Module 1 – Linear Systems and Span
Topic 1 – Systems of Linear Equations
Lesson 1 – Solution Sets of Linear Equations

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05/16/2022

1 Topics

We will explore the following concepts:

- Systems of Linear Equations
- Elementary Row Operations

2 Learning Objectives:

Students should be able to do the following after watching the video and completing the assigned homework:

- Apply elementary row operations to solve systems of linear equations.

3 A Single Linear Equation

A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

a_1, a_2, ..., a_n and b are the **coefficients**, x_1, x_2, ..., and x_n are the **variables**, and n is the **dimension**, or number of variables.
For example:

- $2x_1 + 4x_2 = 4$ is a line in 2 dimensions
- $3x_1 + 2x_2 + x_3 = 6$ is a plane in 3 dimensions

4 Systems of Linear Equations

When we have one or more linear equations, we have a **Linear System** of equations.
For example, a linear system with two equations is

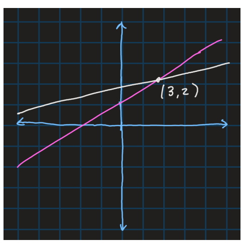
$$\begin{array}{rcrcrcl} x_1 & + & 1.5x_2 & + & \pi x_3 & = & 4 \\ 5x_1 & & & + & 7x_3 & = & 5 \end{array}$$

The set of all possible values of x_1, x_2, $\ldots$, x_n that satisfy all equations is the **solution set**. One point in the solution set is a **solution**.

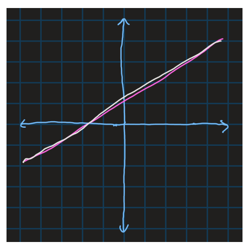
5 Two Variable Case

Consider the following systems. How are they different from each other?

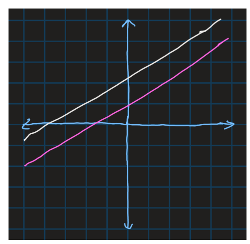
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 2x_2 & = & 1 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 2x_2 & = & 3 \end{array}$$



Non-Parallel Lines
Unique Solution



Identical Lines
Infinite Solutions

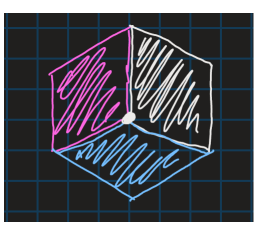


Parallel Lines
No Solutions

6 Three Variable Case

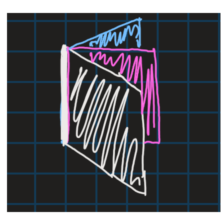
An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}$. The **solution** to a system of **three equations** is the set of points where all three planes intersect:

Planes Intersect At a Point



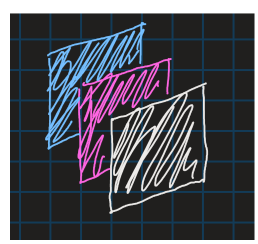
Unique Solution

Planes Intersect On a Line



Infinite Solutions

Parallel Planes



No Solutions

7 Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**:

1. (Replacement/Addition) Add a multiple of one row to another
2. (Interchange) Interchange/swap two rows
3. (Scaling) Multiple a row by a non-zero scalar

Let's use these operations to solve a system of equations:

$$\begin{array}{rrrrrrrr}
R_1 & x_1 & - & 2x_2 & + & x_3 & = & 0 \\
R_2 & & & 2x_2 & - & 8x_3 & = & 8 \\
R_3 & 5x_1 & & & - & 5x_3 & = & 10 \\
\\
R_1 + R_2 \to R_1 & x_1 & + & 0x_2 & - & 7x_3 & = & 8 \\
\frac{1}{2} \times R_2 \to R_2 & & & x_2 & - & 4x_3 & = & 4 \\
R_3 - 5 \times R_1 \to R_3 & & & 10x_2 & - & 10x_3 & = & 10 \\
\\
R_1 & x_1 & & & - & 7x_3 & = & 8 \\
R_2 & & & x_2 & - & 4x_3 & = & 4 \\
R_3 - 10 \times R_2 \to R_3 & & & & & 30x_3 & = & -30
\end{array}$$

Solve $R_3 \implies 30x_3 = -30 \implies x_3 = -1$
Substitute $x_3 = -1$ into $R_1 \implies x_1 - 7(-1) = 8 \implies x_1 + 7 = 8 \implies x_1 = 1$
Substitute $x_3 = -1$ into $R_2 \implies x_2 - 4(-1) = 4 \implies x_2 + 4 = 4 \implies x_2 = 0$
Our solution is the point $(1, 0, -1)$

8 Summary

We explored the following concepts in this video:

- Systems of linear equations
- Elementary row operations
- Applying elementary row operations to solve a linear system

9 Practice 1

Consider the following linear system:

$$
\begin{array}{llcl}
\color{blue}{R_1} & x_1 + x_2 & & = 1 \\
\color{blue}{R_2} & \quad\; x_2 + x_3 & & = 3 \\
\color{blue}{R_3} & \qquad\quad\; x_3 & & = 1
\end{array}
$$

Substitute $x_3 = 1$ into $\color{blue}{R_2} \implies x_2 + 1 = 3 \implies x_2 = 2$
Substitute $x_2 = 2$ into $\color{blue}{R_1} \implies x_1 + 2 = 1 \implies x_1 = -1$
The solution is the point $(-1, 2, 1)$

10 Practice 2

Consider the linear system below:

$$
\begin{array}{lrcl}
\color{blue}{R_1} & x + 2y & = & 4 \\
\color{blue}{R_1} & c_1 x + 6y & = & 1
\end{array}
$$

The coefficient c_1 is a real number. For what value of c_1 are there no solutions to this linear system?

No solution means that the one equation is a scalar multiple of the other, so $k(x + 2y) = c_1 x + 6y$
$\implies kx + 2ky = c_1 x + 6y \implies kx = c_1 x$ and $2ky = 6y$
$\implies k = c_1$ and $2k = 6 \implies k = c_1$ and $k = 3$
So $c_1 = 3$ causes the linear system to have no solutions.