Course: Introduction to Mathematical Thinking

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Week 9-10: Peer-graded Assignment: Test Flight Peer Assessments

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Test Flight Problem Set

TFPS-1

Show that $(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(3m + 5n = 12)$ is either true or false.

Proof: The statement says that there exists a pair *m,n* of natural numbers such that $(3m + 5n = 12)$. Let's call this statement A. We are going to evaluate statement A by simple reasoning and show that we end up with contradictions.

(1) Assume *m*=*n*. Then, by substitution, statement A becomes: $3m + 5m = 8m = 12$. This is false since 12 is not divisible by 8 in $\mathbb{N}$. Therefore, the assumption that *m*=*n* must be false.

(2) Since *m* and *n* are interchangable, we may choose that *n* > *m* such that *n*=*m*+*p* with *p* ∈ $\mathbb{N}$. Then, by substitution statement A becomes:

$3m + 5(m + p) = 8m + 5p = 12$

Since both m and p are natural numbers and therefore by defition are at least 1, the left hand side of statement A is at least 13, which is bigger than 12. Thus ending up again with a false statement A.

Conclusion: by simple reasoning we have shown that evaluating statement A leads to contradictions, so it must be false.

TFPS-2

Show that the statement 'The sum of any five consecutive integers is divisible by 5 (without a remainder)' is either true of false and support your answer by a proof. In mathematical notation this can be stated as:

$$(\forall i \in \mathbb{Z})(n \in \mathbb{N})[\, 5 \mid \sum_{n=1}^{5} (i + n)]$$

Proof: The statement will be proven by simple arithmetic.
Given an arbitrary integer i, the sum of its five consecutives can be written as $(n \in \mathbb{N})$:

$$\sum_{n=1}^{5}(i+n) = (i+1)+(i+2)+(i+3)+(i+4)+(i+5)$$

$$= 5i + (1+2+3+4+5)$$

$$= 5i + 15$$

$$= 5(i+3)$$

which indeed is divisible by 5, without a remainder.
Therefore, the statement is true.

TFPS-3

Say whether the following is true or false and support your answer by a proof: 'For any integer *n*, the number n^2+n+1 is odd.'

Proof: We will prove this statement by simple arithmatic and reasoning. By rewriting we find:

$$n^2+n+1 = n(n+1)+1$$

On the right hand side we see that the number can be rewritten as a product of two factors n and $(n+1)$ enlarged with 1.

(1) Assume (factor) n is even, than factor $(n+1)$ is odd and the product is also even, since even times odd always results in an even number. Thus adding 1 results in an odd number: if n is even, the number n^2+n+1 is odd.

(2) Assume (factor) n is odd, than factor $(n+1)$ is even and the product is also even, since odd times even always results in an even number. Thus adding 1 results in an odd number. Therefore, also if n is odd, the number n^2+n+1 is odd.

Thus, since we have chosen n arbitrarely, it has been proven that in for all integers n the number number n^2+n+1 is odd. The statement is true. Mathematically written:

$$(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})[\,(\,n^2+n+1\,) = 2m+1]$$

TFPS-4

Prove that every odd natural number is one of the form 4*n*+1 or 4*n*+3, where n is an integer.

Proof: We will prove this statement by using a simple trick, namely starting from an arbitrarily chosen natural number and dividing that by 4.

Let *m* by an arbitrary natural number and divide this by 4. Only if *m* is a multiple of 4 it can be written as $m = 4n$ where *n* also is a natural number: 4|*m*.

In all other cases *m* is not divisible by 4 and division by 4 will result in a remainder of 1, 2 or 3:

$$(m = 4n + 1) \quad \vee \quad (m = 4n + 2) \quad \vee \quad (m = 4n + 3)$$

Clearly, in the cases that $(m = 4n)$ and $(m = 4n + 2)$ the number *m* is even.

From that it obviously follows that in case $(m = 4n + 1)$ or $(m = 4n + 1)$ *m* is odd.

Since we started with chosing *m* to be any arbitrary natural number, we hereby have proven that any odd natural number can be expressed in the form 4*n*+1 or 4*n*+3. Q.E.D.

TFPS-5

Prove that for any integer *n*, at least one of the integers *n*, *n*+2, *n*+4 is divisible by 3.

Proof: We will prove this statement by simple arithmetic.

(1) Let *n* by an arbitrary integer and divide this by 3. Only if *n* is a multiple of 3 it can be written as $n = 3m$ where *m* also is a integer : 3|*n*.

In all other cases *n* is not divisible by 3 and division by 3 will result in a remainder of 1 or 2:

$$(n = 3m + 1) \quad \vee \quad (n = 3m + 2)$$

(2) Clearly, in the case $(n = 3m)$ the number *n* is divisible by 3, which - by the way - clearly is not the case for every n.

(3) So, it remains to be proven that for the other two possible cases, i.e.

$$(n = 3m + 1) \quad \vee \quad (n = 3m + 2)$$

$(n + 2)$ or $(n + 4)$ is divisible by 3.

(4) In the case that division by 3 leads to a remainder of 1 $(n = 3m + 1)$ by simply adding 2 at both sides of the identity leads to:

$$n + 2 = (3m + 1) + 2 = 3m + 3 = 3(m + 1)$$

Which is clearly divisible by 3. Since it suffices that only one of *n*, *n*+2, *n*+4 is divisible by 3, we don't have to evaluate whether *n*+4 is divisible by 3 in this case.

(5) In the case that division by 3 leads to a remainder of 2 $(n = 3m + 2)$ by simply adding 4 at both sides of the identity leads to:

$$n + 4 = (3m + 4) + 2 = 3m + 6 = 3(m + 2)$$

Which is clearly divisible by 3. Since it suffices that only one of *n*, *n*+2, *n*+4 is divisible by 3, we don't have to evaluate whether *n*+2 is divisible by 3 in this case.

(6) Since we started with chosing *n* to be any arbitrary integer and since we have evaluated all possible representations of n in terms of a product of 3 with a

remainder, we hereby have proven that at least one ot the integers *n*, *n*+2, *n*+4 can be divided by 3. Q.E.D.

TFPS-6

Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof: We are going to proof this statement by using the result of the previous assignment in which it was proven that: 'for any integer *n*, at least one of the integers *n*, *n*+2, *n*+4 is divisible by 3.'

(1) The sequence of primes are natural numbers. The set of natural numbers is a subset of the set of integers. Therefore, the statement 'for any integer *n*, at least one of the integers *n*, *n*+2, *n*+4 is divisible by 3.' also holds for natural numbers.

(2) A prime triple can be written as a set {*p*, *p*+2, *p*+4} with *p* the first prime of this triple. As it is true for any natural number n, it is also true for the prime triple {*p*, *p*+2, *p*+4} that at least one of the numbers in this set is divisible by 3.

(3) By definition of a prime it follows than not at least one of the numbers in the triple set is divisible by 3, but also only one, and furthermore that this number must be the prime 3 itself.

(4) Since 3 is also the smallest prime that can be part of a prime triple, it is hereby proven that the triple {3,5,7} is the smallest triple of primes that are separated 2 apart.

(5) Also by definition of a prime it follows that {3,5,7} is not only the smallest prime triple, but also the only one. This follows from step (2): if another, larger, prime triple would exist, than at least one of the primes in that triple is divisible by 3 and that goes against the defition of a prime. Therefore it is proven that {3,5,7} is the only existing prime triple.

TFPS-7

Prove that for any natural number *n*, $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

Proof: The proof is by induction in two steps: (1) first prove the identity for *n*=1; (2) then prove that if the identity holds for any arbitrary *n*, it follows that it holds for *n*+1

Let A(*n*) be the identity to be proven. Then it can be rewritten as:

$$A(n) = \sum_{k=1}^{n} 2^k = 2^{n+1} - 2$$

(1) Let *n*=1, does the identity hold for A(1) ? The left hand side of the identity A(1) gives us $2^1=2$ and the right hand side give us $2^{1+1} - 2 = 2^2 - 2 = 4-2 = 2$, which is identical. Therefore, for *n*=1 the identity holds.

(2) Furthermore, can it be proven that $(\forall n \in \mathbb{N})[A(n) => A(n+1)]$?
This is the so-called induction step.
Starting with the left hand side of the identity $A(n)$ we can write

$$A(n+1) = \sum_{k=1}^{n+1} 2^k = (\sum_{k=1}^{n} 2^k) + 2^{n+1}$$

by extracting the last (n+1)th term from the sequence. Then by definition we can substitute the right hand side summation by $A(n)$:

$$A(n+1) = \sum_{k=1}^{n+1} 2^k = A(n) + 2^{n+1}$$

Then, using the induction step, we can replace $A(n)$ by the left hand side of the identity $A(n)$:

$$A(n+1) = \sum_{k=1}^{n+1} 2^k = [2^{n+1} - 2] + 2^{n+1}$$

$$= 2.2^{n+1} - 2 = 2^{n+2} - 2$$

which again by induction equals the right hand side of the identity of $A(n+1)$.

Therefore, it is proven by induction that the identity

$$A(n) = \sum_{k=1}^{n} 2^k = 2^{n+1} - 2$$

holds for every natural number *n*. Stated otherwise: $(\forall n \in \mathbb{N})[A(n)]$.

TFPS-8

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit *L* as $n \to \infty$, then for any fixed number *M* > 0, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit *ML*.

Proof: We will prove this by using the definition of a limit of a sequence along with the identity that for any fixed number *M* > 0, the sequence $\{Ma_n\}_{n=1}^{\infty} = M\{a_n\}_{n=1}^{\infty}$.
In essence, the sequence can be one of natural numbers, integers, rationals or reals. Since the *n* is often used to indicate a natural number, let's assume it is a sequence of natural numbers.

(1) The definition of a limit *L* as $n \to \infty$ for a sequence $\{a_n\}_{n=1}^{\infty}$ goes as follows:

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall m \in \mathbb{N} | m \geq n) \; [\, |a_m - L| < \varepsilon]$$

Or, in words: let $\varepsilon > 0$ be given (and arbitrarily small), then we have to find an *n* such that

$$(\forall m \in \mathbb{N} | m \geq n) \; [\, |a_m - L| < \varepsilon]$$

This can be proven by picking any *n* such that $|a_n - L| < \varepsilon$, then by choosing $m \geq n$ it follows that $|a_m - L| < |a_n - L| < \varepsilon$, which proves that the sequence has limit *L* as $n \to \infty$.

(2) Consequently for a sequence $\{Ma_n\}_{n=1}^{\infty}$, assuming to have a yet unknown limit *F* it follows that:

$$(\forall \varepsilon' > 0)(\exists n \in \mathbb{N})(\forall m \in \mathbb{N}|m \geq n)\ [\ |Ma_m - F| < \varepsilon']$$

Or, in words: let $\varepsilon' > 0$ be given (and arbitrarily small), then we have to find an *n* such that

$$(\forall m \in \mathbb{N}|m \geq n)\ [\ |Ma_m - F| < \varepsilon']$$

Since *M*>0 this can be rewritten as:

$$(\forall m \in \mathbb{N}|m \geq n)\ [M\ \left|a_m - \frac{F}{M}\right| < \varepsilon']$$

or

$$(\forall m \in \mathbb{N}|m \geq n)\ [\ \left|a_m - \frac{F}{M}\right| < \frac{\varepsilon'}{M}]$$

Now by substitution of $\varepsilon' = M\varepsilon$ which is allowed since ε can have any value apart from 0 (*M*>0) and $F = ML$ we end up with the same condition for the limit *L* for the original sequence $\{a_n\}_{n=1}^{\infty}$, which is already proven in step (1).

Thus, it has been proven that the limit of the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit *ML* as $n \to \infty$.

TFPS-9

Give an example of a family of intervals (of the real line) $A_n, n = 1,2 \ldots$ such that $A_{n+1} \subset A_n$, for all *n* and their intersection

$$\bigcap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\} = \emptyset$$

Prove that your example has the stated property.

<u>Example</u>: Let the family of intervals A_n be defined by the union of open intervals where the greatest lower boundary and largest upper boundary are identical and given by the natural numbers *k* = 1,2,….,

$$A_n = \bigcup_{k=1}^{\infty}(k,k) - \bigcup_{k=1}^{n}(k,k) = \bigcup_{k=n+1}^{\infty}(k,k) \qquad , where\ k \in \mathbb{N}$$

<u>Proof</u>:

(1) First it has to be proven that $(\forall n)[\ A_{n+1} \subset A_n\ ]$.

(2) This can be done by using the definition for A_n. From the definition it follows that for any arbitrary *n* from 1 onwards, the set A_n contains all intervals that the set A_{n+1} contains plus one extra, i.e. the interval (*n*+1,*n*+1). Therefore, $A_{n+1} \subset A_n$.

(3) Then, it has to be proven that

$$\bigcap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\} = \emptyset$$

Or in words: there is no $x \in \mathbb{R}$ that is an element of all intervals contained in the family A_n. The proof of this is very simple, since all intervals of A_n are empty, there can be no element present in any interval of family A_n, so also the intersection of all intervals of family A_n is empty. Q.E.D.

TFPS-10

Give an example of a family of intervals (of the real line) $A_n, n = 1,2$ ... such that $A_{n+1} \subset A_n,$ for all *n* and their intersection consists of a single real number *r*

$$\bigcap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\} = \{r\}$$

Prove that your example has the stated property.

Example: Let the family of intervals A_n be defined by the union of a closed interval $[\pi, \pi]$ along with the union of open intervals where the greatest lower boundary and largest upper boundary are identical and given by the natural numbers *k* = 1,2,….,

$$A_n = [\pi, \pi] \ \cup \bigcup_{k=n+1}^{\infty} (k, k) \qquad , where\ k \in \mathbb{N}$$

Proof:

(1) First it has to be proven that $(\forall n)[\, A_{n+1} \subset A_n \,]$.

(2) This can be done by using the definition for A_n. From the definition it follows that for any arbitrary *n* from 1 onwards, the set A_n contains all intervals that the set A_{n+1} contains plus one extra, i.e. the interval (*n*+1,*n*+1). Therefore, $A_{n+1} \subset A_n$.

(3) Then, it has to be proven that

$$\bigcap_{n=1}^{\infty} A_n = \{x|(\forall n)(x \in A_n)\} = \{r\}$$

Or in words: there is only one real number $r \in \mathbb{R}$ that is an element of all intervals contained in the family A_n. The proof of this is very simple, since all intervals of A_n are empty, except for the first closed interval, which contains the element $\{\pi\}$, which is a real number. This one element is present in all A_n, so the intersection of all intervals of family A_n contains this one element $\{\pi\}$. Q.E.D.