CS 237: Probability in Computing

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Lecture 22: Queueing Theory and Discrete-Event Simulation

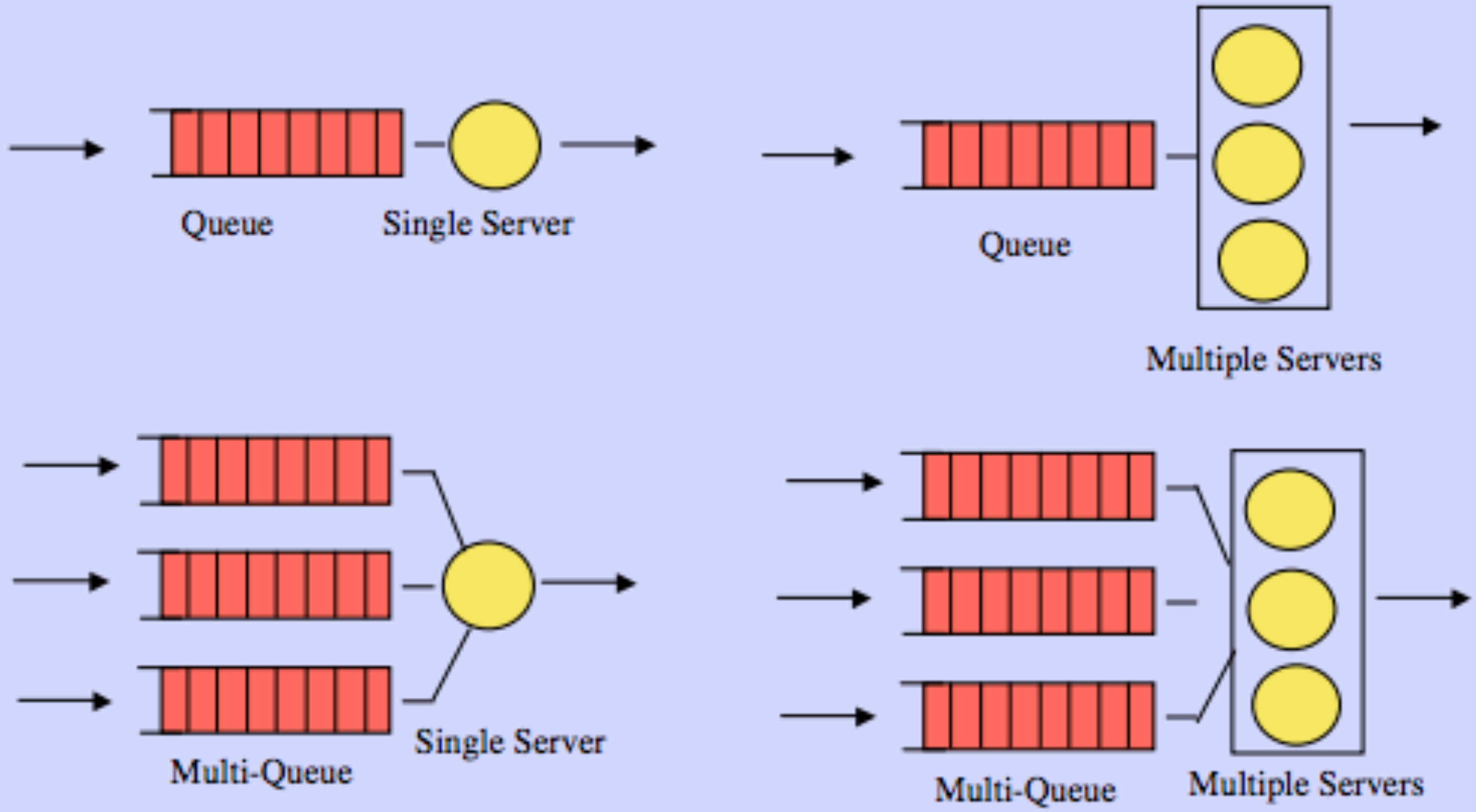
- Queueing Theory

Note: The QT slides are due to one Harry Perros who has good taste in ideas but bad taste in slide background colors….

- Queueing theory deals with the analysis of queues (or waiting lines) where customers wait to receive a service.
- Queues abound in everyday life!
  - *Supermarket checkout*
  - *Traffic lights*
  - *Waiting for the elevator*
  - *Waiting at a gas station*
  - *Waiting at passport control*
  - *Waiting at a a doctor's office*
  - *Paperwork waiting at somebody's office to be processed*

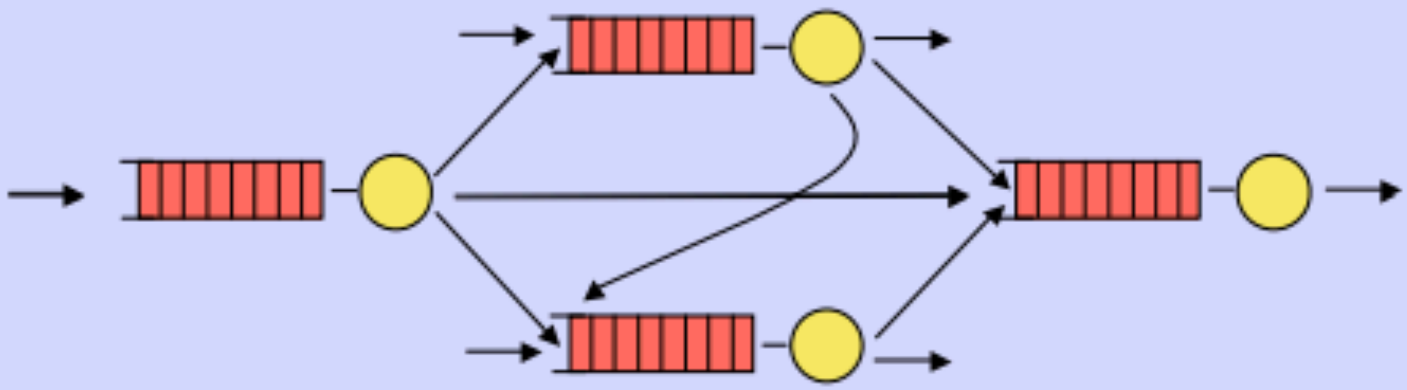
- There are also queues that we cannot see (unless we use a software/hardware system), such as:
  - *Streaming a video:* Video is delivered to the computer in the form of packets, which go through a number of routers. At each router they have to waiting to be transmitted out
  - *Web services:* A request issued by a user has to be executed by various software components. At each component there is a queue of such requests.
  - *On hold at a call center*

Notation - single queueing systems



Notation - Networks of queues

Tandem queues



Arbitrary topology of queues

Parameters of interest

You define a queueing system by specifying the following:

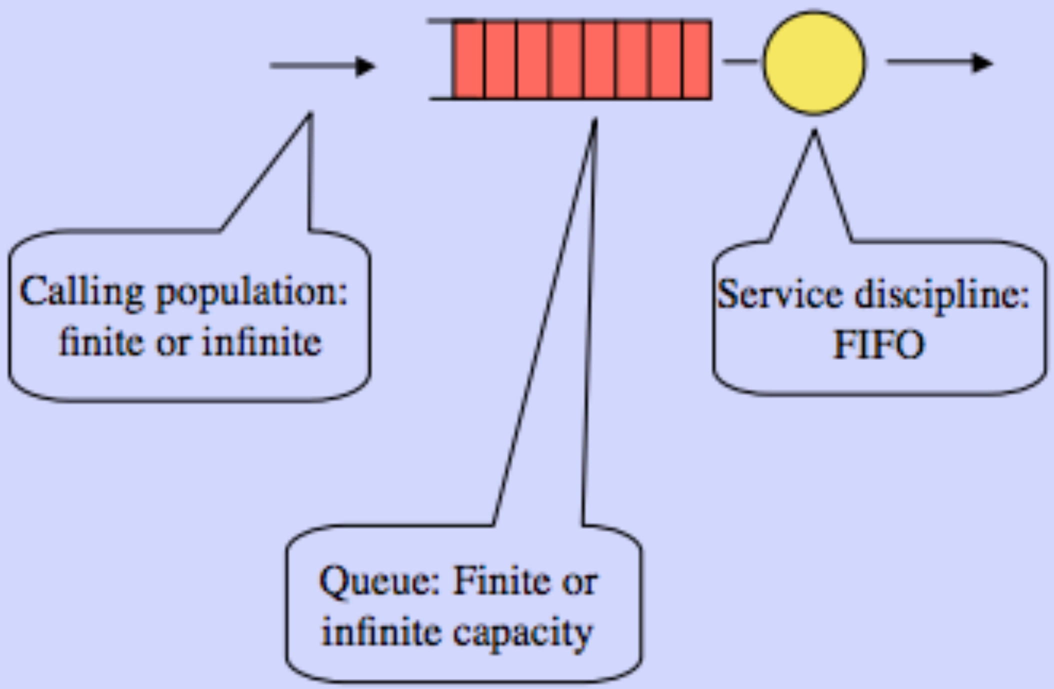
- *Service discipline:* **How is the queue organized, i.e., FIFO, Priority Queue, etc. (typically FIFO queue).**
- *How many servers?* **(typically 1)**
- *How many queues?* **(typically 1)**
- *Distribution of arrivals:* **Poisson (with exponential inter-arrival times) or general (any distribution) with some mean and standard deviation.**
- *Distribution of service times* **(how long does each task need the server): Typically Exponential with some mean.**

Measures of interest

- *Wait time:* **How long does a task wait in the queue?**
- *Mean wait time (per task).*
- *Percentile of wait time:* **What percent of tasks wait more than period of time t?**
- *Mean queue length (= average number of tasks waiting).*
- *Server utilization:* **What percentage of time is server busy?**
- *System throughput:* **How many tasks complete per unit time?**

One can also characterize these in terms of distribution, e.g., distribution of the queue length.

The single server queue

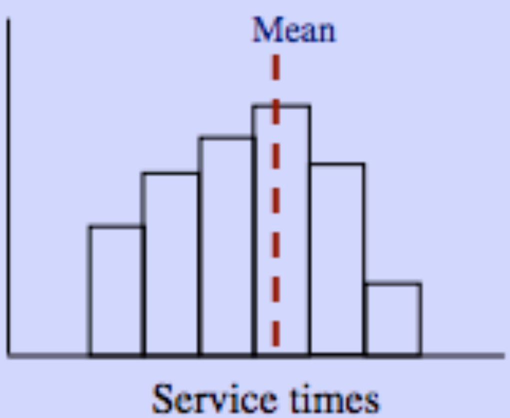


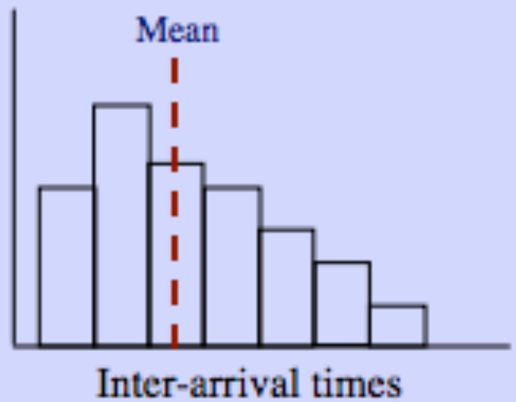
Queue formation

- A queue is formed when customers arrive faster than they can get served.

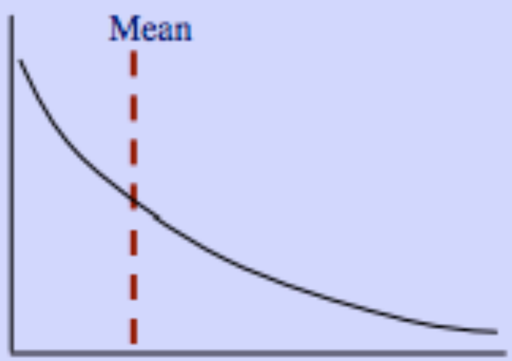
- Examples:
  - Service time = 10 minutes, a customer arrives every 15 minutes ---> No queue will ever be formed!
  - Service time = 15 minutes, a customer arrives every 10 minutes ---> Queue will grow for ever (bad for business!)

- Service times and inter-arrival times are rarely constant.
- From real data we can construct a histogram of the service time and the inter-arrival time.





- If real data is not available, then we assume a theoretical distribution.
- A commonly used theoretical distribution in queueing theory is the exponential distribution.



The M/M/1 queue

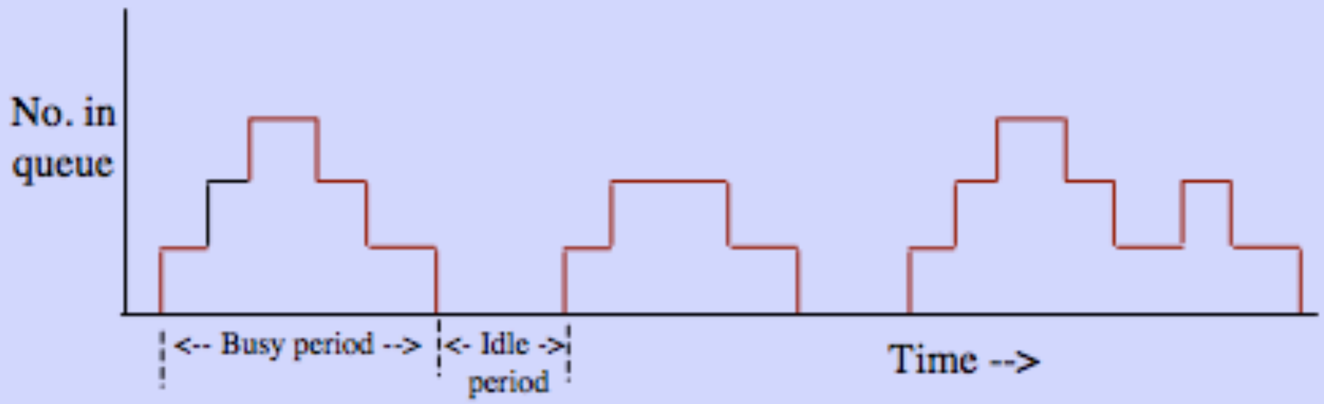
- M implies the exponential distribution (Markovian)
- The M/M/1 notation implies:
  - *a single server queue*
  - *exponentially distributed inter-arrival times*
  - *exponentially distributed service times.*
  - *Infinite population of potential customers*
  - *FIFO service discipline*

Stability condition

- A queue is stable, when it does not grow to become infinite over time.
- The single-server queue is stable if on the average, the service time is less than the inter-arrival time, i.e.

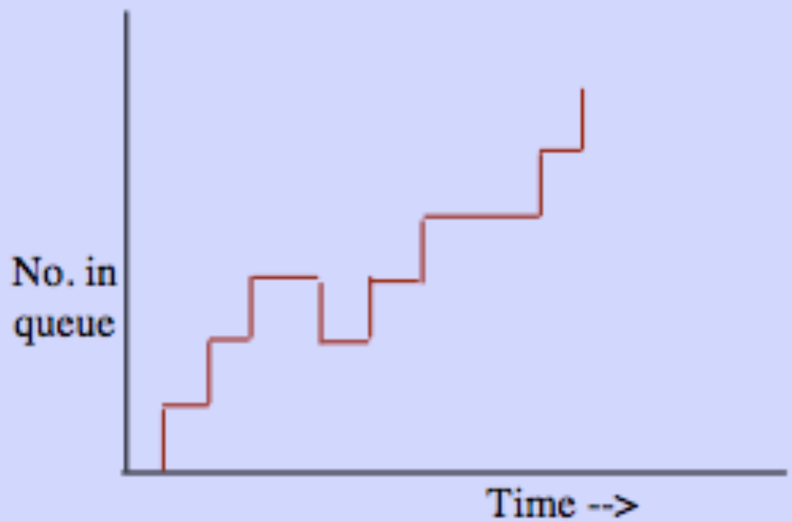
$$\boxed{\textit{mean service time} < \textit{mean inter-arrival time}}$$

Behavior of a stable queue
Mean service time < mean inter-arrival time



When the queue is stable, we will observe busy and idle periods continuously alternating

Behavior of an unstable queue
Mean service time > mean inter-arrival time



Queue continuously increases..

This is the case when a car accident occurs on the highway

Arrival and service rates: definitions

- *Arrival rate is the mean number of arrivals per unit time = 1/ (mean inter-arrival time)*
  - If the mean inter-arrival = 5 minutes, then the arrival rate is 1/5 per minute, i.e. 0.2 per minute, or 12 per hour.
- *Service rate is the mean number of customers served per unit time = 1/ (mean service time)*
  - If the mean service time = 10 minutes, then the service rate is 1/10 per minute, i.e. 0.1 per minute, or 6 per hour.

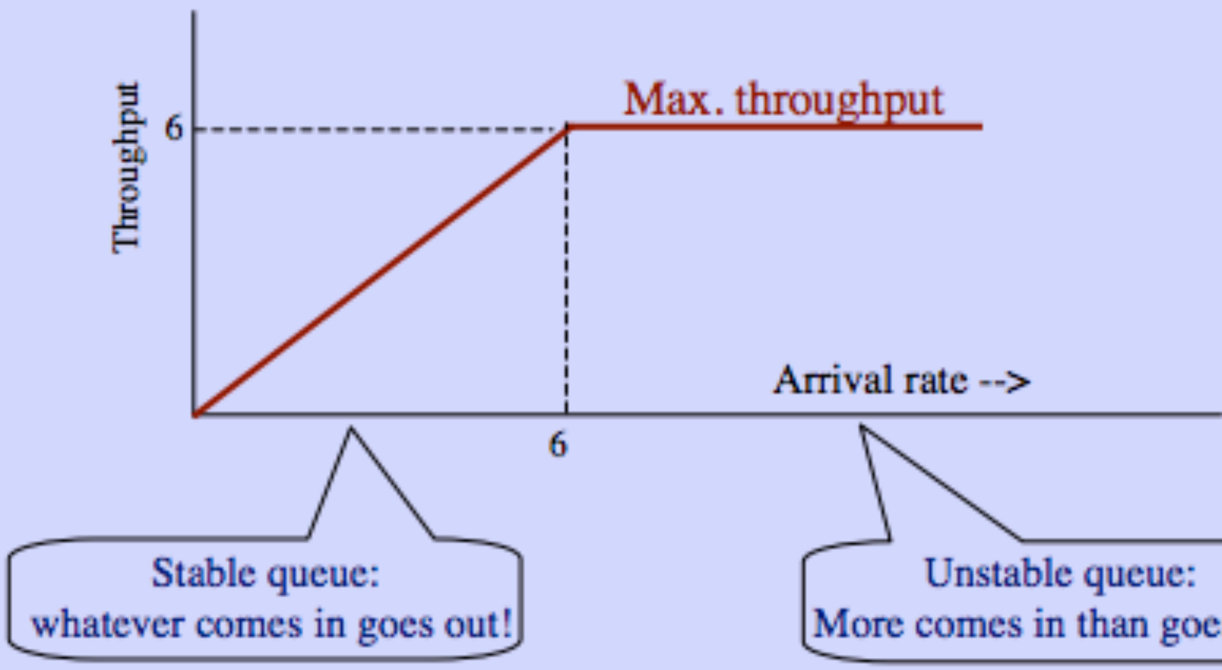
Throughput

- This is average number of completed jobs per unit.
- Example:
  - The throughput of a production system is the average number of finished products per unit time.
- Often, we use the *maximum throughput* as a measure of performance of a system.

Throughput of a single server queue

- This is the average number of jobs that depart from the queue per unit time (after they have been serviced)
- Example: The mean service time =10 mins.
  - What is the maximum throughput (per hour)?
  - What is the throughput (per hour) if the mean inter-arrival time is:
    - 5 minutes ?
    - 20 minutes ?

Throughput vs the mean inter-arrival time. Service rate = 6

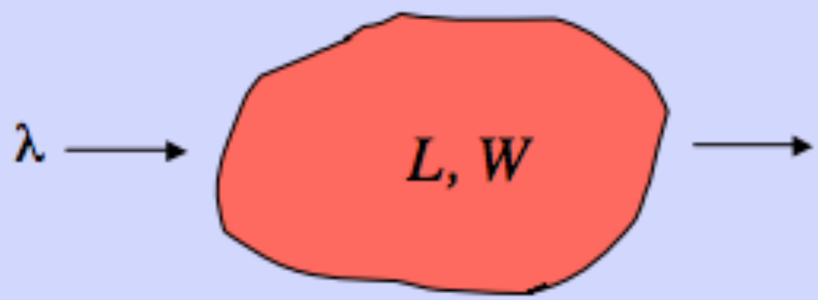


Server Utilization= Percent of time server is busy = (arrival rate) x (mean service time)

- Example:
  - Mean inter-arrival = 5 mins, or arrival rate is 1/5 = 0.2 per min. Mean service time is 2 minutes
  - Server Utilization = Percent of time the server is busy:

    0.2x2=0.4 or 40% of the time.
  - Percent of time server is idle?
  - Percent of time no one is in the system (either waiting or being served)?

Little's Law



Denote the mean number of customers in the system as L and the mean waiting time in the system as W. Then:

$$\lambda W = L$$