CS 237: Probability in Computing

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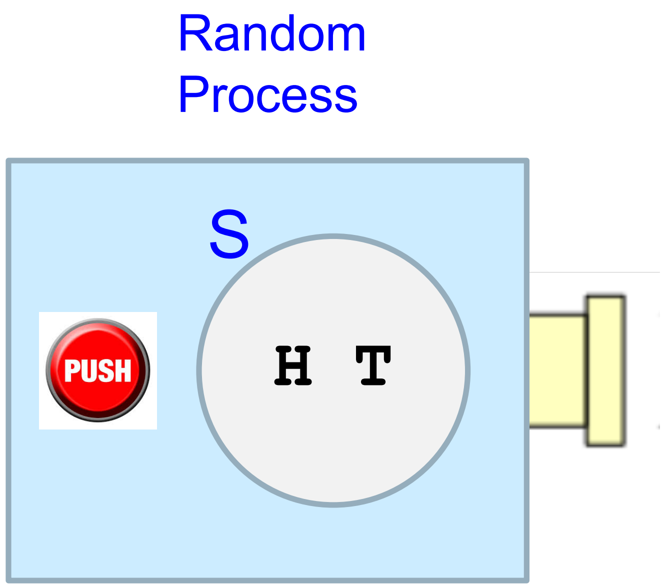
Lecture 7:

- Random Variables
- Discrete Random Variables
- Probability Distributions of Random Variables
- Expressions and functions of Random Variables
- Expected value of a Random Variable

Random Experiments and Random Variables

A Random Experiment is a process that produces uncertain outcomes from a well-defined sample space.



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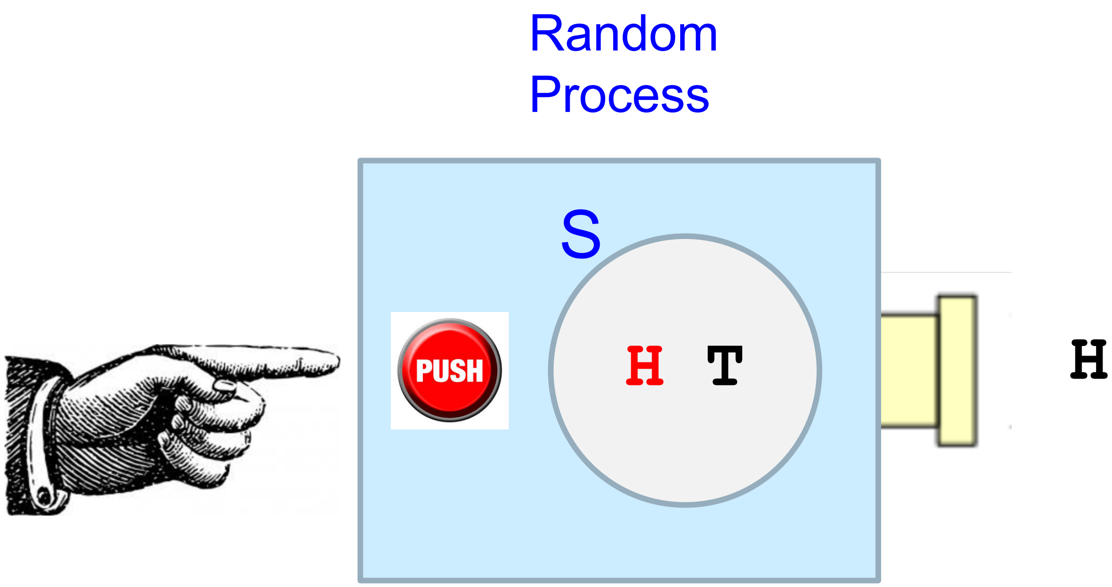


Random Variables

In order to formalize this notion, the notion of a Random Variable has been developed. A Random Variable X is a function from a sample space S into the reals:

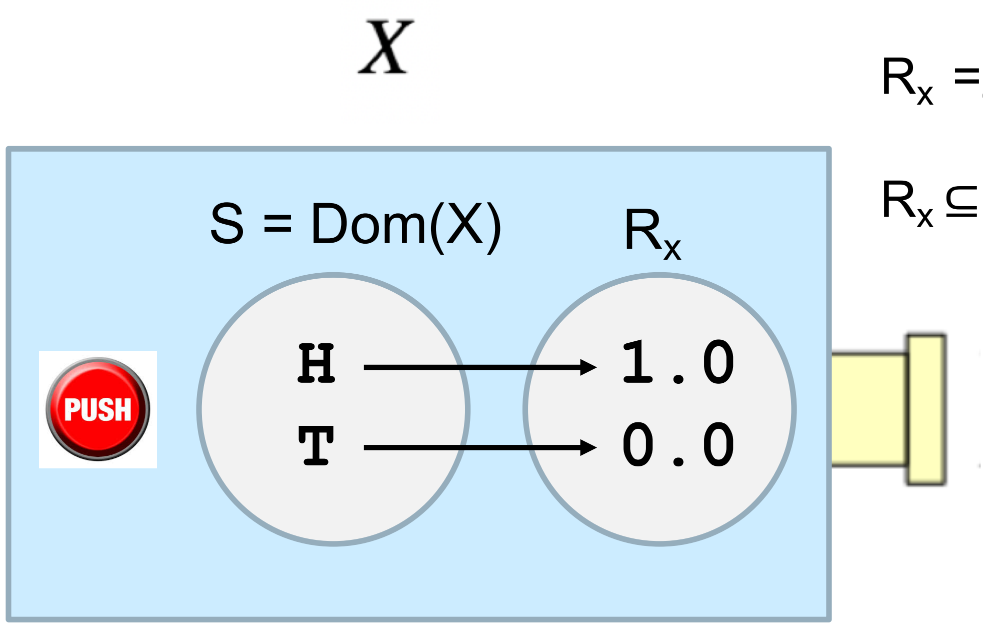
$$X : S \to \mathcal{R}$$

Now when an outcome is requested, the sample point is translated into a real number:

S = Domain(X)

$R_x =_{def}$ Range(X)

$R_x \subseteq \mathcal{R}$



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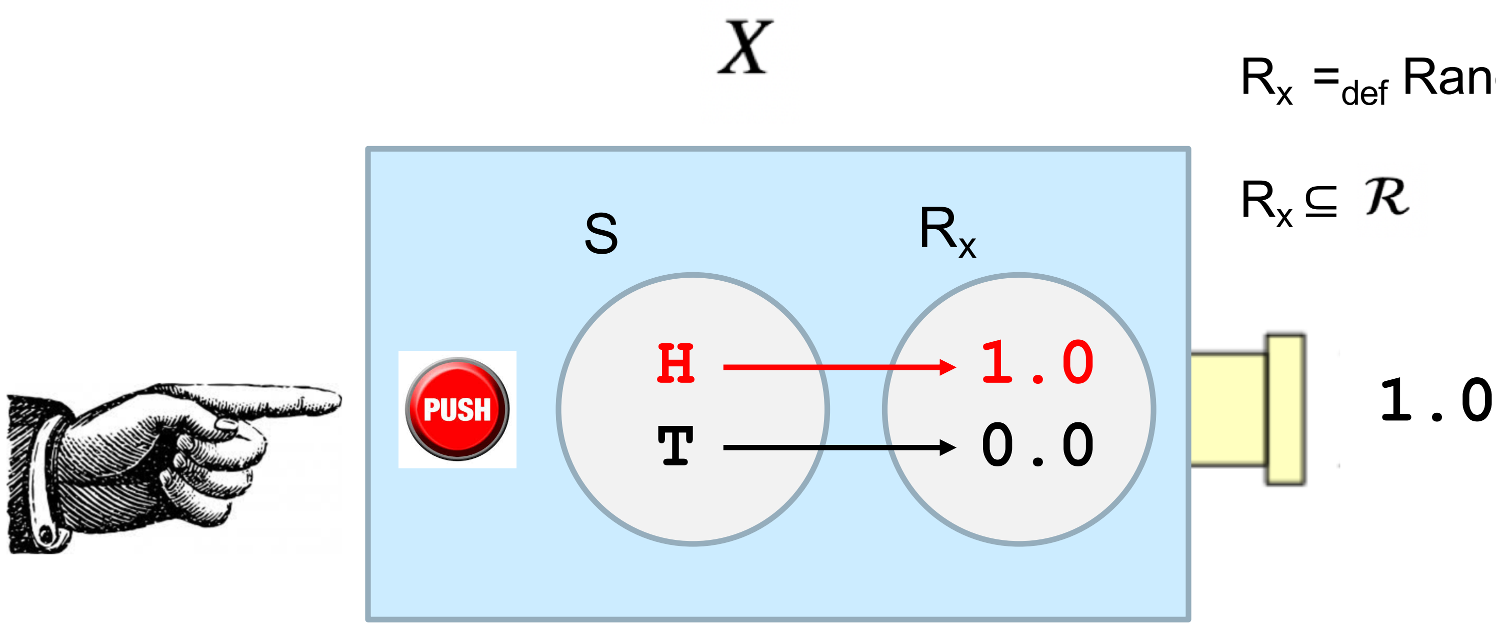
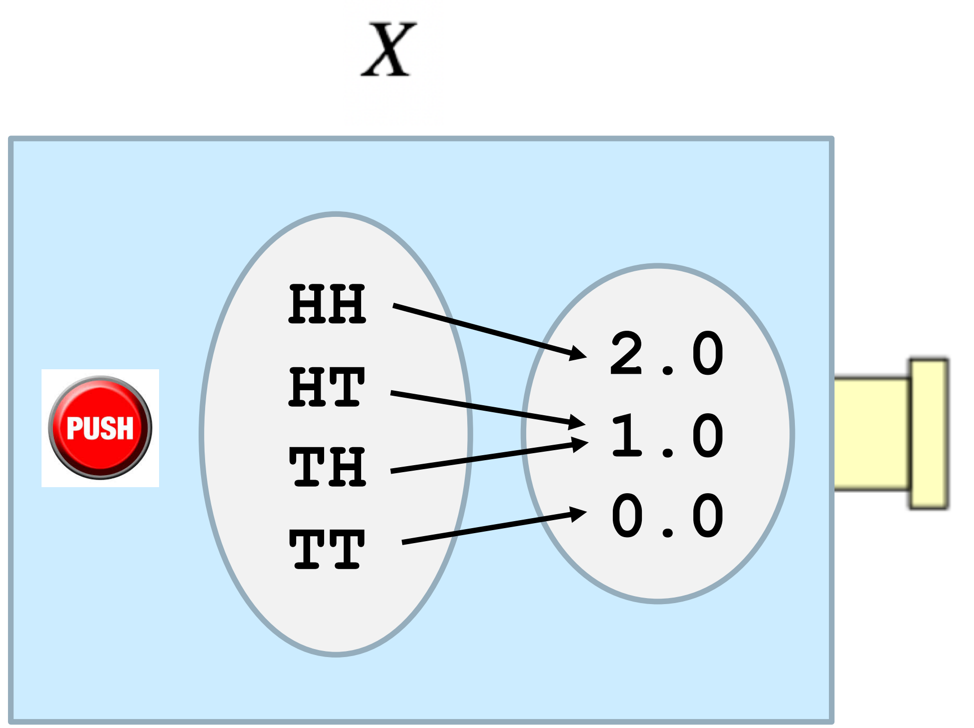


Random Variables

This may seem awkward, but it helps to explain the difference between random experiments whose literal outcomes are not numbers, but which are translated into numbers for clarity.

Example: X = “the number of heads which appear when two fair coins are flipped.”



Random Variables

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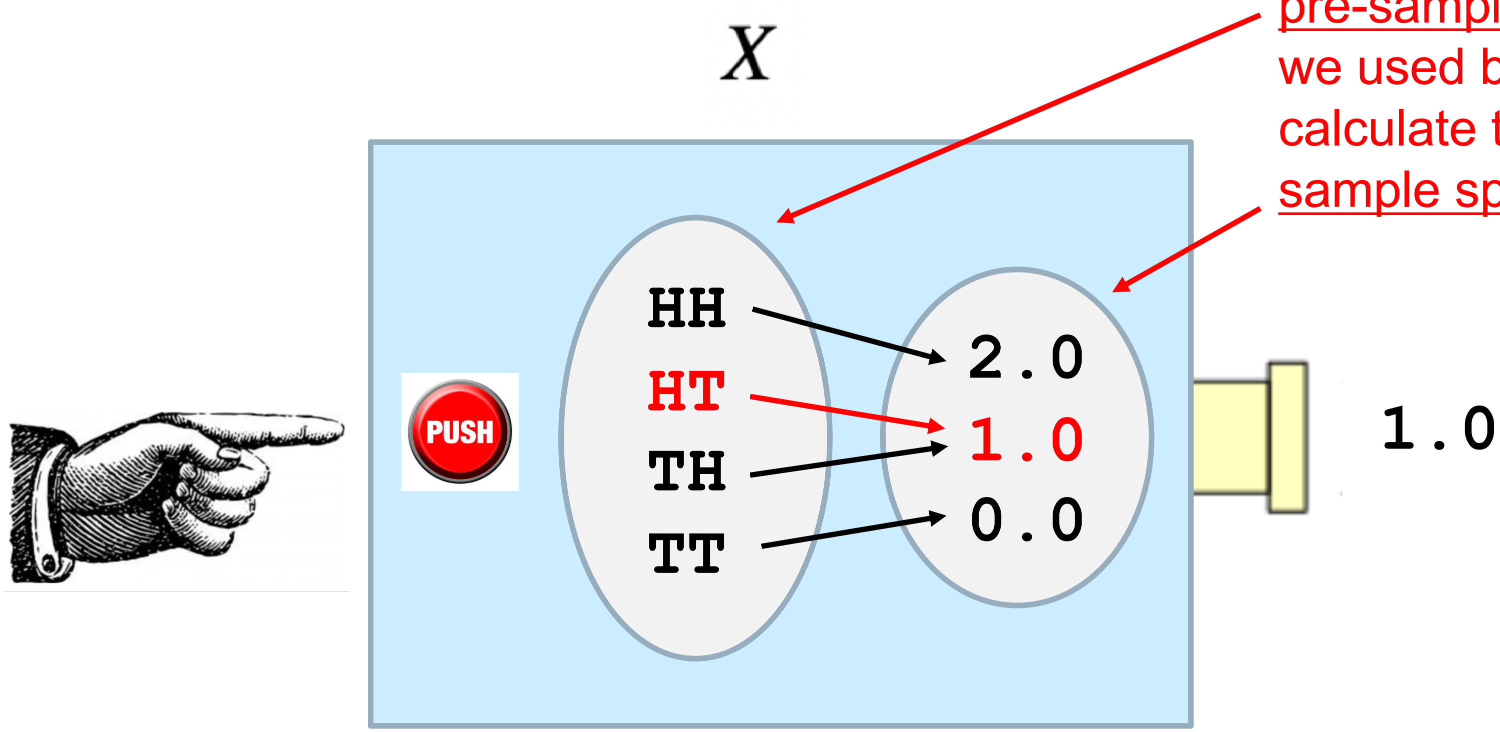
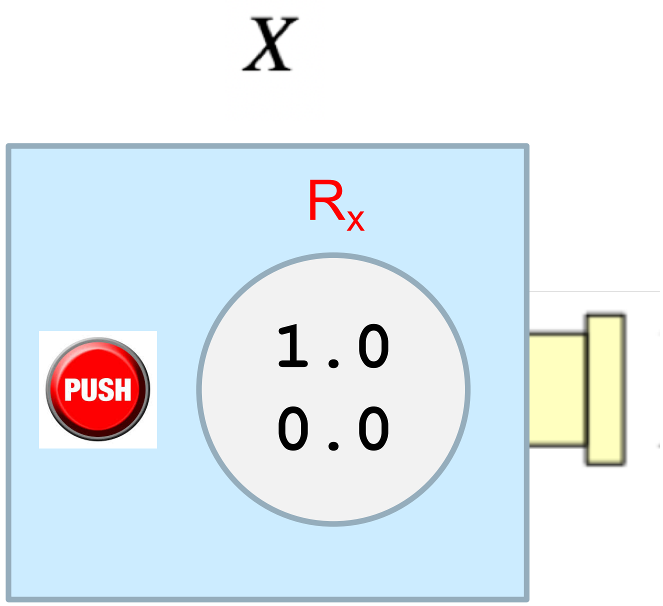
Example: X = "the number of heads which appear when two fair coins are flipped.

This is like the <u>pre-sample space</u> we used before to calculate the actual <u>sample space</u>.



Random Variables

In general, in this class we will call the possible outputs R_x, since this the symbol used in your textbook, although you could just think of it as the sample space from which the outputs are drawn.



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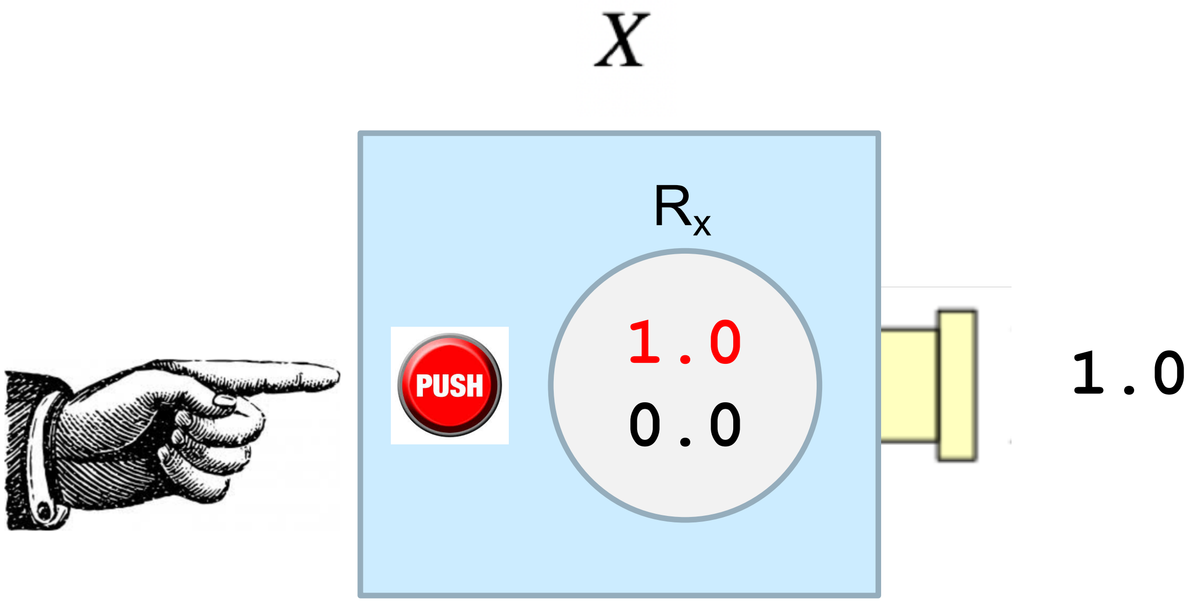

Discrete vs Continuous Random Variables

A random variable X is called discrete if R_x is finite or countably infinite:

Example of finite random variable:

X = "the number of dots showing after rolling two dice"

R_X = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }

Example of countably infinite random variable:

Y = "the number of flips of a coin until a head appears"

R_Y = { 1, 2, 3, ... }

A random variable is called continuous if Rx is uncountable. Example:

Z = "the distance of a thrown dart from the center of a circular target of 1 meter radius"

R_Z = [0.0 .. 1.0]

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For several weeks we will only consider discrete random variables!

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R_Z = [0.0 .. 1.0)

Discrete Random Variables: Probability Distributions

The Probability Distribution of a discrete random variable X is a function f from the range of X into $\mathcal{R}$:

$$f_X : R_X \to [0..1]$$

such that (i) $\forall a \in R_x \quad f_X(a) \geq 0$

(ii) $$\sum_{a \in R_x} f_X(a) = 1.0$$

If there is no possibility of confusion we will write f instead of f_X.

Discrete Random Variables: Probability Distributions

To specify a random variable precisely, you simply need to give the range R_X and the probability distribution f:

Examples:

X = "The number of dots showing on a thrown die"

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

$$f_X = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$

For simplicity, we simply list the values in Range(f_X) corresponding to the listing of R_X.

Y = "The number of tosses of a fair coin until a head appears"

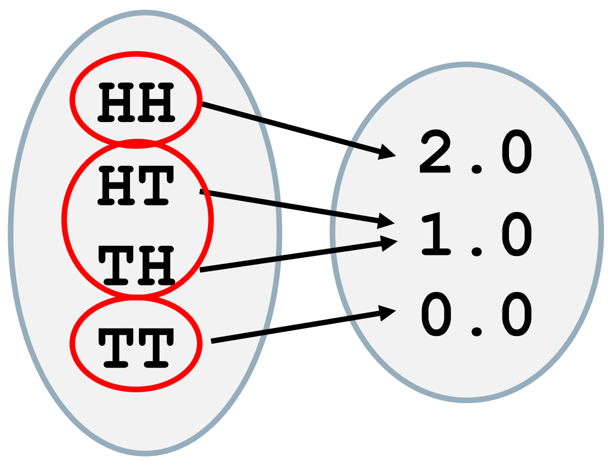
$$R_Y = \{ 1, 2, 3, \ldots \}$$

$$f_Y = \{ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \}$$

Discrete Random Variables: Probability Distributions

How does this relate to our first definition of a probability space, events, probability function, etc., etc. ??

<u>Probability Space</u>	<u>Random Variable X</u>
Sample Space	R_X
Event	Countable Union of Intervals on real line
Probability Function	Probability Distribution f_X



Discrete Random Variables: Probability Distributions

We will emphasize the distributions of random variables, using graphical representations (as in HW 1) to help our intuitions.

Example:

$$f_X : R_X \to [0..1]$$

such that (i) $\forall a \in R_x \;\; f_X(a) \geq 0$

(ii) $\sum_{a \in R_x} f_X(a) = 1.0$

Y = "The number of tosses of a fair coin until a head appears"

$$R_Y = \{\ 1, 2, 3,\ \ldots\ \}$$

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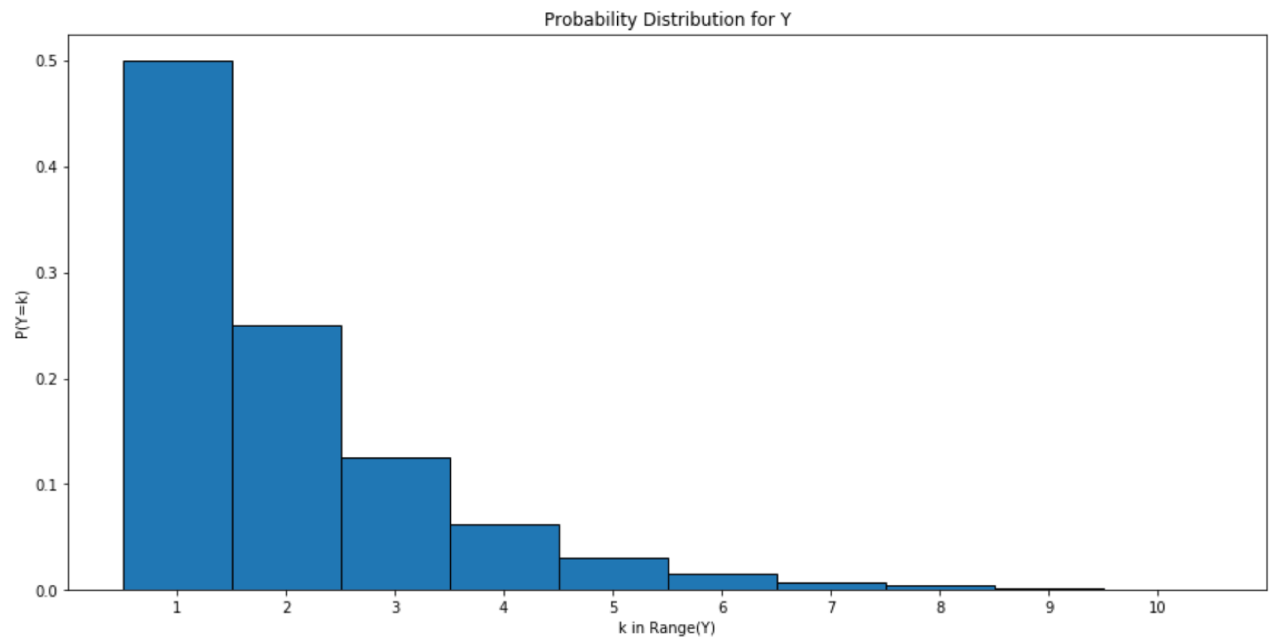


Discrete Random Variables

Notation:

$$P(X = k) \quad =_{\text{def}} \quad f_X(k)$$

$$P(X \leq k) \quad =_{\text{def}} \quad \sum_{a \leq k} f_X(a)$$

$$P(k \leq X \leq m) \quad =_{\text{def}} \quad \sum_{k \leq a \leq m} f_X(a)$$

Example:

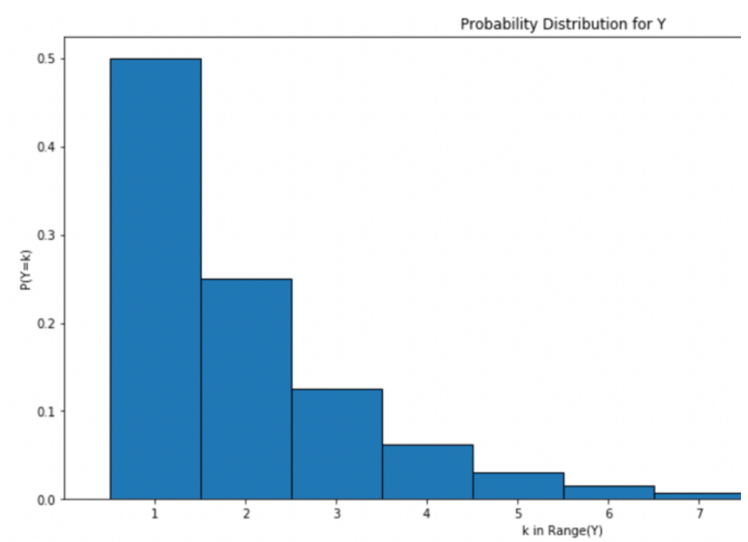
$$P(Y = 4) \quad = \frac{1}{16}$$

$$P(Y < 4) \quad = \frac{7}{8}$$

$$P(2 \leq Y \leq 4) \quad = \frac{7}{16}$$

$$R_Y = \{\ 1, 2, 3,\ \ldots\ \}$$

$$f_Y = \{\ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ \ldots \}$$



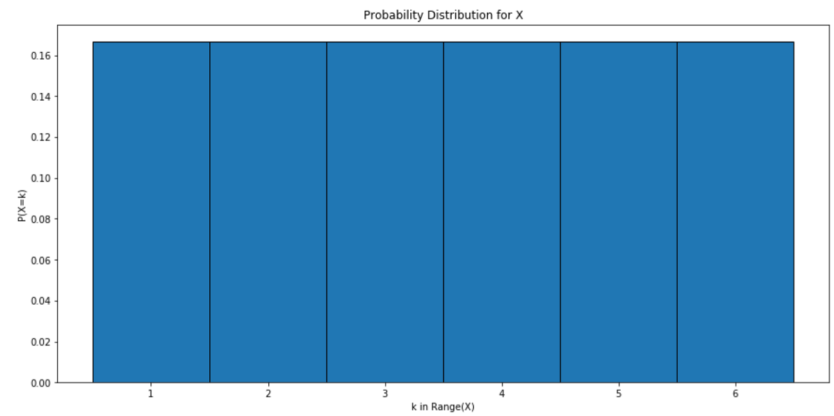
Functions of Discrete Random Variables

New random variables can be created by functions or expressions involving old random variables. But you have to be careful!

Example: X = "the number of dots on a thrown die"

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

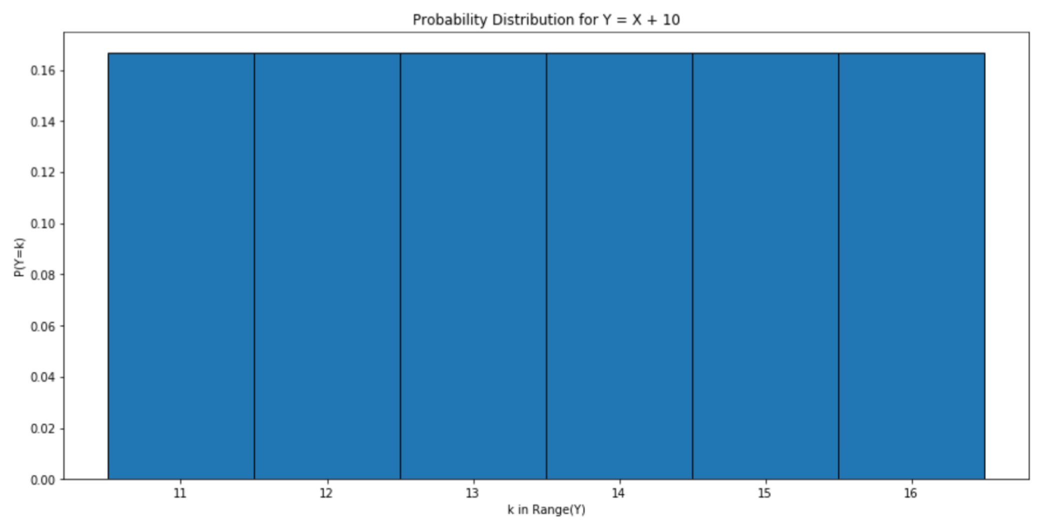
$$f_X = \{\ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\ \}$$



Let Y = X + 10

$$R_Y = \{\ 11, 12, 13, 14, 15, 16\ \}$$

$$f_Y = \{\ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\ \}$$

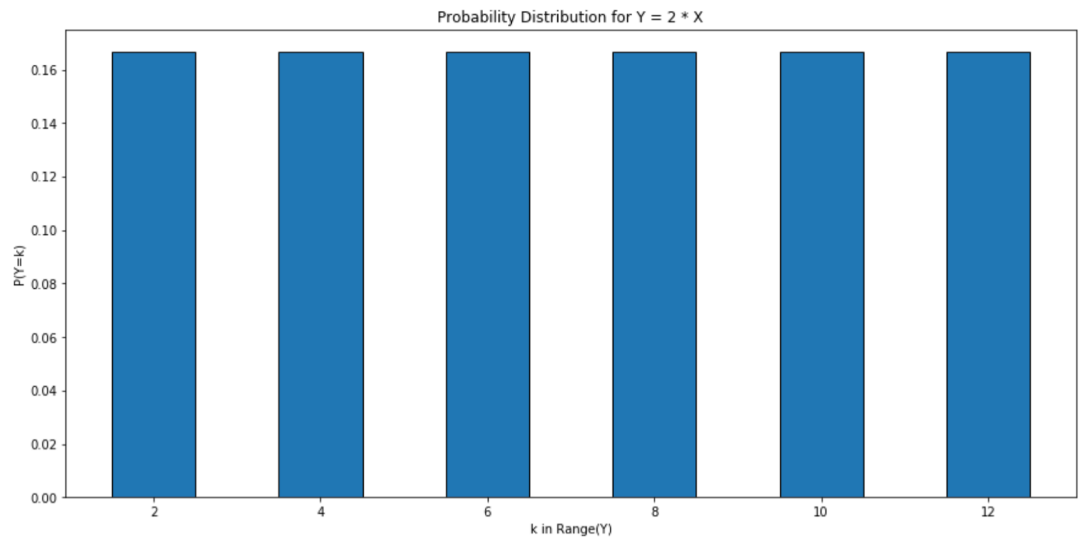


Functions of Discrete Random Variables

Let Y = 2 * X

$$R_Y = \{\, 2, 4, 6, 8, 10, 12 \,\}$$

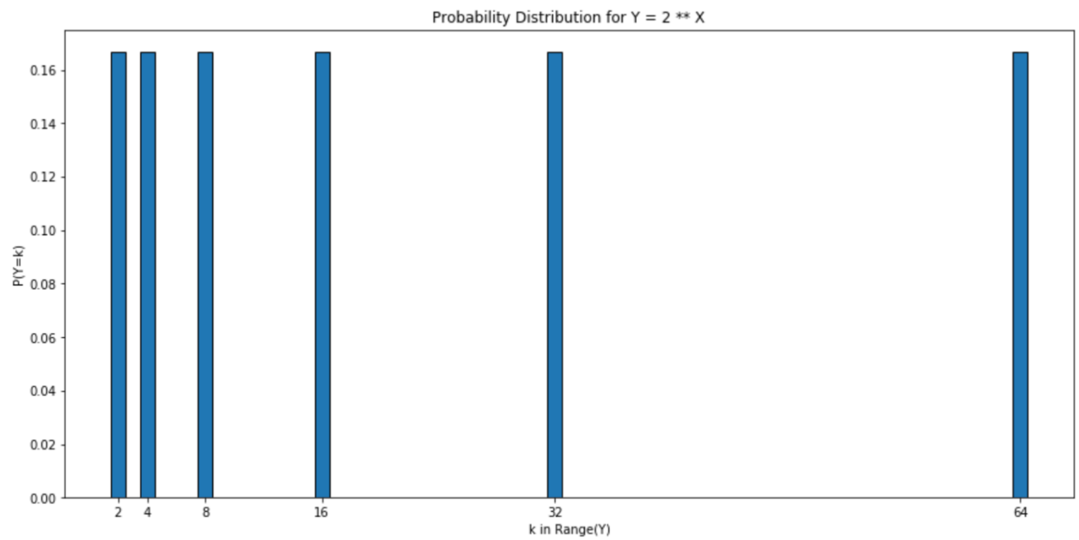
$$f_Y = \{\, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \,\}$$



Let Y = 2^X

$$R_Y = \{\, 2, 4, 8, 16, 32, 64 \,\}$$

$$f_Y = \{\, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \,\}$$



Functions of Discrete Random Variables

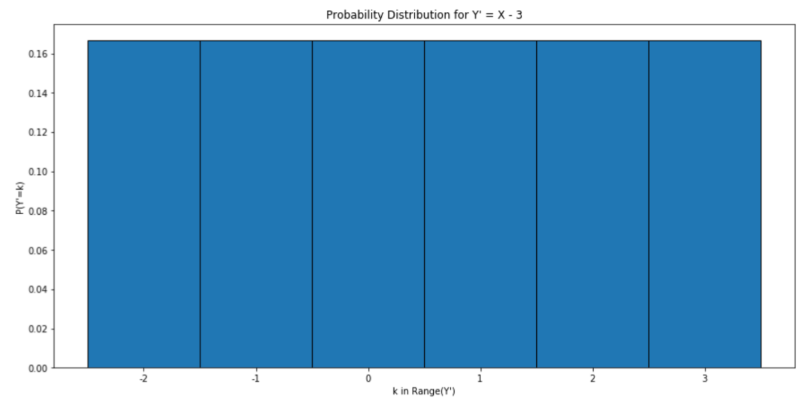
Why did I say you have to be careful?   Two main reasons...

One, the function of a random variable may combine outcomes...

Example: Let Y' = X – 3 and let Y = | X - 3 |

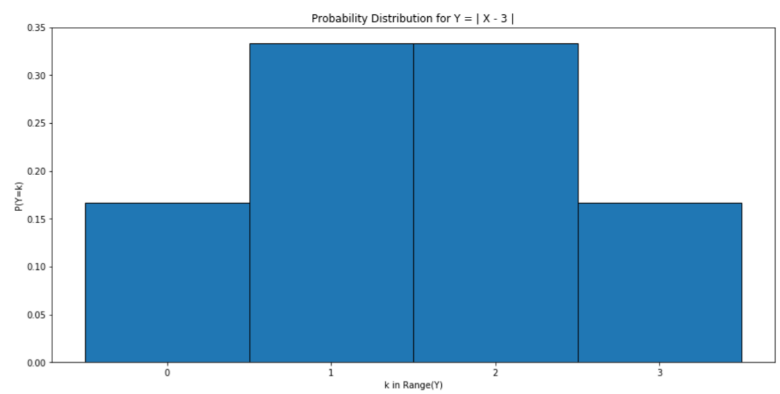
$$R_{Y'} = \{ -2, -1, 0, 1, 2, 3 \}$$

$$f_{Y'} = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$



$$R_Y = \{ 0, 1, 2, 3 \}$$

$$f_Y = \{ \tfrac{1}{6}, \tfrac{2}{6}, \tfrac{2}{6}, \tfrac{1}{6} \}$$



Functions of Discrete Random Variables

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

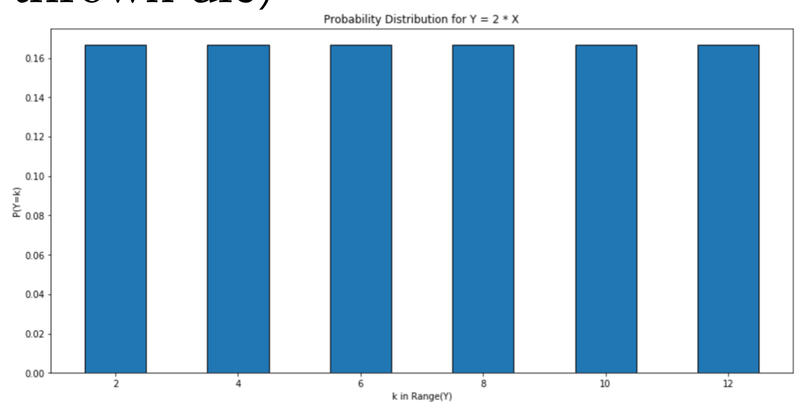
$$f_X = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$

Two, you have to be careful when a random variable is used more than once, since each occurrence refers to a potentially different random outcome!

Let Y = 2 * X (twice the dots showing on a thrown die)

$$R_Y = \{ 2, 4, 6, 8, 10, 12 \}$$

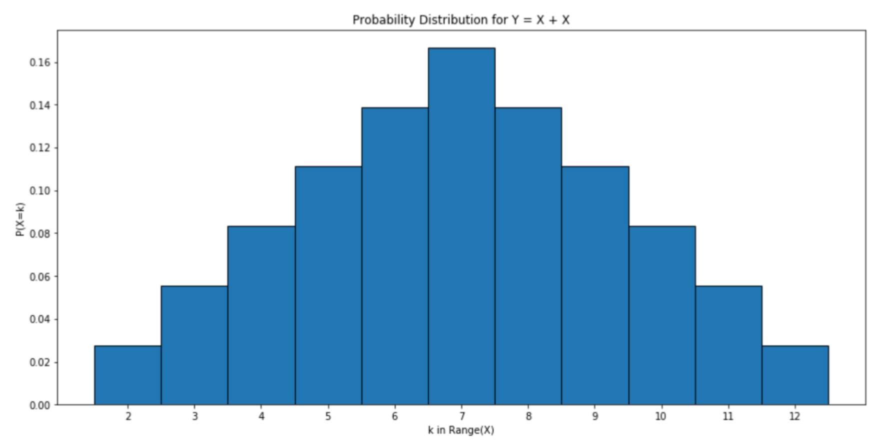
$$f_Y = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$



Let Y = X + X (sum of the dots showing on two thrown dice)

$$R_Y = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$$

$$f_Y = \{ \tfrac{1}{36}, \tfrac{2}{36}, \tfrac{3}{36}, \tfrac{4}{36}, \tfrac{5}{36}, \tfrac{6}{36}, \tfrac{5}{36}, \tfrac{4}{36}, \tfrac{3}{36}, \tfrac{2}{36}, \tfrac{1}{36} \}$$



Discrete Random Variables: Expected Value

A fundamental way of characterizing a collection of real numbers is the average or mean value of the collection:

Example: The mean/average of { 2, 4, 6, 9 } = 21/4 = 5.7

The corresponding notion for a random variable X is the Expected Value:

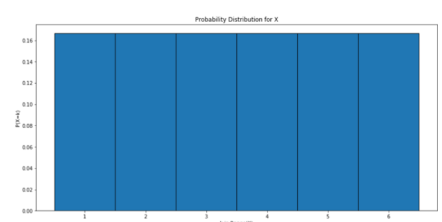
$$E(X) =_{\text{def}} \sum_{a \in R_x} a * f(a)$$

Example: X = "the number of dots showing on a single thrown die"

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

$$f_X = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$

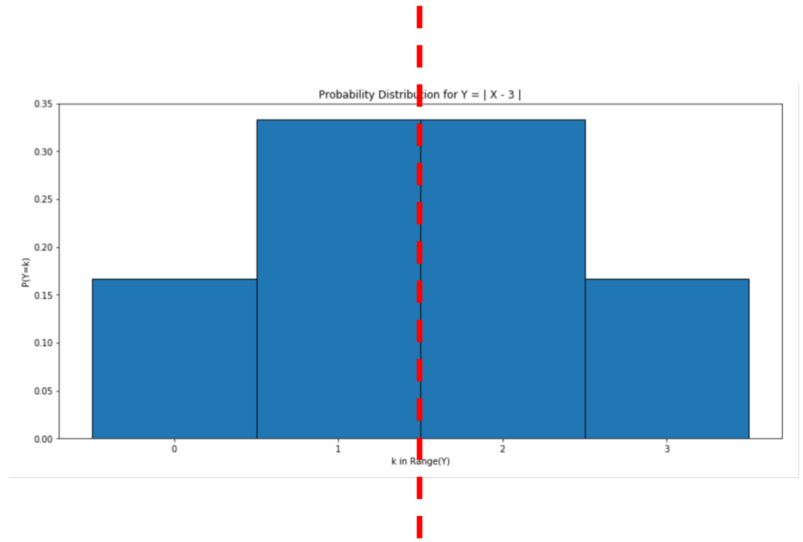
$$E(X) = \sum_{k \in R_X} \frac{k}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$



Discrete RandomVariables: Expected Value

Example: Y = | X – 3 |

$E(\,Y\,)$



$$R_Y = \{\,0, 1, 2, 3\,\}$$

$$f_Y = \{\,\tfrac{1}{6}, \tfrac{2}{6}, \tfrac{2}{6}, \tfrac{1}{6}\,\}$$

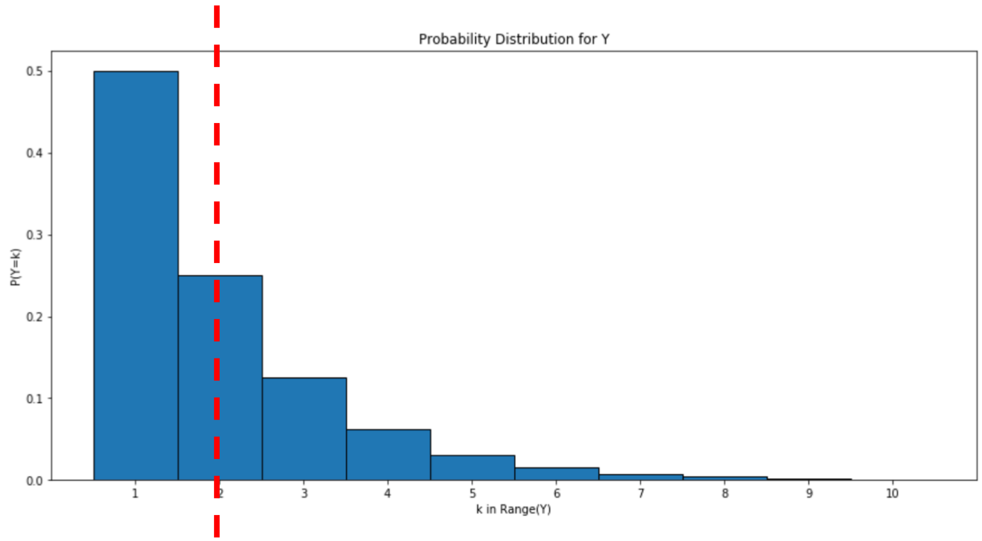
$$E(\,Y\,) \;=\; \sum_{k\,\in\,R_Y} k * f(k) \;=\; \frac{0}{6} + \frac{1}{3} + \frac{2}{3} = \frac{3}{6} \;=\; 1.5$$

Discrete RandomVariables: Expected Value

Example: Y = "tosses of a fair coin until a heads appears"

$$R_Y = \{\ 1, 2, 3,\ \ldots\ \}$$

$$f_Y = \{\ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8},\ \ldots\}$$



$$E(\,Y\,) \;=\; \sum_{k \in R_Y} k * f(k) \;=\; \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} \ldots = 2.0$$

Expected Value and Fair Games

The expected value of a random variable is a **one-number summary of its behavior**. It describes what you can expect as the limiting behavior over many trials.

A good example of what this means occurs with games in which you win or lose money on each round or trial. Such a game can be modeled by a random variable: X = "the amount you win (+) or lose (-)." A game is **fair** if E(X) = 0.

Example: The rules of "**Chuck-a-luck**" are as follows. The player makes a bet on any number 1 through 6 and then three dice are thrown. If 1, 2, or 3 dice show the same number as the player's choice, then he or she wins back the original bet plus 1, 2, or 3 times the original bet.

- So if you bet \$1 on 4 and the dice roll 2, 4, and 6, you get back 1+1=2 for a net win of \$1.
- If you bet \$1 on 2 and the dice roll 2, 6, and 2, you get back 1 + 2 = 3 for a net win of \$2.
- If you bet \$1 on 5 and no 5's show and you lose \$1.

Expected Value and Fair Games

To analyze Chuck-a-Luck, suppose the player always bets \$1 on each round (or trial) of the game. Let

X = "net win or loss for one round."

Then $R_X = \{ -1, 1, 2, 3 \}$

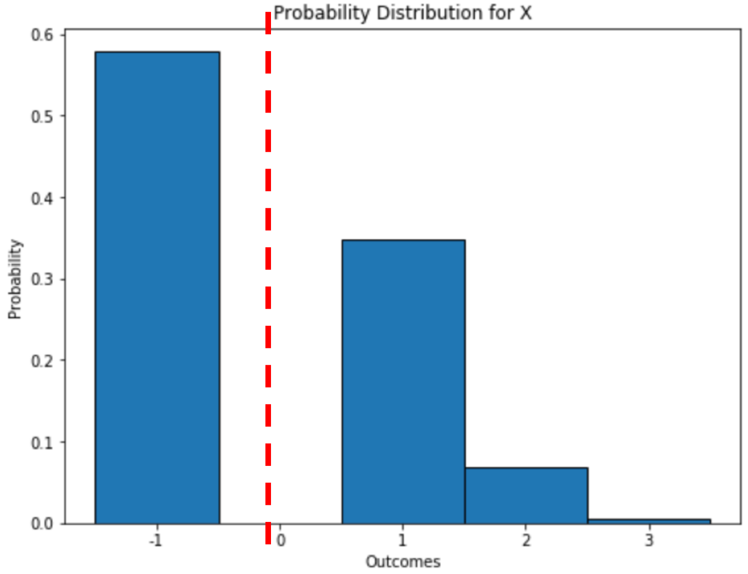
$$P(X = -1) = \binom{3}{0} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} = 0.5787$$

$$P(X = 1) = \binom{3}{1} * \frac{1}{6} * \frac{5}{6} * \frac{5}{6} = 0.3472$$

$$P(X = 2) = \binom{3}{2} * \frac{1}{6} * \frac{1}{6} * \frac{5}{6} = 0.0694$$

$$P(X = 3) = \binom{3}{3} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} = 0.0047$$

$$f_X = \{ 0.5787,\ 0.3472,\ 0.0694,\ 0.0047 \}$$



On average you lose about 8 cents per round, so Chuck-a-luck is not fair:

$$E(X) = \sum_{k \in R_X} k * f_X(k) = -0.5787 + 0.3472 + 2 * 0.0694 + 3 * 0.0047 = -0.0786$$