

# Class 1: Modelling with seasonality and the frequency domain

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PRESS RECORD

## Learning outcomes

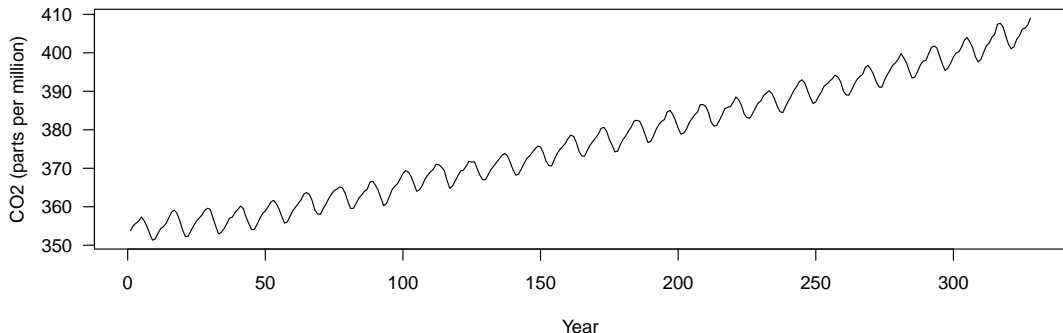
- ▶ Understand how to fit seasonal models in forecast and JAGS
- ▶ Understand seasonal differencing and sARIMA models
- ▶ Know the difference between time and frequency domain models and be able to implement a basic Fourier model

## Seasonal time series

- ▶ So far we haven't covered how to deal with data that are *seasonal* in nature
- ▶ These data generally fall into two categories:
  1. Data where we know the frequency or frequencies (e.g. monthly data on a yearly cycle, frequency = 12)
  2. Data where we want to estimate the frequencies (e.g. climate time series, animal populations, etc)
- ▶ The former are easier, and there are many techniques for inducing seasonal behaviour
- ▶ The latter are much more interesting. The ACF and PACF can help, but we can usually do much better by creating a *power spectrum*

## An example seasonal series

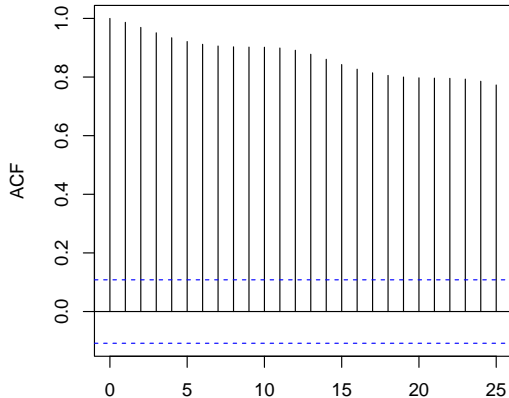
```
C02 = read.csv(file = '../data/C02.csv',  
               na.strings = -99.99)  
C02_1990 = C02[C02$year >= 1990, ]  
with(C02_1990, plot(1:nrow(C02_1990), C02_ppm, type = 'l',  
                   ylab = 'C02 (parts per million)',  
                   xlab = 'Year', las = 1))
```



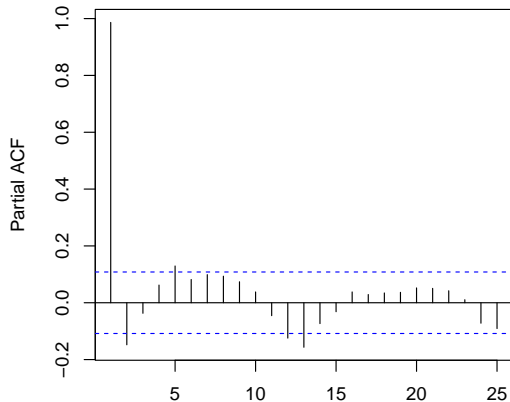
## ACF and PACF

```
par(mfrow = c(1, 2))  
acf(CO2_1990$CO2_ppm)  
pacf(CO2_1990$CO2_ppm)
```

Series CO2\_1990\$CO2\_ppm



Series CO2\_1990\$CO2\_ppm



## Seasonal time series 1: including seasonality as a covariate

- The simplest way is to include month as a covariate in a regression type model

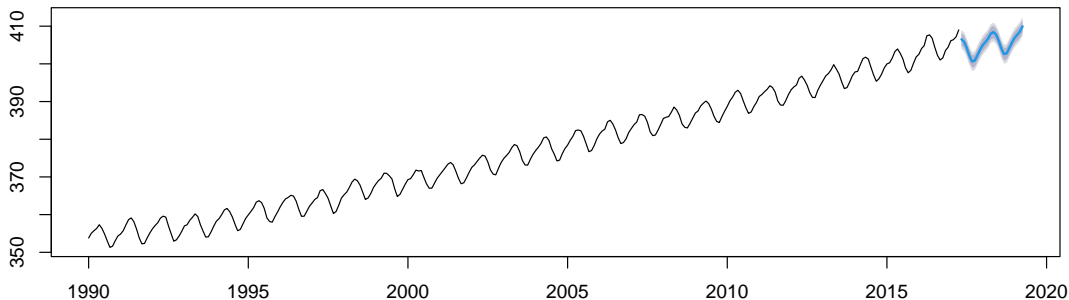
```
C02_1990$mfac = model.matrix(~ as.factor(C02_1990$month) - 1)
colnames(C02_1990$mfac) = month.abb
lm(C02_ppm ~ year + mfac - 1, data = C02_1990)
```

```
##
## Call:
## lm(formula = C02_ppm ~ year + mfac - 1, data = C02_1990)
##
## Coefficients:
##      year      mfacJan      mfacFeb      mfacMar      mfacApr      mfacMay      mfacJun
##    1.936   -3502.265   -3501.467   -3500.562   -3499.231   -3498.793   -3499.453
##  mfacJul      mfacAug      mfacSep      mfacOct      mfacNov      mfacDec
## -3501.077   -3503.160   -3504.667   -3504.467   -3503.029   -3501.590
```

## Forecasts

```
C02_ts = ts(C02_1990$C02_ppm, frequency = 12,  
            start = c(1990, 1))  
s_model_1 = tslm(C02_ts ~ trend + season)  
plot(forecast(s_model_1, h = 24))
```

**Forecasts from Linear regression model**



## What is the time series model doing here?

- ▶ This is just a regression model, so that:

$$y_t = \beta_t \text{year}_t + \gamma_1 \text{Jan}_t + \gamma_2 \text{Feb}_t + \gamma_3 \text{Mar}_t + \dots + \gamma_{12} \text{Dec}_t + \epsilon_t$$

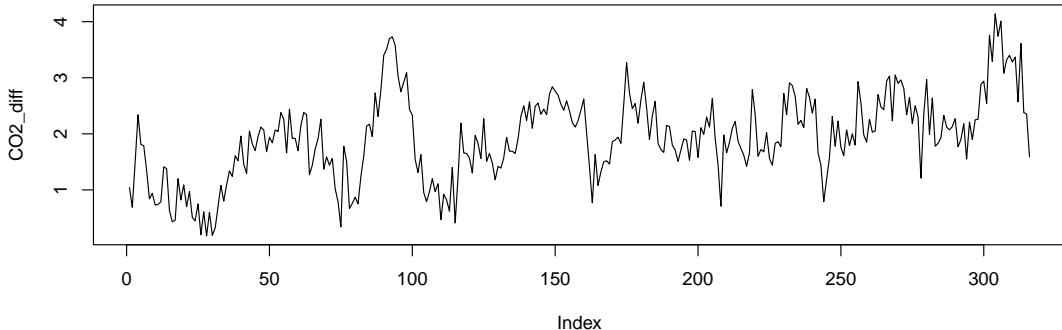
- ▶ You can do this using `lm` or using `forecast`'s special function for linear regression forecasting `tslm`
- ▶ The `tslm` function is clever because it can automatically create the seasonal indicator variables
- ▶ (Remember that when dealing with indicator variables you have to drop one factor level for the model to fit if you want to include an intercept)



## Seasonal time series 2: seasonal differencing

- ▶ We have already met methods which difference the data (possibly multiple times) at lag 1
- ▶ We can alternatively create a seasonal difference by differencing every e.g. 12th observation

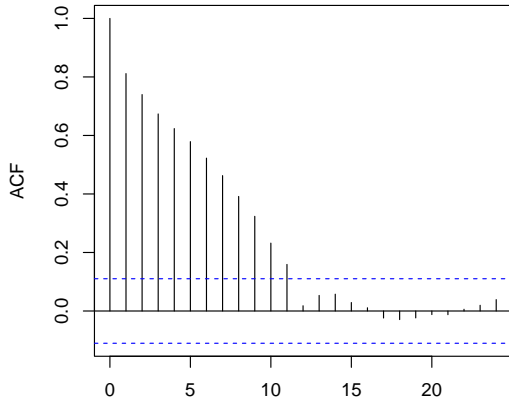
```
CO2_diff = diff(CO2_1990$CO2_ppm, lag = 12)  
plot(CO2_diff, type = 'l')
```



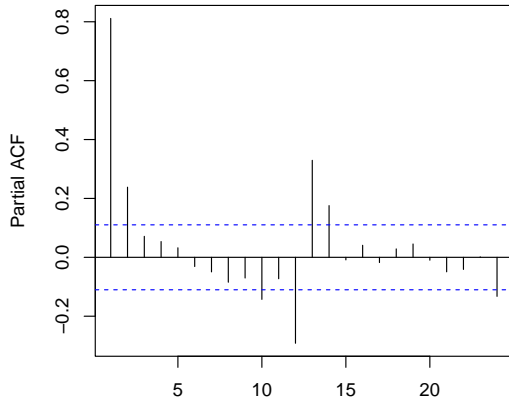
## Differenced acf and pacf

```
par(mfrow = c(1, 2))  
acf(CO2_diff, na.action = na.pass)  
pacf(CO2_diff, na.action = na.pass)
```

Series CO2\_diff



Series CO2\_diff



## Fit an ARIMA model with a seasonal difference

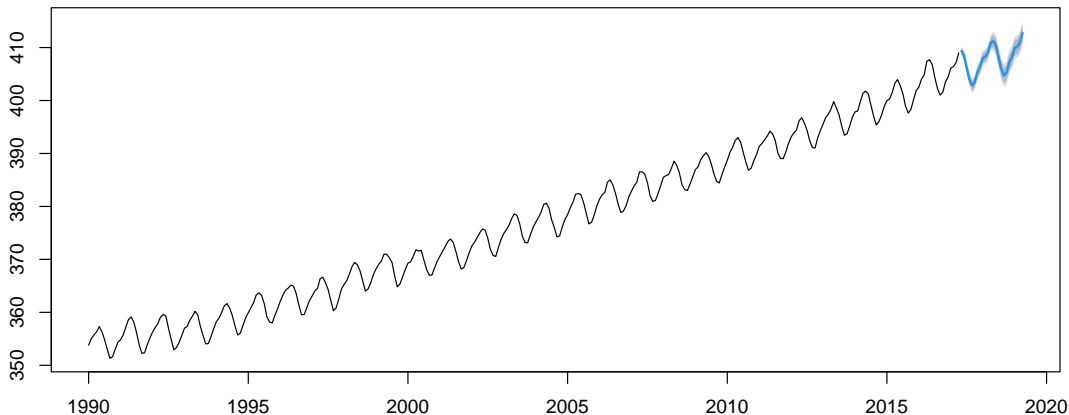
```
CO2_1990_ts = ts(CO2_1990$CO2_ppm, frequency = 12,  
                 start =c(1990, 1))  
Arima(CO2_1990_ts, order = c(1, 0, 0),  
      seasonal = c(0, 1, 0),  
      include.drift = TRUE)
```

```
## Series: CO2_1990_ts  
## ARIMA(1,0,0)(0,1,0)[12] with drift  
##  
## Coefficients:  
##          ar1    drift  
##      0.8129  0.1588  
## s.e.  0.0325  0.0107  
##  
## sigma^2 = 0.1943:  log likelihood = -185.12  
## AIC=376.23   AICc=376.31   BIC=387.5
```

## Forecasts from seasonally differenced series

```
s_model_2 = Arima(CO2_1990_ts, order = c(1, 0, 0),  
                  seasonal = c(0, 1, 0), include.drift = TRUE)  
plot(forecast(s_model_2, h = 24))
```

**Forecasts from ARIMA(1,0,0)(0,1,0)[12] with drift**



## A full seasonal arima model

- ▶ We previously met the ARIMA specification where:

$$\text{diff}^d(y_t) = \text{constant} + \text{AR terms} + \text{MA terms} + \text{error}$$

- ▶ We can extend this to include seasonal differencing and *seasonal AR and MA* terms to create a seasonal ARIMA or sARIMA model
- ▶ For example:

$$y_t - y_{t-12} = \alpha + \beta y_{t-1} + \gamma y_{t-12} + \epsilon_t$$

- ▶ This is a sARIMA(1, 0, 0)(1, 1, 0)<sub>12</sub> model

## Fitting sARIMA models in forecast

```
auto.arima(CO2_1990_ts)
```

```
## Series: CO2_1990_ts
```

```
## ARIMA(0,1,1)(1,1,2)[12]
```

```
##
```

```
## Coefficients:
```

```
##          ma1      sar1      sma1      sma2
```

```
##      -0.3888  -0.7684  -0.1020  -0.6482
```

```
## s.e.   0.0582   0.4837   0.4891   0.4246
```

```
##
```

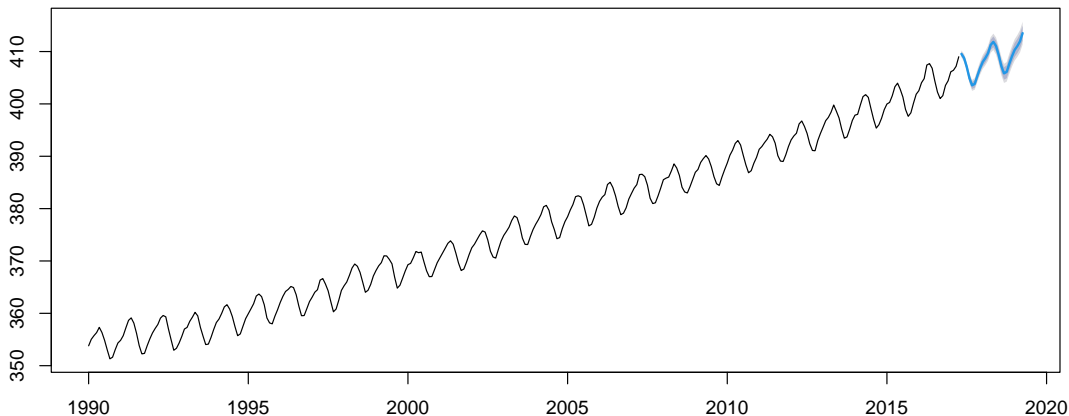
```
## sigma^2 = 0.1137:  log likelihood = -103.27
```

```
## AIC=216.55   AICc=216.74   BIC=235.31
```

## Plotting forecasts

```
s_model_3 = auto.arima(CO2_1990_ts)
plot(forecast(s_model_3, h = 24))
```

**Forecasts from ARIMA(0,1,1)(1,1,2)[12]**



## A simple sARIMA model with JAGS

```
model_code = '  
model  
{  
  # Likelihood  
  for (t in (s+1):T) {  
    y[t] ~ dnorm(mu[t], sigma^-2)  
    mu[t] <- alpha + beta * y[t-1] + gamma * y[t-s]  
  }  
  
  # Priors  
  alpha ~ dnorm(0, 10^-2)  
  beta ~ dnorm(0, 10^-2)  
  gamma ~ dnorm(0, 10^-2)  
  sigma ~ dunif(0, 100)  
}  
'
```



## Fitting a $\text{sARIMA}(1, 0, 0)(1, 0, 0)_{12}$ model in JAGS

```
s_model_4 = jags(data = list(y = C02_ts, s = 12,  
                             T = length(C02_ts)),  
                 parameters.to.save = c('alpha', 'beta',  
                                       'gamma', 'sigma'),  
                 model.file=textConnection(model_code))
```

```
print(s_model_4)
```

```
## Inference for Bugs model at "4", fit using jags,  
## 3 chains, each with 2000 iterations (first 1000 discarded)  
## n.sims = 3000 iterations saved  
##
```

	mu.vect	sd.vect	2.5%	25%	50%	75%	97.5%	Rhat	n.eff
## alpha	-6.458	0.881	-8.155	-7.076	-6.457	-5.861	-4.727	1.001	3000
## beta	0.189	0.025	0.141	0.172	0.189	0.205	0.238	1.002	1000
## gamma	0.833	0.025	0.782	0.816	0.833	0.849	0.882	1.002	1100
## sigma	0.600	0.024	0.555	0.584	0.599	0.616	0.650	1.001	3000
## deviance	571.736	2.952	568.197	569.629	571.031	573.058	579.316	1.001	3000

```
##
```

```
## For each parameter, n.eff is a crude measure of effective sample size,
```

```
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).17/27
```

## Multiple seasonality

- ▶ Very occasionally you come across multiple seasonality models
- ▶ For example you might have hourly data over several months with both hourly and monthly seasonality
- ▶ `forecast` has a special function for creating multiple series time series: `msts`

```
x = msts(taylor, seasonal.periods=c(48,336),  
        start=2000+22/52)
```

- ▶ The above is half-hourly data so has period 48 half-hours and 336 hours, i.e. weekly ( $336/48 = 7$ )
- ▶ `forecast` has some special functions (notably `tbats`) for modelling multi seasonality data

## Frequency estimation

## Methods for estimating frequencies

- ▶ The most common way to estimate the frequencies in a time series is to decompose it in a *Fourier Series*

- ▶ We write:

$$y_t = \alpha + \sum_{k=1}^K [\beta_k \sin(2\pi t f_k) + \gamma_k \cos(2\pi t f_k)] + \epsilon_t$$

- ▶ Each one of the terms inside the sum is called a *harmonic*. We decompose the series into a sum of sine and cosine waves rather than with AR and MA components
- ▶ Each sine/cosine pair has its own frequency  $f_k$ . If the corresponding coefficients  $\beta_k$  and  $\gamma_k$  are large we might believe this frequency is important

## Estimating frequencies via a Fourier model

- ▶ It's certainly possible to fit the model in the previous slide in JAGS, as it's just a linear regression model with clever explanatory variables
- ▶ However, it can be quite slow to fit and, if the number of frequencies  $K$  is high, or the frequencies are close together, it can struggle to converge
- ▶ More commonly, people repeatedly fit the simpler model:

$$y_t = \alpha + \beta \sin(2\pi t f_k) + \gamma \cos(2\pi t f_k) + \epsilon_t$$

for lots of different values of  $f_k$ . Then calculate the *power spectrum* as  $P(f_k) = \frac{\beta^2 + \gamma^2}{2}$ . Large values of the power spectrum indicate important frequencies

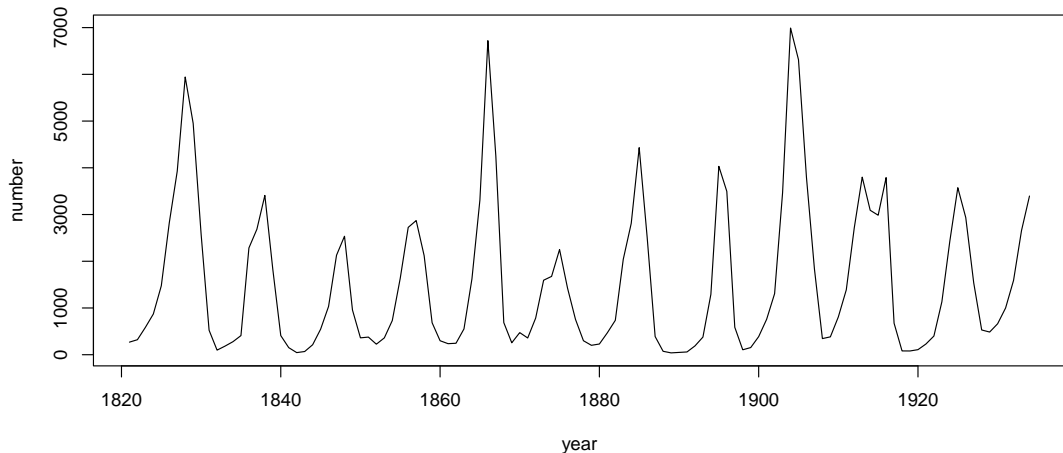
- ▶ It's much faster to do this outside of JAGS, using other methods, but we will stick to JAGS

## JAGS code for a Fourier model

```
model_code = '  
model  
{  
  # Likelihood  
  for (t in 1:T) {  
    y[t] ~ dnorm(mu[t], sigma^-2)  
    mu[t] <- alpha + beta * cos(2*pi*t*f_k) +  
              gamma * sin(2*pi*t*f_k )  
  }  
  P = (beta^2 + gamma^2) / 2  
  
  # Priors  
  alpha ~ dnorm(0, 10^-2)  
  beta ~ dnorm(0, 10^-2)  
  gamma ~ dnorm(0, 10^-2)  
  sigma ~ dunif(0, 100)  
}
```

## Example: the Lynx data

```
lynx = read.csv('../data/lynx.csv')  
plot(lynx, type = 'l')
```



## Code to run the JAGS model repeatedly

```
periods = 5:40
K = length(periods)
f = 1/periods
Power = rep(NA,K)

for (k in 1:K) {
  curr_model_data = list(y = as.vector(lynx[,2]),
                        T = nrow(lynx),
                        f_k = f[k],
                        pi = pi)

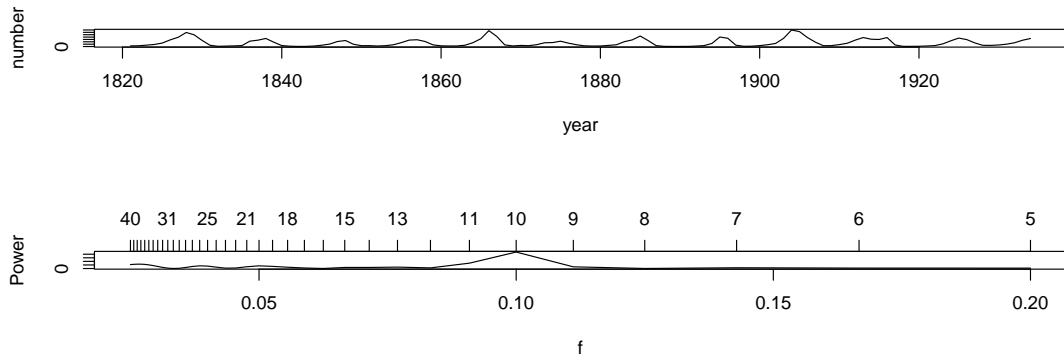
  model_run = jags(data = curr_model_data,
                  parameters.to.save = "P",
                  model.file=textConnection(model_code))

  Power[k] = mean(model_run$BUGSoutput$sims.list$P)
}
```



## Plotting the periodogram

```
par(mfrow = c(2, 1))  
plot(lynx, type = 'l')  
plot(f, Power, type='l')  
axis(side = 3, at = f, labels = periods)
```



## Bayesian vs traditional frequency analysis

- ▶ For quick and dirty analysis, there is no need to run the full Bayesian model, the R function `periodogram` in the TSA package will do the job, or `findfrequency` in `forecast` which is even simpler
- ▶ However, the big advantage (as always with Bayes) is that we can also plot the uncertainty in the periodogram, or combine the Fourier model with other modelling ideas (e.g. ARIMA)
- ▶ There are much fancier versions of frequency models out there (e.g. Wavelets, or frequency selection models) which can also be fitted in JAGS but require a bit more time and effort
- ▶ These Fourier models work for continuous time series too

# Summary

- ▶ We now know how to fit models for seasonal data via seasonal factors, seasonal differencing, and sARIMA models
- ▶ We can fit these using `forecast` or JAGS
- ▶ We've seen a basic Fourier model for estimating frequencies via the Bayesian periodogram