak^\theta - \delta k) - c)$$

The maximization problem FOC is:

$$\frac{1}{c} = V'(k)$$

By the envelope theorem:

$$\rho V'(k) = V'(k)(\theta A k^{\theta-1} - \delta) + V''(k)(Ak^{\theta} - \delta k - c)$$

From the FOC:

$$V''(k) = -\frac{1}{c(k)^2}c'(k)$$

Combining the expressions above, we get the differential equation:

$$c(k)(\theta A k^{\theta-1} - (\delta + \rho) = (Ak^{\theta} - \delta k - c(k))c'(k)$$

We also have in the Steady State:

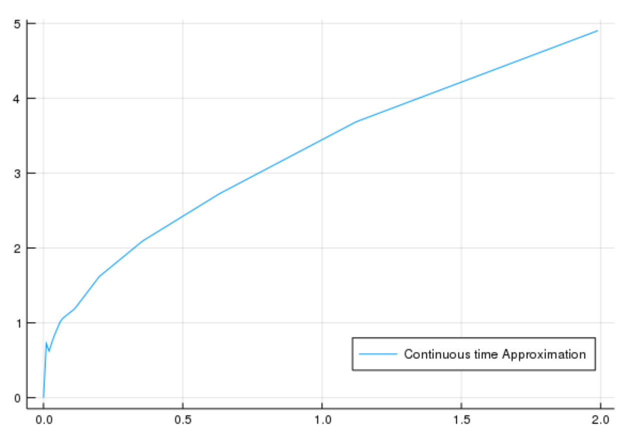
$$k_{ss} = \left(\frac{\delta + \rho}{\theta A}\right)^{\frac{1}{\theta-1}}$$

$$c_{ss} = Ak_{ss}^{\theta} - \delta k_{ss}$$

I don't really see how having $\delta = 1$ would help me having a test case. Candler's chapter in Marimon Scott book claims that we do not have closed form solutions for arbitrary parameters, but we do have when utility is CRRA with parameter 0.5, which is not our case here. So, unfortunately I will not have a comparison. The functional equation is thus:

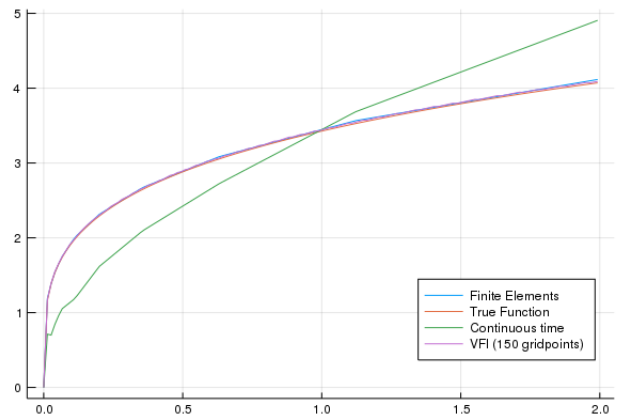
$$F(c)(k) = c(k)(\theta A k^{\theta-1} - (\delta + \rho) - (Ak^{\theta} - \delta k - c(k))c'(k) = 0$$

We use the FInite element method as before:



Question 3

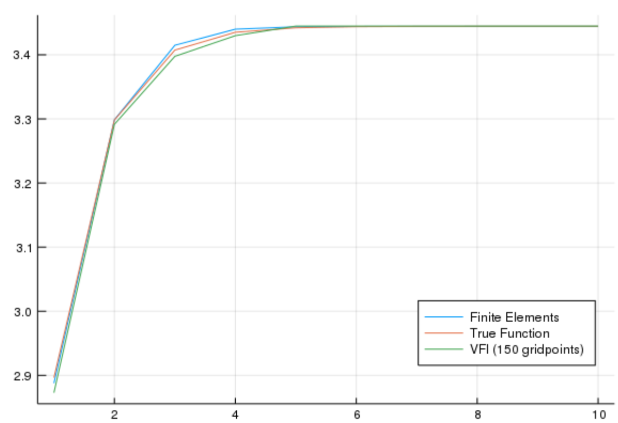
Below, I plot the consumption policy functions for the discrete and continuous time case above and compare them with the VFI policy functions I found when doing HW1.



As we can see above, all the methods produce good results when compared to the true function (I'm using $\delta = 1$). I'm not sure if the continuous time case is comparable, since the model is using different assumptions and not only a different solution method. The problem with the VFI method is that I used gridpoints andthe function cannot be evaluated at points outside the grid.

Question 4

Below a plot of the consumption time series generated by each model starting at $0.5k_{ss}$, given the problems found above, I don't plot the continuous time. As we can se, there very small are differences in the time series when compared to the true function.


3.4
3.3
3.2
3.1
3.0
2.9
2
4
6
8
10
Finite Elements
True Function
VFI (150 gridpoints)