

Name:

## CS 237—Midterm Solutions

Summer 2017

Note: My solutions will provide all the relevant calculations and the formulae, diagrams, etc. I will also provide the exact result (using Wolfram Alpha!); this of course was not necessary for full credit.

You must complete 5 of the 6 problems on this exam for full credit. Each problem is worth 20 points. Please leave blank, or draw an X through, or write “Do Not Grade,” on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers.** No calculators allowed, and you may leave *complicated* arithmetic expressions uncomputed, but please do multiply  $1/2 * 1/2$  to get  $1/4$  if the occasion presents itself.

**Problem One.** Find the mean  $E(X)$ , variance  $\text{Var}(X)$ , and standard deviation  $\sigma_X$  of EACH of the following distributions. In this case, I would like you to do all the math and show the answers.

(A)  $\text{Rng}(X) = \{ 2, 3, 11 \}$      $f_X = \{ 1/3, 1/2, 1/6 \}$

(B)  $\text{Rng}(X) = \{ 1, 2, 3, 4 \}$      $f_X = \{ 0.4, 0.1, 0.2, 0.3 \}$

Solution: This is problem 5.15 from Schaum's.

**Problem Two.** Suppose A, B, C, D, E, F, G, and H are 8 people who must sit together at a round table. Assume, as usual, that two arrangements that are rotations of each other are to be considered to be the same.

Show all work! You do not need to calculate complicated formulae. Each of these questions is independent and represents separate problems.

(A) How many possible seating arrangements are there for these 8 people?

Solution: There are  $8!$  permutations, and 8 possible rotations, so

$$8! / 8 = 7! = 5040 \text{ arrangements}$$

Or think about it this way: Pick one person, say A, and assign them a seat. Then there are  $7!$  ways to arrange everyone else. This eliminates the notion of rotations.

(B) Suppose A and B must sit together in a group (i.e., there can not be any other person between these two) and, independently, C and D must sit together. How many arrangements are there?

Solution: We consider each of AB and CD to be “multipeople”, which gives us 6 people or  $6!/6 = 5! = 120$  arrangements. But AB can be in  $2! = 2$  orders (A to left of B or vice versa) and same for CD. So this gives us  $120 * 2 * 2 = 480$  arrangements.

(C) Suppose E and F can NOT be seated next to each other. How many arrangements are there?

Solution: From the  $8!/8$  arrangements, we must subtract those where E and F sit next to each other. Following the same argument as in (B), there are  $2! * 7!/7$  such arrangements, so we have

$$8!/8 - 2! * 7!/7 = 3600 \text{ arrangements}$$

(D) Suppose G and H insist that exactly one person sits between them. How many arrangements are there?

Solution: As in (B), we have there is a “multi-person” of two people, this time with 1 person between them. There are  $2!$  permutations of G and H, and  $6!$  possible choices for the remaining seats. This gives us  $6!$  possible arrangements of people (including the multi-person), and so we have (removing the duplications from rotation):

$$12 * 6!/6 = 1440 \text{ arrangements.}$$

Again, you can eliminate duplications due to rotations by picking one person, say G, and assigning them a seat. Then there are 2 seats that H can sit in (one seat away clockwise or one seat away counter-clockwise), and  $6!$  ways to arrange the other people. This is  $2 * 6! = 1440$  arrangements.

**Problem Three.** Suppose you draw 5 cards from a standard deck of cards, without replacement. What is the probability that you get 2 **red** cards of the same denomination and 3 cards of the same denomination of **any color or suit** (but of course different from the first denomination)?

**Solution:** As usual, we construct the hand using combinations, and divide by the total number of 5-card hands:

Choose a denomination:  $C(13,1)$  -- now there are only two red cards of this denomination

Choose a second denomination:  $C(12,1)$





















































Choose 3 cards from that denomination:  $C(4,3)$

Total number of five card hands:  $C(52,5)$

Thus:

$$\frac{C(13,1) * C(12,1) * C(4,3)}{C(52,5)} \quad [ = 0.00024 ]$$

Hearts and Diamonds are RED, and Spades and Clubs are Black. Denomination refers to Ace, 2, 3, etc.

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Spades													
Hearts													
Diamonds													
Clubs													



**Problem Four.** Two 4-sided dice are rolled.

(A) Let  $A$  = "the sum of the dots showing on the two rolls is odd" and  $B$  = "both tosses were greater than 1." Are  $A$  and  $B$  independent? Be precise.

**Solution:** Here is the sample space, where outcomes in  $A$  are boldface, and outcomes in  $B$  have a grey background:

Outcomes from Two 4-sided Dice

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Thus,  $P(A) = 8/16 = 1/2$  and  $P(B) = 9/16$  and  $P(A \text{ and } B) = 4/16 = 1/4$ , and  $P(A) \cdot P(B) = 9/32$ . But  $1/4 = 8/32 \neq 9/32$  and hence  $A$  and  $B$  are **NOT independent**.

Alternately, you can calculate  $P(A) = 8/16 = 1/2$  and  $P(A|B) = 4/9$  and observe that the probability of  $A$  has changed if you know  $B$ .

(B) Let  $A$  = "the two rolls showed the same number" and  $B$  = "the second toss was greater than 2." Are  $A$  and  $B$  independent? Be precise.

**Solution:** Here is the sample space, where outcomes in  $A$  are boldface, and outcomes in  $B$  have a grey background:

Outcomes from Two 4-sided Dice

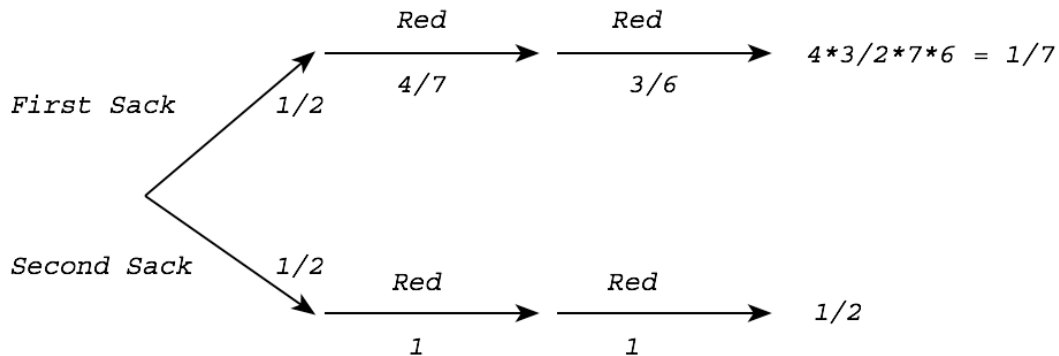
	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Thus,  $P(A) = 4/16 = 1/4$  and  $P(B) = 8/16 = 1/2$  and  $P(A \text{ and } B) = 2/16 = 1/8$ , and so  $P(A) \cdot P(B) = 1/8 = P(A \text{ and } B)$  and hence  $A$  and  $B$  **ARE independent**.

Alternately, you can calculate Thus,  $P(A) = 4/16 = 1/4$  and  $P(A|B) = 2/8 = 1/4$ . Hence the probability does not change when you know  $B$  has happened.

**Problem Five.** A sack contains 4 red and 3 black balls. A second sack contains 3 red balls. A sack is selected at random and 2 balls are withdrawn from it at random, without replacement, and all found to be red. What is the probability that the first sack was the one selected?

(A) Draw a tree diagram to analyze the situation and provide the answer.



If we know that both balls selected were red, then we MUST be in one of the two endpoints shown. Therefore, the probability that we took the top path is the percentage of the total probability due to the first path, or  $(1/7) / (1/7 + 1/2) = (1/7) / (9/14) = 2/9$ .

(B) Formalize the problem in terms of conditional probabilities and Bayes's Theorem, and show how the answer can be derived through manipulation of the formulae. You should, of course, get the same answer as in (A)!

Let us write down all the probabilities we know for  $A$  = selected the first sack, and  $B$  = withdrew two red balls:

$$P(A) = 1/2 \quad P(B) = 1/7 + 1/2 = 9/14 \quad P(A \text{ and } B) = 1/7 \quad P(B | A) = 4/7 * 3/6 = 2/7$$

Note that this last probability is the probability, after selecting the first sack, that furthermore getting two red balls. Also note that

$$P(B | A) = P(A \text{ and } B) / P(A) = (1/7) / (1/2) = 2/7$$

Now,  $P(A | B)$  can be calculated two ways, one as

$$P(A | B) = P(A \text{ and } B) / P(B) = (1/7) / (9/14) = 2/9$$

or as

$$P(A | B) = P(B | A) * P(A) / P(B) = (2/7) * (1/2) / (9/14) = 2/9.$$

**Problem Six.** A classroom consists of 20 students and the teacher decides to play various games where the students have to be separated into two or three teams.

(A) Suppose the teacher needs to separate them into two teams of any size, as long as no team has 0 members. How many ways to do this are there?

**Solution:** Let us denote the set of all students as S. The power set (set of all subsets) of S contains  $2^{20} = 1,048,576$  sets, and each of these corresponds to a possible way T of selecting one of the teams (the other set S - T then being determined). But we must remove the empty set and the set consisting of all the students, so there are  $2^{20} - 2 = 1,048,574$  ways left. But these over-count by a factor of 2, since any choice of two teams T and S-T will be counted twice, once when you pick T (and the other team is S-T), and once when you pick S-T (and the other team is T). This gives us a mere

$$(2^{20} - 2) / 2 = 524,287 \text{ ways.}$$

(B) Suppose the teacher wants to split the class into two equal sized teams. How many ways are there of doing this?

**Solution:** Now each team must have 10 students. There are  $C(20,10)$  ways of choosing one team, and then the other team is determined. But again, you double count in the same manner as explained above. So you have

$$C(20,10) / 2 = 92,378 \text{ ways.}$$

(C) Suppose the teacher wants to separate them into three teams of size 8, 7, and 5. How many ways can this be done?

**Solution:** Now there is no problem with over-counting because when we for example choose a team of size 8, it can not be confused with either of the other teams. Hence we have  $C(20,8)$  ways of choosing the team of 8, then  $C(12,7)$  ways of choosing a team of 7 from the remaining students, and then the last team of 5 is determined.

$$C(20,8)*C(12,7) = 99,768,240$$

(D) Suppose the teacher wants to separate them into three teams of size 8, 6, and 6. How many ways can this be done?

**Solution:** Now we can choose the team of 8 in  $C(20,8)$  ways, and a team of 6 in  $C(12,6)$  ways, and the second team of 6 is then determined. However, we will over-count by a factor of 2 because the two teams of 6 can not be distinguished (so this is a combination of the techniques of (B) and (C)). So we have

$$C(20,8)*C(12,6) / 2 = 58,198,140 \text{ ways.}$$