Homework 4 - João Guilherme Santos Lazzaro

Question 1

The problem is:

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \overset{\infty}{\underset{t=0}{}} \beta^t \log(c_t)$$

s.t.

$$c_t + k_{t+1} = Ak_t^{\theta} + (1-\delta)k_t$$

The FOCs imply that:

$$\frac{c_{t+1}}{c_t} = \beta(A\theta k_{t+1}^{\theta-1} + 1 - \delta)$$

Hence,

$$k_{ss} = \quad \frac{1-\beta(1-\delta)}{\beta A\theta}^{\frac{1}{\theta-1}}$$

The functional equation is:

$$F(c)(k) = 1 - \beta\frac{c(k)}{c(Ak^{\theta} + (1-\delta)k - c(k))}(A\theta(A\theta k^{\alpha} + (1-\delta)k - c(k))^{\theta-1} + 1 - \delta) = 0$$

And, in the test case $\delta = 1$, we know the closed form solution:

$$c(k) = (1-\beta\theta)Ak^{\theta}$$

Let us now define some parameters and find the Steady State value of capital:

```
In [59]:  sing Plots, Optim, LinearAlgebra, FastGaussQuadrature

          Defining parameters
           = 0.25
           = 0.9
           = 1
           = (1-β*(1-δ))/(θ * β) #This will normalize the SS to 1
          ss = ((1- β*(1-δ))/(β*A*θ))^(1/(θ-1))
```

```
Out[59]: 1.0
```

We will use the finite element method with the piecewise linear base function φ_i defined below (and in the lecture notes):

In [60]:
```
Finite elements Piecewise Linear function:
unction φi(x,X,i::Int)
   #x: point to evaluate the function
   #X: Vector with elements nodes
   #i: Which element in the function
   if i>1 && i<length(X) #i is not in a boundary
       if X[i-1]<=x<=X[i]
           f = (x-X[i-1])/(X[i]-X[i-1])
       elseif X[i]<=x<=X[i+1]
           f = (X[i+1]-x)/(X[i+1]-X[i])
       else
           f = 0
       end
   elseif i==1 #i is in the boundary(1)
       if X[i]<=x<=X[i+1]
           f = (X[i+1]-x)/(X[i+1]-X[i])
       else
           f = 0
       end
   elseif i==length(X) #i is in the top boundary
       if X[i-1]<=x<=X[i]
           f = (x-X[i-1])/(X[i]-X[i-1])
       else
           f=0
       end
   end
   return f
nd
```

Out[60]:
```
φi (generic function with 1 method)
```

The elements node $k_i \in [0,2]$ with their distance increasing exponentially since it is known that the consuption function is less linear close to 0. We have 15 nodes.

In [61]:
```
Defining the elements:
In this case, we are putting more elements at low values of k because the fun
tion is known to be "less linear" in that region.
 = zeros(15)
or i=2:length(K)
   global K
   K[i] = K[i-1] +0.0005*exp(0.574*(i-2))
nd
```

Out[61]:
```
15-element Array{Float64,1}:
 0.0
 0.0005
 0.0013876771423281366
 0.0029636185603518304
 0.0057614729092074785
 0.010728655615292541
 0.019547164715211198
 0.03520314263003213
 0.06299805010159346
 0.11234385817285451
 0.19995014996199395
 0.35548235545269025
 0.6316071228726379
 1.1218264118113592
 1.9921393268498697
```

We will approximate the consumption functio $c(k)$ by:

$$c^n(k;\alpha) = \sum_{i=1}^{n} \alpha_i \varphi_i(k)$$

In [62]:
```
unction cn(k,α;K=K)
This function is piecewise linear approximation for a function with arguments

To get a good approximations one need to find the parameters α
K are the elements nodes. It is optionally defined here for efficiency.
       n = length(K)
       c = 0
       for i = 1:n
           c = c + α[i]*φi(k,K,i)
       end
       return c
nd
```

Out[62]:
```
cn (generic function with 1 method)
```

The Residual equation is:

$$R(k;\alpha) = F(c^n(k;\alpha))$$
$$= 1 - \beta \frac{{}_{i=1}^{n} \alpha_i \varphi_i(k)}{{}_{i=1}^{n} \alpha_i \varphi_i(Ak^\theta + (1-\delta)k - {}_{i=1}^{n} \alpha_i \varphi_i(k))} (A\theta(A\theta k^\alpha + (1-\delta)k - \overset{n}{\underset{i=1}{}} \alpha_i \varphi_i(k))^{\theta-1} + 1 -$$

```
In [63]:  capital policy function from Bugdet constraint
          olk(k,α) = min(max(eps(),A*k.^θ+(1-δ)*k-cn(k,α)),K[end])
          min max are needed to avoid NaNs and other numerical instabilities

          unction residual(k,α)
             #This function is specific for the deterministic growth model.
             #Residual function comes from FOCs
             #cn below is an approximation for consumption
                 R = cn(k,α)/cn(polk(k,α),α) * β * (A*θ*polk(k,α)^(θ-1)+1-δ)- 1
             return R
          nd

Out[63]:  residual (generic function with 1 method)
```

Since the finite element method is a Galerkin method, the base function is the weight we use to minimize the weighted residual:

$$\min_{\alpha} \;\; {}_{0}^{\bar{k}} \varphi_i(k) R(k;\alpha) dk$$

To calculate the integral, we use Gauss-Legendre method with 3 nodes per element. Also note, that we impose the boundary condition $\alpha_1 = 0$ since this imply $c(0) = 0$.

In [64]:
```
unction integra(k,α;K=K)
This function calculates the function that will be integrated:
integra(k;α):= ϕi(k)R(k;α), where ϕi are the weights and R is the residual fu
ction
In the Finite element methods, the weights are the same as the approximating
functions
   T=zeros(length(K))
   for i=1:length(K)
       T[i] = φi(k,K,i)*residual(k,α)
   end
   return T
nd

This function calculates the integral (the norm of the integrated functions),
s a functions of the parameters to minimized
We define that way since this is the format accepted by the solver:
mini(α):= ∫integra(k;α)dk
odes, weights = gausslegendre(3*(length(K)-1)) #Gauss Legendre nodes and weig
ts,this function is just a Quadrature table
unction mini(α;nodes=nodes,weights=weights,K=K)
   if length(α)<length(K)
       α = vcat(0,α)
   end
   #g = quadgk.(integra,K[1],K[end])[1] #Integral
   #See Judd's book pg 261 on numerical integration and the gausslegendre for
ula:
   gaussleg = zeros(length(K))
   for j=1:length(nodes)
       gaussleg .+= (K[end]-K[1])/2 .* weights[j] .* integra((nodes[j] .+1).*
       (K[end]-K[1])/2 .+ K[1],α)
   end
   return norm(gaussleg,1)
nd
```

Out[64]:
```
mini (generic function with 1 method)
```

Let's start the minimization procedure. We use the BFGS method which is a Quasi-Newton method. It does not calculates the Hessian as a Newton procedure would do, but it approximates the Hessian by a positive definite approximation which is easier to compute and invert (see Judd's book). Note that this minimization step is extremely sensitive to initial conditions. A grid search or another procedure might be needed to determine the initial parameters.

In [65]:
```
Setting initial conditions
nitial =  ones(length(K)-1) .* range(0.35, stop = 3.5, length = length(K)-1)

Check if the initial conditions are somewhat close to the true parameters.
ini(initial)
```

Out[65]:
```
0.6613235916442249
```

In [66]:
```
Here we start the minimization procedure we want to find α:= argmin mini(α)

Solver stuff
lower and upper bound of the parameters:
ower = zeros(length(initial))
pper = Inf*ones(length(initial))
Optimizer method is BFGS, see Judd's book page 114 for an explanation:
nner_optimizer = BFGS()

Solver:
la = optimize(mini,lower,upper,initial, Fminbox(inner_optimizer))
Parameters
 = vcat(0,bla.minimizer) #we need to concatenate to include the boundary cond
tion
```

Out[66]:
```
15-element Array{Float64,1}:
 0.0
 0.37442113219372514
 0.6640748582488243
 0.8539733791826457
 0.9252970761844344
 1.1458792909060764
 1.29154740222145
 1.4961818748748827
 1.730224876203169
 2.0077754086540636
 2.3141705183580985
 2.6711610637841803
 3.085257330000326
 3.565426495237902
 4.116321573422392
```

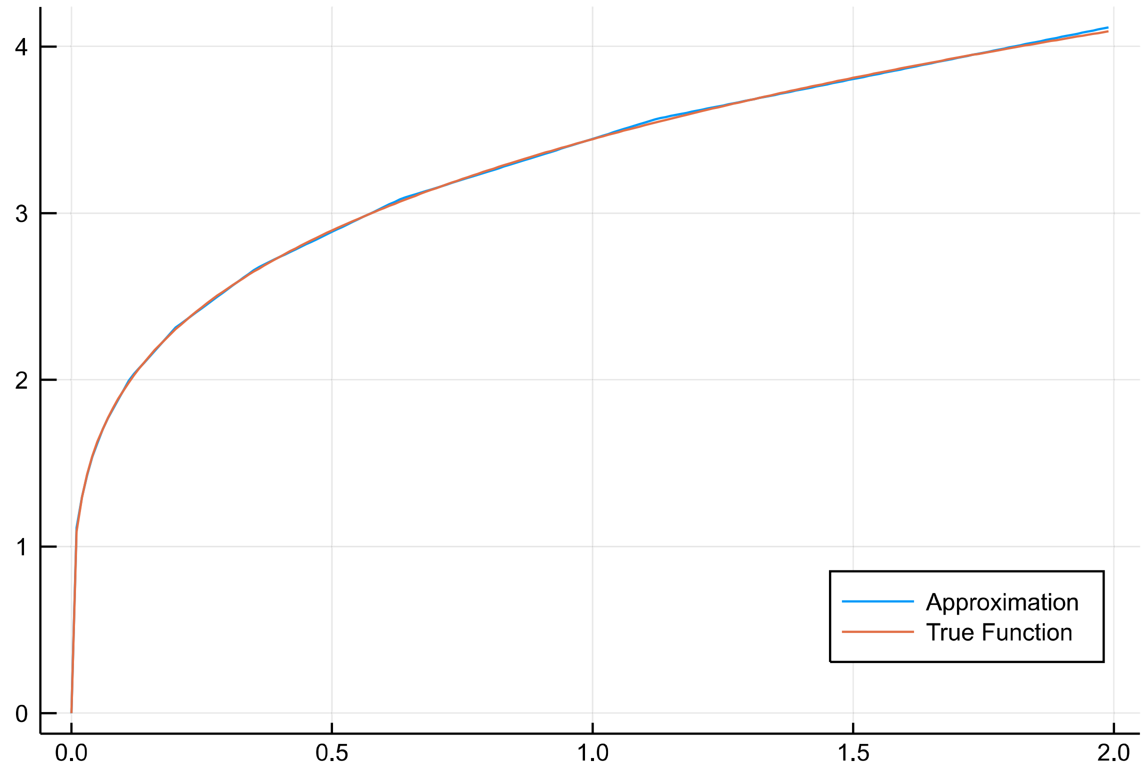
In [67]:
```
Checking if the residual is low at the solution
ini(α)
```

Out[67]:
```
8.706464890141266e-6
```

Below, we plot the approximate consumption function and the true one. As we can see, the approximation is barely indistinguible from the true function.

In [68]:
```
Plotting
=K[1]:0.01:K[end]

(k) = (1-β*θ)*A*k^θ
olkplot(k) = polk(k,α)
nplot(k) = cn(k,α)
lot(k,[cnplot.(k),c.(k)],label=["Approximation" "True Function"],legend=:bott
mright)
```

Out[68]:



Now, we do the same procedure for a case with $\delta = 0.05$. Note that now we can't compare to a known formula, but the shape should be similar to the one above.

In [43]:
```
New parameters
 = 0.05
 = (1-β*(1-δ))/(θ * β) #This will normalize the SS to 1

ss = ((1- β*(1-δ))/(β*A*θ))^(1/(θ-1))

olk(k,α) = min(max(eps(),A*k.^θ+(1-δ)*k-cn(k,α)),K[end]) #capital policy func
ion from Bugdet constraint

unction residual(k,α) #Residual function comes from FOCs
       R = β * cn(k,α)/cn(polk(k,α),α)  * (A*θ*polk(k,α)^(θ-1)+1-δ)- 1
   return R
nd
```

Out[43]:
```
residual (generic function with 1 method)
```

In [44]:
```
nitial = ones(length(K)-1)*0.0035
or i=2:length(initial)
   global initial
   initial[i] = initial[i-1] +0.000265*exp(0.589*(i-2))
nd
nitial
initial = ones(length(K)-1)*0.1.*range(0.03,stop = 8,length = length(K)-1)
ini(initial)
```

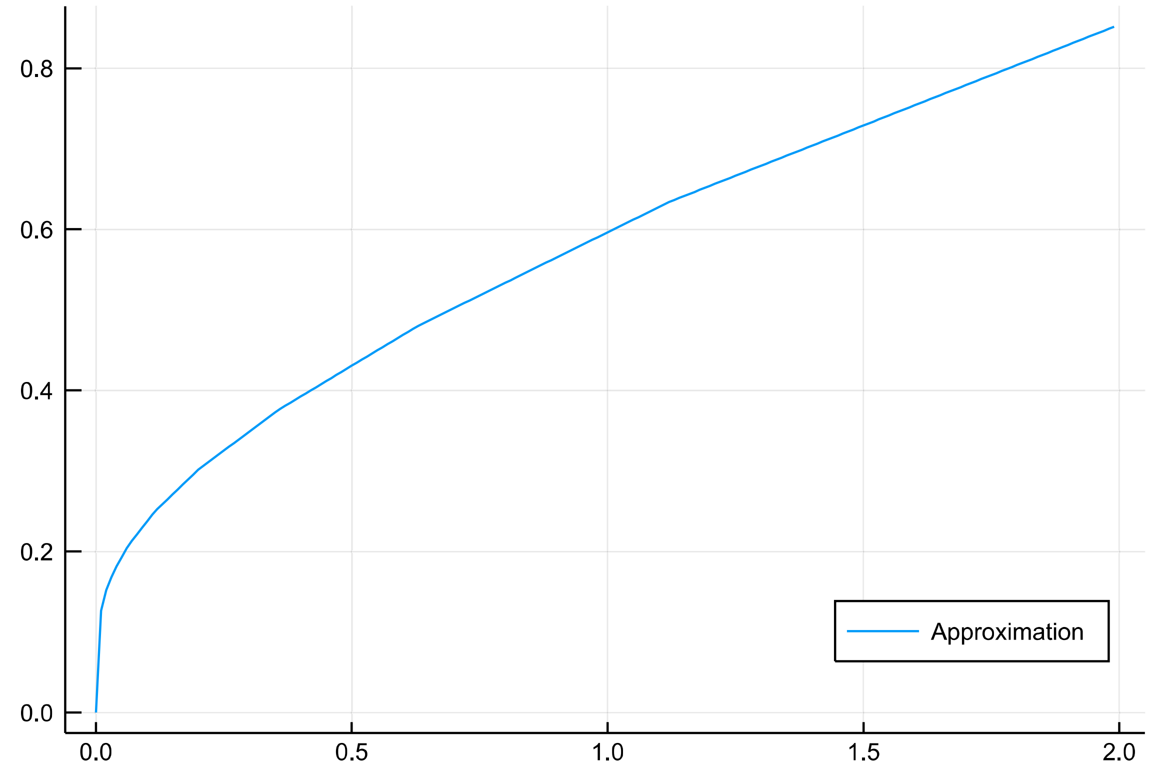
Out[44]:
```
0.560172602119699
```

In [45]:
```
Solver stuff
lower and upper bound of the parameters:
ower = zeros(length(initial))
pper = Inf*ones(length(initial))
Optimizer method is BFGS, see Judd's book page 114 for an explanation:
nner_optimizer = BFGS()

Solver:
la = optimize(mini,lower,upper,initial, Fminbox(inner_optimizer))
Parameters
 = vcat(0,bla.minimizer)
```

Out[45]:
```
15-element Array{Float64,1}:
 0.0
 0.01239167265675882
 0.09230419026395284
 0.021151728251957085
 0.115504132550029
 0.12844822914611462
 0.15135403379323256
 0.17588563782962213
 0.20737801051704224
 0.24777147103586775
 0.3013961059132636
 0.3754319853414289
 0.4808309867565117
 0.6344669728788809
 0.852052586048369
```

In [46]:
```
Plotting
=K[1]:0.01:K[end]

n2plot(k) = cn(k,α)
lot(k,[cn2plot.(k)],label=["Approximation"],legend=:bottomright)
```

Out[46]:



This seems a fair approximation to the consumption function.

Question 2

The Problem is:

$$\max \int_0^\infty e^{-\rho t}\log(c)\,dt$$

s.t.

$$dk = (Ak^\theta - \delta k)dt - c$$

The Bellman's equation for this problem is:

$$\rho V(k) = \max \log(c) + V'(k)(Ak^\theta - \delta k) - c)$$

The maximization problem FOC is:

$$\frac{1}{c} = V'(k)$$

By the envelope theorem:

$$\rho V'(k) = V'(k)(\theta A k^{\theta-1} - \delta) + V''(k)(Ak^{\theta} - \delta k - c)$$

From the FOC:

$$V''(k) = -\frac{1}{c(k)^2} c'(k)$$

Combining the expressions above, we get the differential equation:

$$c(k)(\theta A k^{\theta-1} - (\delta + \rho) = (Ak^{\theta} - \delta k - c(k))c'(k)$$

We also have in the Steady State:

$$\begin{aligned} k_{ss} &= \left(\frac{\delta+\rho}{\theta A}\right)^{\frac{1}{\theta-1}} \\ c_{ss} &= Ak_{ss}^{\theta} - \delta k_{ss} \end{aligned}$$

I don't really see how having $\delta = 1$ would help me having a test case. Candler's chapter in Marimon & Scott book claims that we do not have closed form solutions for arbitrary parameters, but we do have when utility is CRRA with parameter 0.5, which is not our case here. So, unfortunately I will not have a comparison.

The functional equation is thus:

$$F(c)(k) = c(k)(\theta A k^{\theta-1} - (\delta + \rho) - (Ak^{\theta} - \delta k - c(k))c'(k) = 0$$

Let's define some parameters:

```
In [47]:  Defining parameters
           = 0.25
           = -log(0.9)
           = 1
           = (δ + ρ)/θ  #This will normalize the SS to 1
          ss = ((δ+ρ)/(A*θ))^(1/(θ-1))

Out[47]: 1.0
```

We'll use the finite element methods, so all functions defined above are useful. We need only first to define the derivative of the piecewise linear function:

In [48]:
```
unction derivφi(x,X,i)
   if i>1 && i<length(X) #i is not in a boundary
       if X[i-1]<=x<=X[i]
           f = 1/(X[i]-X[i-1])
       elseif X[i]<=x<=X[i+1]
           f = -1/(X[i+1]-X[i])
       else
           f = 0
       end
   elseif i==1
       if X[i]<=x<=X[i+1]
           f = -1/(X[i+1]-X[i])
       else
           f = 0
       end
   elseif i==length(X)
       if X[i-1]<=x<=X[i]
           f = 1/(X[i]-X[i-1])
       else
           f=0
       end
   end
   return f
nd
```

Out[48]:
```
derivφi (generic function with 1 method)
```

In [49]:
```
Defining the elements:
 = zeros(15)
or i=2:length(K)
   global K
   K[i] = K[i-1] +0.0005*exp(0.574*(i-2))
nd
```

Out[49]:
```
15-element Array{Float64,1}:
 0.0
 0.0005
 0.0013876771423281366
 0.0029636185603518304
 0.005761472909207478
 0.010728655615292541
 0.019547164715211198
 0.03520314263003213
 0.06299805010159346
 0.11234385817285451
 0.19995014996199395
 0.35548235545269025
 0.6316071228726379
 1.1218264118113592
 1.9921393268498697
```

In this way, we can define the derivative of the consumption function:

In [50]:
```
unction derivcn(k,α;K=K)
This function is piecewise linear approximation for the derivative of a funct
on with arguments k
To get a good approximations one need to find the parameters α
K are the elements nodes. It is optionally defined here for efficiency.
       n = length(K)
       c = 0
       for i = 1:n
           c = c + α[i]*derivφi(k,K,i)
       end
       return c
nd
```

Out[50]:
```
derivcn (generic function with 1 method)
```

Naturally, we'll need to redefine the residual function:

In [51]:
```
nclude("finite_elements_functions.jl")
Defining the residual function from Euler equation:
unction residual(k,α)
   #This function is specific for the deterministic growth model.
   #Residual function comes from FOCs
   #cn is an approximation for consumption
   #derivcn is an approxiamtion for the derivative of consumption.
       R = θ*A*k^(θ-1)-(δ+ρ) - derivcn(k,α)/cn(k,α) * (A*k^θ-δ*k-cn(k,α))
   return R
nd
```

Out[51]:
```
residual (generic function with 1 method)
```

We can proceed with the maximization process as before:

In [52]:
```
Setting initial conditions

nitial = ones(length(K)-1)*0.05
or i=2:length(initial)
   global initial
   initial[i] = initial[i-1] +0.165*exp(0.08*(i-2))
nd
nitial
Check if the initial conditions are somewhat close to the true parameters.
ini(initial)
```

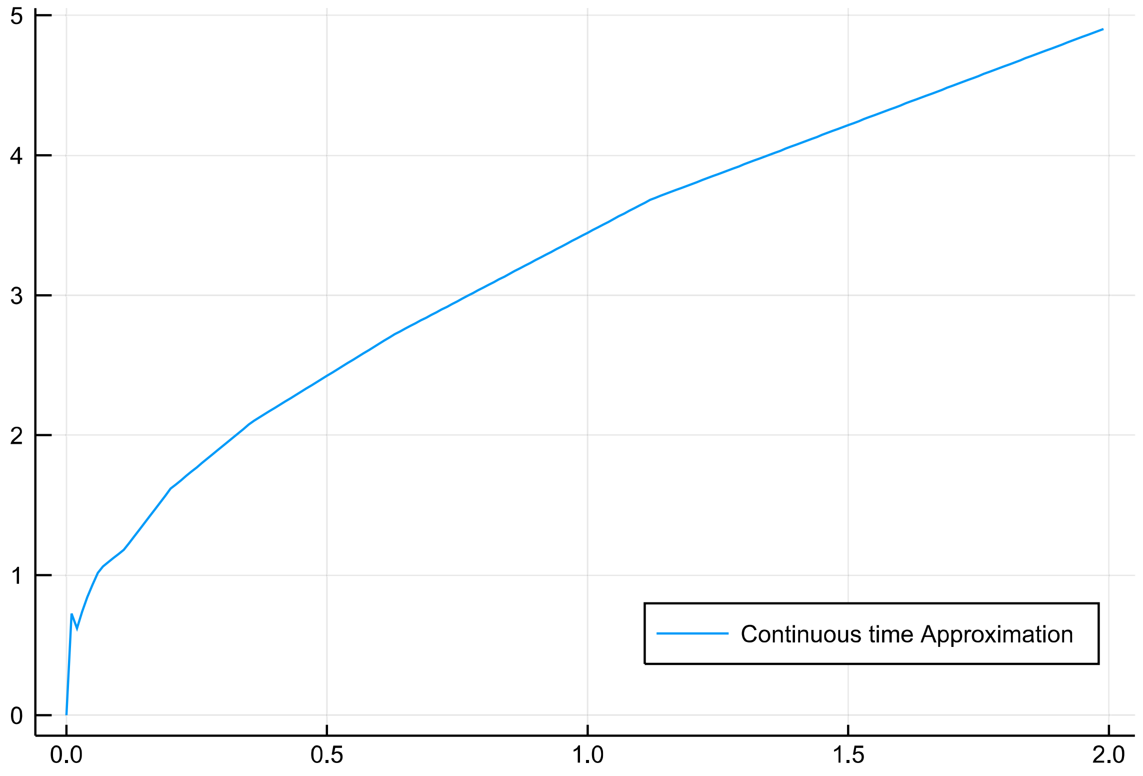
Out[52]:
```
2.052594240189306
```

In [53]:
```
Solver stuff:
   #Lower and upper bound of the parameters:
   lower = zeros(length(initial))
   upper = Inf*ones(length(initial))
   #Optimizer method is BFGS, see Judd's book page 114 for an explanation:
   inner_optimizer = BFGS()


Solver:
la = optimize(mini,lower,upper,initial, Fminbox(inner_optimizer))
Parameters
cont = vcat(0,bla.minimizer)
```

Out[53]:
```
15-element Array{Float64,1}:
 0.0
 0.19396272687684454
 0.42643297440642164
 0.659358606555276
 0.5615689980755327
 0.7529694430997189
 0.6153895256123474
 0.7995275668295624
 1.0422042487340248
 1.1889787934266627
 1.617465258477728
 2.0926639905250934
 2.7250222831048774
 3.6847649563027844
 4.906164184156026
```

In [54]:
```
Plotting
=K[1]:0.01:K[end]

c(k) = (1-β*ϑ)*A*k^ϑ
ontplot(k) = cn(k,αcont)
lot(k,[contplot.(k)],label=["Continuous time Approximation"],legend=:bottomri
ht)
```

Out[54]:



This looks like a good consumption function!

Question 3

It has been a while since the last time I opened the HW 1 codes. Sometimes a person should not revolve his past, but let's do it!

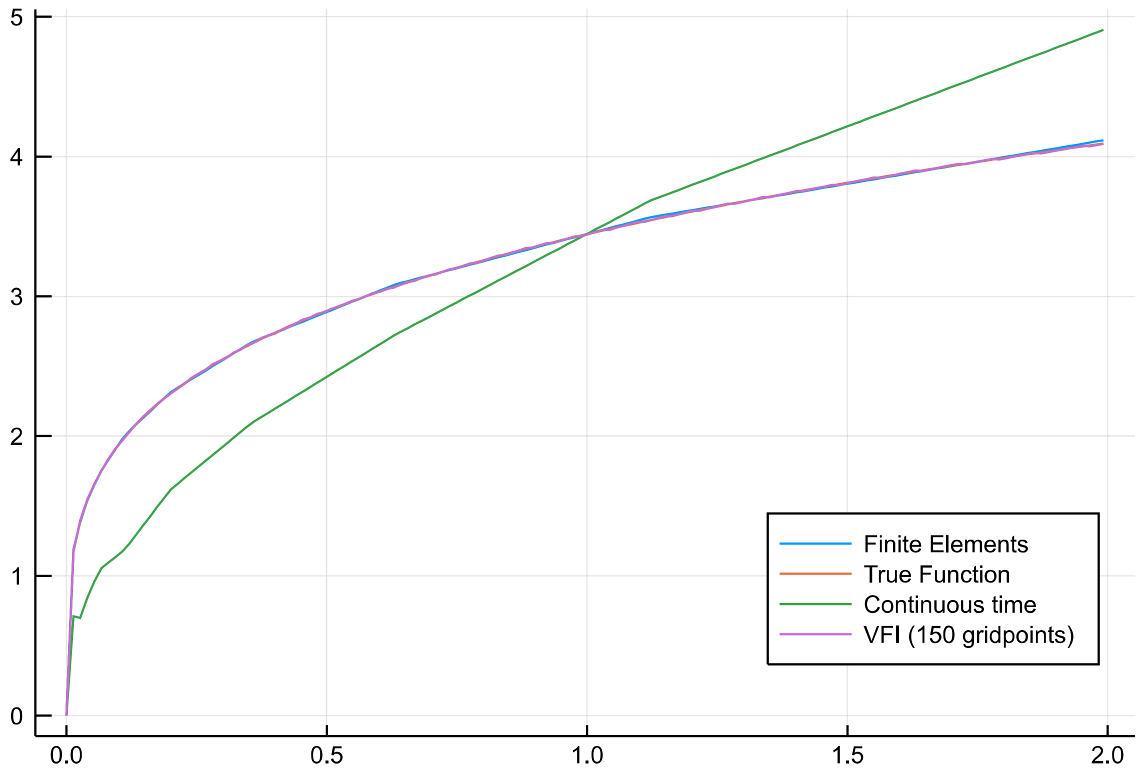
In [69]:
```
Get the VFI objects, see HW1 or the file "run_VFI.jl" to adjust parameters.
nclude("HW1_stuff\\run_VFI.jl")
```

Out[69]:
```
([-75.1926; 10.9757; … ; 12.5878; 12.5899], [1; 26; … ; 90; 90], [150; 150; …
; 150; 150], [0.000542535; 1.17705; … ; 4.08135; 4.09023], 2.220446049250313e
-16:0.013370062596307849:1.9921393268498697, 0.0:0.006711409395973154:1.0, 7.
307295035053245, 1)
```

Below, I plot the consumption policy functions for the discrete and continuous time case above and compare them with the VFI policy function I found when doing HW1.

In [70]:
```
Plotting the consumption policies
lot(k,[cnplot.(k),c.(k),contplot.(k),policy_c],label=["Finite Elements" "True
unction" "Continuous time" "VFI (150 gridpoints)"],legend=:bottomright)
```

Out[70]:



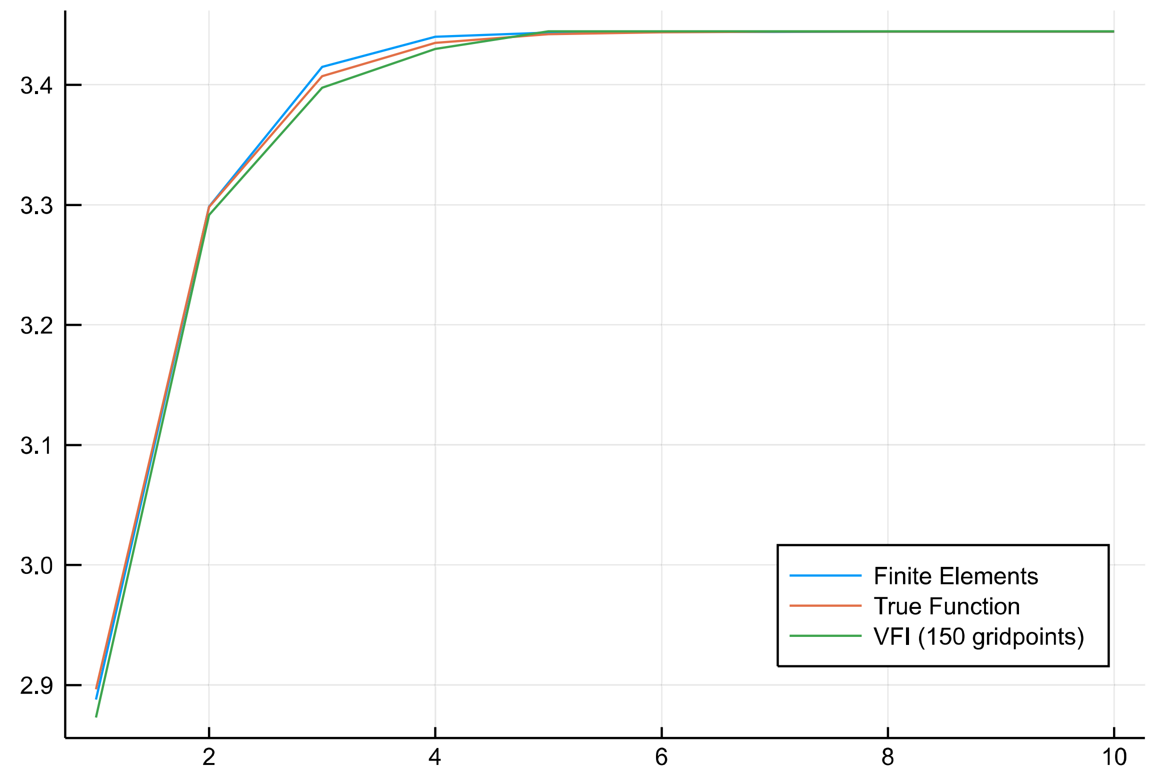
As we can see above, all the methods produce good results when compared to the true function (I'm using $\delta = 1$). I'm not sure if the continuous time case is comparable, since the model is using different assumptions and not only a different solution method. The problem with the VFI method is that I used gridpoints andthe function cannot be evaluated at points outside the grid.

Question 4

Below a plot of the consumption time series generated by each model starting at $0.5k_{ss}$, given the problems found above, I don't plot the continuous time. As we can se, there very small are differences in the time series when compared to the true function.

In [71]:
```
_finiteelements=0.5*ones(10)
_VFI_index = Int(floor(KVFI/4))*ones(Int,10)
rue_k =  0.5*ones(10)

or i=2:10
   global k_finiteelements, k_VFI_index
   k_finiteelements[i] = polkplot(k_finiteelements[i-1])
   k_VFI_index[i] = policy_k[k_VFI_index[i-1]]
   true_k[i] = β*θ*A*true_k[i-1]^θ
nd
_VFI = policy_c[k_VFI_index]
_finiteelements = cnplot.(k_finiteelements)
rue_c = c.(true_k)


lot(1:10,[c_finiteelements,true_c,c_VFI],label=["Finite Elements" "True Funct
on" "VFI (150 gridpoints)"],legend=:bottomright)
```

Out[71]:


Finite Elements
True Function
VFI (150 gridpoints)