CS 237 — Midterm Exam

Fall 2019

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write "Do Not Grade," on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers**. No calculators allowed, and *you may leave complicated formulae uncomputed,* but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

Problem One

Consider the following events for a family with children:

A = There are children of both genders (girls and boys)
B = There is at most one boy

Show that A and B are independent if the family has exactly three children. Show all work!

Solution: (This is Schaum's Problem 4.29.) There are two ways to solve this, depending on whether you thought of a family as a sequence or a set.

Sequence:
   Sample space = { BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG }
   A = { BBG, BGB, BGG, GBB, GBG, GGB }   B = { BGG, GBG, GGB, GGG}
   A and B = { BGG, GBG, GGB }
   Thus: P(AB) = 3/8 = P(A)*P(B) = 6/8 * 4/8

Set:
   Sample space = { {B,B,B }, {B,B,G }, {B,G,G}, {G,G,G} }
   A = { {B,B,G }, {B,G,G} }   B = { {B,G,G}, {G,G,G} }
   A and B = { {B,G,G} }
   Thus: P(AB) = 1/4 = P(A)*P(B) = ½ * ½

Problem Two

In this problem, various random experiments will be described and you must give the name and any parameters of the distribution involved and perhaps some other information. Be precise! You may simply give formulae for any calculations required.

Wayne and Roman play a dart game in which they each throw a dart and try to hit a bulls-eye. Wayne's probability of hitting the bulls-eye is 1/3 and Roman's probability of hitting the bulls-eye is 1/4. We may assume that these events are independent.

(A) Suppose Wayne and Roman both shoot at the target. Let X = "1 if they both hit the target and 0 if they don't."

What is the distribution of X?

Give the name and any relevant parameters for each distribution.

$$X \sim Bernoulli(\, p = 1/12 \,)$$

What is Var(X)?

$$Var(X) = (11/12)*(\, 1/12 \,) = 11/144$$

(B) Now suppose Wayne shoots at the target 10 times a day, for practice (but he never gets any better at it, and his probability of hitting the bulls-eye is always 1/3). Z = "the number of bulls-eyes he hits in a given week."

There are two reasonable answers:

What is the distribution of Z? $\qquad$ Z ~ Binomial(N=70,p=1/3)	What is the distribution of Z? $\qquad$ Z ~ Poi(70/3)
What is the probability that he gets 5 bulls-eyes next weekend? $\quad \binom{20}{5} \cdot (1/3)^5 \cdot (2/3)^{15} = 0.1457$	What is the probability that he gets 5 bulls-eyes next weekend? $\quad \dfrac{e^{-20/3}(20/3)^5}{5!} = 0.1397$

(C) Suppose Wayne and Roman play a new game in which in each round they both shoot at the target, and they stop when they both hit the target on the same round. Let Y = "the number of rounds until they stop."

What is the distribution of Y?

$$Y \sim Geometric(\, p = 1/12)$$

What is E(Y)?

$$E(Y) = \, 1/(1/12) = 12$$

Problem Three

You have a sack containing 4 red balls and 3 black balls. Consider each of these situations separately (you start each case with a sack of the 7 red and black balls). In each case you are selecting 3 balls *without replacement.* Show all work.

(A) Suppose you select 2 balls from the sack and observe that they are the same color. What is the probability that if you select a 3rd ball it will be a different color from the first two?

S = same color   D = third different

P(BB) = 3*2/ (7*6) = 6/42

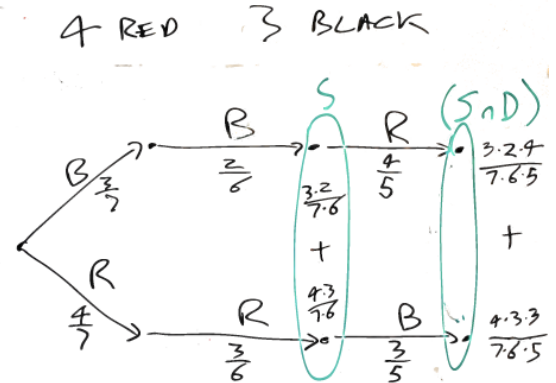P(RR) = 4*3/ (7*6) = 12/42

P(S) = 6/42 + 12/42 = 18/42 = 9/21 = 3/7

P(BBR) = 3*2*4/ (7*6*5) = 24/210

P(RRB) = 4*3*3/ (7*6*5) = 36/210

P(S and D) = 24/210 + 36/210 = 60/210 = 2/7

P(D | S) = P(S and D) / P(S) = (60/210)/(18/42) = (2/7)/(3/7) = 2/3
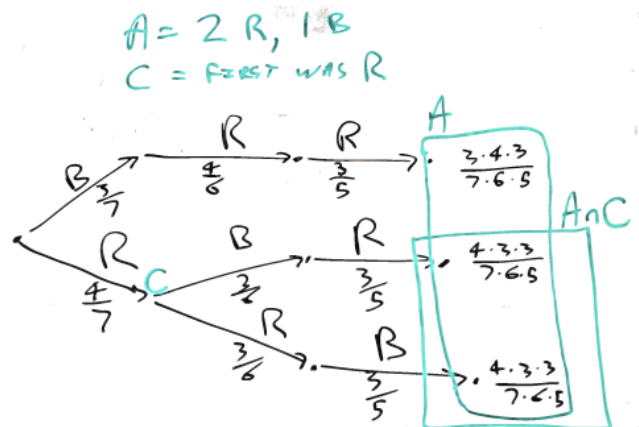
(B) Suppose you select 3 balls from the sack and observe that there are 2 red and 1 black. You did not notice which colors came out in which order (they are now just a *set* of 3 balls). What is the probability that the first ball you selected was in fact red?

A= 2 red and 1 black

P(A) = P(BRR) + P(RBR) + P(RRB)

= 3 * (4*3*3 / (7*6*5)) = 108/210

C = first ball is red

P(C and A) = P(RBR) + P(RRB)

= 2 * (4*3*3 / (7*6*5)) = 71/210

P(C | A) = P(C and A) / P(A) = (71/210)/(108/210) = 71/108 = 2/3

Problem Four

To play a game, the player rolls a single die.

If the die shows 4, 5, or 6 dots, the player wins that many dollars (e.g., roll a 5, get $5), but if the die shows, 1, 2, or 3, the player loses $10.

Let X = the amount of money the player wins (will be negative if he loses money).

(A) Give the range R_X and PDF f_X for the appropriate random variable X.

R_X = { -10, 4, 5, 6 }

f_X = { 1/2, 1/6, 1/6, 1/6 }

(B) Find the expected value E(X) of the game.

E(X) = -10/2 + 4/6 + 5/6 + 6/6 = -30/6 + 15/6 = -15/6 = $ 2.50

(C) Find the variance Var(X) of the game.

$Var(X) = E(X^2) - E(X)^2$

$= [100/2 + 16/6 + 25/6 + 36/6] - (5/2)^2$

$= 377/6 - 25/4 = 754/12 - 75/12 = 679/12 = 56.5833$

Problem Five

Suppose a standard set of 52 playing cards is randomly shuffled and you are dealt a hand of seven (7) cards, without replacement (as is usual in card games). For the following, just give the appropriate formula.

(A) What is the probability that you get 4 cards of one denomination and 3 cards of a different denomination, so there are two denominations represented in all? [Example: 4 Jacks and 3 Aces]

$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}}{\binom{52}{7}}$$

(B) What is the probability that you get 3 of one suit, 3 of a different suit, and 1 more card with a suit different from the other 6 cards (so there are three suits represented in all, for example, 3 Spades, 3 Hearts, and one Diamond)?

$$\frac{\binom{4}{1}\binom{13}{3}\binom{3}{1}\binom{13}{3}\binom{26}{1}}{2\cdot\binom{52}{7}} \quad \text{or} \quad \frac{\binom{4}{2}\binom{13}{3}\binom{13}{3}\binom{26}{1}}{\binom{52}{7}}$$

Problem Six

In the following assume that we are dealing with Poisson processes and so potentially you could use the Poisson or the Exponential distribution. (You also could possibly use other distributions as needed.) For each of the following, give the distribution involved (including any parameters) and give the formula to solve the problem.

(A) Suppose that every 24 hours, on average, I get one spam call on my iPhone. What is the probability that I get exactly 4 spam calls tomorrow between 12 noon and 12 midnight?

Distribution: Assuming the time unit is 12 hours we are counting using the Poisson:

$$X \sim \text{Poi}(\lambda = 0.5)$$

since we can scale the rate parameter to fit the question.

Formula: $P(X = 4) = \dfrac{e^{-0.5}0.5^4}{4!} = 0.0016$

(only the formula is necessary)

(B) I drop my iPhone, on average, once a week. Suppose I do not drop it for 4 days in a row, what is the probability that I finish out the last 3 days of the week without dropping my iPhone?

Distribution: This is counting arrivals (dropped phone) at a rate of 1/week. Since arrivals are independent, the previous 4 days do not matter. Scaling to arrivals in a 3 day period, we have

$$X \sim \text{Poi}(3/7)$$

Formula: $P(X = 0) = \dfrac{e^{-3/7}(3/7)^0}{0!} = e^{-3/7} = 0.6514$

(C) Suppose my iPhone is expected to last 10,000 hours before the battery fails. What is the probability it lasts more than 7,000 but less than 12,000 hours?

Distribution: This is time measuring arrivals of failures, so we have an expected value of 5000 hours, which means the rate parameter is 1/5000 arrivals per hour:

$$X \sim \text{Exp}(\lambda = 1/10{,}000)$$

Formula:
$$P(7{,}000 < X < 12{,}000) = P(X > 7000) - P(X > 12{,}000)$$

$$= e^{\frac{-7000}{10{,}000}} - e^{\frac{-12{,}000}{10{,}000}} = 0.1954$$