

Introduction to Limits

Notes

John Guzauckas

June 9, 2022

1 Objectives

At the end of this sequence, and after some practice, you should be able to:

- Use a calculator to determine right and left hand limits.
- Identify right and left hand limits based on graphs.
- Determine if a limit exists based on values of right and left hand limits.
- Understand that the limit does not depend on the value of a function at the point of interest.

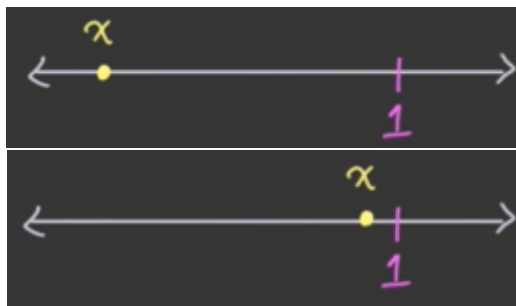
2 Moving Closer and Closer

Calculus is all about functions.

You probably know that a function f takes an input x and gives an output $f(x)$, which we could write as $x \mapsto f(x)$.

But in calculus, we aren't concerned with just a single input and its output, we want to consider a whole range of inputs. We want to know what happens as the input moves/varies.

An example: As x moves closer to 1 from the left: $x \rightarrow 1^-$



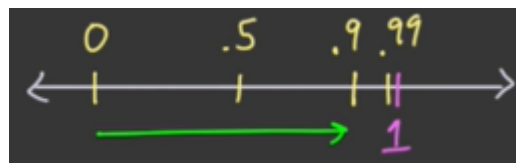
It is important to note that when we say x moves closer to 1, we do not mean that x will ever equal 1, we are only concerned with values of x that are near 1. In a function f , we know that as we change our input x , our output $f(x)$ is going to change as well.

With limits, we ask the question, does the output $f(x)$ move close to something as the input x moves close to something?

For our example, let's take the function $\frac{\sqrt{3-5x+x^2+x^3}}{x-1}$.

To do this, we could take values for x that are getting closer to 1 from the left to use as inputs for f .

There are an infinite amount of values we could choose for x , but we will start with 4 simpler values.



x	$f(x)$
0	$f(0) \approx -1.73$
0.5	$f(0.5) \approx -1.87$
0.9	$f(0.9) \approx -1.97$
0.99	$f(0.99) \approx -1.997$

Reading the values in this table, we can see that as x approaches 1 from the left, $f(x)$ approaches -2 .

Now we can try to find what f approaches as x approaches 1 from the right side.

x	$f(x)$
2	$f(2) \approx 2.23$
1.5	$f(1.5) \approx 2.12$
1.1	$f(1.1) \approx 2.02$
1.01	$f(1.01) \approx 2.003$

This time, reading the values in the table, we can see that as x approaches 1 from the right, $f(x)$ approaches 2.

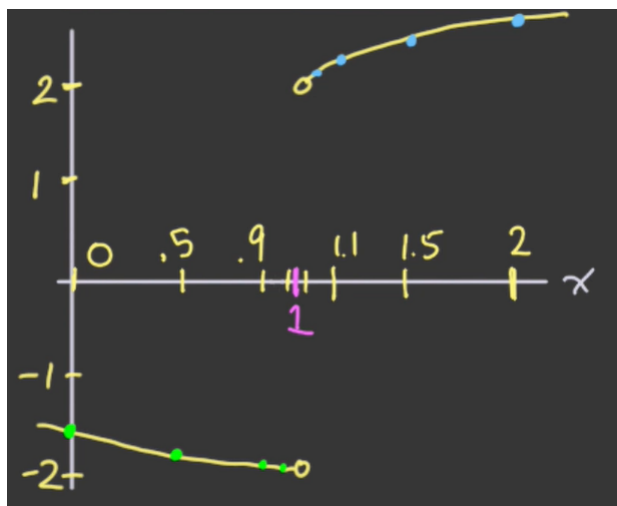
That means that in this example, the direction we were approaching from with x affected where $f(x)$ was approaching.

3 One-Sided Limits

From the last section, we know that given the function $f(x) = \frac{\sqrt{3-5x+x^2+x^3}}{x-1}$,

- As $x \rightarrow 1^-$, $f(x) \rightarrow -2$
- As $x \rightarrow 1^+$, $f(x) \rightarrow 2$

We can plot the points we had in the tables from the last section on the coordinate plane to get a better picture of what is happening with our function:



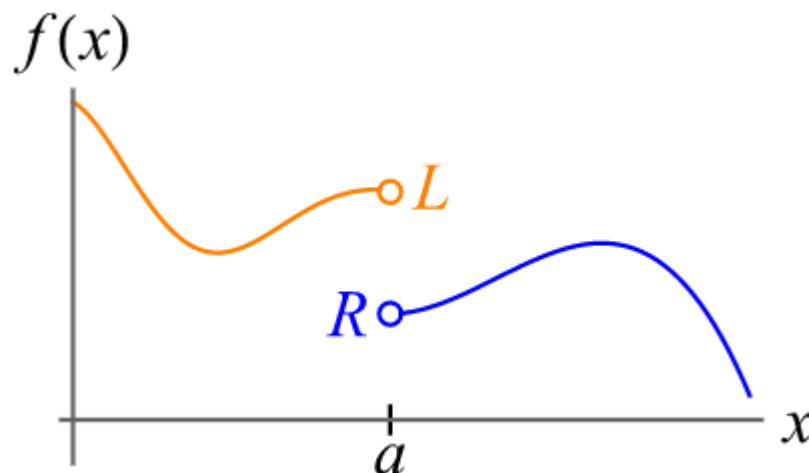
We call this phenomena where a function approaches a value a **limit**

We could rephrase our earlier statements to use our new vocabulary and some new notation:

- The limit of $f(x)$ as $x \rightarrow 1^-$ is $-2 \implies \lim_{x \rightarrow 1^-} f(x) = -2$
- The limit of $f(x)$ as $x \rightarrow 1^+$ is $2 \implies \lim_{x \rightarrow 1^+} f(x) = 2$

We call the limit as x approaches a value from the left the **left-hand limit** and the limit as x approaches a value from the right the **right-hand limit**.

4 Definitions of Right-Hand and Left-Hand Limits



Suppose $f(x)$ gets really close to R for values of x that get really close to (but are not equal to) a from the right. Then we say R is the **right-hand limit** of the function $f(x)$ as x approaches a from the right. We can write this two different ways:

$$f(x) \rightarrow R \text{ as } x \rightarrow a^+$$

$$\lim_{x \rightarrow a^+} f(x) = R$$

If $f(x)$ gets really close to L for values of x that get really close to (but are not equal to) a from the left, we say that L is the **left-hand limit** of the function $f(x)$ as x approaches a from the left. We can also write this two different ways:

$$f(x) \rightarrow L \text{ as } x \rightarrow a^-$$

$$\lim_{x \rightarrow a^-} f(x) = L$$

Let's explore the right and left-hand limits of a few more functions.

1. In this problem, we'll examine the function $g(x) = \frac{x}{\tan(2x)}$ as $x \rightarrow 0^\pm$.

Here is a table of values of $g(x)$ as x approaches 0 from the right:

x	$g(x)$
1.0	-0.458
0.5	0.321
0.1	0.493
0.05	0.498
0.01	0.4999

These data suggest that $\lim_{x \rightarrow 0^+} g(x) = 0.5$.

Find the left-hand limit:

- (a) As $x \rightarrow 0^-$, $g(x)$ gets closer and closer to a particular number L ($g(x) \rightarrow L$)
- (b) As $x \rightarrow 0^-$, $g(x)$ gets bigger and bigger without bound ($g(x) \rightarrow +\infty$)
- (c) As $x \rightarrow 0^-$, $g(x)$ gets bigger and bigger in the negative direction without bound ($g(x) \rightarrow -\infty$)
- (d) As $x \rightarrow 0^-$, $g(x)$ approaches neither a finite number L , nor $+\infty$, or $-\infty$.

x	$g(x)$
-1.0	-0.457
-0.5	0.321
-0.1	0.493
-0.05	0.498
-0.01	0.4999

The values follow the same trend as the right-hand limit, so $\lim_{x \rightarrow 0^-} g(x) = 0.5$. This means that (a) is the correct answer, as it is getting closer and closer to a particular number L .

2. In this problem, we'll examine the function $h(x) = \frac{|x| + \sin(x)}{x^2}$ as $x \rightarrow 0^\pm$.

Here is a table of values of $h(x)$ for values of x that are close to zero on the left:

x	$h(x)$
-1.0	0.159
-0.5	0.082
-0.1	0.017
-0.05	0.002
-0.01	0.0002

These data suggest that $\lim_{x \rightarrow 0^-} h(x) = 0$

Find the right-hand limit:

- (a) As $x \rightarrow 0^+$, $h(x)$ gets closer and closer to a particular number L ($h(x) \rightarrow L$)
- (b) As $x \rightarrow 0^+$, $h(x)$ gets bigger and bigger without bound ($h(x) \rightarrow +\infty$)
- (c) As $x \rightarrow 0^+$, $h(x)$ gets bigger and bigger in the negative direction without bound ($h(x) \rightarrow -\infty$)
- (d) As $x \rightarrow 0^+$, $h(x)$ approaches neither a finite number L , nor $+\infty$, or $-\infty$.

x	$h(x)$
1.0	1.841
0.5	3.918
0.1	19.983
0.05	39.992
0.01	199.998
0.001	1999.999

The values continue to scale upwards, so $h(x) \rightarrow +\infty$. This means that (b) is the correct answer, as it is getting bigger and bigger without bound.

3. In this problem, we'll examine the function $j(x) = \sin(\frac{13}{x})$, as x approaches 0 from the right. Use a calculator to figure out what $\lim_{x \rightarrow 0^+} j(x)$ might be:

- (a) As $x \rightarrow 0^+$, $j(x)$ gets closer and closer to a particular number L ($j(x) \rightarrow L$)
- (b) As $x \rightarrow 0^+$, $j(x)$ gets bigger and bigger without bound ($j(x) \rightarrow +\infty$)
- (c) As $x \rightarrow 0^+$, $j(x)$ gets bigger and bigger in the negative direction without bound ($j(x) \rightarrow -\infty$)
- (d) As $x \rightarrow 0^+$, $j(x)$ approaches neither a finite number L , nor $+\infty$, or $-\infty$.

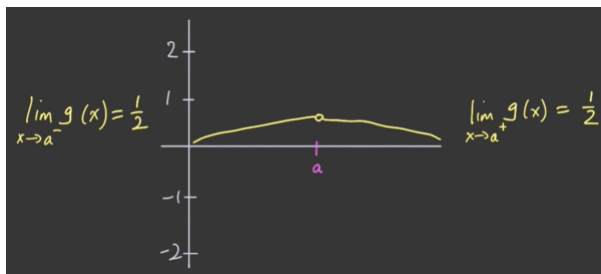
x	$j(x)$
1.0	0.420
0.5	0.763
0.1	-0.930
0.05	0.683
0.01	-0.581
0.001	0.089

This data does not have a clear trend, as it keeps bouncing around values. This must mean that (d) is the correct answer, as $j(x)$ does not approach a finite number L , $+\infty$, or $-\infty$.

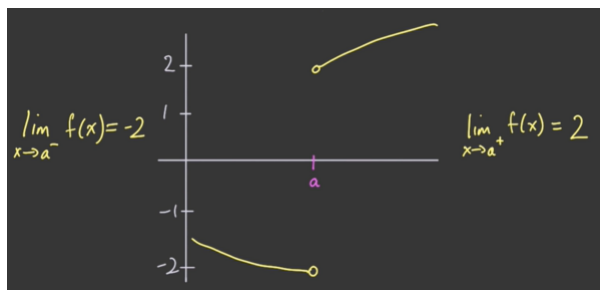
5 Possible Limit Behaviors

There are many possible behaviors of limits.

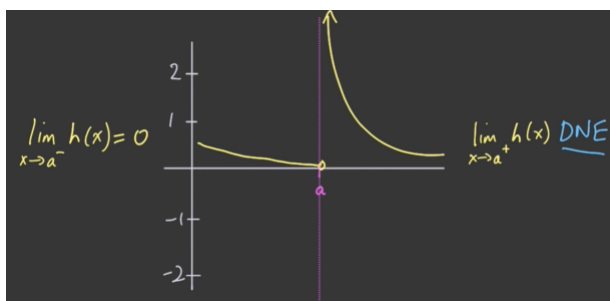
- The right-hand and left-hand limits may both exist and be equal.



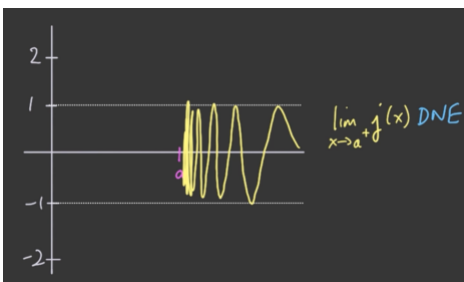
- The right-hand and left-hand limits may both exist, but may fail to be equal.



- A right- and/or left-hand limit could fail to exist due to blowing up to $\pm\infty$. (Example: Consider the function $\frac{1}{x}$ near $x = 0$.) In this case, we either say the limit blows up to infinity, or that the limit does not exist because ∞ is not a real number!



- A right- and/or left-hand limit could fail to exist because it oscillates between many values and never settles down. In this case, we say the limit does not exist.



6 Sample Limit Questions

1. Suppose $\lim_{x \rightarrow a^+} f(x)$ exists and equals R . Must $\lim_{x \rightarrow a^-} f(x)$ exist?

No, $\lim_{x \rightarrow a^-} f(x)$ does not have to exist, as right- and left-hand limits do not have to be equal, and one could not exist while the other does.

2. Suppose that $\lim_{x \rightarrow a^+} f(x) = R$ and $\lim_{x \rightarrow a^-} f(x) = L$. Must $R = L$?

No, the right- and left-hand limits do not have to be the same value even if they both exist.

3. Suppose that $\lim_{x \rightarrow a^+} f(x)$ is some number R . Must $f(a) = R$?

No, the limit never checks the actual value of $f(a)$, so the limit can tend towards a value even if the function has a hole/removable point there.

4. Suppose that $f(a) = K$. Must $\lim_{x \rightarrow a^+} f(x) = K$?

No, the point (a, K) could be a removable point from the function, which the limit would ignore.