Notes, Zarconia Game

J.A. Gyamfi

1 On strategy 1: Just roll once

The player rolls two fair dices with values ranging from 1 to 10 (extrema included). Let d_1 and d_2 be the outcome of the dices. The score rules are:

1. player earns a basic score of $d_1 + d_2$;
2. if $|d_1 - d_2| = 1$, then player earns an additional score of $d_1 + d_2$;
3. if d_1 and d_2 are different and correspond to an extremum of the range of values of the dices, then player earns an additional score of $d_1 + d_2$;
4. if $d_1 = d_2$, then player incurs a penalty of d_1^2.

I want to generalize this game assuming the value of the dices range from 1 to N, extrema included, where $N >= 1$.

Let $E[s]$ be the expectation value of score.

1.1 N=1

In the case whereby $N = 1$, the only possible outcome of rolling the two dices is $(d_1, d_2) = (1, 1)$. Thus,

$$E[s] = p(1,1) \cdot (1+1) - p(1,1) \cdot 1^2 = p(1,1) = 1 \ , \tag{1}$$

where $p(1,1)$ is the probability of obtaining the outcome $(1,1)$.

1.2 N>1

For $N > 1$,

$$\begin{aligned}
E[s] =& 2 \sum_{d_1 \geq d_2} p(d_1, d_2) \cdot (d_1 + d_2) && \text{(2a)} \\
&+ 2 \sum_{|d_1 - d_2| = 1} p(d_1, d_2) \cdot (d_1 + d_2) + 2p(1, N) \cdot (N+1) \delta_{N,2} && \text{(2b)} \\
&- \sum_{d_1 = d_2} p(d_1, d_2) \cdot (d_1 + d_2) && \text{(2c)} \\
&- \sum_{d_1 = d_2} p(d_1, d_2) \cdot (d_1 \cdot d_2) && \text{(2d)}
\end{aligned}$$

The factor 2 appears in Eq. (2a) and (2b) because the two dices are indistinguishable at this point (i.e. outcome $(d_1, d_2) = (d_2, d_1)$), and we are also assuming in Eq. (2a) that $d_1 \geq d_2$.

The terms in Eq. (2b) represent the expression for the bonus. The first term of Eq. (2b) represents the expression for the bonus according to rule 2, while the second term refers to the bonus according to rule 3. We introduced the Kronecker delta in the second term of Eq. (2b) because, for the case $N = 2$, rule points 2 and 3 above coincide. So the purpose of the Kronecker delta is to count the bonus in the case $N = 2$ only once.

For any given $N > 1$, for a given d_1, the number of times the outcome (d_1, d_2) appears, where $d_1 = d_2$, is only one in the sample space. But with the term in Eq. (2a), we have counted each of these cases twice, so the term in Eq. (2c) is supposed to correct that. The last term, Eq. (2d), is the penalty cost.

For any given $N > 1$,

$$p(d_1, d_2) = \frac{1}{N^2} \tag{3}$$

with any of the following conditions: i) $d_1 \geq d_2$, ii) $|d_1 - d_2| = 1$, iii) $d_1 = d_2$. These are the conditions under which the sums in the terms of Eq. (2) are to be done. Thus, we may rewrite the latter equation as,

$$\begin{aligned} E[s] =& \frac{2}{N^2} \sum_{d_1 \geq d_2} (d_1 + d_2) && \text{(4a)} \\ &+ \frac{2}{N^2} \sum_{|d_1 - d_2| = 1} (d_1 + d_2) + \frac{2}{N^2} \cdot (N+1)\delta_{N,2} && \text{(4b)} \\ &- \frac{1}{N^2} \sum_{d_1 = d_2} (d_1 + d_2) && \text{(4c)} \\ &- \frac{1}{N^2} \sum_{d_1 = d_2} (d_1 \cdot d_2) \ . && \text{(4d)} \end{aligned}$$

One can show that:

$$\sum_{d_1 \geq d_2} (d_1 + d_2) = \frac{N}{2}(N+1)^2 \tag{5}$$

$$\sum_{|d_1 - d_2| = 1} (d_1 + d_2) + (N+1)\delta_{N,2} = N(N+1) - 3\delta_{N,2} \tag{6}$$

$$\sum_{d_1 = d_2} (d_1 + d_2) = N(N+1) \tag{7}$$

$$\sum_{d_1 = d_2} (d_1 \cdot d_2) = \frac{N(N+1)(2N+1)}{6} \ . \tag{8}$$

Inserting these expressions into Eq. (4) yields the following final expression for $E[s]$:

$$\boxed{E[s] = \frac{N+1}{N} \cdot \left(\frac{4N+11}{6}\right) - \frac{3}{2}\delta_{N,2}} \ . \tag{9}$$