Introduction to Limits
Notes

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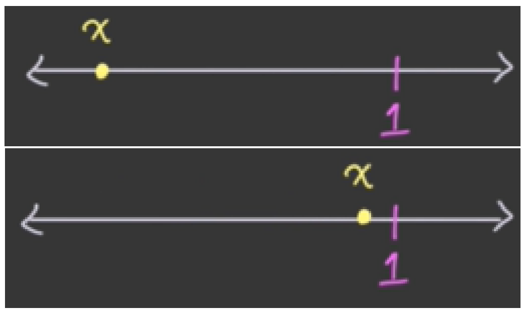
1 Objectives

At the end of this sequence, and after some practice, you should be able to:

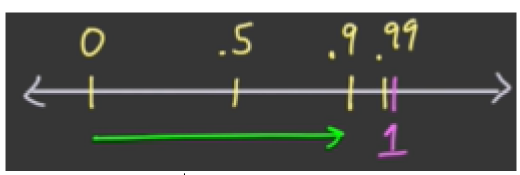
- Use a calculator to determine right and left hand limits.
- Identify right and left hand limits based on graphs.
- Determine if a limit exists based on values of right and left hand limits.
- Understand that the limit does not depend on the value of a function at the point of interest.

2 Moving Closer and Closer

Calculus is all about functions.
You probably know that a function f takes an input x and gives an output $f(x)$, which we could write as $x \mapsto f(x)$.
But in calculus, we aren't concerned with just a single input and it's output, we want to consider a whole range of inputs. We want to know what happens as the input moves/varies.
An example: As x moves closer to 1 from the left: $x \to 1^-$



It is important to note that when we say x moves closer to 1, we do not mean that x will ever equal 1, we are only concerned with values of x that are near 1.
In a function f, we know that as we change our input x, our output $f(x)$ is going to change as well.
With limits, we ask the question, does the output $f(x)$ move close to something as the input x moves close to something?
For our example, lets take the function $\frac{\sqrt{3-5x+x^2+x^3}}{x-1}$.
To do this, we could take values for x that are getting closer to 1 from the left to use as inputs for f.
There are an infinite amount of values we could choose for x, but we will start with 4 simpler values.



x	$f(x)$
0	$f(0) \approx -1.73$
0.5	$f(0.5) \approx -1.87$
0.9	$f(0.9) \approx -1.97$
0.99	$f(0.99) \approx -1.997$

Reading the values in this table, we can see that as x approaches 1 from the left, $f(x)$ approaches -2.
Now we can try to find what f approaches as x approaches 1 from the right side.

x	$f(x)$
2	$f(2) \approx 2.23$
1.5	$f(1.5) \approx 2.12$
1.1	$f(1.1) \approx 2.02$
1.01	$f(1.01) \approx 2.003$

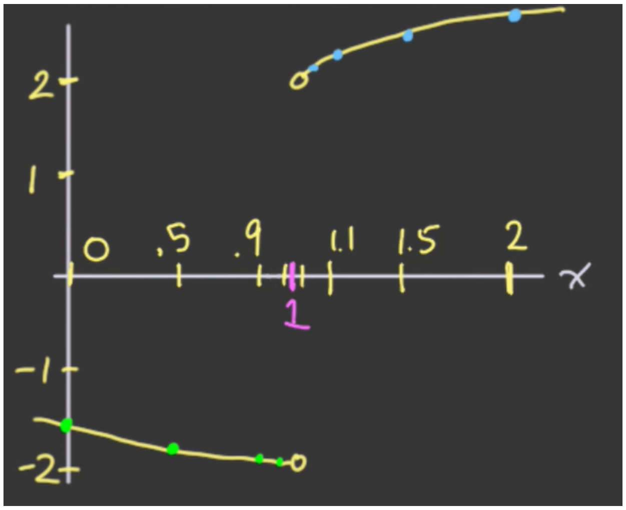
This time, reading the values in the table, we can see that as x approaches 1 from the right, $f(x)$ approaches 2.
That means that in this example, the direction we were approaching from with x affected where $f(x)$ was approaching.

3 One-Sided Limits

From the last section, we know that given the function $f(x) = \frac{\sqrt{3-5x+x^2+x^3}}{x-1}$,

- As $x \to 1^-$, $f(x) \to -2$
- As $x \to 1^+$, $f(x) \to 2$

We can plot the points we had in the tables from the last section on the coordinate plane to get a better picture of what is happening with our function:

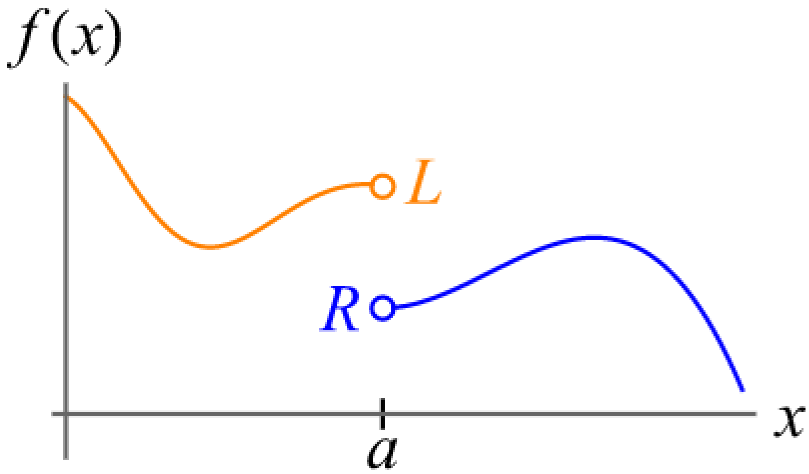


We call this phenomena where a function approaches a value a **limit**
We could rephrase our earlier statements to use our new vocabulary and some new notation:

- The limit of $f(x)$ as $x \to 1^-$ is $-2 \implies \lim_{x \to 1^-} f(x) = -2$
- The limit of $f(x)$ as $x \to 1^+$ is $2 \implies \lim_{x \to 1^+} f(x) = 2$

We call the limit as x approaches a value from the left the **left-hand limit** and the limit as x approaches a value from the right the **right-hand limit**.

4 Definitions of Right-Hand and Left-Hand Limits



Suppose $f(x)$ gets really close to R for values of x that get really close to (but are not equal to) a from the right. Then we say R is the **right-hand limit** of the function $f(x)$ as x approaches a from the right. We can write this two different ways:

$$f(x) \to R \text{ as } x \to a^+$$

$$\lim_{x \to a^+} f(x) = R$$

If $f(x)$ gets really close to L for values of x that get really close to (but are not equal to) a from the left, we say that L is the **left-hand limit** of the function $f(x)$ as x approaches a from the left. We can also write this two different ways:

$$f(x) \to L \text{ as } x \to a^-$$

$$\lim_{x \to a^-} f(x) = L$$

Let's explore the right and left-hand limits of a few more functions.

1. In this problem, we'll examine the function $g(x) = \frac{x}{\tan(2x)}$ as $x \to 0^{\pm}$.

   Here is a table of values of $g(x)$ as x approaches 0 from the right:

x	$g(x)$
1.0	-0.458
0.5	0.321
0.1	0.493
0.05	0.498
0.01	0.4999

These data suggest that $\lim_{x \to 0^+} g(x) = 0.5$.

Find the left-hand limit:

(a) As $x \to 0^-$, $g(x)$ gets closer and closer to a particular number L ($g(x) \to L$)

(b) As $x \to 0^-$, $g(x)$ gets bigger and bigger without bound ($g(x) \to +\infty$)

(c) As $x \to 0^-$, $g(x)$ gets bigger and bigger in the negative direction without bound ($g(x) \to -\infty$)

(d) As $x \to 0^-$, $g(x)$ approaches neither a finite number L, nor $+\infty$, or $-\infty$.

x	$g(x)$
-1.0	-0.457
-0.5	0.321
-0.1	0.493
-0.05	0.498
-0.01	0.4999

The values follow the same trend as the right-hand limit, so $\lim_{x \to 0^-} g(x) = 0.5$. This means that (a) is the correct answer, as it is getting closer and closer to a particular number L.

2. In this problem, we'll examine the function $h(x) = \frac{|x|+\sin(x)}{x^2}$ as $x \to 0^{\pm}$.

Here is a table of values of $h(x)$ for values of x that are close to zero on the left:

x	$h(x)$
-1.0	0.159
-0.5	0.082
-0.1	0.017
-0.05	0.002
-0.01	0.0002

These data suggest that $\lim_{x \to 0^-} h(x) = 0$

Find the right-hand limit:

(a) As $x \to 0^-$, $h(x)$ gets closer and closer to a particular number L ($h(x) \to L$)

(b) As $x \to 0^-$, $h(x)$ gets bigger and bigger without bound ($h(x) \to +\infty$)

(c) As $x \to 0^-$, $h(x)$ gets bigger and bigger in the negative direction without bound ($h(x) \to -\infty$)

(d) As $x \to 0^-$, $h(x)$ approaches neither a finite number L, nor $+\infty$, or $-\infty$.

x	$h(x)$
1.0	1.841
0.5	3.918
0.1	19.983
0.05	39.992
0.01	199.998
0.001	1999.999

The values continue to scale upwards, so $h(x) \to +\infty$. This means that (b) is the correct answer, as it is getting bigger and bigger without bound.

3. In this problem, we'll examine the function $j(x) = \sin(\frac{13}{x})$, as x approaches 0 from the right. Use a calculator to figure out what $\lim_{x \to 0^+} j(x)$ might be:

   (a) As $x \to 0^+$, $j(x)$ gets closer and closer to a particular number L ($j(x) \to L$)

   (b) As $x \to 0^-$, $j(x)$ gets bigger and bigger without bound ($j(x) \to +\infty$)

   (c) As $x \to 0^-$, $j(x)$ gets bigger and bigger in the negative direction without bound ($j(x) \to -\infty$)

   (d) As $x \to 0^-$, $j(x)$ approaches neither a finite number L, nor $+\infty$, or $-\infty$.

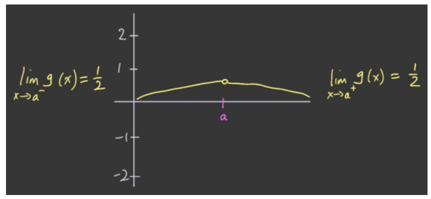
x	$j(x)$
1.0	0.420
0.5	0.763
0.1	-0.930
0.05	0.683
0.01	-0.581
0.001	0.089

This data does not have a clear trend, as it keeps bouncing around values. This must mean that (d) is the correct answer, as $j(x)$ does not approach a finite number L, $+\infty$, or $-\infty$.

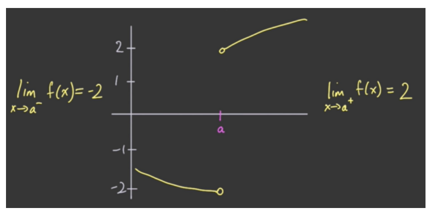
5 Possible Limit Behaviors

There are many possible behaviors of limits.

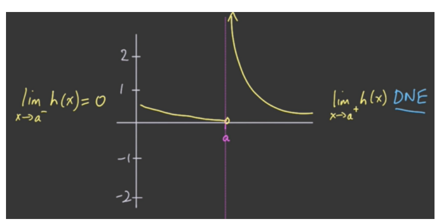
- The right-hand and left-hand limits may both exist and be equal.



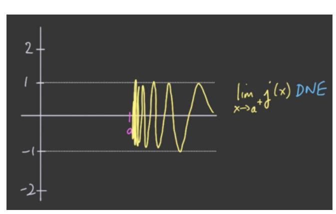
- The right-hand and left-hand limits may both exist, but may fail to be equal.



- A right- and/or left-hand limit could fail to exist due to blowing up to $\pm\infty$. (Example: Consider the function $\frac{1}{x}$ near $x = 0$.) In this case, we either say the limit blows up to infinity, or that the limit does not exist because ∞ is not a real number!



- A right- and/or left-hand limit could fail to exist because it oscillates beteen many values and never settles down. In this case, we say the limit does not exist.



6 Sample Limit Questions

1. Suppose $\lim_{x \to a^+} f(x)$ exists and equals R. Must $\lim_{x \to a^-} f(x)$ exist?

   No, $\lim_{x \to a^-} f(x)$ does not have to exist, as right- and left-hand limits do not have to be equal, and one could not exist while the other does.

2. Suppose that $\lim_{x \to a^+} f(x) = R$ and $\lim_{x \to a^-} f(x) = L$. Must $R = L$?

   No, the right- and left-hand limits do not have to be the same value even if they both exist.

3. Suppose that $\lim_{x \to a^+} f(x)$ is some number R. Must $f(a) = R$?

   No, the limit never checks the actual value of $f(a)$, so the limit can tend towards a value even if the function has a hole/removable point there.

4. Suppose that $f(a) = K$. Must $\lim_{x \to a^+} f(x) = K$?

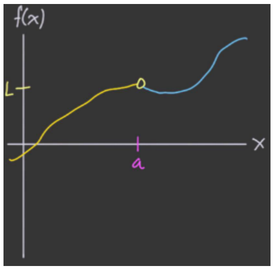
   No, the point (a, K) could be a removable point from the function, which the limit would ignore.

7 The Overall Limit

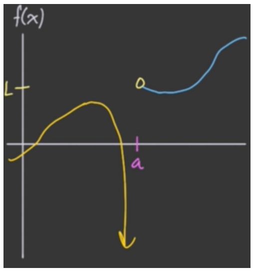
The overall limit takes into account both the right- and left-hand limits to make a statement about the function for all values of x close to the value a.

We can make the statement $\lim_{x \to a} f(x) = L$ for the overall limit iff $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

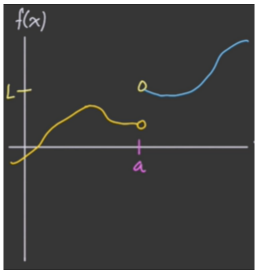
This means that the overall limit only exists at a value if both the left- and right-hand limits go to the same value. Here is a graphical example:



Sometimes the overall limit doesn't exist even if one of the left- or right-hand limits exists. If one of the one-sided limits doesn't exist, then the overall limit does not exist. Here is a graphical example:



Even if both the left- and right-hand limits exist, if they are not equal, then the overall limit does not exist. Here is a graphical example:



8 Definition of the Limit

8.1 Basic Definition of the Limit

If a function $f(x)$ approaches some value L, as x approaches a from *both the right and the left*, then **the limit** of $f(x)$ exists and equals L.

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ then $\lim_{x \to a} f(x) = L$.

8.2 Formal Definition of the Limit

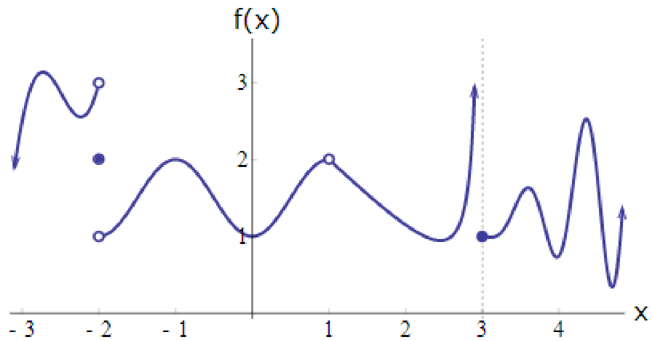
Formally, the statement $\lim_{x \to a} f(x) = L$ is defined as:

For all $\epsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Explanation: The distance between two numbers y and z is given by $|y - z|$. Thus, the very last part of the definition is saying that the distance from $f(x)$ to L is less than ϵ; one should think of ϵ as representing a small distance. This close distance occurs if $0 < |x - a| < \delta$; that is, if x is within some distance δ from a, but not necessarily if that distance is 0 (we don't care about $x = a$ itself).

The "for all" and "there exists" clauses have to do with how small these distances need to get. We want $f(x)$ to eventually get arbitrarily close to L, so this statement needs to be satisfied no matter how small ϵ gets. Given any choice of ϵ, we can satisfy the condition $|f(x) - L| < \epsilon$ as long as x gets close enough to a; the proximity required is measured by δ.

9 Estimate Limits



1. Determing the following:

$$\lim_{x \to -2^-} f(x) = 3$$
$$\lim_{x \to -2^+} f(x) = 1$$
$$\lim_{x \to -2} f(x) = DNE$$
$$f(-2) = 2$$

2. Determine the following:

$$\lim_{x \to 1^-} f(x) = 2$$
$$\lim_{x \to 1^+} f(x) = 2$$
$$\lim_{x \to 1} f(x) = 2$$
$$f(1) = DNE$$

3. Determine the following:

$$\lim_{x \to 3^-} f(x) = DNE$$
$$\lim_{x \to 3^+} f(x) = 1$$
$$\lim_{x \to 3} f(x) = DNE$$
$$f(3) = 1$$

10 Sample Limit Questions 2

1. If we know $f(a)$ exists, this means that $\lim_{x \to a} f(x)$ exists

   False, because the function could have a removable point at $x = a$ that would allow $f(a)$ to exist, but would not be included in the limit.

2. Suppose that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = 3$. Which of the following must be true?

   (a) $\lim_{x \to a} f(x) = 3$

   (b) $f(a) = 3$

   (c) Both must be true

   (d) Neither is necessarily true.

   The correct answer is (a), as the requirement for an overall limit being equal to a value is that the two one-sided limits both go to that same value. The answer is not (c) because (b) is false, as the function could have a removable point at the value $x = a$, making $f(a) \neq 3$.

3. The floor function $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. Determine the following:

$$\lim_{x \to 2^-} \lfloor x \rfloor = 1$$

$$\lim_{x \to 2^+} \lfloor x \rfloor = 2$$

$$\lim_{x \to 2} \lfloor x \rfloor = DNE$$

$$\lfloor 2 \rfloor = 2$$

11 Limit Laws

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:

- $\lim_{x \to a} [f(x) + g(x)] = L + M$ (Limit Law for Addition)
- $\lim_{x \to a} [f(x) - g(x)] = L - M$ (Limit Law for Subtraction)
- $\lim_{x \to a} [f(x) \times g(x)] = L \times M$ (Limit Law for Multiplication)
- $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$ if $M \neq 0$ (Limit Law for Division)