Name

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CS 237—Midterm Exam

Fall 2018

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write "Do Not Grade," on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention—no exceptions! If answers are on the back of the page please tell me so. **Circle final answers**. No calculators allowed, and *you may leave complicated formulae uncomputed,* but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

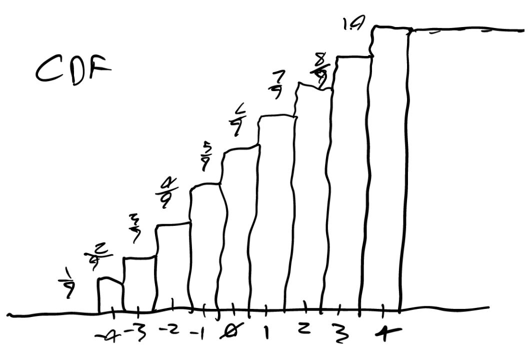
Problem One

Suppose a discrete random variable **X** is distributed according to the uniform discrete distribution in the range [-4, -3, -2, -1, 0, 1, 2, 3, 4], that is,

$$f_X(k) = \begin{cases} 1/9 & \text{if } -4 \le k \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, let the random variable $\mathbf{Z} = \mathbf{X}^2$.

(a) Draw the CDF of **X**.



(b) Give the range R_Z and probability function f_X for **Z**.

$R_Z = \{0, 1, 4, 9, 16\}$

$f_Z = \{\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\}$

(c) What is E[**Z**]? Show all work.

$$E[Z] = \frac{2}{9} + \frac{8}{9} + \frac{18}{9} + \frac{32}{9} = \frac{60}{9} = \frac{20}{3}$$

(d) What is Var(**X**)? Show all work.

$$Var(X) = E(X^2) - E(X)^2$$
$$= E(Z) - \emptyset$$
$$= \frac{20}{3}$$

$E(X) = \emptyset$
symmetric!

Problem Two Let X be a random variable defined as follows. Toss a die: if the number of dots showing is odd, let Y = 1, otherwise let Y = 3.

(A) What distribution does X follow? Give the range R_X and probability distribution f_X for X.

This is Bernoulli(0.5):

$$R_X = \{1, 3\}$$
$$f_X = \{\tfrac{1}{2}, \tfrac{1}{2}\}$$

(B) Calculate explicitly (not just by quoting a formula) the expected value E(X) and show every step. You may leave the result as a fraction.

$$E(X) = \frac{1}{2} + \frac{3}{2} = 2$$

(C) Calculate the variance Var(X) and standard deviation σ_X of X. Again, do not just quote a formula, but show explicitly all calculations. You may leave the result as a fraction.

$$\begin{aligned} VAR(X) &= E(X^2) - E(X)^2 \\ &= \left(\frac{1}{2} + \frac{9}{2}\right) - 2^2 \\ &= 5 - 4 = 1 \end{aligned}$$

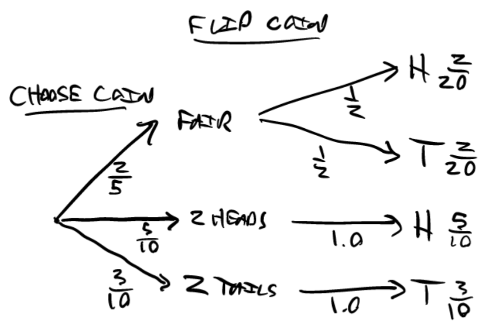
$$\sigma_X = 1$$

Problem Three

A box contains 10 coins where 5 coins have a head on each side, 3 coins have a tail on each side and 2 are fair coins (head and tail with 50% chance of each when tossed).

A tree diagram is the best way to start this problem.

(A) Suppose a coin is chosen at random and tossed. Find the probability that a head appears.



$P(H) = \frac{2}{20} + \frac{5}{10} = 0.6 \qquad P(T) = \frac{2}{20} + \frac{3}{10} = 0.4$

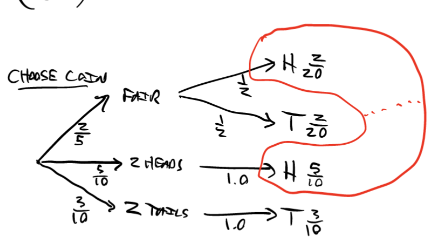
(B) Suppose we repeat the experiment in (A) 10 times. What is the probability that at least 7 heads appear? (You may just give the formula.)

THIS IS BINOMIAL $X \sim B(10, \frac{6}{10})$

$$\sum_{k=7}^{10} \binom{10}{k} (0.6)^k (1-0.6)^{10-k}$$

(C) Suppose a coin is selected at random and tossed. If a head appears, find the probability that the coin was fair, i.e., one with a head on one side and tail on the other.

(C)



$P(\text{FAIR} \mid H) = \frac{P(\text{BOTH})}{P(H)}$

$= \frac{\frac{2}{20}}{\frac{2}{20} + \frac{5}{10}} = \frac{2}{12} = \frac{1}{6}$

Problem Four

Wayne and Lenka are playing a game in which each has a fair coin, and they flip the coins at the same time, and keep doing so until they have the same face showing (both heads or both tails). A round is one simultaneous flip. For example, , they might have the following:

L: H T H
W: T H H (three rounds = three flips)

or they might have:

L: H
W: H (one round = one flip)

(A) If X = the number of rounds the game lasts, what is the distribution of X? Be absolutely precise.

THIS IS $X \sim$ GEOMETRIC$\left(\frac{1}{2}\right)$

$R_X = \{1, 2, \ldots\}$ $\quad f_X = \{\frac{1}{2}, \frac{1}{4}, \ldots\}$

(B) What is the expected number of rounds when they play this game?

$$E(X) = \frac{1}{p} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

(C) What is the probability that the game lasts more than 1, but less than 6 rounds?

(C) $\underline{A}$ $P(1 < X < 6) = P(X > 1) - P(X > 5)$

$= \left(1-\frac{1}{2}\right)^1 - \left(1-\frac{1}{2}\right)^5$

$= \frac{1}{2} - \frac{1}{32} = \frac{15}{32}$

1 2 3 4 5 6 ...
$\underbrace{\frac{1}{2}}_{\frac{1}{2}}$ $\underbrace{\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}\ \frac{1}{32}}_{\frac{15}{32}}$ $\underbrace{\frac{1}{64} \ldots}_{\frac{1}{32}}$

(D) What is the probability that if the game lasts exactly 7 rounds, that Wayne has landed heads more times than Lenka? You may calculate it or guess, but if you guess you must give a reason for your guess.

(D) GAME IS SYMMETRIC (COULD EXCHANGE HEADS & TAILS WITHOUT CHANGING PROBS)
SO $\frac{1}{2}$

Problem Five

Suppose X ~ Uniform(0,10), i.e., it produces a random real number between 0 and 10 according to the following graph of the Probability Density Function $f_X(a)$ (the y axis has been left off on purpose):

(a) Give of $f_X(a)$ as a mathematical formula.

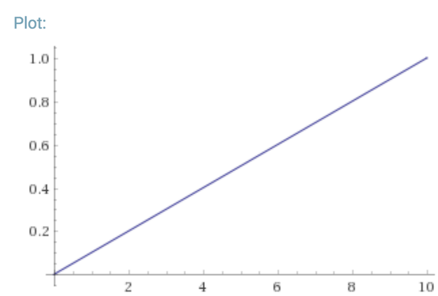
$$f_X(a) = \begin{cases} \frac{1}{10} & \text{FOR } 0 \le a \le 10 \\ 0 & \text{OTHERWISE} \end{cases}$$

(b) Calculate the Cumulative Distribution Function $F_X(a)$ using integrals and draw the graph.

$$F_X(a) = \int_{\cancel{-\infty}\; 0}^{a} \frac{1}{10}\, dx$$

$$= \frac{1}{10}x \Big|_0^a$$

$$= \begin{cases} 1 & \text{FOR } a > 10 \\ \frac{a}{10} & \text{FOR } 0 \le a \le 10 \\ 0 & \text{FOR } a < 0 \end{cases}$$



(c) Calculate E(X) using integrals (you can probably guess what it is, but I want you to derive it using integrals).

$$(c)\quad E(X) = \int_{\cancel{-\infty}\; 0}^{\cancel{\infty}\; 10} x \cdot \frac{1}{10}\, dx$$

$$= \frac{1}{20}x^2 \Big|_0^{10} = \frac{100}{20} = 5.0$$

CHECK:

$$\frac{\left(\frac{1}{20}x^2\right)}{dx} = \frac{2}{20}x = \frac{1}{10}x \checkmark$$

Problem Six

There are 9 students in a class, and they need to be divided into 3 teams to play a contest.

You may leave these answers as formulae.

(a) Suppose that they need to be divided into 3 equal-sized teams named "Reds", "Blues", and "Blacks." How many ways can this be done?

(a)

$$\underbrace{\binom{9}{3}}_{\text{REDS}}\underbrace{\binom{6}{3}}_{\text{BLUES}}\underbrace{\binom{3}{3}}_{\text{BLACKS}} = 1680$$

NO DOUBLE COUNTING!

OR

$$\frac{9!}{3!\,3!\,3!}$$

(b) Now suppose they need to be divided into one team of 5, and two teams of 2 each. How many ways can this be done?

(b)

DOUBLE COUNTING

$$\frac{\binom{9}{5}\binom{4}{2}\binom{2}{2}}{2!} = \frac{756}{2} = 378$$

OR

$$\frac{9!}{5!\,2!\,2!\,2!}$$

(c) Now suppose you need to divide the class into 3 equal-sized teams, and for each team you need to select a captain. How many ways can this be done? (Realize that once you select the members of each team, there are multiple ways to select the captains of each team.)

(c)

PICK FIRST TEAM — PICK CAPTAIN — ETC — ETC

$$\frac{\binom{9}{3}\binom{3}{1}\binom{6}{3}\binom{3}{1}\binom{3}{3}\binom{3}{1}}{3!}$$

SINCE SAME SIZE & NO NAMES, ELIMINATE ALL PERMUTATIONS OF 3