Choose Your Calculus Adventure
Diagnostic Questions

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June 7, 2022

1 Algebra Problems

1. What is the slope of the line through the points $(3, -5)$ and $(1, -1)$?

$$(3, -5) = (x_1, y_1), (1, -1) = (x_2, y_2)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-5)}{1 - 3} = \frac{4}{-2} = -2$$

The slope of the line through is the points $(3, -5)$ and $(1, -1)$ is -2.

2. The lines $3x + 2y = 7$ and $x - 3y = 6$ intersect at a point with what coordinates?

$$3x + 2y = 7 \Longrightarrow 3x = 7 - 2y \Longrightarrow x = \frac{7 - 2y}{3}$$
$$x - 3y = 6 \Longrightarrow \frac{7 - 2y}{3} - 3y = 6$$
$$\Longrightarrow 7 - 2y - 9y = 18 \Longrightarrow -11y = 11 \Longrightarrow y = -1$$
$$x = \frac{7 - 2y}{3} \Longrightarrow x = \frac{7 - 2(-1)}{3} \Longrightarrow \frac{7 + 2}{3} \Longrightarrow x = \frac{9}{3} \Longrightarrow x = 3$$

The lines $3x + 2y = 7$ and $x - 3y = 6$ intersect at the point $(3, -1)$

3. Which of the following expressions is equivalent to $\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}$?

   (a) $x-y$
   (b) $x+y$
   (c) x^2+y^2
   (d) $\frac{1}{x}+\frac{1}{y}$
   (e) $\frac{1}{x^2}+\frac{1}{y^2}$
   (f) $\frac{xy}{x+y}$
   (g) $\frac{x+y}{xy}$
   (h) None of the above

$$\left(\frac{1}{x}+\frac{1}{y}\right)^{-1} \Longrightarrow \left(\frac{y}{xy}+\frac{x}{xy}\right)^{-1} \Longrightarrow \left(\frac{y+x}{xy}\right)^{-1} \Longrightarrow \frac{xy}{x+y}$$

   So $\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}$ is equivalent to (f) $\frac{xy}{x+y}$.

4. List all possible solutions to the equation $x^3-x^2-2x=0$.

$$x^3-x^2-2x=0 \Longrightarrow x(x^2-x-2)=0 \Longrightarrow x(x+1)(x-2)=0$$
$$x=0$$
$$x+1=0 \Longrightarrow x=-1$$
$$x-2=0 \Longrightarrow x=2$$

   So the possible solutions to $x^3-x^2-2x=0$ are $-1,0,2$.

5. A 0.25mL sample of water drawn from a 5 liter flask contains 1.25×10^8 bacteria. Give the approximate number of bacteria in the flask, expressing your answer in scientific notation.

   5 liters is the same as 5000mL. We can find our multiplier by dividing the total volume by the sample volume.

$$x=\frac{5000}{0.25} \Longrightarrow x=20000$$

$$1.25\times10^8\times20000 \Longrightarrow 1.25\times10^8\times2\times10^4$$
$$\Longrightarrow 1.25\times2\times10^8\times10^4 \Longrightarrow 2.5\times10^{12}$$

   The total number of bacteria in the flask is approximately 2.5×10^{12}.

6. For what value of the constant a will the following system of linear equations have no solution?

$$\begin{aligned} 6x - 5y &= 3 \\ 3x + ay &= 1 \end{aligned}$$

A system of linear equations will have no solutions given they have the same slope but different y-intercepts.

$$6x - 5y = 3 \implies -5y = 3 - 6x \implies y = \frac{6}{5}x - \frac{3}{5}$$

$$3x + ay = 1 \implies ay = 1 - 3x \implies y = \frac{-3}{a}x + \frac{1}{a}$$

So $m_1 = \frac{6}{5}$ and $m_2 = \frac{-3}{a}$.

$$m_1 = m_2 \implies \frac{6}{5} = \frac{-3}{a} \implies 6a = -15 \implies a = -2.5$$

We also know that the y-intercepts cannot be equal.

$$\frac{3}{5} \neq \frac{1}{-2.5} \implies -7.5 \neq 5\checkmark$$

So the constant $a = -2.5$ will result in the system having no solutions.

7. Find the value of the constant a for which the polynomial $x^3 + ax^2 - 1$ will have -1 as a root.

$$(-1)^3 + a(-1)^2 - 1 = 0 \implies -1 + a - 1 = 0 \implies a - 2 = 0 \implies a = 2$$

When $a = 2$, the polynomial will have -1 as a root.

8. If $a_n = \frac{x^n}{2^n n!}$, find $\frac{a_{n+1}}{a_n}$.

   (a) $\frac{x^{n+1}}{2^{n+1}(n+1)}!$

   (b) $\frac{nx^n}{2^n(n+1)}$

   (c) $\frac{nx}{2^n(n+1)}$

   (d) $\frac{x}{2^n(n+1)}$

   (e) $\frac{x}{2(n+1)}$

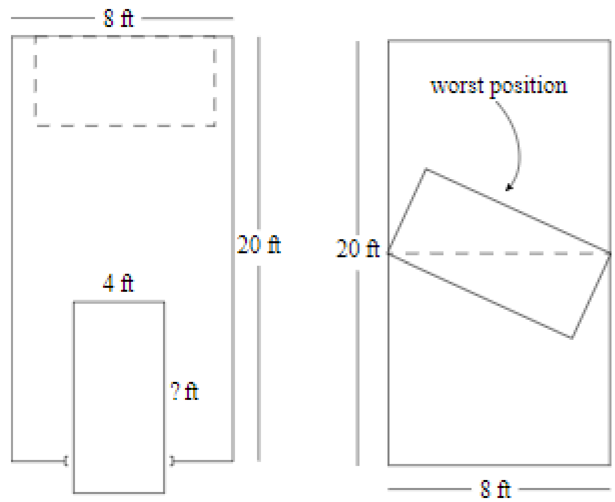
$$\frac{a_{n+1}}{a_n} \implies \frac{\frac{x^{n+1}}{2^{n+1}(n+1)!}}{\frac{x^n}{2^n n!}} \implies \frac{x^{n+1}}{2^{n+1}(n+1)!} \times \frac{2^n n!}{x^n} \implies \frac{x^{n+1}2^n n!}{x^n 2^{n+1}(n+1)!}$$

$$\implies \frac{x^n x 2^n n!}{x^n 2^n 2(n+1)n!} \implies \frac{x}{2(n+1)}$$

$\frac{a_{n+1}}{a_n}$ is equivalent to (e) $\frac{x}{2(n+1)}$.

2 Geometry Problems

1. A bed that is 4 feet wide must enter through a door (that is just over 4 feet wide) along the 8 foot wall of a 8 by 20 foot room. What is the largest length of a bed that can be rotated to fit with the longest side along the 8 foot long wall opposite the door?

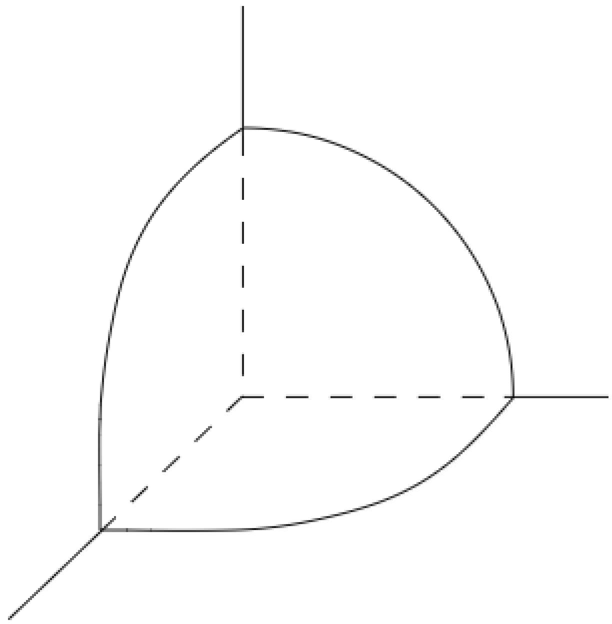


According to the Pythagorean Theorem, the diagonal length of the bed must be the sum of the squares of its length and width. We know the width of the bed is 4 feet and the diagonal needs to stop at 8 feet.

$$4^2 + l^2 = 8^2 \Longrightarrow l^2 = 64 - 16 \Longrightarrow l^2 = 48 \Longrightarrow l = \sqrt{48} \Longrightarrow l = 2\sqrt{12}$$

The largest length of a bed is $l = 2\sqrt{12} \approx 6.93$ feet

2. The four-sided solid shown was the part of the solid sphere (of radius 2, centered at the origin) in the first octant. Find its total surface area.



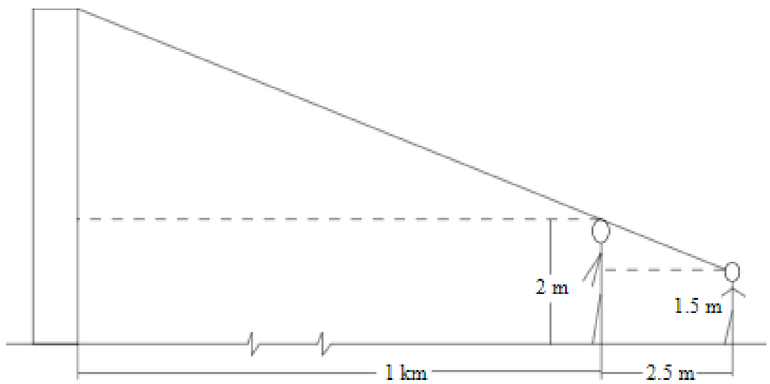
To find the surface area, we find the area of each of the four sides of the solid. The outer curved side is an eighth of the total surface area for a sphere. The inner walls are each one quarter of the area of a circle with the same radius as the sphere, which with three sides, makes three quarters of the area of the circle.

$$SA = \frac{1}{8} \times 4\pi r^2 + \frac{3}{4}\pi r^2 \Longrightarrow SA = \frac{1}{2}\pi(2)^2 + \frac{3}{4}\pi(2)^2$$

$$\Longrightarrow SA = \frac{1}{2} \times 4\pi + \frac{3}{4} \times 4\pi \Longrightarrow SA = 2\pi + 3\pi \Longrightarrow SA = 5\pi$$

The surface area of the first octant of this sphere is 5π.

3. To estimate the height of a skyscraper 1km in the distance, Jenny finds that if her friend Steve stands 2.5 meters away, the top of his head just lines up with the topc of the building. Steve is 2 meters tall, and Jenny's eye is 1.5 meters from the ground. How high is the building?



To calculate the height, we need to calculate the height of the triangle made by Steve's head and the building. We know the length of the base of the triangle is 1km, but we need additional information in order to calculate the height. To do this, we can use the triangle between Jenny and Steve, as it is similar to the larger triangle. In similar triangles, the ratios of corresponding sides must remain constant. To calculate the ratio, we can use the lengths of the bases of the similar triangles.

$$r = \frac{1000}{2.5} \Longrightarrow r = 400$$

Now we can set up a ratio using the heights of the similar triangles

$$400 = \frac{h}{0.5} \Longrightarrow h = 200$$

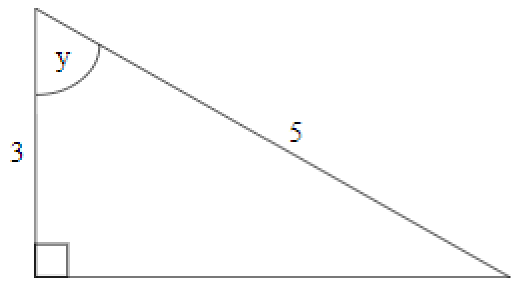
This is not the final height of the building, as we have not accounted for Steve's height yet.

$$h = 200 + 2 \Longrightarrow h = 202$$

The height of the building is 202 meters.

3 Trigonometry

1. In the given right triangle, what is $\tan(y)$?



$$\tan(y) = \frac{\text{opposite}}{\text{adjacent}}$$

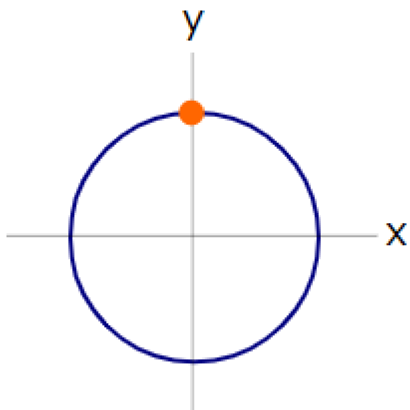
In the given triangle, we know the adjacent side and the hypotenuse, so we must calculate the opposite side using the pythagorean theorem.

$$a^2 + b^2 = c^2 \Longrightarrow 3^2 + b^2 = 5^2 \Longrightarrow b^2 = 16 \Longrightarrow \sqrt{b^2} = \sqrt{16} \Longrightarrow b = 4$$

$$\tan(y) = \frac{\text{opposite}}{\text{adjacent}} \Longrightarrow \tan(y) = \frac{4}{3}$$

For this right triangle, $\tan(y) = \frac{4}{3} \approx 1.33$.

2. A horse runs counterclockwise (anticlockwise) around the circular track of radius 400m a constant speed, starting at the marked point. It completes one lap in three minutes. What is its y-coordinate after one minute?



The horse completes one lap in three minutes, which means in one minute, it completes one-third of a lap.

$$\theta = \frac{1}{3} \times 2\pi + \frac{\pi}{2} \implies \theta = \frac{2\pi}{3} + \frac{\pi}{2}$$

$$\implies \theta = \frac{4\pi + 3\pi}{6} \implies \theta = \frac{7\pi}{6}$$

On a unit circle, the coordinate associated with the angle $\frac{7\pi}{6}$ is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2},\right)$. To scale our y-coordinate, we multiply by the radius

$$y = 400 \times -\frac{1}{2} \implies y = -200$$

The y-coordinate of the horse after one minute is $y = -200$.

3. Find the smallest possible solution to the equation $\sin(2x) = \frac{1}{2}$; here x is in radians.

$$\sin(2x) = \frac{1}{2} \Longrightarrow \sin^{-1}(\sin(2x)) = \sin^{-1}\left(\frac{1}{2}\right)$$

The smallest value for $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

$$2x = \frac{\pi}{6} \Longrightarrow x = \frac{\pi}{12}$$

The smallest possible solution to $\sin(2x) = \frac{1}{2}$ is $x = \frac{\pi}{12}$.

4. A line with slope $\frac{1}{2}$ makes an acute angle θ with the x axis. What is $\sin(\theta)$?

A line with a slope of $\frac{1}{2}$ makes a right-triangle with a vertical line at $x = 2$. The created triangle has a length of 2 and a height of 1.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

We currently know that the opposite side of our triangle is the height of 1, but we need to calculate the hypotenuse using the pythagorean theorem.

$$a^2 + b^2 = c^2 \Longrightarrow 1^2 + 2^2 = c^2 \Longrightarrow c^2 = 5\sqrt{c^2} = \sqrt{5} \Longrightarrow c = \sqrt{5}$$

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \Longrightarrow \sin(\theta) = \frac{1}{\sqrt{5}}$$

$$\Longrightarrow \sin(\theta) = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \Longrightarrow \sin(\theta) = \frac{\sqrt{5}}{5}$$

Given the line creating θ, $\sin(\theta) = \frac{\sqrt{5}}{5}$

5. By using the trigonometric identity $\cos(2x) = \cos^2(x) - \sin^2(x)$, and other identities, find the positive expression for $\sin\left(\frac{A}{2}\right)$ in terms of $\cos(A)$.

We know that $\cos^2(A) = 1 - \sin^2(A)$.

$$\cos(2A) = 1 - \sin^2(A) - \sin^2(A) \Longrightarrow \cos(2A) = 1 - 2\sin^2(A)$$

Next step is to divide the input angle by 2, resulting in

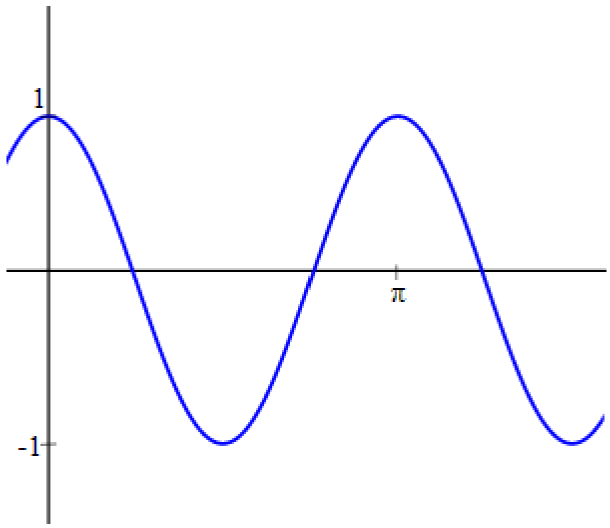
$$\cos(A) = 1 - 2\sin^2\left(\frac{A}{2}\right) \Longrightarrow \cos(A) - 1 = -2\sin^2\left(\frac{A}{2}\right)$$

$$\Longrightarrow \frac{1 - \cos(A)}{2} = \sin^2\left(\frac{A}{2}\right) \Longrightarrow \sqrt{\frac{1 - \cos(A)}{2}} = \sqrt{\sin^2\left(\frac{A}{2}\right)}$$

$$\Longrightarrow \sin\left(\frac{A}{2}\right) = \sqrt{\frac{1 - \cos(A)}{2}}$$

Using the trigonometric identities, we know $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{1-\cos(A)}{2}}$

6. The graph below represents which of the these functions?



(a) $\sin(x)$
(b) $\cos(x)$
(c) $\sin\left(\frac{x}{2}\right)$
(d) $\cos\left(\frac{x}{2}\right)$
(e) $\sin(2x)$
(f) $\cos(2x)$

Looking at the graph, we can see that it is a cos function that has not been vertically or horizontally shifted, which limits our options to (b), (d), and (f).

We can see that the period of the cos function has been modified (shrunk), which eliminates option (b).

A shrinking period is a result of a scale increase of the input, which means that the correct function is (f) $\cos(2x)$.

4 Logarithms and Exponentials

1. If $\log_{10}(a) = 4.2$ and $\log_{10}(b) = 0.5$, what is $\log_{10}(ab)$?

$$\log_{10}(ab) = \log_{10}(a) + \log_{10}(b) \Longrightarrow \log_{10}(ab) = 4.2 + 0.5 \log_{10}(ab) = 4.7$$

2. If $2^a = \frac{\sqrt(8)}{4^3}$, what is a?

$$2^a = \frac{\sqrt(8)}{4^3} \Longrightarrow 2^a = \frac{\left(2^3\right)^{\frac{1}{2}}}{\left(2^2\right)^3} \Longrightarrow 2^a = \frac{2^{\frac{3}{2}}}{2^6} \Longrightarrow 2^a = 2^{\frac{3}{2}-6} \Longrightarrow 2^a = 2^{-\frac{9}{2}}$$

$$\Longrightarrow \log_2\left(2^a\right) = \log_2\left(2^{-\frac{9}{2}}\right) \Longrightarrow a = -\frac{9}{2} \Longrightarrow a = -4.5$$

3. Which of the following is equal to $\sqrt{\frac{x^{16}(1+x^2)}{9}}$?

   (a) $\frac{x^4(1+x)}{3}$
   (b) $\frac{x^8(1+x)}{3}$
   (c) $\frac{x^4(1+x^2)^{0.5}}{3}$
   (d) $\frac{x^8(1+x^2)^{0.5}}{3}$
   (e) $\frac{x^4(1+x^2)}{3}$
   (f) $\frac{x^8(1+x^2)}{3}$
   (g) None of the above

$$\sqrt{\frac{x^{16}(1+x^2)}{9}} \Longrightarrow \left(\frac{x^{16}(1+x^2)}{9}\right)^{\frac{1}{2}} \Longrightarrow \frac{x^{16\times\frac{1}{2}}(1+x^2)^{\frac{1}{2}}}{3^{2\times\frac{1}{2}}}$$

$$\Longrightarrow \frac{x^8(1+x^2)^{0.5}}{3}$$

$\sqrt{\frac{x^{16}(1+x^2)}{9}}$ is equivalent to (d) $\frac{x^8(1+x^2)^{0.5}}{3}$

4. Solve for x: $\log_{10}\left[(x+1)^2\right] = 2$

$$\log_{10}\left[(x+1)^2\right] = 2 \Longrightarrow 2\log_{10}[x+1] = 2 \Longrightarrow \log_{10}[x+1] = 1$$

$$\Longrightarrow 10^{\log_{10}[x+1]} = 10^1 \Longrightarrow \pm(x+1) = 10$$

$$\Longrightarrow -x-1 = 10 \Longrightarrow -x = 11 \Longrightarrow x = -11$$

$$\Longrightarrow x+1 = 10 \Longrightarrow x = 9$$

5. A pot of water (at sea level) is boiling; the heat is turned off at time $t = 0$, and 2 minutes later the water temperature has fallen to 80∘C. If the temperature T (in ∘C) is expressed in terms of time t (in minutes) by the law $T = Ae^{-kt}$ Find the values of the constants A and k.

   So we know $T = 100$ when $t = 0$ and $T = 80$ when $t = 2$.

$$100 = Ae^{-k\times 0} \Longrightarrow 100 = Ae^0 \Longrightarrow A = 100$$

$$80 = 100e^{-2k} \Longrightarrow 0.8 = e^{-2k} \Longrightarrow \ln(0.8) = \ln\left(e^{-2k}\right)$$

$$\Longrightarrow \ln(0.8) = -2k \Longrightarrow k = -\frac{\ln(0.8)}{2}$$

   So we know $A = 100$ and $k = -\frac{\ln(0.8)}{2} \approx 0.11$.

5 Limits

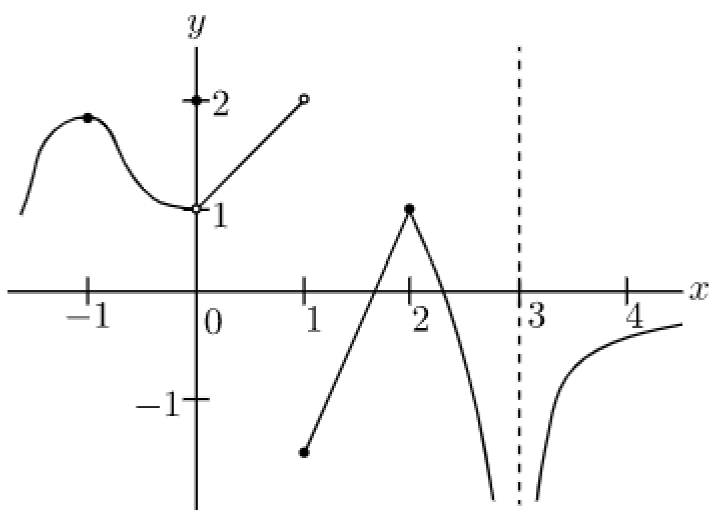
1. What is $\lim_{x\to\infty} \frac{x^2-4}{2+x-4x^2}$

   In a limit where the largest power is the same in the numerator and denominator, the limit is the ratio of the coefficients of those largest power terms, discarding all other terms

$$\lim_{x\to\infty} \frac{x^2-4}{2+x-4x^2} \Longrightarrow \lim_{x\to\infty} \frac{x^2}{-4x^2} \Longrightarrow \lim_{x\to\infty} -\frac{1}{4} \Longrightarrow -\frac{1}{4}$$

   So we know $\lim_{x\to\infty} \frac{x^2-4}{2+x-4x^2} = -\frac{1}{4}$.

2. The graph of a function f is shown below. If the limit as $x \to b$ exists and f is not continuous at b, then what is the value of b?



   The function f fulfills these requirements at $b = 0$. Since the function approaches the same value ($y = 1$) from both sides, the limit as $x \to 0$ exists. Since the function has a hole at $x = 0$, it is not continuous.

3. What is $\lim_{x\to 3} \frac{\frac{6}{x}-2}{3-4x+x^2}$?

If we try to evaluate the limit by plugging in $x = 3$, we get an undefined value due to division by 0. Instead, we can simplify the function by factoring:

$$\frac{\frac{6}{x}-2}{3-4x+x^2} \implies \frac{-2(1-\frac{3}{x})}{(1-\frac{3}{x})(x^2-x)} \implies \frac{-2}{x^2-x}$$

With the simplification, we can now plug in $x = 3$ to calculate the limit.

$$\lim_{x\to 3} \frac{\frac{6}{x}-2}{3-4x+x^2} = \frac{-2}{(3)^2-3} \implies \lim_{x\to 3} \frac{\frac{6}{x}-2}{3-4x+x^2} = \frac{-2}{6}$$
$$\implies \lim_{x\to 3} \frac{\frac{6}{x}-2}{3-4x+x^2} = -\frac{1}{3}$$

4. Which of the following functions have a *removable discontinuity* at $x = 2$.

   (a) $f(x) = \frac{x^2-x-2}{x-2}$

   (b) $f(x) = \frac{1}{(x-2)^2}$

   (c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

   (d) $f(x) = \begin{cases} x^3-1 & x > 2 \\ -x^2 & x \leq 2 \end{cases}$

   We can start by evaluating (a) $f(x) = \frac{x^2-x-2}{x-2}$

$$f(x) = \frac{x^2-x-2}{x-2} \implies f(x) = \frac{(x-2)(x+1)}{x-2}$$

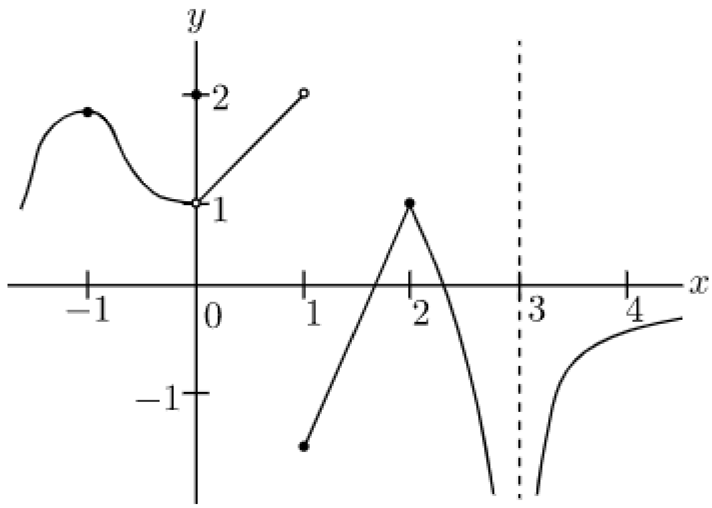
   Since both the numerator and denominator have a factor of $x-2$, the function will have a hole/removable discontinuity at $x - 2 = 0 \implies x = 2$.

   (b) $f(x) = \frac{1}{(x-2)^2}$ only has a factor of $x-2$ in the denominator, which will result in an asymptote, but not a removable discontinuity.

   (c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$ will have a removable discontinuity at $x = 2$ due to the inclusion of the equation from (a), and the point being redefined by the piecewise function to $y = 3$.

   (d) $f(x) = \begin{cases} x^3-1 & x > 2 \\ -x^2 & x \leq 2 \end{cases}$ is not a removable discontinuity because the two-sided limit is not equal at $x = 2$.

5. Identify the left-hand limit $\lim_{x \to 1^-} f(x)$ based on the graph of $f(x)$ shown below.



We can see the left-hand limit $\lim_{x \to 1^-} f(x) = 2$ as coming from the left side, the graph is heading towards the hole at $y = 2$.

6. Identify the right-handed limit $\lim_{x \to -1^+} \frac{x^2-1}{|x+1|}$

To identify this limit we can plug in values close to the limit to see the result:

$$x = 0 \Longrightarrow \frac{(0)^2 - 1}{|0+1|} = \frac{-1}{1} = -1$$

$$x = -0.5 \Longrightarrow \frac{(-0.5)^2 - 1}{|-0.5+1|} = \frac{-0.75}{0.5} = -1.5$$

$$x = -0.9 \Longrightarrow \frac{(-0.9)^2 - 1}{|-0.9+1|} = \frac{-0.19}{0.1} = -1.9$$

$$x = -0.99 \Longrightarrow \frac{(-0.99)^2 - 1}{|-0.99+1|} = \frac{-0.199}{0.01} = -1.99$$

The trend these follow suggests that $\lim_{x \to -1^+} \frac{x^2-1}{|x+1|} = -2$.

6 Derivatives

1. What is $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}\right)}{h}$

   This is the limit definition for the derivative of $\cos(x)$ evaluated at $\frac{\pi}{6}$. First, we compute the derivative:

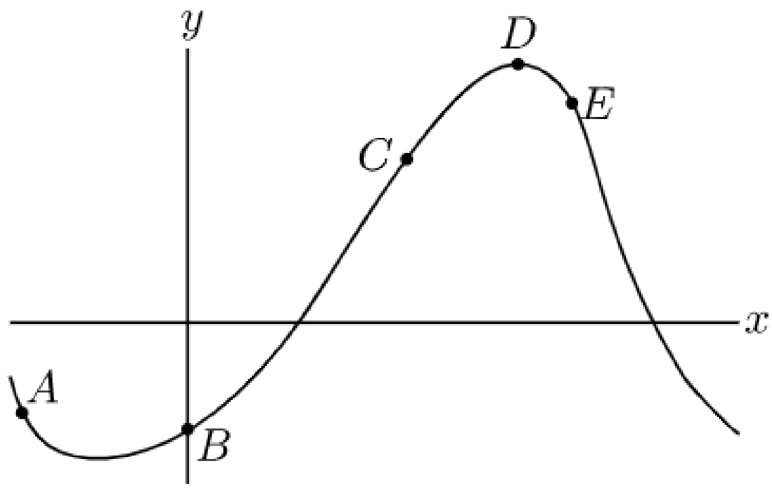
   $$\frac{d}{dx}\cos(x) = -\sin(x)$$

   Now, we plug in $x = \frac{\pi}{6}$

   $$-\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

   So $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}\right)}{h} = -\frac{1}{2}$

2. At which of the five points on the graph are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both negative?

   At (A), the slope is negative, so the first derivative is negative, but the function is concave up, so the second derivative is positive.

   At (B), the slope is positive, so the first derivative is positive.

   At (C), the slope is positive, so the first derivative is positive.

   At (D), the slope is 0, so the first derivative is 0.

   At (E), the slope is negative, so the first derivative is negative, and the function is concave down, so the second derivative is also negative.

3. What is the average rate of change of the function $f(x) = x^4 - 5x$ between $x = 0$ and $x = 3$? Average rate of change is calculated by the formula $\frac{f'(b)-f'(a)}{b-a}$, so we need to calculate the derivative $f'(x)$:

$$f'(x) = \frac{d}{dx}\left(x^4 - 5x\right) \Longrightarrow f'(x) = 4x^3 - 5$$

In this problem, $b = 3$ and $a = 0$, which we can plug into the average rate of change formula:

$$\frac{f'(3) - f'(0)}{3 - 0} \Longrightarrow \frac{\left(4(3)^3 - 5\right) - \left(4(0)^3 - 5\right)}{3} \Longrightarrow \frac{(4 \times 27 - 5) - (0 - 5)}{3}$$

$$\Longrightarrow \frac{103 + 5}{3} \Longrightarrow \frac{108}{3} \Longrightarrow 36$$

The average rate of change of $f(x) = x^4 - 5x$ between $x = 0$ and $x = 3$ was 36.

4. The position of a particle moving along a line is $p(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$. For what values of t is the speed of the particle increasing?

   (a) $3 < t < 4$ only
   (b) $t > 4$ only
   (c) $t > 5$ only
   (d) $0 < t < 3$ and $t > 5$
   (e) $3 < t < 4$ and $t > 5$

   Given the position function, the speed/velocity function is the first derivative and the acceleration function is the second derivative. The speed of the particle is increasing when the acceleration function is positive. First, calculating the second derivative:

$$a(t) = p''(t) \Longrightarrow a(t) = \frac{d^2}{dx^2}\left(2t^3 - 24t^2 + 90t + 7\right)$$

$$\Longrightarrow a(t) = \frac{d}{dx}\left(6t^2 - 48t + 90\right) \Longrightarrow a(t) = 12t - 48$$

   Now we find the inflection points, which is where the second derivative changes sign:

$$a(t) = 0 \Longrightarrow 12t - 48 = 0 \Longrightarrow 12t = 48 \Longrightarrow t = 4$$

   With the inflection point of $t = 4$ determined, we can check a t-value on either side of it to check the concavity.

$$t = 3 \Longrightarrow a(3) \Longrightarrow 12(3) - 48 \Longrightarrow 36 - 48 \Longrightarrow -12 < 0$$

$$t = 5 \Longrightarrow a(5) \Longrightarrow 12(5) - 48 \Longrightarrow 60 - 48 \Longrightarrow 12 > 0$$

   Since the function is concave up for $t > 4$, we know that the correct answer is (b), the speed of the particle is increasing on the interval $t > 4$ only.

5. Evaluate the limit $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$. Calculating this limit literally leads to an indeterminate form as both the numerator and denominator tend towards ∞. Due to the indeterminate form, we can apply L'Hospital's rule, where our limit is equivalent to the limit with the derivatives of the numerator and denominator.

$$\frac{d}{dx} \ln(x) \Longrightarrow \frac{1}{x}$$

$$\frac{d}{dx} x^2 \Longrightarrow 2x$$

Now we can apply L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{\times}}{2x} \Longrightarrow \lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} \Longrightarrow \lim_{x\to\infty} \frac{\ln(x)}{x^2} = 0$$

6. If f is differentiable at $x = a$, which of the following must be true?

   (a) f is continuous at $x = a$

   (b) $\lim_{x\to a} f(x)$ exists

   (c) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists

   (d) $f'(a)$ is defined.

   (e) $f''(a)$ is defined.

   A function that is differentiable at a point must also be continuous at the point, so (a) is true.

   Since according to (a), the function must be continuous at $x = a$, meaning the limit of $f(x)$ at that value must exist, so (b) is also true.

   (c) is the limit definition of the derivative at a point, which must exist if the function is differentiable at $x = a$, so (c) is true.

   If a function is differentiable at a point $x = a$, then the derivative's value at that point must be defined, so (d) is true.

   A function being differentiable does not imply that it is twice differentiable, so (e) is not true.

7. Let $f(x) = x^3 + 5x^2 - 7x - 1$. What is $f'(1)$?

$$f'(x) = \frac{d}{dx}\left(x^3 + 5x^2 - 7x - 1\right) \Longrightarrow f'(x) = 3x^2 + 10x - 7$$

$$f'(1) = 3(1)^2 + 10(1) - 7 \Longrightarrow f'(1) = 3(1) + 10 - 7 \Longrightarrow f'(1) = 6$$

8. Let $g(x) = x^2 e^x$. What is $g'(1)$

   In order to take this derivative, we will need the product rule: $\frac{d}{dx}(a(x)b(x)) = a'(x)b(x) + a(x)b'(x)$

$$a(x) = x^2 \Longrightarrow a'(x) = \frac{d}{dx}\left(x^2\right) \Longrightarrow a'(x) = 2x$$

$$b(x) = e^x \Longrightarrow b'(x) = \frac{d}{dx}\left(e^x\right) \Longrightarrow b'(x) = e^x$$

   Now applying the product rule:

$$g'(x) = \frac{d}{dx}\left(x^2 e^x\right) \Longrightarrow \; g'(x) = 2x \times e^x + x^2 \times e^x$$

$$\Longrightarrow g'(x) = 2xe^x + x^2 e^x \Longrightarrow g'(x) = e^x\left(2x + x^2\right)$$

$$g'(1) = e^1\left(2(1) + (1)^2\right) \Longrightarrow g'(1) = e(2+1) \Longrightarrow g'(1) = 3e$$

9. Suppose that $f(x) = g(5x)$ for all x, and that both functions are differentiable. Which of the following is necessarily true?

   (a) $f'(1) = g'(1)$
   (b) $f'(5) = g'(1)$
   (c) $f'(1) = g'(5)$
   (d) $5f'(1) = g'(1)$
   (e) $5f'(1) = g'(5)$
   (f) $f'(1) = 5g'(1)$
   (g) $f'(1) = 5g'(5)$

   If we apply the chain rule to our two functions, we would get $f'(x) = 5g'(5x)$

   (a), (c), and (f) might not necessarily be true as if $x = 1$, then we only know that $f'(1) = 5g'(5)$, but does prove (g) to be necessarily true.

   (b) might not necessarily be true as if $x = 5$, then we only know that $f'(5) = 5g'(25)$.

   (d) and (e) might not necessarily be true, as modifying our chain rule result would be $5f'(x) = 25g'(5x)$, which when $x = 1$ would be $5f'(1) = 25g'(5)$.

10. Let $f(t) = \frac{\ln(5t+1)}{\sqrt{t+1}}$. What is $f'(0)$?

In order to take this derivative, we will need the quotient rule:
$\frac{d}{dx}\left(\frac{a(x)}{b(x)}\right) = \frac{a'(x)b(x)-a(x)b'(x)}{(b(x))^2}$

$$a(t) = \ln(5t+1) \Longrightarrow a'(t) = \frac{d}{dt}(\ln(5t+1))$$

$$\Longrightarrow a'(t) = \frac{d}{dt}(5t+1)\frac{d}{dt}(\ln(5t+1)) \Longrightarrow a'(t) = 5\left(\frac{1}{5t+1}\right)$$

$$\Longrightarrow a'(t) = \frac{5}{5t+1}$$

$$b(t) = \sqrt{t+1} \Longrightarrow b'(t) = \frac{d}{dt}(\sqrt{t+1})$$

$$\Longrightarrow b'(t) = \frac{d}{dt}(t+1)\frac{d}{dt}\left((t+1)^{\frac{1}{2}}\right) \Longrightarrow b'(t) = 1\left(\frac{1}{2}(t+1)^{-\frac{1}{2}}\right)$$

$$\Longrightarrow b'(t) = \frac{1}{2\sqrt{t+1}}$$

Now applying the quotient rule:

$$f'(t) = \frac{d}{dt}\left(\frac{\ln(5t+1)}{\sqrt{t+1}}\right) \Longrightarrow f'(t) = \frac{\frac{5}{5t+1} \times \sqrt{t+1} - \ln(5t+1) \times \frac{1}{2\sqrt{t+1}}}{\left(\sqrt{t+1}\right)^2}$$

$$\Longrightarrow f'(t) = \frac{\frac{5\sqrt{t+1}}{5t+1} + \frac{\ln(5t+1)}{2\sqrt{t+1}}}{t+1} \Longrightarrow f'(t) = \frac{5\sqrt{t+1}}{(5t+1)(t+1)} + \frac{\ln(5t+1)}{2\sqrt{t+1}(t+1)}$$

$$\Longrightarrow f'(t) = \frac{5}{(5t+1)\sqrt{t+1}} + \frac{\ln(5t+1)}{2(t+1)^{\frac{3}{2}}}$$

$$f'(0) = \frac{5}{(5(0)+1)\sqrt{0+1}} + \frac{\ln(5(0)+1)}{2(0+1)^{\frac{3}{2}}} f'(0) = \frac{5}{(0+1)\sqrt{1}} + \frac{\ln(0+1)}{2(1)^{\frac{3}{2}}}$$

$$f'(0) = \frac{5}{1(1)} + \frac{\ln(1)}{2(1)} \Longrightarrow f'(0) = \frac{5}{1} + \frac{0}{2} \Longrightarrow f'(0) = 5 + 0 \Longrightarrow f'(0) = 5$$