ECON 8185 - HW 2

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The model is:

$$\max_{c_t,x_t,l_t} E\sum_{t=0}^{\infty}\beta^t\frac{\left(c_t l_t^{\psi}\right)^{1-\sigma}}{1-\sigma}N_t$$

S.T.

$$\begin{gathered}c_t+(1+\tau_{xt})x_t=r_tk_t+(1-\tau_{ht})w_th_t+T_t\\ N_{t+1}k_{t+1}=[(1-\delta)k_t+x_t]N_t\\ h_t+l_t=1\\ S_t=PS_{t-1}+Q\epsilon_t,\quad S_t=[\ln z_t,\tau_{ht},\tau_{xt},\ln g_t]\\ c_t,x_t\geq 0\end{gathered}$$

Where $N_t=(1+\gamma_n)^t$ and firm tecnhology is $Y_t=K_t^{\theta}(Z_tL_t)^{1-\theta}$. γ_z is the rate of labor-augmenting technical progress. The stochastic shocks hitting this economy affect technology, tax rates, and government spending and the stochastic processes are modeled as a VAR(1) process. The resource constraint is $Y_t=N_t(c_t+x_t+g_t)$. We must work with detrended variables, we should use "hat" variables, but typing this is time consuming so from now on all variables should be understood as detrended unless I say something about them.

Defining some parameters:

In [1]:
```
using Plots, NLsolve, ForwardDiff, DataFrames, LinearAlgebra, QuantEcon, Plots
, Optim, Statistics
#Parameters:
δ = 0.0464   #depreciation rate
θ = 0.35  #capital share of output
β = 0.9722  #Discouting
σ = 1  #Elasticity of Intertemporal Substitution
ψ = 3    #Labor parameter
γn= 0.015    #Population growth rate
γz= 0.016   #Productivitu growth rate
gss = 0.01 #average g
τxss = 0.05 #average τx
τhss = 0.05 #average τh
zss = 0.0 #average z (z is in logs)


#Parameters to be estimated in the next homework
#Autocorrelations
ρg = 0.9
ρx = 0.1
ρh = 0.1
ρz = 0.9

# Cross-correlations
ρzg= 0.0
ρzx = 0.0
ρzh = 0.0
ρhz = 0.0
ρhx = 0.0
ρhg = 0.0
ρxz = 0.0
ρxh = 0.0
ρxg = 0.0
ρgz = 0.0
ρgx = 0.0
ρgh = 0.0

#Variances
σg= 0.001
σx = 0.001
σz = 0.01
σh = 0.01
#Covariances
σzg= 0.0
σzx = 0.00
σzh = 0.00
σhx = 0.00
σhg = 0.00
σxg = 0.00
```

Out[1]: 0.0

The detrended FOC's of this model are:

$$c_t + (1+\gamma_z)(1+\gamma_n)k_{t+1} - (1-\delta)k_t + g_t = y_t = k_t^{\theta}(z_t h_t)^{1-\theta}$$
$$\psi \frac{c_t}{1-h_t} = (1-\tau_{ht})(1-\theta)\left(\frac{k_t}{h_t}\right)^{\theta} z_t^{1-\theta}$$
$$c_t^{-\sigma}(1-h_t)^{\psi(1-\sigma)}(1+\tau_{xt}) = \beta(1+\gamma_z)^{-\sigma} E_t c_{t+1}^{-\sigma}(1-h_{t+1})^{\psi(1-\sigma)}\left(\theta k_{t+1}^{\theta}(z_{t+1}h_{t+1})^{1-\theta} + (1-\delta)(1+\right.$$

Substituting for c, we get an equation for k, and one for h. Below, I find the Steady State values:

In [2]:
```
#Function with the FOCs
zss = exp(zss)

function SS!(eq,vector::Vector)
    k,h, c= vector
    eq[1]=k/h-((1+τxss)*(1-β*(1+γz)^(-σ)*(1-δ))/(β*(1+γz)^(-σ)*θ*zss^(1-θ)) )^
(1/(θ-1))
    eq[2]=c-( (k/h)^(θ-1)*zss^(1-θ) -(1+γz)*(1+γn)+1-δ)*k+gss
    eq[3]=ψ*c-( (1-τhss)*(1-θ)*(k/h)^θ *zss^(1-θ))*(1-h)
end

SteadyState = nlsolve(SS!, [3,0.25,.4],ftol = :1.0e-20)
kss,hss,css = SteadyState.zero
```

Out[2]:
```
3-element Array{Float64,1}:
 1.6707265229440968
 0.22844261316599446
 0.31866093597940304
```

In [3]:
```
#GDP
    yss = kss^(θ)*(zss*hss)^(1-θ)
    xss = (1+γz)*(1+γn)*kss-(1-δ)*kss
```

Out[3]:
```
0.12971520724137942
```

Question 1

a) Iterate on Bellman’s equation

Due to the curse of dimensionality, I won't pursue this path.

b) Map it to a linear quadratic problem

Recall from lecture notes, we have to map the original problem into the following LQ problem:

$$\max_{\{u_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (X_t' Q X_t + u_t' R u_t + 2X_t' W u_t)$$

s.t.

$$X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1}$$

X_0 given.

In this case, we have: $u_t = [k_{t+1}, h_t]'$

Note that in this problem X_t may be decomposed as:

$$X_t = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_t$$

Where X_1 are the individual states, X_2 are the aggregate or exogenous states with known laws of motion, and X_3 are the aggregate states with laws of motion that are unknown and need to be computed in equilibrium. We have that:

$X_1 = [1, k_t]'$, $X_2 = [\tau_{xt}, \tau_{ht}, g_t]$ and $X_3 = [K_t, H_t]$

Finally, rewrite $y_t = [\tilde{X}_1, \tilde{X}_2]'$ and the problem constraint becomes:

$$y_{t+1} = \tilde{A}_y y_t + \tilde{B}_y \tilde{u}_t + A_z \tilde{X}_{3t}$$

Where tilde variables are the undiscounted counterpart of each variable. Matrices A_y, B_y, Q, R and W may be found by second and first order Taylor expansions of the utility function and constraints. Matrix A_z is for now unknown. Following the methods in the lecture notes (using Big K, little k trick) this problem may be solved. We mapped it to a LQ problem and we solve it using the modified Vaughan's method in the next section:

c) Apply Vaughan's method.

I use the modified Vaughan method using the log linearized FOC's as in the lecture notes.

Log-linearizing the FOC equations we get the following system of equations:

$$0 = E_t[a_1\tilde{k}_t + a_2\tilde{k}_{t+1} + a_3\tilde{h}_t + a_4\tilde{z}_t + a_5\tilde{\tau}_{ht} + a_6\tilde{g}_t]$$
$$0 = E_t[b_1\tilde{k}_t + b_2\tilde{k}_{t+1} + b_3\tilde{k}_{t+2} + b_4\tilde{h}_t + b_5\tilde{h}_{t+1} b_6\tilde{z}_t + b_7\tilde{\tau}_{xt} + b_8\tilde{g}_t + b_9\tilde{z}_{t+1} + b_{10}\tau_{xt+1} + b_{11}\tilde{g}_{t+1}]$$

Where tilde variables are log deviations from Steady State. Stacking up the equations we get:

$$0 = E_t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & b_5 \end{bmatrix} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{k}_{t+2} \\ \tilde{h}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_4 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{k}_{t+1} \\ \tilde{h}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & 0 & 0 \\ b_6 & 0 & b_7 & b_8 & b_9 & 0 & b_{10} & b_{11} \end{bmatrix}$$

We call the first matrix A_1, and the second A_2. The code below log-linearizes and find these matrices:

In [4]:
```
function loglineq1(vector::Vector)
    k,k1,h,z,τh,g= vector

    c = k^θ  * ((z *h)^(1-θ)) - ((1+γz)*(1+γn)*k1-(1-δ)*k+ g )
    eq =(ψ *c)^(1/θ)  - (k/h)*((1-h)*(1-τh)*(1-θ)*z^(1-θ))^(1/θ)

    return eq
end
function loglineq2(vector::Vector)
    k,k1,k2,h,h1,z,τx,g,z1,τx1,g1 = (vector)
    c = k^θ  * ((z *h)^(1-θ)) - ((1+γz)*(1+γn)*k1-(1-δ)*k+ g )
    c1 = k1^θ * ((z1 *h1)^(1-θ)) - ((1+γz)*(1+γn)*k2-(1-δ)*k1+ g1 )
    eq =  (c^(-σ) *(1-h)^(ψ*(1-σ))*(1+τx)  - (1-δ)*(1+τx1)* β*(1+γz)^(-σ) * c1
^(-σ) * (1-h1)^(ψ*(1-σ)))^(-1/θ) -
       (β*(1+γz)^(-σ) * c1^(-σ) * (1-h1)^(ψ*(1-σ)) * θ*(z1*h1)^(1-θ))^(-1/θ)*
k1
    return eq
end


#Log deviations
T=ForwardDiff.gradient(loglineq1,[kss,kss,hss,zss,τhss,gss])
a =[-kss*T[1]/(kss*T[1]),-kss*T[2]/(kss*T[1]),-hss*T[3]/(kss*T[1]),
-zss*T[4]/(kss*T[1]),-τhss*T[5]/(kss*T[1]),-gss*T[6]/(kss*T[1])]
#if ψ==0
#    a[1],a[2:end]=-1,zeros(5)
#end

T=ForwardDiff.gradient(loglineq2,[kss,kss,kss,hss,hss,zss,τxss,gss,zss,τxss,gs
s])
b = [kss*T[1]/(-kss*T[1]),kss*T[2]/(-kss*T[1]),kss*T[3]/(-kss*T[1]),hss*T[4]/(
-kss*T[1]),
hss*T[5]/(-kss*T[1]),zss*T[6]/(-kss*T[1]),τxss*T[7]/(-kss*T[1]),gss*T[8]/(-kss
*T[1]),
zss*T[9]/(-kss*T[1]),τxss*T[10]/(-kss*T[1]),gss*T[11]/(-kss*T[1])]

A1 = [1 0 0; 0 0 0; 0 b[3] b[5]]
A2 = [0 -1 0; a[1] a[2] a[3]; b[1] b[2] b[4]]
U = [0 0 0 0 0 0 0 0;
a[4] a[5] 0 a[6] 0 0 0 0;
b[6] 0 b[7] b[8] b[9] 0 b[10] b[11]]

A1,A2
```

Out[4]:
```
([1.0 0.0 0.0; 0.0 0.0 0.0; 0.0 -0.951883 0.157952], [0.0 -1.0 0.0; -1.0 1.04
921 -0.306816; -1.0 1.95492 -0.169901])
```

We look for a solution of the form:

$$\tilde{k}_{t+1} = A\tilde{k}_t + BS_t$$
$$Z_t = CX_t + DS_t$$
$$S_t = PS_{t-1} + Q\epsilon_t$$

Where $Z_t = [\tilde{k}_{t+1}, \tilde{h}_t]'$ and S_t are the stochastic exogenous variables. We compute the generalized eigenvalues and eigenvectors for matrices A_1 and $-A_2$ because A_1 is not invertible. Thus, $A_2V = -A_1V\Pi$ and we can get A and C by:

$$A = V_{11}\Pi_{1,1}V_{1,1}^{-1}$$
$$C = V_{2,1}V_{1,1}^{-1}$$

In [5]:
```
eig = eigen(A1,-A2)
V=eig.vectors
Π = eig.values
#Sorting
for j=1:3
for i=1:2
    if eps(Float64)<abs(Π[i+1])<abs(Π[i])
        Π[i],Π[i+1] = Π[i+1],Π[i]
        V[:,i],V[:,i+1] = V[:,i+1],V[:,i]
    elseif abs(Π[i]) < eps(Float64)
        Π[i],Π[end] =Π[end],Π[i]
        V[:,i],V[:,end]=V[:,end],V[:,i]
    end
end
end
if abs(Π[1])>1
    error("All Eigen Values outside unit circle")
end
Π= Diagonal(Π)
```

Out[5]:
```
3×3 Diagonal{Float64,Array{Float64,1}}:
 0.900027   ⋅         ⋅
  ⋅        1.02516    ⋅
  ⋅         ⋅        -2.43595e-17
```

In [6]:
```
A = V[1,1]*Π[1,1]*inv(V[1,1])
C = V[2:end,1]*(V[1,1])
C = hcat(C,zeros(2,1))
```

Out[6]:
```
2×2 Array{Float64,2}:
 0.900027  0.0
 0.437631  0.0
```

In [7]:
```
P = [ρz ρzh ρzx ρzg;
ρhz ρh ρhx ρhg ;
ρxz ρxh ρx ρxg ;
ρgz ρgh ρgx ρg]
Q = [σz σzh σzx σzg;
σzh σh σhx σhg ;
σzx σhx σx σxg ;
σzg σhg σxg σg]
```

Out[7]:
```
4×4 Array{Float64,2}:
 0.01  0.0   0.0    0.0
 0.0   0.01  0.0    0.0
 0.0   0.0   0.001  0.0
 0.0   0.0   0.0    0.001
```

Finally, to get the matrices B and D, we just need to solve a linear system of equations (see Ellen's notes):

In [8]:
```
#=
        function system!(eq,vector::Vector)
            #vector = rand(8)
            #eq= rand(8)
            B=vector[1:4]'
            D2 = vector[5:8]'
            eq[1:4] = a[2].*B .+ a[3].*D2 .+ [a[4] a[5] 0 a[6]]
            eq[5:8] = b[2].*B .+ b[3].*A.*B .+ b[3].*B*P .+ b[4].*D2 .+ b[5].
*C[2].*B .+ b[5].*B*P.+
            [b[6] 0 b[7] b[8]].+[b[9] 0 b[10] b[11] ]*P
         return     eq
        end
        Sol = nlsolve(system!, ones(8),ftol = :1.0e-20, method = :trust_regio
n , autoscale = true)
        D=ones(2,4)
        D[1,:]= Sol.zero[1:4]
        D[2,:]= Sol.zero[5:8] =#


        #Implementing the code solve the system commented out above as a syst
em of linear equations.
        TOP = hcat(a[2]*Matrix{Float64}(I,4,4),   a[3]*Matrix{Float64}(I,4,4
))
        #Everything multiplying B concatenated with stuff multiplying D in th
e first equations
        BOTTOMLEFT =  b[2]*I + b[3].*A *I + b[3]*P + b[5].*C[2]*I + b[5]*P
        #Everything multiplying B in the last equations
        BOTTOM = hcat(BOTTOMLEFT',  b[4]*Matrix{Float64}(I,4,4)) #Concatenate
s with stuff multiplying D
        RHS = - vcat([a[4] a[5] 0 a[6]]',  ([b[6] 0 b[7] b[8]].+[b[9] 0 b[10]
b[11] ]*P)') #Constant terms
        #Solving the system
        BD = (vcat(TOP,BOTTOM)\RHS)[:]
        D=ones(2,4)
        D[1,:]= BD[1:4]
        D[2,:]= BD[5:8]
```

Out[8]:
```
4-element Array{Float64,1}:
 -0.10342712059603074
 -0.07144190072971895
 -0.05304635425032059
 -0.05061918699972633
```

Question 2

For this question I use the solution of question 1-c).

First, I will rewrite the model in the form:

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$
$$Y_t = CX_t + \omega_t$$

Where, $X_t = [k_t, s_t]$, $s_t = [z_t, \tau_{ht}, \tau_{xt}, g_t]$, $Y_t = [y_t, x_t, h_t]$ and as before:

$$s_{t+1} = Ps_t + Q\varepsilon_{t+1}$$

We need to log linearize y, x since we already done it for labor:

In [9]:
```
#Rewritting
A = hcat(vcat(C[1],zeros(4,1)),vcat(D[1,:]',P))
B = hcat(zeros(5,1),vcat(zeros(1,4),Q))


#We have h as function of states. To find the Matrix B, we need to find y and
 x
#as a function of states

function kt1(vector::Vector)
    k,z,τh,τx,g = vector
    tilde = log.([k,z,τh,τx,g]).-log.([kss,zss,τhss,τxss,gss])
    for i = 1:length(tilde)
        if isnan(tilde[i])
            tilde[i] = 0
        end
    end

    k1= A[1,:]' * tilde
    return k1
end


function ht(vector::Vector)
    k,z,τh,τx,g = vector
    tilde = log.([k,z,τh,τx,g]).-log.([kss,zss,τhss,τxss,gss])
    for i = 1:length(tilde)
        if isnan(tilde[i])
            tilde[i] = 0
        end
    end
    h = C[2,1]*(log(k)-log(kss)) + D[2,:]' * tilde[2:end]
    return h
end
```

Out[9]:
```
ht (generic function with 1 method)
```

In [10]:

```
#Log-linearizing y as a function of states
function yt(vector::Vector)
    k,z,τh,τx,g = vector
    h = exp(ht(vector)+log(hss))
    y = k^θ * (z*h)^(1-θ)
    return y
end

#GDP
yss = kss^(θ)*(zss*hss)^(1-θ)

T=ForwardDiff.gradient(yt,[kss,zss,τhss,τxss,gss])
ycoefs = [kss*T[1]/yss,zss*T[2]/yss,τhss*T[3]/yss,τxss*T[4]/yss,gss*T[5]/yss]



#Log linearizing x as function of states
function xt(vector::Vector)
    k,z,τh,τx,g = vector
    k1 = exp(kt1(vector)+log(kss))
    x= (1+γn)*(1+γz)k1 - (1-δ)k

    return x
end
xss = (1+γz)*(1+γn)*kss-(1-δ)*kss
T=ForwardDiff.gradient(xt,[kss,zss,τhss,τxss,gss])
xcoefs = [kss*T[1]/xss,zss*T[2]/xss,τhss*T[3]/xss,τxss*T[4]/xss,gss*T[5]/xss]

#We have the matrix C!
C = [ycoefs[1] ycoefs[2] ycoefs[3] ycoefs[4] ycoefs[5];
xcoefs[1] xcoefs[2] xcoefs[3] xcoefs[4] xcoefs[5];
C[2,1] D[2,1] D[2,2] D[2,3] D[2,4];
0 0 0 0 1]
```

Out[10]:

```
4×5 Array{Float64,2}:
  0.63446    0.582772  -0.0464372  -0.0344801  -0.0329025
 -0.327873   0.29839   -0.148191   -0.206037   -0.273701
  0.437631  -0.103427  -0.0714419  -0.0530464  -0.0506192
  0.0        0.0        0.0         0.0         1.0
```

Now, I simulate the variables with known law of motion:

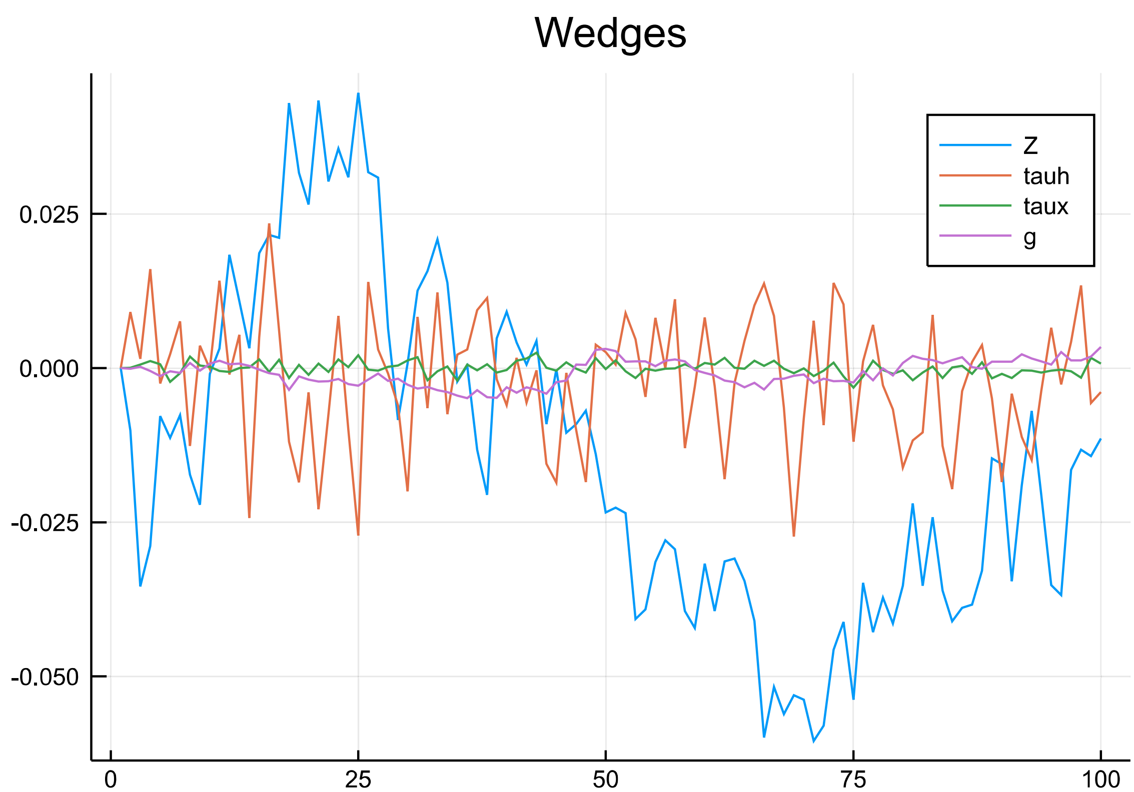
In [11]:

```
#defining the vectors
T=100
X= ones(5,T).* [0,0,0,0,0]
Y = ones(4,T).*[0,0,0,0]
S = randn(5,T)
for t=1:T

    if t>1
    X[:,t] = A*X[:,t-1]+ B*S[:,t]
    end
    Y[:,t] = C*X[:,t]
end

plot([X[2,:],X[3,:],X[4,:],X[5,:]],title ="Wedges", labels = ["Z","tauh","tau
x","g"])
```

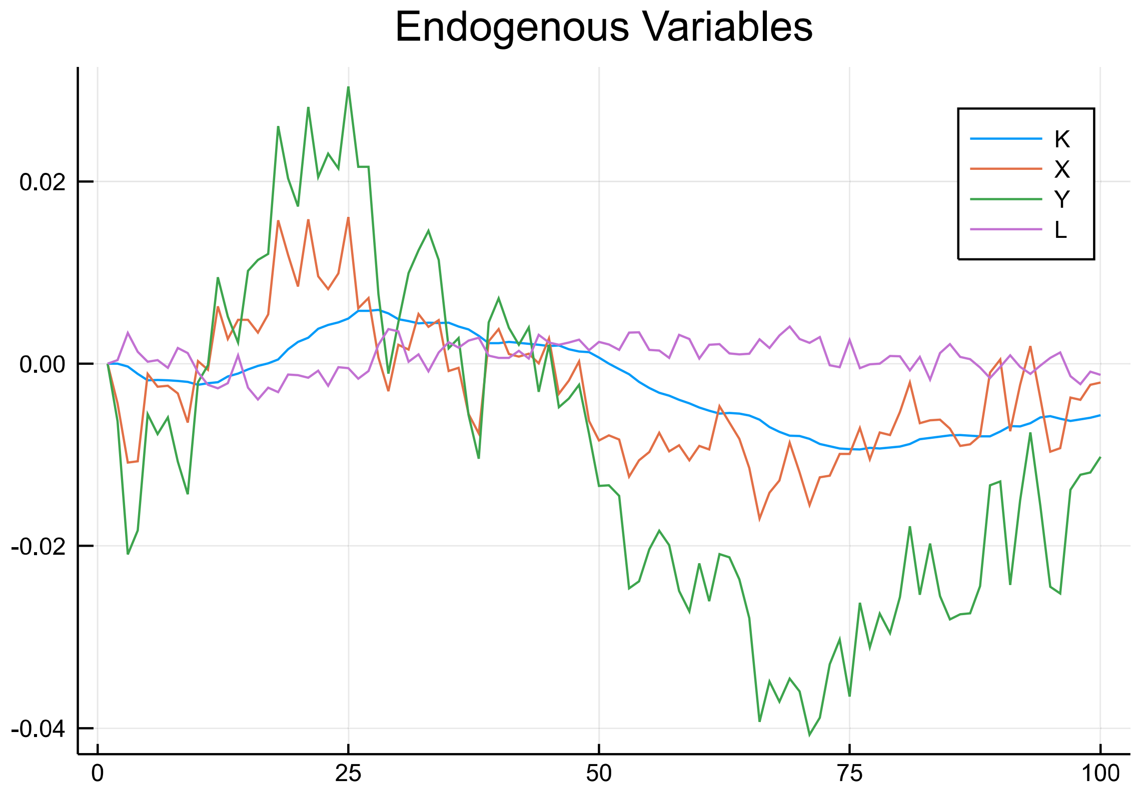
Out[11]:



Below are the endogenous variables:

In [12]:
```
plot([X[1,:],Y[2,:],Y[1,:],Y[3,:]],title = "Endogenous Variables",labels = [
"K","X","Y","L"])
```

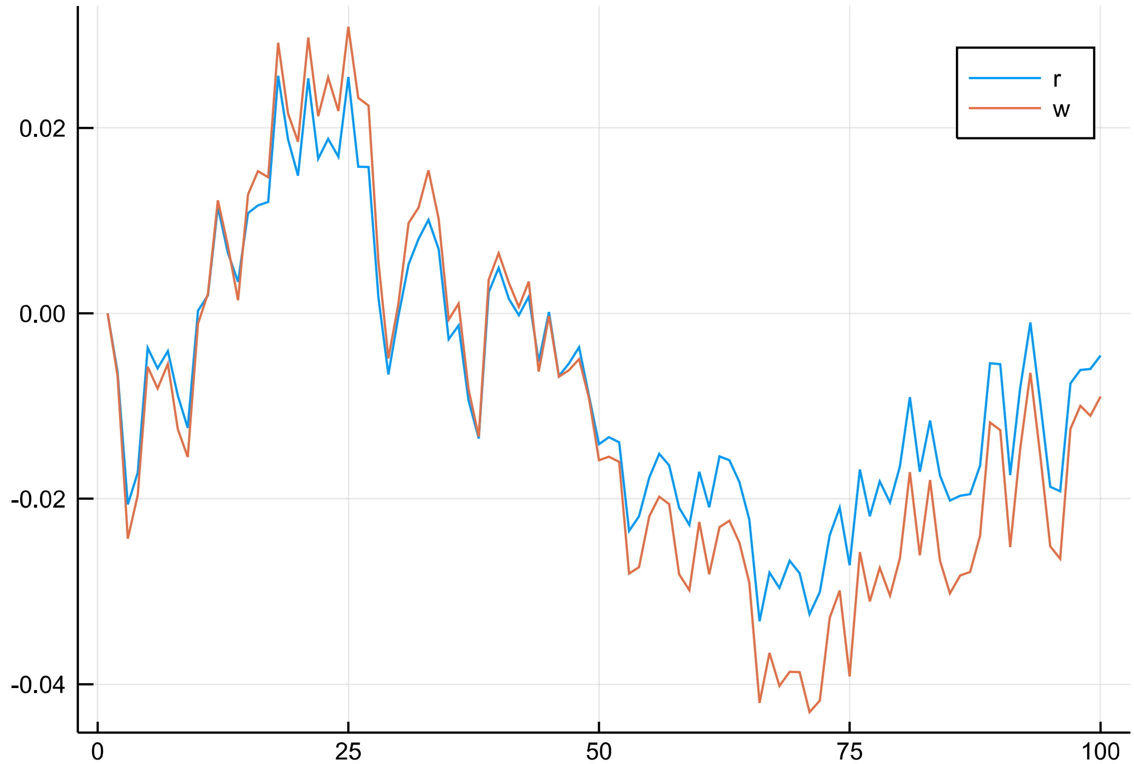
Out[12]:



In the code below, we calculate the factor prices (in log deviations):

In [13]:
```
r = (θ-1) .* X[1,:] .+ (1-θ) .*(X[2,:] .+ Y[3,:])
w = θ.* X[1,:] .+ (1-θ) .* X[2,:] .- θ .* Y[3,:]
plot([r,w], labels = ["r","w"])
```

Out[13]:



The code above is summarized in a function State_Space which returns the matrices A,B,C

In [14]:
```
include("State_Space.jl")
```

Out[14]:
```
State_Space (generic function with 1 method)
```

In [15]:
```
Y
```

Out[15]:
```
4×100 Array{Float64,2}:
 0.0  -0.00632491   -0.0209341    -0.0182986    …  -0.0119489    -0.0102153
 0.0  -0.00434373   -0.0108533    -0.0107298       -0.0023196    -0.00207157
 0.0   0.0004028     0.00336925    0.00131172      -0.00089481   -0.00123216
 0.0  -0.000127274   0.000195276  -0.000460609      0.00184424    0.00343165
```

In [16]:
```
X
```

Out[16]:
```
5×100 Array{Float64,2}:
 0.0   0.0          -0.000327031  -0.00111953   …  -0.00593657   -0.00566425
 0.0  -0.0101334    -0.0353994    -0.0288606       -0.0142915    -0.0114337
 0.0   0.00909905    0.00151298    0.0160519       -0.00567431   -0.00387934
 0.0   3.13228e-5    0.000582698   0.00112821       0.00163903    0.000740905
 0.0  -0.000127274   0.000195276  -0.000460609      0.00184424    0.00343165
```

In [17]:
```
C
```

Out[17]:
```
4×5 Array{Float64,2}:
  0.63446    0.582772  -0.0464372  -0.0344801  -0.0329025
 -0.327873   0.29839   -0.148191   -0.206037   -0.273701
  0.437631  -0.103427  -0.0714419  -0.0530464  -0.0506192
  0.0        0.0        0.0         0.0         1.0
```