Project Two Template

MAT-350: Applied Linear Algebra

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Problem 1

Use the svd() function in MATLAB to compute A_1, the **rank-1 approximation of A.** Clearly state what A_1 is, rounded to 4 decimal places. Also, **compute** the root-mean square error (RMSE) between A and A_1. Solution:

```
% 3 x 3 Matrix A
A = [1 2 3;
     3 3 4;
     5 6 7]
```

```
A = 3×3
     1     2     3
     3     3     4
     5     6     7
```

```
[U,S,V] = svd(A)
```

```
U = 3×3
   -0.2904    0.9504   -0.1114
   -0.4644   -0.2418   -0.8520
   -0.8367   -0.1957    0.5115
S = 3×3
   12.5318         0         0
         0    0.9122         0
         0         0    0.3499
V = 3×3
   -0.4682   -0.8261   -0.3136
   -0.5581    0.0012    0.8298
   -0.6851    0.5635   -0.4616
```

```
% finding the rank-1 approximation of A
A1 = U(:,1:1) * S(1:1,1:1) * V(:,1:1)'
```

```
A1 = 3×3
    1.7039    2.0313    2.4935
    2.7243    3.2477    3.9867
    4.9087    5.8517    7.1832
```

```
A1 = round(A1,4)  % round A1 values to 4 decimals
```

```
A1 = 3×3
    1.7039    2.0313    2.4935
    2.7243    3.2477    3.9867
    4.9087    5.8517    7.1832
```

```
RankA1 = rank(A1) % check the rank
```

```
RankA1 =
3
```

```
% getting RMSE between A and A1
rmse_1 = sqrt(mean((A(:)-A1(:)).^2))
```

```
rmse_1 =
0.3256
```

Problem 2

Use the svd() function in MATLAB to compute A_2, the **rank-2 approximation of** A. Clearly state what A_2 is, rounded to 4 decimal places. Also, **compute** the root-mean square error (RMSE) between A and A_2. Which approximation is better, A_1 or A_2? Explain. Solution:

```
% Computing rank approximation, by adding the first two singular components
A2 = U(:,1)*S(1,1)*V(:,1)' + U(:,2)*S(2,2)*V(:,2)'
```

```
A2 = 3×3
    0.9878    2.0324    2.9820
    2.9065    3.2474    3.8624
    5.0561    5.8515    7.0826
```

```
% Showing rank of A2
RankA2 = rank(A2)
```

```
RankA2 =
2
```

```
% getting RMSE between A and A2
rmse_2 = sqrt(mean((A(:)-A2(:)).^2))
```

```
rmse_2 =
0.1166
```

Explain:

RMSE: accounts for how different my approximation is from the original matrix. It measures the average error between A1 and A2. A smaller RMSE means the approximation is closer to the real matrix and fits it better.

Problem 3

For the 3×3 matrix A, the singular value decomposition is $A = USV'$ where $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$. Use MATLAB to **compute** the dot product $d_1 = dot(\mathbf{u}_1, \mathbf{u}_2)$.

Also, use MATLAB to **compute** the cross product $\mathbf{c} = cross(\mathbf{u}_1, \mathbf{u}_2)$ and dot product $d_2 = dot(\mathbf{c}, \mathbf{u}_3)$. Clearly state the values for each of these computations. Do these values make sense? **Explain**. Solution: $

```
% Extracting three column vectors from U
U1 = U(:,1), U2 = U(:,2), U3 = U(:,3)
```

```
U1 = 3×1
```

```
   -0.2904
   -0.4644
   -0.8367
U2 = 3×1
    0.9504
   -0.2418
   -0.1957
U3 = 3×1
   -0.1114
   -0.8520
    0.5115
```

```
% compute the cross product c = cross(u1,u1); compute the dot product d2 =
dot(c, u3)
c = cross(U1, U2), d2 = dot(c, U3)
```

```
c = 3×1
   -0.1114
   -0.8520
    0.5115
d2 =
1.0000
```

Explain:

The dot product d2 = dot(c, U3) is 1.0000, meaning C and U3 point the same way and both have a length of 1. This shows that U1, U2, and U3 are orthogonal unit vectors. That just means they're all perpendicular to each other and each one has a magnitude of 1. Together they make a right-handed orthonormal basis.

Problem 4

Using the matrix $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$, d*etermine whether or not the columns of* U **span** $\mathbb{R}^3$. **Explain your approach.**

Solution:

```
% Matrix u = [u1, u2 u3]
U = [U1, U2, U3]
```

```
U = 3×3
   -0.2904    0.9504   -0.1114
   -0.4644   -0.2418   -0.8520
   -0.8367   -0.1957    0.5115
```

```

% double check and test the Rank
r = rank(U)
```

```
r =
3
```

```
spans = (r ==3) % check span is = R^3; spans = logical should be 1.
```

```
spans = logical
   1
```

```
% Use RREF to show RREF View
RREF_U = rref(U)
```

```
RREF_U = 3×3
     1     0     0
     0     1     0
     0     0     1
```

Explain:

rank(U) = 3, so they span R^3

Additionally, you can see in echelon form that the matrix reduces to the identity. This means each column is independent, and together they cover the whole 3-D space.

Problem 5

Use the MATLAB imshow() function to load and display the image A stored in the image.mat file, available in the Project Two Supported Materials area in Brightspace. For the loaded image, **derive the value of** k that will result in a compression ratio of $CR \approx 2$. For this value of k, **construct the rank-_k_ approximation of the image**. Solution:

```
% use imshow() to load and display image A
load("/Users/jguida941/Downloads/MAT 350 Project Two MATLAB Image.mat")
imshow(A)
```

```
% get the size of the image
[m,n] = size(A)
```

```
m =
2583
n =
4220
```

```
% 1st I set the target compression ratio (CR) and solve for k
% The formula is CR = (m*n) / (k*(m + n + 1))
% It is rearranged to find k that gives the CR
CR = 2;
k = round((m*n)/(CR*(m+n+1)));
disp(['Calculated k for CR≈', num2str(CR), ' is ', num2str(k)])
```

```
Calculated k for CR≈2 is 801
```

```
% make the rank-k version of the image
[U,S,V] = svd(double(A),'econ')
```

```
U = 2583×2583
    0.0106    0.0360    0.0006   -0.0032    0.0032   -0.0041    0.0066   -0.0022 ···
    0.0105    0.0361    0.0006   -0.0030    0.0035   -0.0049    0.0062   -0.0020
    0.0105    0.0362    0.0006   -0.0034    0.0037   -0.0042    0.0064   -0.0025
    0.0105    0.0362    0.0009   -0.0029    0.0035   -0.0052    0.0056   -0.0028
    0.0106    0.0361    0.0011   -0.0034    0.0035   -0.0046    0.0061   -0.0022
    0.0106    0.0363    0.0011   -0.0031    0.0030   -0.0049    0.0061   -0.0031
    0.0106    0.0364    0.0008   -0.0032    0.0032   -0.0043    0.0057   -0.0033
    0.0106    0.0365    0.0006   -0.0029    0.0033   -0.0050    0.0052   -0.0031
    0.0106    0.0366    0.0007   -0.0033    0.0031   -0.0040    0.0053   -0.0033
    0.0106    0.0368    0.0009   -0.0030    0.0034   -0.0044    0.0052   -0.0032
    0.0106    0.0367    0.0009   -0.0029    0.0034   -0.0044    0.0051   -0.0036
    0.0106    0.0365    0.0009   -0.0030    0.0031   -0.0047    0.0049   -0.0037
    0.0106    0.0367    0.0007   -0.0034    0.0030   -0.0047    0.0045   -0.0028
    0.0106    0.0367    0.0004   -0.0035    0.0036   -0.0048    0.0050   -0.0038
    0.0106    0.0369    0.0003   -0.0030    0.0032   -0.0041    0.0047   -0.0039
      ⋮

S = 2583×2583
10^5 ×
    4.0600         0         0         0         0         0         0         0 ···
         0    0.8702         0         0         0         0         0         0
         0         0    0.4169         0         0         0         0         0
         0         0         0    0.4104         0         0         0         0
         0         0         0         0    0.3405         0         0         0
         0         0         0         0         0    0.2992         0         0
         0         0         0         0         0         0    0.2550         0
         0         0         0         0         0         0         0    0.2268
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
      ⋮

V = 4220×2583
    0.0130   -0.0044    0.0358   -0.0028    0.0085   -0.0177   -0.0128    0.0163 ···
    0.0130   -0.0045    0.0357   -0.0024    0.0079   -0.0184   -0.0134    0.0162
    0.0130   -0.0045    0.0359   -0.0025    0.0078   -0.0181   -0.0124    0.0168
    0.0130   -0.0046    0.0361   -0.0030    0.0087   -0.0185   -0.0125    0.0158
    0.0130   -0.0045    0.0366   -0.0032    0.0095   -0.0182   -0.0116    0.0149
    0.0129   -0.0046    0.0369   -0.0034    0.0095   -0.0185   -0.0112    0.0151
```

```
    0.0130   -0.0047    0.0372   -0.0046    0.0104   -0.0178   -0.0103    0.0141
    0.0130   -0.0048    0.0377   -0.0045    0.0097   -0.0179   -0.0108    0.0145
    0.0130   -0.0046    0.0375   -0.0049    0.0094   -0.0170   -0.0102    0.0147
    0.0130   -0.0045    0.0379   -0.0043    0.0090   -0.0166   -0.0095    0.0142
    0.0130   -0.0042    0.0382   -0.0045    0.0090   -0.0162   -0.0094    0.0135
    0.0130   -0.0042    0.0383   -0.0041    0.0085   -0.0161   -0.0100    0.0141
    0.0129   -0.0039    0.0380   -0.0043    0.0083   -0.0165   -0.0096    0.0133
    0.0129   -0.0035    0.0382   -0.0037    0.0080   -0.0162   -0.0095    0.0135
    0.0128   -0.0032    0.0386   -0.0037    0.0073   -0.0158   -0.0099    0.0122
      ⋮
```

```
A_compressed = U(:,1:k)*S(1:k,1:k)*V(:,1:k)'
```

```
A_compressed = 2583×4220
   26.4896   27.2541   30.5810   28.9530   23.3828   25.7705   35.1037   29.3968 ···
   32.6831   34.0733   28.4258   30.5682   27.6882   28.4547   35.6629   31.2002
   35.5230   30.8250   18.7994   19.9743   19.8952   17.0400   24.6764   26.0911
   33.6440   29.7702   26.1173   29.8858   26.5095   15.9739   24.7017   25.8886
   27.7165   26.0012   30.5018   36.7547   35.3434   29.8915   34.5078   25.1416
   27.3996   25.5183   26.6092   29.4463   25.5795   28.8615   33.9147   23.0624
   32.0484   32.4889   27.5089   22.7250   20.3684   25.0803   33.3388   26.7700
   26.0954   32.2044   27.4183   18.1894   21.2836   28.1417   31.7244   26.2813
   23.2187   25.5412   22.1689   25.1362   29.2165   30.3222   34.5845   30.5812
   21.3048   20.9797   19.1568   25.5860   26.9030   24.8220   30.0068   30.1700
   21.0570   20.5597   20.4413   28.6261   24.3885   24.3385   26.3504   21.4459
   22.1100   23.3674   24.9443   30.5527   21.3032   17.7011   22.2625   23.0917
   27.2234   29.9587   25.7000   27.7856   24.1104   21.8076   25.1441   25.7437
   26.3002   27.8818   23.8573   29.6187   30.6100   28.4090   26.3002   21.7899
   23.0872   23.2337   20.9303   25.9354   26.7234   24.2094   26.1100   20.1887
      ⋮
```

```
% show both images side by side to double check
figure
subplot(1,2,1), imshow(A,[]), title('Original Image is:')
subplot(1,2,2), imshow(A_compressed,[]), title(['Rank:', num2str(k), '
Approximation:'])
```


```
% final check

% same image size, and rank should equal k
size(A), size(A_compressed), rank(A_compressed), k
```

```
ans = 1×2
        2583        4220
ans = 1×2
        2583        4220
ans =
801
k =
801
```

Explain:

SVD was used to rebuild the image using the top k singular values that give a compression ratio of around CR = 2. This allows the resulting image to keep most of its detail while also reducing data size. The rank-k version proves the image can be rebuilt with far fewer values without losing key structure.

Problem 6

Display the image and compute the root mean square error (RMSE) between the approximation and the original image. Make sure to include a copy of the approximate image in your report. Solution:

```
close all
% Displayed with imshow(A, []) using double precision
% Converting to uint8 is unnecessary in modern MATLAB and slightly alters
% the RMSE due to quantization. The display works correctly without
conversion
imshow(A_compressed, [])
```

```
RMSEk = norm(double(A)-A_compressed,'fro')/sqrt(m*n)
```

```
RMSEk =
3.1539
```

```
title(sprintf('Rank: %d Approximation, RMSE = %.4f', k, RMSEk))
set(gcf,'Position',[100 100 700 600])  % fit window
```

Rank: 801 Approximation, RMSE = 3.1539

Problem 7

Repeat Problems 5 and 6 for $CR \approx 10$, $CR \approx 25$, and $CR \approx 75$. **Explain** what trends you observe in the image approximation as CR increases and provide your recommendation for the best CR based on your observations. Make sure to include a copy of the approximate images in your report. Solution:

```
% repeat the problems 5, 6
% with CR = 10, 25, and 75
```

```
% precompute once
[m,n] = size(A);
[U,S,V] = svd(double(A),'econ');

% CR = 10
CR = 10;                            % set target compression ratio to 10
k = round((m*n)/(CR*(m+n+1)));     % solve for k using CR formula

% build rank-k approximation using the first k singular values
A10 = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';

% calculate RMSE to measure the differance in this image compared to the
original
RMSE10 = norm(double(A)-A10,'fro')/sqrt(m*n);

% display the new compressed image and display CR  and RMSE for the title
figure, imshow(A10,[])
title(['CR ≈ 10, RMSE = ', num2str(RMSE10)]);set(gcf,'Position',[100 100
700 600]) % fit in title % CR = 25
```

CR ≈ 10, RMSE = 8.2118

```
CR = 25; k = round((m*n)/(CR*(m+n+1)));
A25 = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
RMSE25 = norm(double(A)-A25,'fro')/sqrt(m*n);
figure, imshow(A25,[]), title(['CR ≈ 25, RMSE = ', num2str(RMSE25)]);
set(gcf,'Position',[100 100 700 600]) % fit in title % CR = 25
```

```
% CR = 75
CR = 75; k = round((m*n)/(CR*(m+n+1)));
A75 = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
RMSE75 = norm(double(A)-A75,'fro')/sqrt(m*n);
figure, imshow(A75,[]), title(['CR ≈ 75, RMSE = ', num2str(RMSE75)]);
set(gcf,'Position',[100 100 700 600]) % fit in title % CR = 25
```

CR ≈ 75, RMSE = 18.2656

```
% Display RMSE values for rank-1 and rank-2 approximations
disp(['RMSE for rank-1 approximation: ', num2str(rmse_1)]);, disp(['RMSE
for rank-2 approximation: ', num2str(rmse_2)]);
```

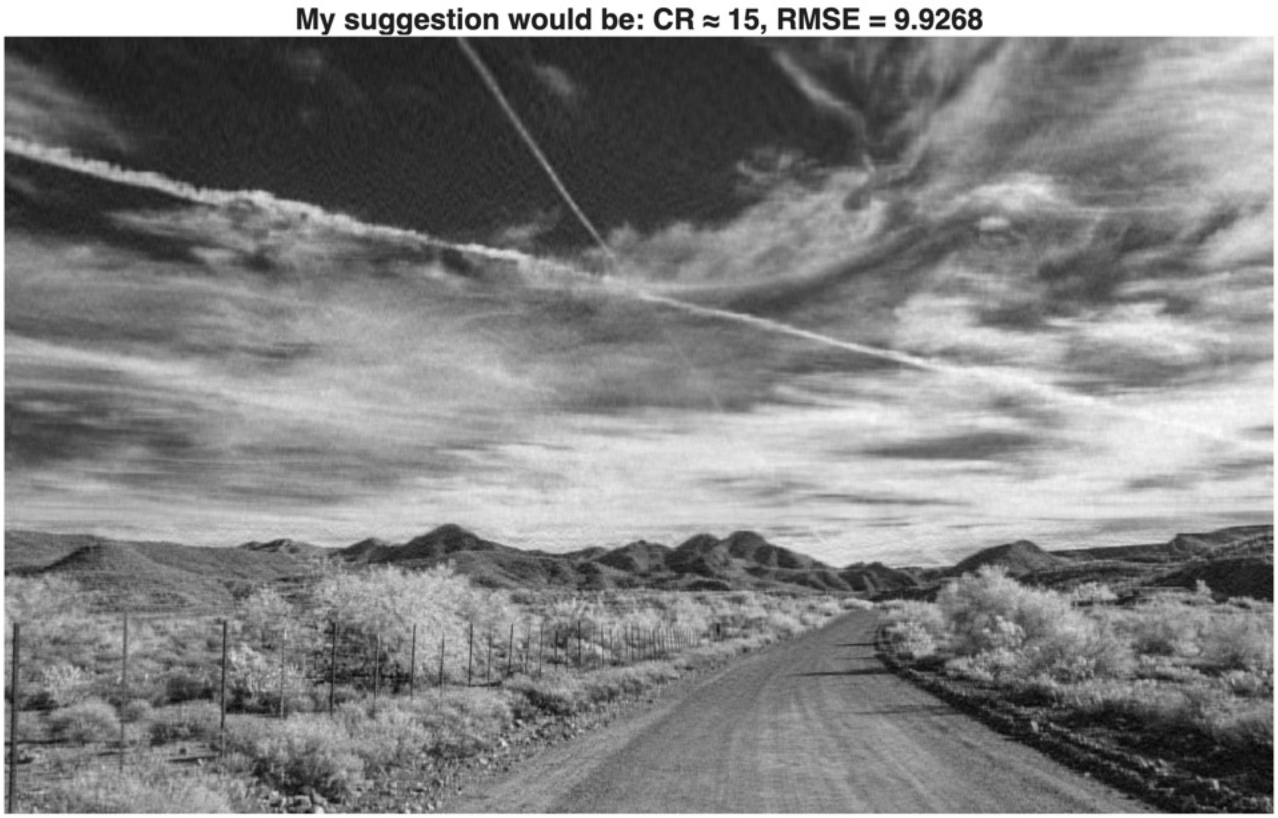
```
RMSE for rank-1 approximation: 0.32565
RMSE for rank-2 approximation: 0.11664
```

```
disp('My suggestion would be CR ≈ 15:')
```

```
My suggestion would be CR ≈ 15:
```

```
CR = 15;
k15 = round((m*n)/(CR*(m+n+1)));
A15 = U(:,1:k15)*S(1:k15,1:k15)*V(:,1:k15)';
RMSE15 = norm(double(A)-A15,'fro')/sqrt(m*n);
figure; imshow(A15,[]);
set(gcf,'Position',[100 100 700 600]);
```

```
title(['My suggestion would be: CR ≈ 15, RMSE = ', num2str(RMSE15)]);
```




```
disp(['RMSE for CR ≈ 15: ', num2str(RMSE15)]);
```

```
RMSE for CR ≈ 15: 9.9268
```

Explain:

At a low compression rate (like CR=10), the image remains sharper and more detailed but the trade off is more storage.

A medium compression rate (CR=25) reduces the file size, but the trade off is itt introduces some blurring and texture loss.

At a high rate (CR=75), the image becomes heavily blurred showing the trade-off between size and quality.

To find a better balance, I calculated CR≈15 using the same formula. This value keeps more singular components than CR=25, so edges and fine details stay clearer, while the file size is still much smaller than at CR=10.

Based on the results, CR≈15 provides the best compromise between visual quality and compression efficiency.

But if I had to choose only between CR=10, CR=25, and CR=75, I would pick CR=25. It keeps most of the image structure while still reducing the file size more effectively than CR=10.