

# CS 237: Probability in Computing

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## Lecture 6:

- Poker Probability: finish with analysis of standard poker hands
- Counting subsets
- Powerset and its relationship to  $\binom{N}{K}$
- Counting Partitions and the unordering principle!

Next Time: Discrete Random Variables, Read Chapter 5.1 – 5.3

# Finite Combinatorics and Probabilities

## Poker Probabilities

Example set of 52 poker playing cards

Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

A poker hand is 5 cards (a set) chosen without replacement.

How many possible poker hands?

$$|S| = \binom{52}{5} = 2,598,960$$

# Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

**Example:** What is the probability that you get a hand with 3 red cards and 2 black cards?

**Solution:**

$$\frac{\binom{26}{3} \binom{26}{2}}{\binom{52}{5}} = 0.3251$$

By the way, Wolfram Alpha is the way to go when doing these problems...

 **WolframAlpha** computational intelligence

$C(26,3)*C(26,2) / C(52,5)$

Assuming "C" is a math function | Use as a [unit](#) instead

**Input:**  
$$\binom{26}{3} \times \frac{\binom{26}{2}}{\binom{52}{5}}$$

---

**Exact result:**  
$$\frac{1625}{4998}$$

---

**Decimal approximation:**  
0.325130052020808323329331732693077230892356942777110844337...

# Poker Probability

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs	♣ A	♣ 2	♣ 3	♣ 4	♣ 5	♣ 6	♣ 7	♣ 8	♣ 9	♣ 10	♣ J	♣ Q	♣ K
Diamonds	♦ A	♦ 2	♦ 3	♦ 4	♦ 5	♦ 6	♦ 7	♦ 8	♦ 9	♦ 10	♦ J	♦ Q	♦ K
Hearts	♥ A	♥ 2	♥ 3	♥ 4	♥ 5	♥ 6	♥ 7	♥ 8	♥ 9	♥ 10	♥ J	♥ Q	♥ K
Spades	♠ A	♠ 2	♠ 3	♠ 4	♠ 5	♠ 6	♠ 7	♠ 8	♠ 9	♠ 10	♠ J	♠ Q	♠ K

**Problem:** What is the probability that a five-card hand has at least 3 Diamonds?

**Solution:** You need to separate this problem into cases, and might as well choose 3, 4, or 5 Diamonds, and for each find the probability and sum:

$$P(3 \text{ Diamonds}) = \frac{\binom{13}{3} \binom{39}{2}}{\binom{52}{5}} = 0.0815$$

$$P(4 \text{ Diamonds}) = \frac{\binom{13}{4} \binom{39}{1}}{\binom{52}{5}} = 0.0107$$

$$P(5 \text{ Diamonds}) = \frac{\binom{13}{5}}{\binom{52}{5}} = 0.0005$$

These sum to 0.0928.

# Poker Probability

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs	♣	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Diamonds	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Hearts	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Spades	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠

**Problem:** What is the probability of a Pair, 3-of-a-Kind, and 4-of-a-Kind?

**Solution:** First choose the denomination of the 2, 3 or 4 of a kind, then the suits of those cards, then the remaining cards of different denominations, again choosing the denomination, then the suits::

Pair:

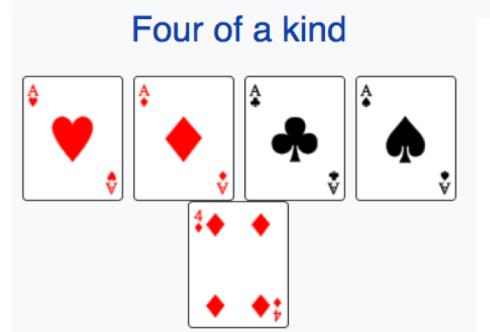
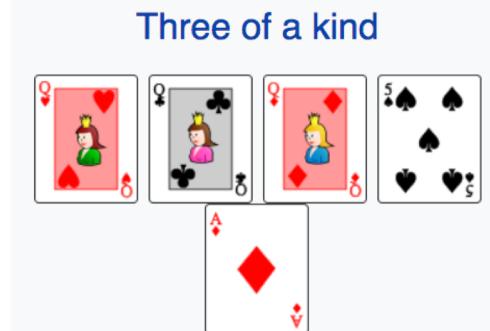
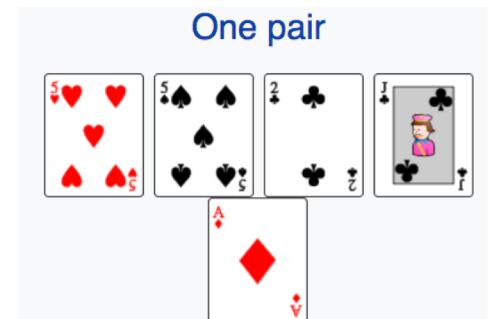
$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}} = 0.4226$$

3-of-a-Kind:

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2}{\binom{52}{5}} = 0.0211$$

4-of-a-Kind:

$$\frac{\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}} = 0.00024$$



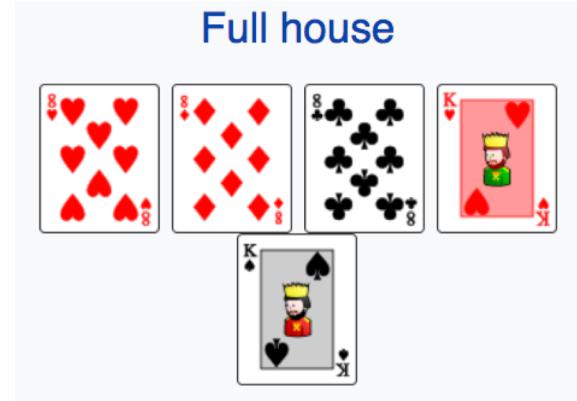
**Problem:** Suppose in our class we have 160 students with 90 men and 70 women. I want to choose teams for an in-class demo with 5 men, 5 women, and also a scorekeeper (who can be anyone not on a team, man or woman). How many ways can I choose the teams and scorekeeper?

**Solution:** First choose the 5 men from the 90, then the 5 women from the 70, then one scorekeeper from the 150 people not on teams:

$$\binom{90}{5} * \binom{70}{5} * \binom{150}{1} = 79,787,790,884,062,800$$

# Poker Probability

**Problem:** What is the probability of a Full House (3 of one denomination and 2 of another)?

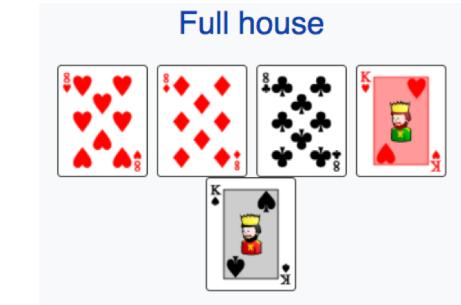


**Solution:** First choose the denomination of the 3, then those 3 suits, then the denomination of the 2, then those 2 suits:

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}} = 0.00144$$

We will consider one more hand, Two Pair, after a review question and some consideration of partitions.

# Poker Probability



Question: Why is this not "choose the two ranks, then the suits for each"?

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}} = 0.00144$$

$$\frac{\binom{13}{2} \binom{4}{3} \binom{4}{2}}{\binom{52}{5}}$$

It's the difference between sequences and sets!

Number of sets of  
2 from 13:  $C(13, 2)$ :

Number of sequences  
of 2 from 13,  $P(13, 2)$ :

$$\binom{13}{2} = 78 \quad P(13, 2) = 13 \cdot 12 = 156$$

# Counting Sets: Power Set

The **Power Set** of a set  $S$  is the set of all subsets ( = set of all events ):

$$\mathcal{P}(S) \quad =_{\text{def}} \quad \{ A \mid A \subseteq S \}$$

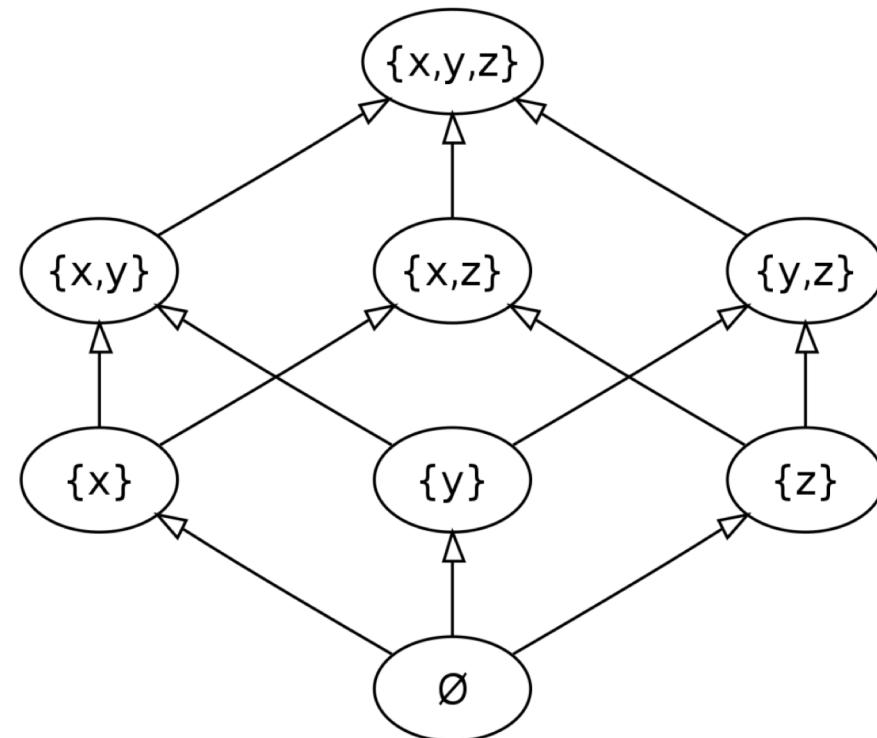
the cardinality of Power Set:  $|\mathcal{P}(S)| = 2^{|S|}$

This is easy to see if we consider the **enumeration** of all sequences of  $\{ T, F \}$  of length  $|S|$ , stating which elements of  $S$  are in the subset:

$\{ x, y \}$

$\begin{matrix} x & y & z \\ T & T & F \end{matrix}$

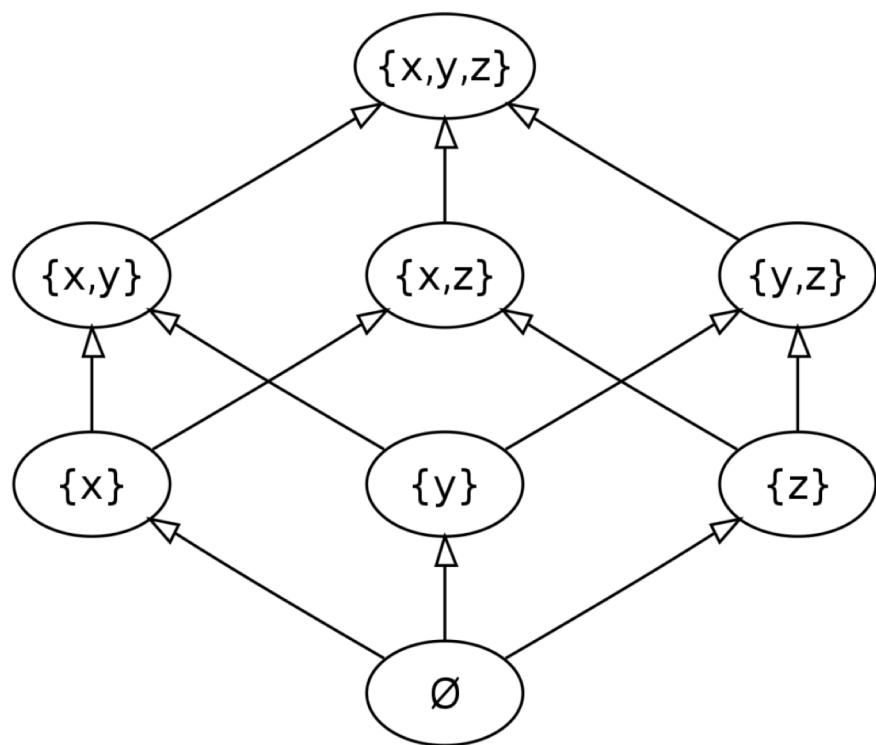
$$|\{T, F\}|^{|S|} = 2^{|S|}$$



# Counting Sets: Power Set and Combinations

There is of course a strong connection between the power set and combinations:

$C(N, K)$  = how many subsets of size  $K$  from a set of size  $N$ .

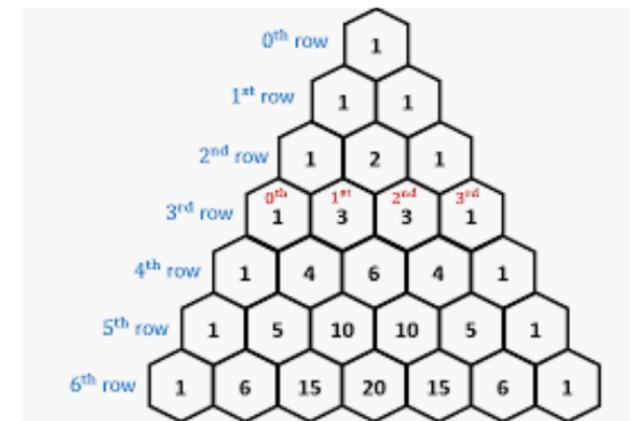


$$\binom{3}{3} = 1$$

$$\binom{3}{2} = 3$$

$$\binom{3}{1} = 3$$

$$\binom{3}{0} = 1$$



$$|\mathcal{P}(S)| = 2^{|S|} = \sum_{k=0}^{|S|} \binom{|S|}{k}$$

# Counting Sets: Power Set and Combinations

**Problem:** A pizza shop claims they serve “more than 1000 kinds of pizza.” You investigate and find they offer 10 different toppings (including cheese and tomato sauce among the 10). Is their claim correct? What about if we insist that a pizza must have cheese and tomato sauce at the very least?

**Solution:** Technically, yes, if you include all possible combinations of toppings, including cheese or no cheese and tomato sauce or no tomato sauce:

$$2^{10} = 1024$$

But this is a little funny, as it includes the empty set (no toppings, just bare crust!).

If you insist that “pizza” must have cheese and tomato sauce, then we have only

$$2^8 = 256$$

# Counting Sets: Partitions

A **partition** of a set  $S$  is a set of disjoint subsets which include every member of  $S$ :

$$S = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$$

**Partitions:**

- $\{ \{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10\} \}$
- $\{ \{1\}, \{2, 4\}, \{3, 6, 8\}, \{5, 7, 9\}, \{10\} \}$
- $\{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\} \}$

**Not :**

- $\{ \{1, 2, 3, 4\}, \{5, 6, 7, 8, 9\} \}$  Doesn't have the 10 !
- $\{ \{1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\} \}$  Not disjoint !

# Counting Sets: Partitions

Counting partitions: always a good idea to try some examples first ...

**Problem.** Suppose we have five students  $\{ A, B, C, D, E \}$

We want to divide them into two teams of 3 and 2 people each. How many ways can we do this?

Team of 3

A B C

A B D

A B E

A C D

A C E

A D E

B C D

B C E

B D E

C D E

Team of 2

D E

C E

C D

B E

B D

B C

B E

A D

A C

A B

$$\binom{5}{3} = 10$$

Note: Once we have chosen the team of 3, the other team is determined!

For each one of these, there is only one set of 3 and set of 2, e.g.,

$\{ \{A,B,C\}, \{D,E\} \}$

# Counting Sets: Partitions

Now let's try 2 teams of 2:

**Problem.** Suppose we have four students  $\{ A, B, C, D \}$

We want to divide them into two teams of 2 people each. How many **ways** can we do this? Is it this?

$$\binom{4}{2} = 6$$

Team of 2

A B

A C

A D

B C

B D

C D

Team of 2

C D

B D

B C

A D

A C

A B

# Counting Sets: Partitions

Now let's try 2 teams of 2:

Problem. Suppose we have four students  $\{ A, B, C, D \}$

We want to divide them into two teams of 2 people each. How many **ways** can we do this?

$$\binom{4}{2} = 6$$

Not correct! We have overcounted by a factor of 2.

Team of 2

A B  
A C  
A D  
B C  
B D  
C D

Team of 2

C D  
B D  
B C  
A D  
A C  
A B

As sets, these are the same way of building a partition:

$\{ \{A,B\}, \{C,D\} \}$  is same set of sets as:

$\{ \{C,D\}, \{A,B\} \}$

Same WAY of dividing into 2 teams!

# Counting Sets: Partitions

Now let's try 3 teams of 2:

Problem. Suppose we have six students  $\{ A, B, C, D, E, F \}$

We want to divide them into 3 teams of 2 people each. How many ways can we do this?

$$\binom{6}{2} \binom{4}{2} = 15 * 6 = 90$$

Team of 2

A B

....

A B

....

C D

....

C D

....

E F

....

E F

Team of 2

C D

E F

A B

E F

A B

C D

Team of 2

E F

C D

E F

A B

C D

A B

Note: Once we have chosen the first 2 teams of 2, the last team is determined!

All these “ways” of dividing into 3 teams of equal size are the same!

Overcounting by  $P(3,3) = 3!$ , correct answer is:

$$\frac{\binom{6}{2} \binom{4}{2}}{3!} = \frac{90}{6} = 15$$

The Unordering Principle strikes again!

# Counting Sets: Partitions

**Problem.** Suppose we have 15 students and want to divide them into

2 teams of 3,  
4 teams of 2, and  
a single student who will be referee.

How many ways of doing this are there?

**Solution:** Use multi-nomial coefficients to remove the duplicates among teams you can't distinguish by size:

$$\frac{\binom{15}{3} \binom{12}{3} \binom{9}{2} \binom{7}{2} \binom{5}{2} \binom{3}{2}}{2! * 4!} = \frac{2,270,268,000}{48} = 47,297,250$$

# Counting Sets: Partitions

Now suppose we distinguish the teams by NAME.

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each called "Attackers" and "Defenders." How many ways can we do this?

$$\binom{4}{2} = 6$$

Attackers

A B  
A C  
A D  
B C  
B D  
C D

Defenders

C D  
B D  
B C  
A D  
A C  
A B

Now there is no overcounting!  
Switching attackers and defenders  
gives you a different way.  
There are no duplicate ways.

This may seem obscure, but think  
about experiments involving a  
"test group" (who take a new  
drug) and a "control group" (who  
take a placebo). Switching the  
groups makes a difference!

# Counting Sets: Partitions

**Problem.** Suppose we have 15 students and want to divide them into

- 2 teams of 3, named “MIT Attackers” and “Harvard Attackers”
- 4 teams of 2, all defenders (all unnamed); and
- a single student who will be referee.

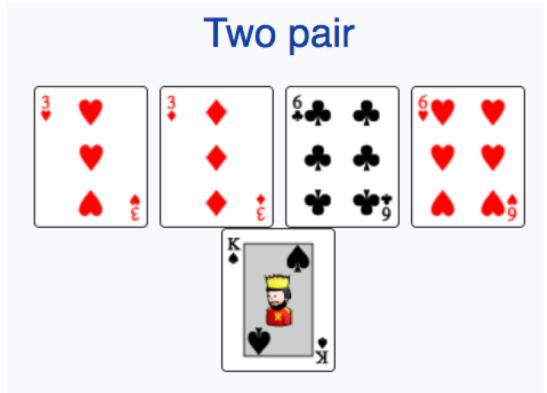
How many **ways** of doing this are there?

**Solution:** Use multinomial coefficients to remove the duplicates among teams you can't distinguish by size or name:

$$\frac{\binom{15}{3} \binom{12}{3} \binom{9}{2} \binom{7}{2} \binom{5}{2} \binom{3}{2}}{4!} = \frac{2,270,268,000}{24} = 94,594,500$$

# Poker Probability -- One Last Time:

**Problem:** What is the probability of Two Pair (2 of one denomination and 2 of different denominations)?



**Solution:** First choose the denomination of the first pair, then those 2 suits, then the denomination of the second pair, then those 2 suits, then the remaining card:

$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{1} \binom{4}{2} \binom{11}{1} \binom{4}{1}}{\binom{52}{5}} = 0.09508$$

But wait.... This doesn't correspond to the web page OR our experiments, which seem to suggest it is too high by a factor of 2. What is wrong?

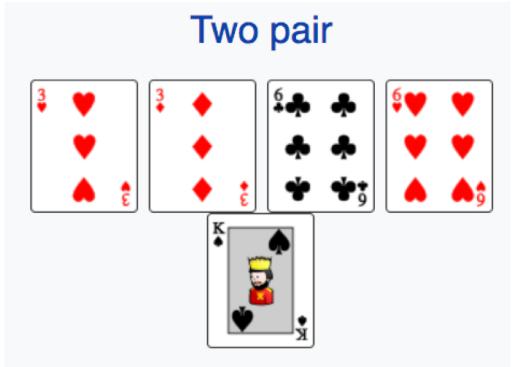
## Problem 5 (C): What is probability of Two Pairs in Poker?

```
In [98]: # probability should be close to analytical value of 0.047539
seed(0)
num_trials = 10**5
print('Probability of two pairs in poker is ' + str(probability_of
Probability of two pairs in poker is 0.04618
```

Two pair	858	123,552	4.7539%

# Poker Probability One Last Time:

Just another example of **you-know-what**, in this case, overcounting the two pairs:



**2D      2H      3C      3D      5S**  
**3C      3D      2D      2H      5S**

These are the same hand, but would be counted twice!

So we could divide by  $2!$  to get the right number:

$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{1} \binom{4}{2} \binom{11}{1} \binom{4}{1}}{2 * \binom{52}{5}} = 0.04754$$

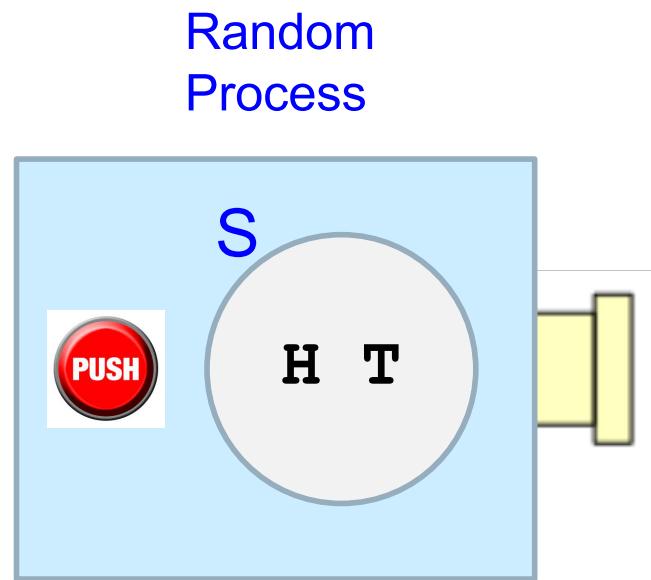
OR we could choose a set of 2 ranks  
to get the two pairs:

$$\frac{\binom{13}{2} \binom{4}{2}^2 \binom{44}{1}}{\binom{52}{5}}$$

Same problem as: you have 52 students, and want to select 2 teams of 2, plus a referee.

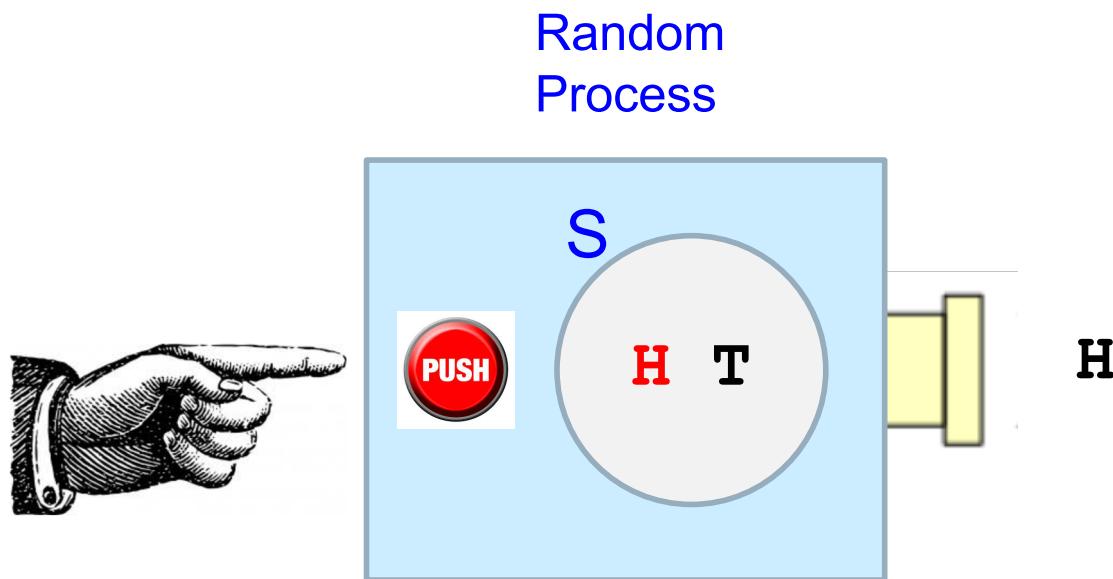
# Random Experiments and Random Variables

A Random Experiment is a process that produces uncertain outcomes from a well-defined sample space.



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A Random Experiment is a process that produces uncertain outcomes from a well-defined sample space.



# Random Variables

In order to formalize this notion, the notion of a Random Variable has been developed. A **Random Variable**  $X$  is a function from a sample space  $S$  into the reals:

$$X : S \rightarrow \mathcal{R}$$

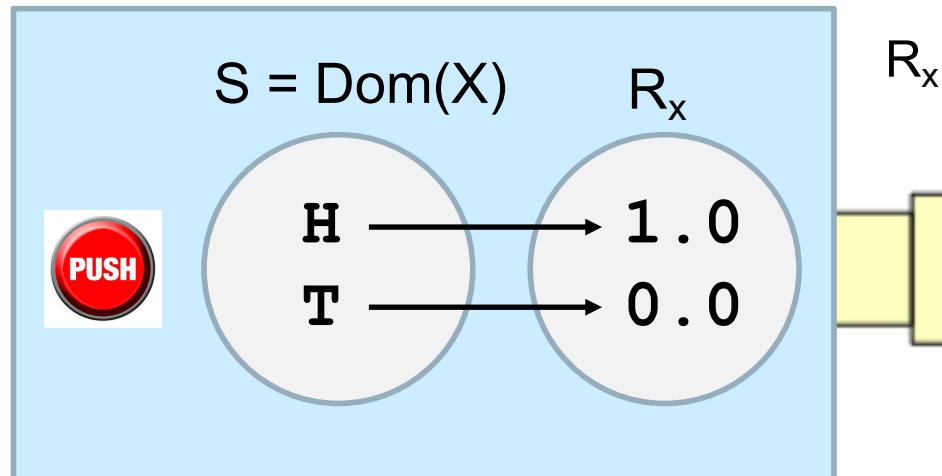
Now when an outcome is requested, the sample point is translated into a real number:

$$S = \text{Domain}(X)$$

$$X$$

$$R_x =_{\text{def}} \text{Range}(X)$$

$$R_x \subseteq \mathcal{R}$$



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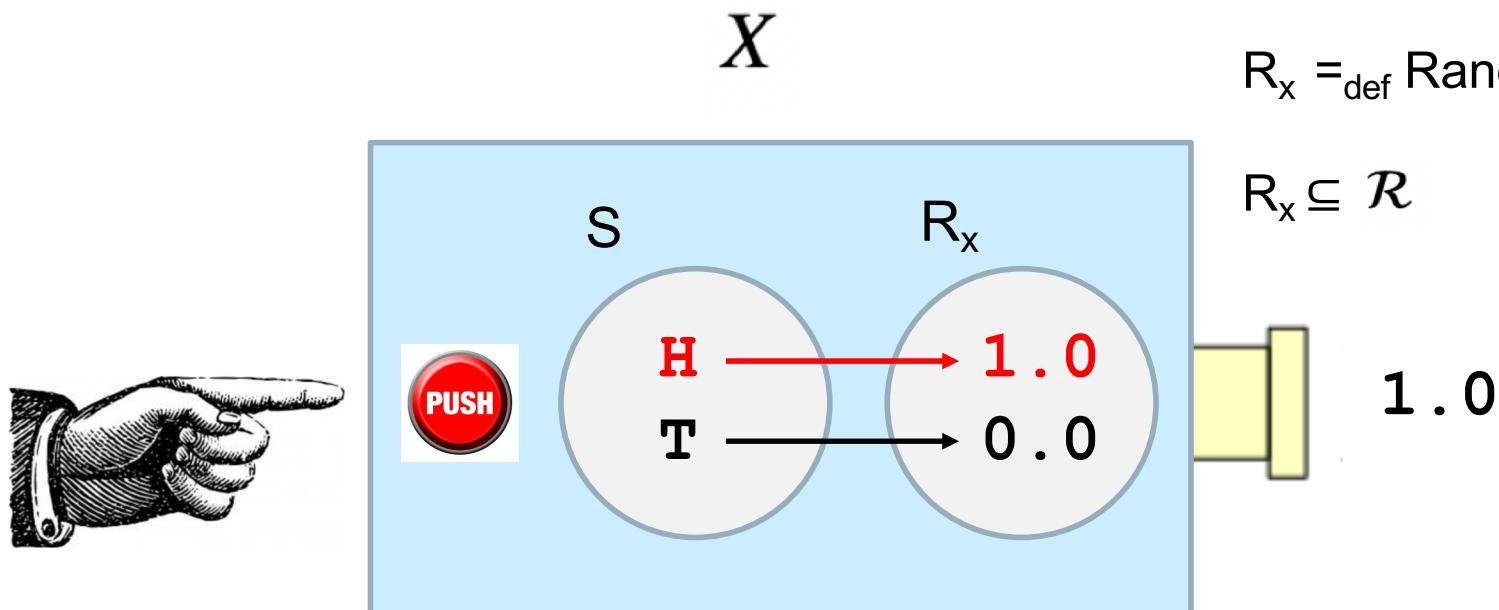
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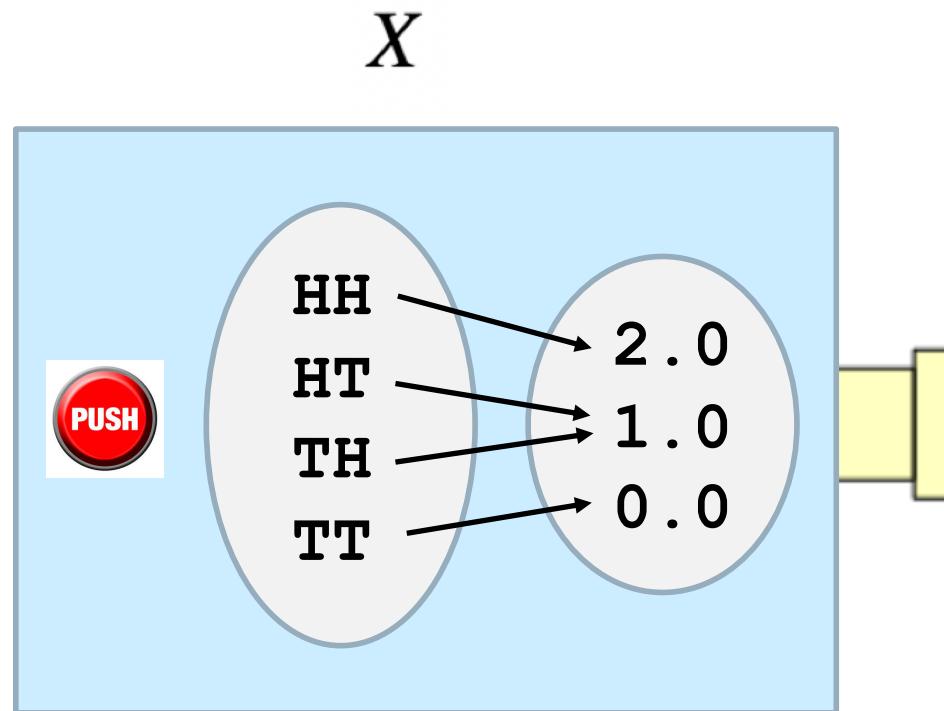
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# Random Variables

This may seem awkward, but it helps to explain the difference between random experiments whose literal outcomes are not numbers, but which are translated into numbers for clarity.

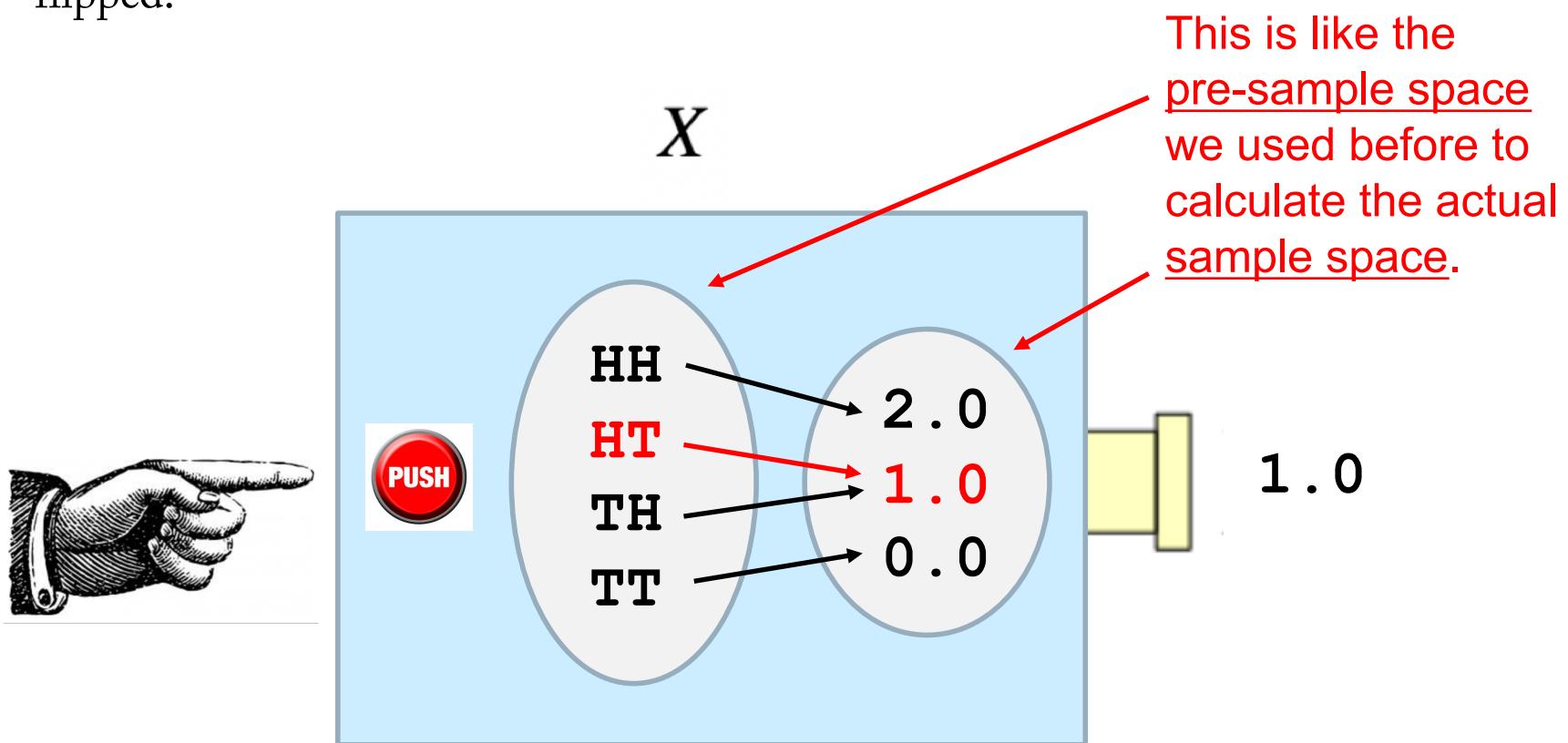
**Example:**  $X$  = “the number of heads which appear when two fair coins are flipped.”



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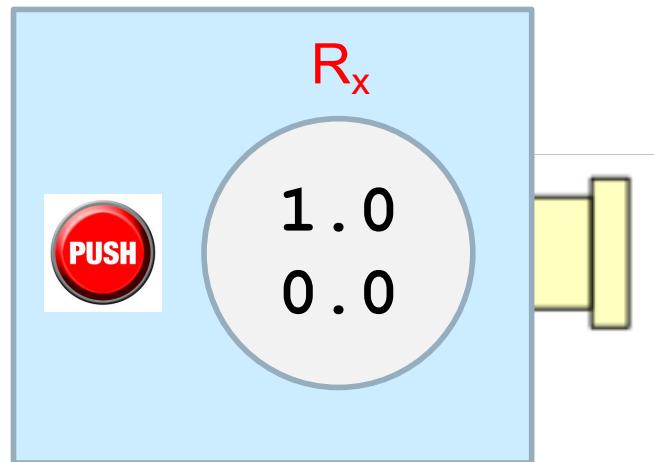
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# Random Variables

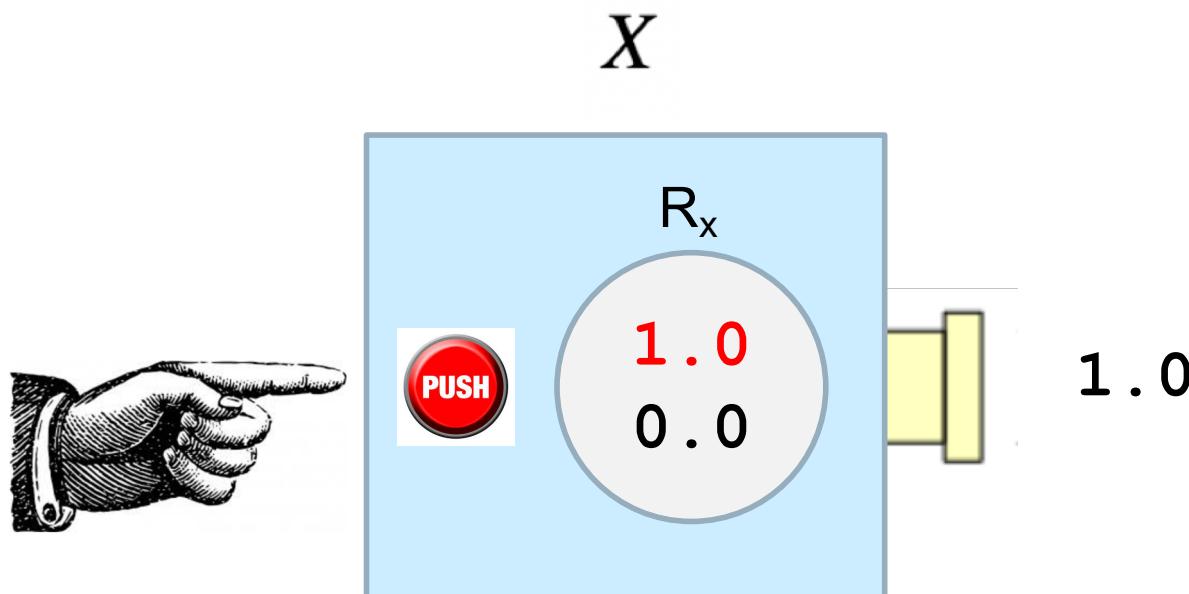
In general, in this class we will call the possible outputs  $R_x$ , since this is the symbol used in your textbook, although you could just think of it as the sample space from which the outputs are drawn.

$X$



# Random Variables

In general, in this class we will call the possible outputs  $R_x$ , since this is the symbol used in your textbook, although you could just think of it as the sample space from which the outputs are drawn.



# Discrete vs Continuous Random Variables

A random variable  $X$  is called **discrete** if  $R_x$  is **finite or countably infinite**:

Example of finite random variable:

$X$  = “the number of dots showing after rolling two dice”

$$R_x = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$$

Example of countably infinite random variable:

$Y$  = “the number of flips of a coin until a head appears”

$$R_y = \{ 1, 2, 3, \dots \}$$

A random variable is called **continuous** if  $R_x$  is **uncountable**. Example:

$Z$  = “the distance of a thrown dart from the center of a circular target of 1 meter radius”

$$R_z = [ 0.0 .. 1.0 ]$$

# Discrete vs Continuous Random Variables

A random variable  $X$  is called **discrete** if  $R_x$  is **finite or countably infinite**:

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For several weeks we will only consider discrete random variables!

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$$R_z = [ 0.0 .. 1.0 ]$$

# Discrete Random Variables: Probability Distributions

The Probability Distribution Function (PDF) of a discrete random variable  $X$  is a function  $f$  from the range of  $X$  into  $\mathcal{R}$  :

$$f_X : R_X \rightarrow [0..1]$$

such that (i)  $\forall a \in R_x \quad f_X(a) \geq 0$

(ii)  $\sum_{a \in R_x} f_X(a) = 1.0$

If there is no possibility of confusion we will write  $f$  instead of  $f_X$ .

# Discrete Random Variables: Probability Distributions

To specify a random variable precisely, you simply need to give the range  $R_X$  and the pdf  $f$ :

Examples:

$X$  = “The number of dots showing on a thrown die”

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

$$f_X = \left\{ \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right\}$$

For simplicity, we simply list the values in  $\text{Range}(f_X)$  corresponding to the listing of  $R_X$ .

$Y$  = “The number of tosses of a fair coin until a head appears”

$$R_Y = \{1, 2, 3, \dots\}$$

$$f_Y = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$$

# Discrete Random Variables: Probability Distributions

How does this relate to our first definition of a probability space, events, probability function, etc., etc. ??

Probability Space

Sample Space

Event

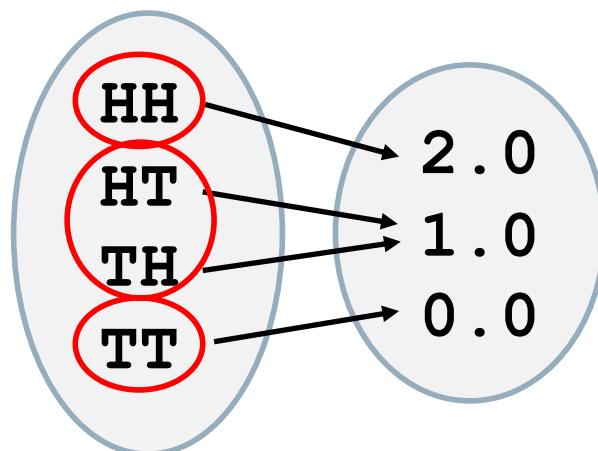
Probability Function

Random Variable X

$R_X$

Interval on real line

Probability Distribution  $f_X$



For continuous random variables there are additional conditions about events having to be the countable sum of intervals on the real number line.

# Discrete Random Variables: Probability Distributions

We will emphasize (starting next week) the distributions of random variables, using graphical representations (as in HW 01) to help our intuitions.

**Example:**

$Y$  = “The number of tosses of a fair coin until a head appears”

$$R_Y = \{ 1, 2, 3, \dots \}$$

$$f_Y = \{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \}$$

