CS 237: Probability in Computing

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Lecture 8:

- Review: Expected value of a Random Variable
- Variance and Standard Deviation of a Random Variable
- Standardized Random Variables
- Cumulative Distribution Functions
- [If time] Introduction to Probability Distributions

Discrete RandomVariables: Expected Value

A fundamental way of characterizing a collection of real numbers is the average or mean value of the collection:

Example: The mean/average of { 2, 4, 6, 9 } = 21/4 = 5.7

The corresponding notion for a random variable X is the Expected Value:

$$E(X) =_{\text{def}} \sum_{a \in R_x} a * f(a)$$

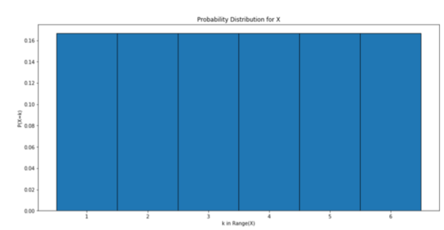
Alternate notation: $\mu_X = E(X)$

Example: X = "the number of dots showing on a single thrown die"

$R_X = \{1, 2, 3, 4, 5, 6\}$

$f_X = \{ \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \}$

$$E(X) = \sum_{k \in R_X} \frac{k}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$



Expected Value and Fair Games

The expected value of a random variable is a **one-number summary of its behavior**. It describes what you can expect as the limiting behavior over many trials.

A good example of what this means occurs with games in which you win or lose money on each round or trial. Such a game can be modeled by a random variable: X = "the amount you win (+) or lose (-)." A game is **fair** if E(X) = 0.

Example: The rules of "**Chuck-a-luck**" are as follows. The player makes a bet on any number 1 through 6 and then three dice are thrown. If 1, 2, or 3 dice show the same number as the player's choice, then he or she wins back the original bet plus 1, 2, or 3 times the original bet.

- So if you bet $1 on 4 and the dice roll 2, 4, and 6, you get back 1+1=2 for a net win of $1.
- If you bet $1 on 2 and the dice roll 2, 6, and 2, you get back 1 + 2 = 3 for a net win of $2.
- If you bet $1 on 5 and no 5's show and you lose $1.

Expected Value: Basic Properties

There are **two important things** to remember about E(X):

- **It may not exist (may be infinite!)**
- **Linearity of Expectation**

Example where E(X) is infinite (the "St.Petersburg Paradox"): Consider the following game: you roll a coin until heads appears and I give you 2^K dollars, where K is the number of rolls. Thus,

X = "the number of rolls until heads appears" $\qquad$ $Y = 2^X$

and we seek E(Y).

$$E(Y) = 2^1 \frac{1}{2} + 2^2 \frac{1}{4} + 2^3 \frac{1}{8} + \ldots + 2^k \frac{1}{2^k} + \ldots$$

$$= 1 + 1 + 1 + \ldots$$

This will happen for countably infinite RVs when the sequence does not converge.

In order to make this game fair, I would have to charge you an infinite amount of money to play, even though you only win a finite amount of money each round!

Expected Value: Basic Properties

Theorem (Linearity of Expectation)

For any random variable **X** and real numbers **a** and **b**,

$$E(a * X + b) = a * E(X) + b$$

Proof:

$$\begin{aligned}
E(aX + b) &= \sum_{k \in R_X} (a * k + b) * f_X(k) \\
&= \sum_{k \in R_X} (a * k * f_X(k)) + (b * f_X(k)) \\
&= \sum_{k \in R_X} (a * k * f_X(k)) \; + \sum_{k \in R_X} (b * f_X(k)) \\
&= a * \sum_{k \in R_X} (k * f_X(k)) \; + \; b * \sum_{k \in R_X} f_X(k) \\
&= a * E(X) \; + \; b * 1.0 \\
&= a * E(X) \; + \; b
\end{aligned}$$

This will make many calculations involving expected value MUCH easier!

(Obvious) Corollary: For any constant b, E(b) = b.

Discrete Random Variables: Expected Value

But how useful is expected value in making the following decision?

Consider two games. **Which one would you prefer to play?**

Game One: For \$1 per round, you can flip a coin, and I'll give you \$6 (your bet back plus \$5) if heads appears, and nothing if tails appears (so you lose your bet). Call this the random variable X_1 = net payout.

$$E(X_1) = -1 \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} = \$2.00$$

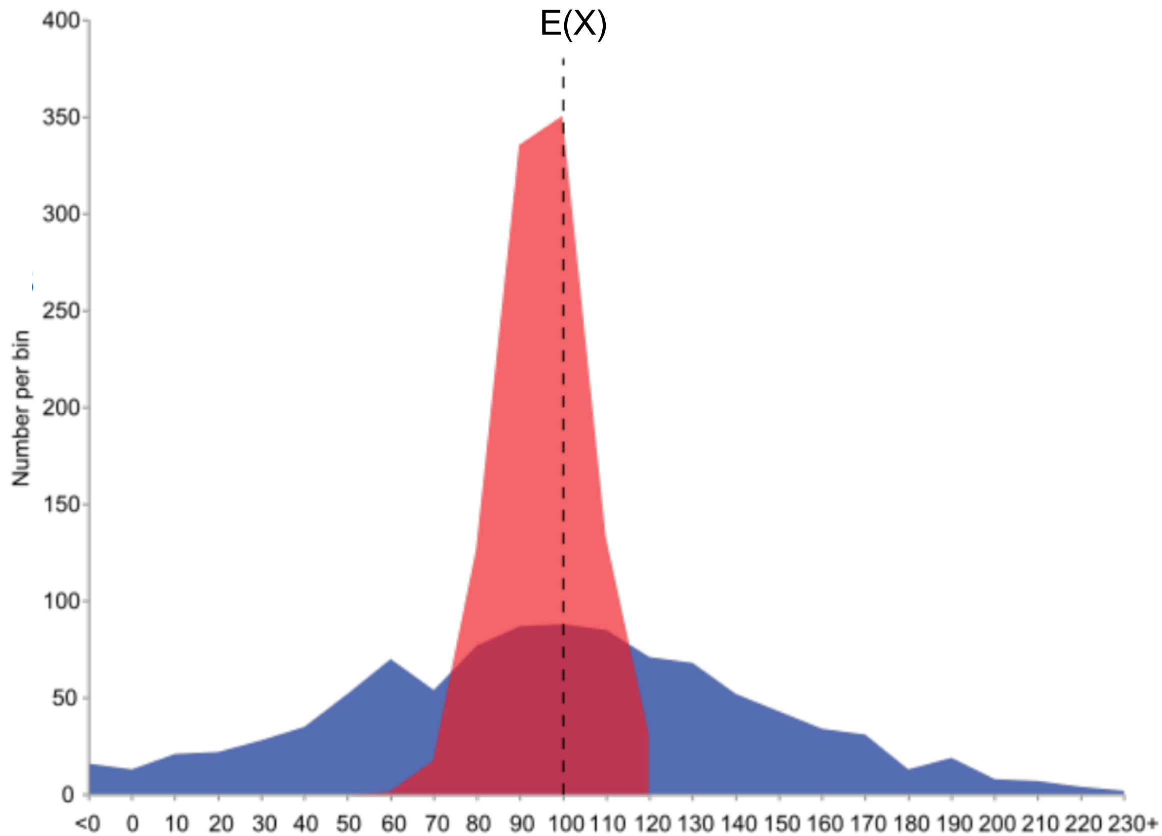
Game Two: For \$1 per round, you can flip a coin 20 times, and if you get 20 heads, I'll give you \$3,145,728, else nothing. Call this the random variable X_2 .

$$E(X_2) = -1 \cdot \frac{2^{20} - 1}{2^{20}} + \$3,145,727 \cdot \frac{1}{2^{20}} = \$2.00$$

Oh, and by the way, I only have time for a single round....

Discrete RandomVariables: Variance

The question is: How much does X vary from E(X)? How spread out is the probability distribution around the expected value?



Discrete Random Variables: Variance

The **Variance** of a random variable, Var(X), is the expected deviation from E(X).

But what precisely is "deviation"?

First (doomed) attempt: **deviation = distance from expected value**

deviation = X – E(X) Var(X) = E[X – E(X)]

Example: X_1 = "Flip a coin and return the number of heads showing"
X_2 = "Flip a coin and return 100 * the number of heads showing"

$R_{X_1} = \{0, 1\}$ $f_{X_1} = \{\frac{1}{2}, \frac{1}{2}\}$ $E(X) = 0.5$

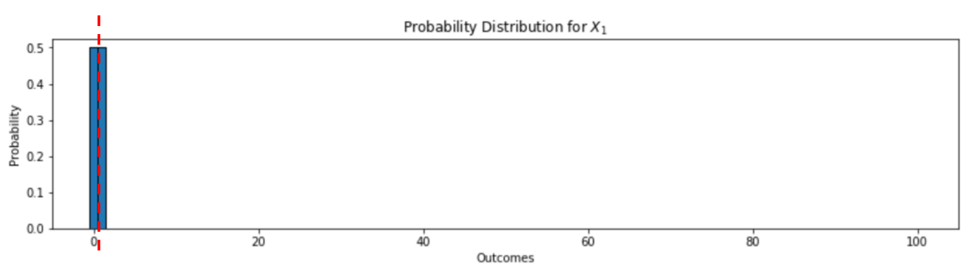
$R_{X_2} = \{0, 100\}$ $f_{X_2} = \{\frac{1}{2}, \frac{1}{2}\}$ $E(X) = 50$

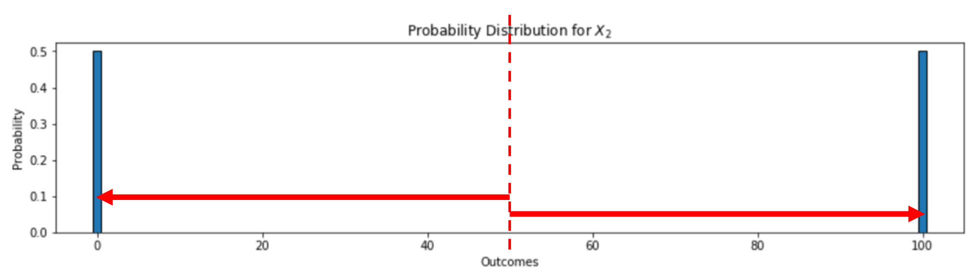
$R_{X_1-0.5} = \{-0.5, 0.5\}$ $f_{X_1-0.5} = \{\frac{1}{2}, \frac{1}{2}\}$ $E(X_1 - 0.5) = E(X_1) - 0.5 = 0.0$

$R_{X_2-50} = \{-50, 50\}$ $f_{X_2-50} = \{\frac{1}{2}, \frac{1}{2}\}$ $E(X_2 - 50) = E(X_2) - 50 = 0.0$

Note that E(X) is a constant:

$E(X - E(X)) = E(X) - E(X) = 0.0$





Discrete RandomVariables: Variance

Example: X_1 = "Flip a coin and return the number of heads showing"
X_2 = "Flip a coin and return 100 * the number of heads showing"

Second attempt: deviation = absolute value of distance from E(X)

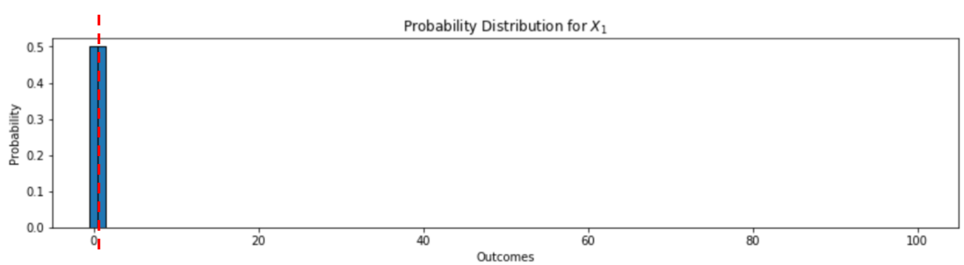
deviation = | X – E(X) | Var(X) = E[| X – E(X) |]

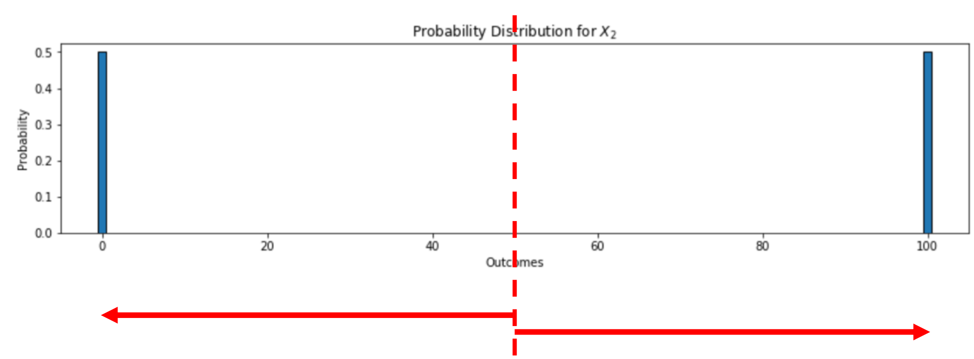
$$R_{X_1} = \{\, 0,\ 1 \,\} \qquad f_{X_1} = \{\, \tfrac{1}{2}, \tfrac{1}{2} \,\} \qquad E(X) = 0.5$$

$$R_{X_2} = \{\, 0,\ 100 \,\} \qquad f_{X_2} = \{\, \tfrac{1}{2}, \tfrac{1}{2} \,\} \qquad E(X) = 50$$

$$R_{|X_1-0.5|} = \{\, 0.5 \,\} \qquad f_{|X_1-0.5|} = \{\, 1.0 \,\} \qquad E(\,|\,X_1 - 0.5\,|\,) = 0.5$$

$$R_{|X_2-50|} = \{\, 50 \,\} \qquad f_{|X_2-50|} = \{\, 1.0 \,\} \qquad E(\,|\,X_2 - 50\,|\,) = 50$$





Discrete RandomVariables: Variance

Example: X_1 = "Flip a coin and return the number of heads showing"
X_2 = "Flip a coin and return 100 * the number of heads showing"

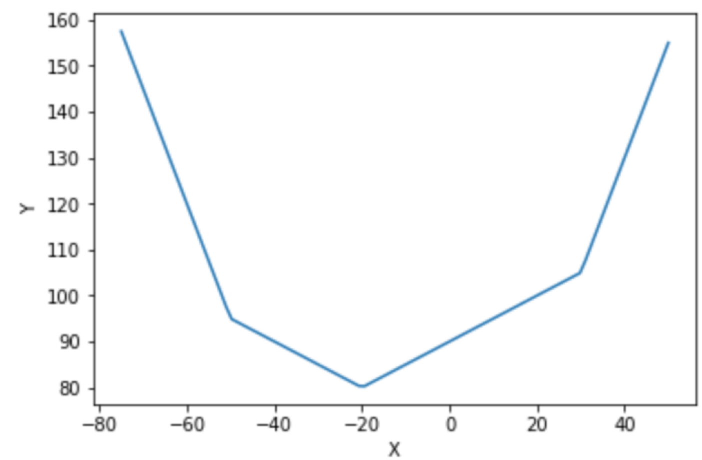
Second attempt: deviation = absolute value of distance from E(X)

deviation = $| X - E(X) |$ $\qquad$ $Var(X) = E[\, | X - E(X) | \,]$

Looks promising! What's wrong with that?

Ugh! Requires case analysis and blows up with an exponential number of cases, and resulting in functions that are not continuous; such "piece-wise" functions are very hard to work with!

$$f(x) = |x - 30| + |x + 50| + |x/2 + 10|$$



Discrete Random Variables: Variance

Ok, finally, here is the best definition:

$$Var(X) =_{def} E[(X - \mu_X)^2]$$

Alternate notation for expected value:

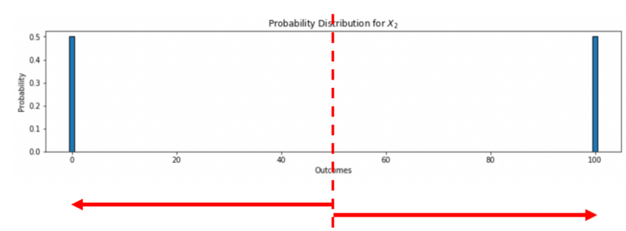
$$\mu_X = E(X)$$

or just μ if **X** is obvious.

This is the standard definition and has several advantages:

- It it much easier to work with mathematically;
- Like the absolute value, it gives only positive values.

But it gives results which are not very intuitive!



$R_{X_1} = \{0, 1\}$ $\quad f_{X_1} = \{\frac{1}{2}, \frac{1}{2}\}$ $\quad E(X) = 0.5$

$R_{X_2} = \{0, 100\}$ $\quad f_{X_2} = \{\frac{1}{2}, \frac{1}{2}\}$ $\quad E(X) = 50$

$R_{(X_1-0.5)^2} = \{0.25\}$ $\quad f_{(X_1-0.5)^2} = \{1.0\}$ $\quad E[(X_1 - 0.5)^2] = \underline{0.25}$

$R_{(X_2-50)^2} = \{2500\}$ $\quad f_{(X_2-50)^2} = \{1.0\}$ $\quad E[(X_2 - 50)^2] = \underline{2500}$

And what about the units?
If these are dollars, then this is 2500 squared dollars...

Discrete Random Variables: Standard Deviation

Therefore a more common measure of spread around the mean is the Standard Deviation:

$$\sigma_X =_{def} \sqrt{Var(X)}$$

$$R_{X_1} = \{\, 0,\ 1 \,\} \qquad f_{X_1} = \{\, \tfrac{1}{2}, \tfrac{1}{2} \,\} \qquad E(X) = 0.5$$

$$R_{X_2} = \{\, 0,\ 100 \,\} \quad f_{X_2} = \{\, \tfrac{1}{2}, \tfrac{1}{2} \,\} \qquad E(X) = 50$$

$$R_{(X_1-0.5)^2} = \{\, 0.25 \,\} \qquad f_{(X_1-0.5)^2} = \{\, 1.0 \,\} \qquad Var(X_1) = 0.25 \qquad \sigma_{X_1} = 0.5$$

$$R_{(X_2-50)^2} = \{\, 2500 \,\} \qquad f_{(X_2-50)^2} = \{\, 1.0 \,\} \qquad Var(X_2) = 2500 \qquad \sigma_{X_2} = 50$$

This has all the advantages of the variance, plus two more:

- The units are correct; and
- It corresponds to a well-known geometric notion, the Euclidean Distance....

Digression: Distance Metrics

Distance Metrics measure how much two vectors differ from one another.

In two dimensions, our last two definitions of variance correspond to

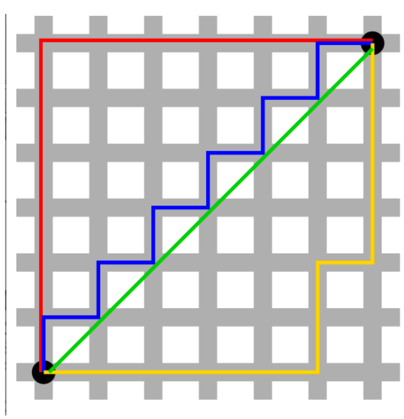
1-norm = SAD = Sum of Absolute Differences (or "Manhatten Distance")

2-norm distance = ED = Euclidean Distance = Square root of sum of squared distances

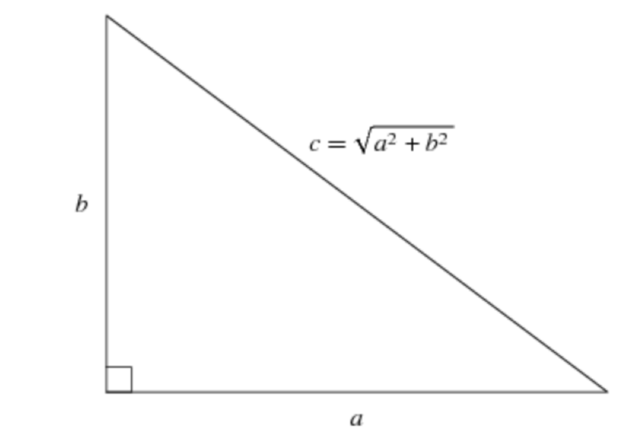
1-norm distance $$= \sum_{i=1}^{n} |x_i - y_i|$$

2-norm distance $$= \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

Pythagorean Theorem



Standard Deviation is consistent with the 2-norm distance metric.



Discrete RandomVariables: Standard Deviation

Useful formulae for the Variance and Standard Deviation:

$$Var(X) = E(X^2) - (\mu_X)^2$$

$$\begin{aligned}
Var(X) &= E[(X - \mu_X)^2] \\
&= E\left[X^2 - 2\mu X + \mu^2\right] \\
&= E(X^2) - E(2\mu X) + E(\mu^2) \\
&= E(X^2) - 2\mu * E(X) + \mu^2 \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

Discrete Random Variables: Standard Deviation

Useful formula for the Variance and Standard Deviation, based on the fact that variance and the standard deviation are NOT linear functions:

Theorem: $Var(aX + b) = a^2 * Var(X)$

Proof:

$$
\begin{aligned}
Var(aX+b) &= E\left[\,((aX+b) - \mu_{aX+b}\,)^2\,\right] \\
&= E\left[\,\Big(\,(aX+b) - (a\mu_X + b)\,\Big)^2\,\right] \\
&= E\left[\,\Big(\,a(X - \mu_X)\,\Big)^2\,\right] \\
&= E\left[\,a^2 * (X - \mu_X\,)^2\,\right] \\
&= a^2 * E\left[\,(X - \mu_X\,)^2\,\right] \\
&= a^2 * Var(X)
\end{aligned}
$$

Corollary:

$$\sigma_{aX+b} = |a| * \sigma_X$$

Discrete RandomVariables: Standard Deviation

Let's apply this idea to our game:

Game One: For \$1 per round, you can flip a coin, and I'll give you \$6 (your bet back plus \$5) if heads appears, and nothing if tails appears (so you lose your bet). Call this the random variable X_1 = net payout.

$$E(X_1) = -1 \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} = \$2.00$$

Game Two: For \$1 per round, you can flip a coin 20 times, and if you get 20 heads, I'll give you \$3,145,728, else nothing. Call this the random variable X_2 .

$$E(X_2) = -1 \cdot \frac{2^{20}-1}{2^{20}} + \$3,145,727 \cdot \frac{1}{2^{20}} = \$2.00$$

Oh, and by the way, I only have time for a single round....

$$E(X_1^2) = \frac{(-1)^2}{2} + \frac{5^2}{2} = \frac{26}{2} = 13$$

$$Var(X_1) = E(X_1^2) - E(X_1)^2 = 13 - 2^2 = 9$$

$$\sigma_{X_1} = \sqrt{9} = 3$$

$$E(X_2^2) = (-1)^2 \cdot \frac{2^{20}-1}{2^{20}} + 3,145,727^2 \cdot \frac{1}{2^{20}} = \frac{3,145,727^2}{2^{20}} + \frac{2^{20}-1}{2^{20}} = 9,437,179$$

$$Var(X_2) = E(X_2^2) - E(X_2)^2 = 9,437,179 - 2^2 = 9,437,175$$

$$\sigma_{X_2} = \sqrt{9437175} = 3071.9985 = \$3,072.00$$

Discrete RandomVariables: Standardized RVs

In order to compare random variables and avoid "comparing apples and oranges," we use the notion of a Standardized Random Variable which has

- E(X) = 0.0
- Var(X) = σ_X = 1.0

In order to create a standard version of a random variable, we simply subtract the expected value (so now the expected value is 0.0) and divide by the standard deviation (so now it is 1.0):

$$X^* =_{def} \frac{X - \mu_X}{\sigma_X}$$

Cumulative Distribution Functions

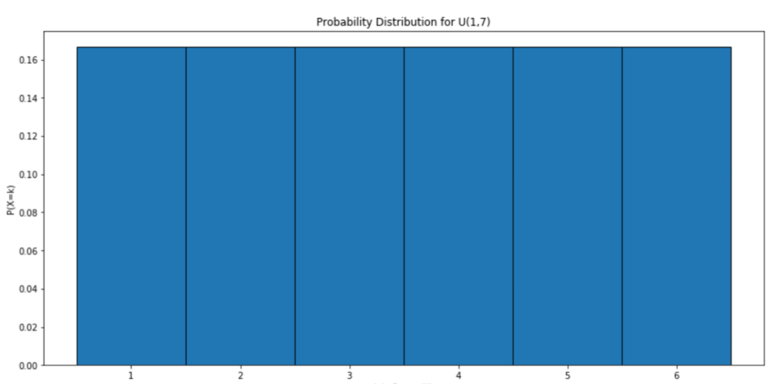
One more topic before considering the standard distributions....

The Cumulative Distribution Function (CDF) for a random variable X shows what happens when we keep track of the sum of the probability distribution from left to right over its range:

$$F_X(k) = P(X \leq k) = \sum_{a \leq k} f_X(a)$$

Example: X = "The number of dots showing on a thrown die"

Probability Distribution Function f_X



Cumulative Distribution Function F_X