Introduction to Limits
Notes

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June 8, 2022

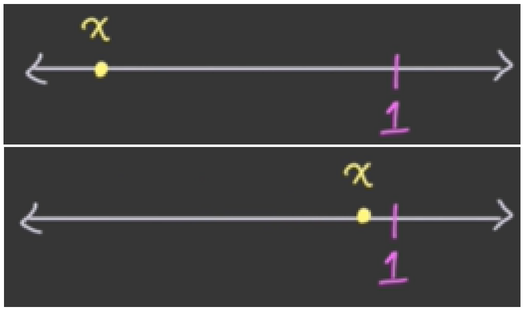
1 Objectives

At the end of this sequence, and after some practice, you should be able to:

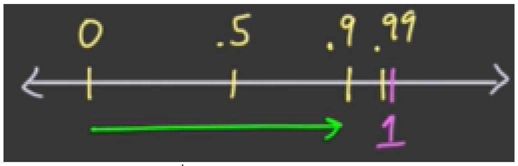
- Use a calculator to determine right and left hand limits.
- Identify right and left hand limits based on graphs.
- Determine if a limit exists based on values of right and left hand limits.
- Understand that the limit does not depend on the value of a function at the point of interest.

2 Moving Closer and Closer

Calculus is all about functions.
You probably know that a function f takes an input x and gives an output $f(x)$, which we could write as $x \mapsto f(x)$.
But in calculus, we aren't concerned with just a single input and it's output, we want to consider a whole range of inputs. We want to know what happens as the input moves/varies.
An example: As x moves closer to 1 from the left: $x \to 1^-$



It is important to note that when we say x moves closer to 1, we do not mean that x will ever equal 1, we are only concerned with values of x that are near 1.
In a function f, we know that as we change our input x, our output $f(x)$ is going to change as well.
With limits, we ask the question, does the output $f(x)$ move close to something as the input x moves close to something?
For our example, lets take the function $\frac{\sqrt{3-5x+x^2+x^3}}{x-1}$.
To do this, we could take values for x that are getting closer to 1 from the left to use as inputs for f.
There are an infinite amount of values we could choose for x, but we will start with 4 simpler values.



x	$f(x)$
0	$f(0) \approx -1.73$
0.5	$f(0.5) \approx -1.87$
0.9	$f(0.9) \approx -1.97$
0.99	$f(0.99) \approx -1.997$

Reading the values in this table, we can see that as x approaches 1 from the left, $f(x)$ approaches -2.
Now we can try to find what f approaches as x approaches 1 from the right side.

x	$f(x)$
2	$f(2) \approx 2.23$
1.5	$f(1.5) \approx 2.12$
1.1	$f(1.1) \approx 2.02$
1.01	$f(1.01) \approx 2.003$

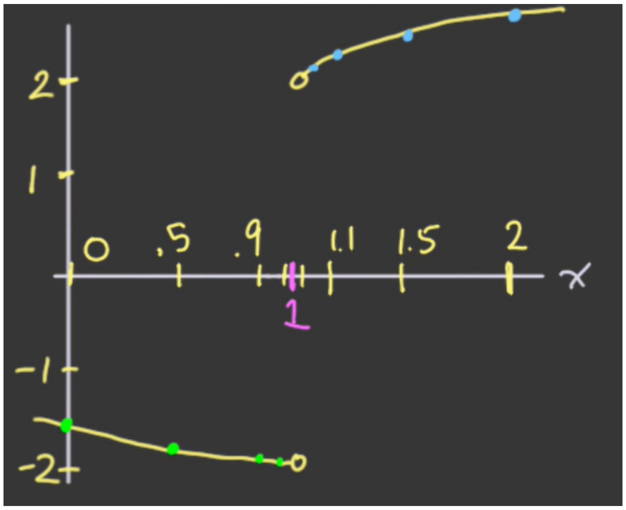
This time, reading the values in the table, we can see that as x approaches 1 from the right, $f(x)$ approaches 2.
That means that in this example, the direction we were approaching from with x affected where $f(x)$ was approaching.

3 One-Sided Limits

From the last section, we know that given the function $f(x) = \frac{\sqrt{3-5x+x^2+x^3}}{x-1}$,

- As $x \to 1^-$, $f(x) \to -2$
- As $x \to 1^+$, $f(x) \to 2$

We can plot the points we had in the tables from the last section on the coordinate plane to get a better picture of what is happening with our function:



We call this phenomena where a function approaches a value a **limit**
We could rephrase our earlier statements to use our new vocabulary and some new notation:

- The limit of $f(x)$ as $x \to 1^-$ is $-2 \Longrightarrow \lim\limits_{x \to 1^-} f(x) = -2$
- The limit of $f(x)$ as $x \to 1^+$ is $2 \Longrightarrow \lim\limits_{x \to 1^+} f(x) = 2$

We call the limit as x approaches a value from the left the **left-hand limit** and the limit as x approaches a value from the right the **right-hand limit**.