

CS 237: Probability in Computing

Wayne Snyder
Computer Science Department
Boston University

Lecture 5:

- Review: Permutations, Unordering Principle
- Combinations
- Applications of combinatorics to probability
- Poker Probability (looking towards lab next week)

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

The Unordering Principle

If there are M ordered **collections** of N elements, then there are $M/N!$ **unordered collections** of the same M elements.

When all elements are distinct, as in our previous example, then obviously, $M/N! = N!/N! = 1$.

The basic idea here is that we are correcting for the **overcounting** when we assumed that the ordering mattered. Therefore we divide by the number of permutations.

This principle also applies to only a part of the collection:

Example: Suppose we have 4 girls and 5 boys, and we want to arrange them in 9 chairs, but we do not care what order the girls are in. How many such arrangements are there?

Answer: There $9!$ permutations, but if we do not care about the order of the (sub)collection of 4 girls, then there are $9!/4! = 15,120$ such sequences.

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

Permutations with Repetitions

If you have N (non-distinct) elements, consisting of m (distinct) elements with multiplicities K_1, K_2, \dots, K_m , that is, $K_1 + K_2 + \dots + K_m = N$, then the number of distinct permutations of the N elements is

$$\frac{N!}{K_1! * K_2! * \dots * K_m!}$$

Example: How many distinct (different looking) permutations of the word “SCIENCE” are there?

Solution: There are 7 letters, with multiplicities:

S, I, N: 1

C: 2

E: 2

Therefore the answer is $\frac{7!}{2! * 2!} = \frac{5040}{4} = 1260$

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

Combinations

When we are selecting without replacement and creating a set, we have a combination.

Canonical Problem: Suppose you have N people and want to choose a committee of K people. How many possible choices are there?

How is this different from a K -Permutation? Suppose you have N people; how many ways of choosing a sequence of K people from these N ?

$$P(N, K) = \frac{N!}{(N - K)!}$$

The difference is between a sequence (K -permutation) and a set (combination), so we have to apply the **unordering principle** and divide by the number of permutations of K people, or $K!$.

Formula:

$$\binom{N}{K} = \frac{P(N, K)}{K!} = \frac{N!}{(N - K)! K!}$$

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

Combinations

$$\binom{N}{K} = \frac{P(N, K)}{K!} = \frac{N!}{(N - K)! K!}$$

Example:

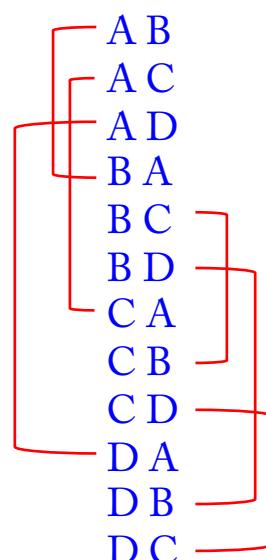
Suppose we have 4 students, { A, B, C, D }, and we want to choose a team of 2 for a hackathon from among these 4. How many ways of doing this are there?

How many sequences of 2 from 4?

$$P(4, 2) = \frac{4!}{(4 - 2)!} = \frac{24}{2} = 12$$

How many sets of 2 from 4?

$$\binom{N}{K} = \frac{N!}{(N - K)! K!} = \frac{4!}{(4 - 2)! 2!} = \frac{24}{2 * 2} = \frac{24}{4} = 6$$



{ A, B }

{ A, C }

{ A, D }

{ B, C }

{ B, D }

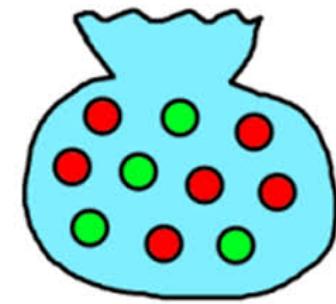
{ C, D }

Unordering sequences
of 2 students collapses
12 sequences to $12/2! =$
6 sets.

Finite Combinatorics and Probabilities

Now let's consider a series of related problems involving permutations and combinations....

Suppose we have a bag containing 4 green balls and 6 red balls.



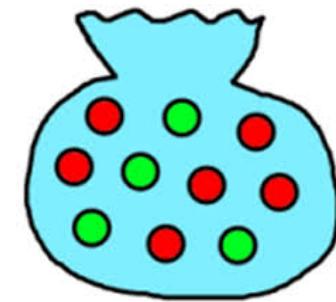
(A) Suppose 3 balls are drawn **with replacement**. What is the probability that you get the **sequence** [R, G, R]?

Recall:
with replacement =>
choices independent

Finite Combinatorics and Probabilities

Now let's consider a series of related problems involving permutations and combinations....

Suppose we have a bag containing 4 green balls and 6 red balls.



(A) Suppose 3 balls are drawn **with replacement**. What is the probability that you get the **sequence** [R, G, R]?

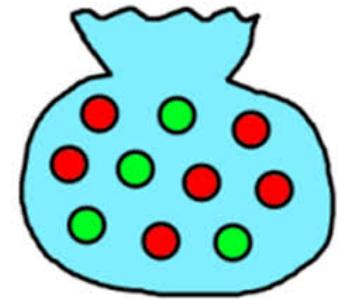
Solution: Since it with replacement, we have independent choices; we simply multiply the probabilities at each choice:

$$\frac{6}{10} * \frac{4}{10} * \frac{6}{10} = \frac{144}{1000} = 0.114$$

Note that ANY sequence of 2 R's and 1 G will have the same probability!

Finite Combinatorics and Probabilities

Suppose we have a bag containing 4 green balls and 6 red balls.

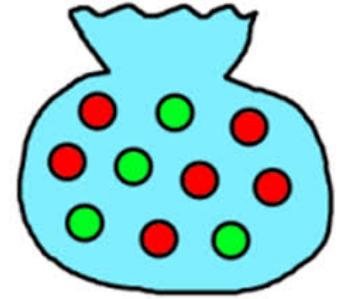


(B) Suppose 3 balls are drawn **without replacement**. What is the probability that you get the **sequence** [R, G, R]?

Recall:

without replacement =>
choices **not** independent

Finite Combinatorics and Probabilities

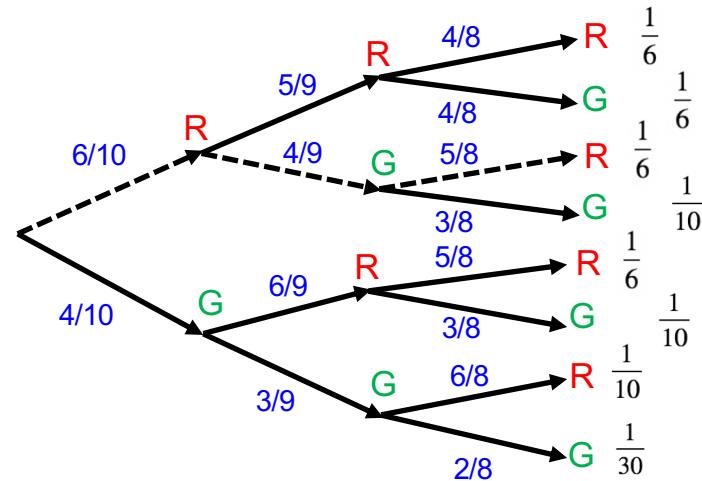


Suppose we have a bag containing 4 green balls and 6 red balls.

(B) Suppose 3 balls are drawn **without replacement**. What is the probability that you get the **sequence** [R, G, R]?

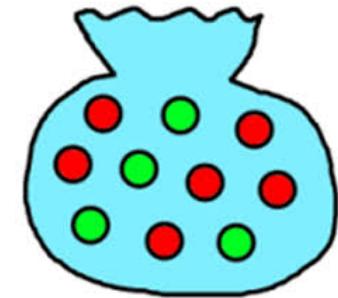
Solution: Since it without replacement, we calculate the (changing) probabilities at each stage:

$$\frac{6}{10} * \frac{4}{9} * \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$$



Again, note that ANY sequence of 2 R's and 1 G will have the same probability!

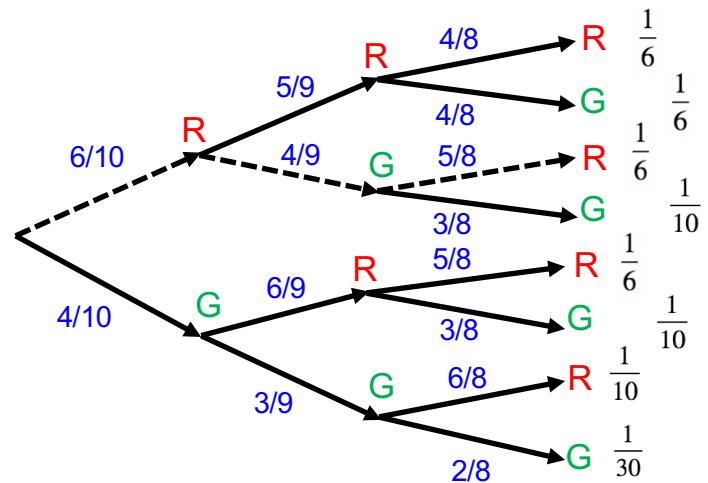
Finite Combinatorics and Probabilities



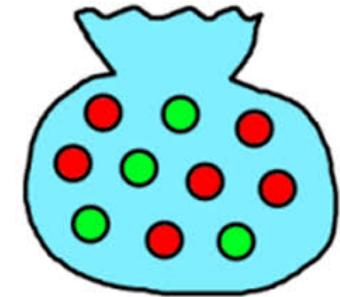
Suppose we have a bag containing 4 green balls and 6 red balls.

(C) Suppose 3 balls are drawn **without replacement**. What is the probability that you get a **set** { R, G, R }?

(Hint: First consider all the possible sequences....)



Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

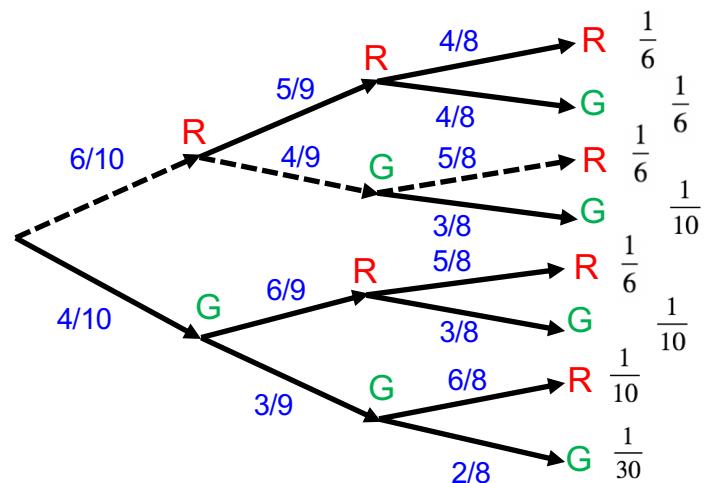
(C) Suppose 3 balls are drawn **without replacement**. What is the probability that you get a **set** { R, R, G }?

Solution 1: Adapt the solution to (B) (sequence =>unorder=> set):
there are three possible ways to get a sequence with 2 R's and 1 G:

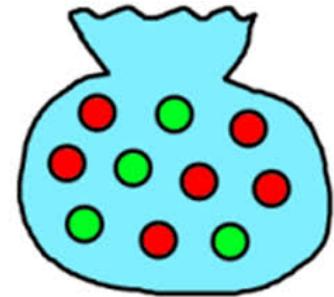
R R G
R G R
G R R

Probability of getting any of these three:

$$1/6 + 1/6 + 1/6 = 3/6 = 0.5$$



Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

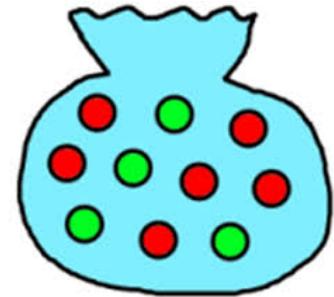
(C) Suppose 3 balls are drawn **without replacement**. What is the probability that you get a **set** { R, R, G }?

(Hint: Forget about sequences entirely, and consider how to do it using combinations, recalling that

$$P(A) = \frac{|A|}{|S|}$$

and think about how to choose the appropriate sets...)

Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

(C) Suppose 3 balls are drawn **without replacement**. What is the probability that you get a **set** { R, R, G }? Let event A = “you get the set { R, R, G }.”

Solution 2: How many possible ways of choosing a set of 3 from the 10 balls?

$$|S| = \binom{10}{3} = \frac{10!}{(10-3)!3!} = 120$$

$$P(A) = \frac{|A|}{|S|}$$

How many ways of choosing a set of 3 balls with 2 R and 1 G?

Note: These choices are independent, so we can multiply:

$$|A| = \binom{6}{2} \binom{4}{1} = 15 * 4 = 60$$

$$\text{So, } P(A) = \frac{\binom{6}{2} \binom{4}{1}}{\binom{10}{3}} = 0.5$$

Finite Combinatorics and Probabilities

Poker Probabilities

Example set of 52 poker playing cards

Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

A poker hand is 5 cards (a set) chosen without replacement.

How many possible poker hands?

$$|S| = \binom{52}{5} = 2,598,960$$

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

(Hint: Construct the hand by calculating how many such hands are possible, by constructing independent parts of the hand, and multiplying....)

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

Solution:

$$\frac{\binom{26}{3} \binom{26}{2}}{\binom{52}{5}} = 0.3251$$

By the way, Wolfram Alpha is the way to go when doing these problems...

 **WolframAlpha** computational intelligence

$C(26,3)*C(26,2) / C(52,5)$

Assuming "C" is a math function | Use as a [unit](#) instead

Input:
$$\binom{26}{3} \times \binom{26}{2} / \binom{52}{5}$$

Exact result:
$$\frac{1625}{4998}$$

Decimal approximation:
0.325130052020808323329331732693077230892356942777110844337...

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

Digression: Notice that you can also calculate this using **sequences** and **permutations**, but it is a bit more complicated, and you have more opportunities to get something wrong...

What is the probability of the exact sequence R R R B B ?

$$\frac{26}{52} * \frac{25}{51} * \frac{24}{50} * \frac{26}{49} * \frac{25}{48} = 0.0325$$

Now unorder it! How many permutations of this sequence of 5 symbols with duplicates?

$$\frac{5!}{3! 2!} = \frac{120}{6 * 2} = 10$$

$$0.03251 * 10 = 0.3251$$

$$\frac{\binom{26}{3} \binom{26}{2}}{\binom{52}{5}} = 0.3251$$

Poker Probability

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs	♣ A	♣ 2	♣ 3	♣ 4	♣ 5	♣ 6	♣ 7	♣ 8	♣ 9	♣ 10	♣ J	♣ Q	♣ K
Diamonds	♦ A	♦ 2	♦ 3	♦ 4	♦ 5	♦ 6	♦ 7	♦ 8	♦ 9	♦ 10	♦ J	♦ Q	♦ K
Hearts	♥ A	♥ 2	♥ 3	♥ 4	♥ 5	♥ 6	♥ 7	♥ 8	♥ 9	♥ 10	♥ J	♥ Q	♥ K
Spades	♠ A	♠ 2	♠ 3	♠ 4	♠ 5	♠ 6	♠ 7	♠ 8	♠ 9	♠ 10	♠ J	♠ Q	♠ K

Problem: What is the probability that a five-card hand has at least 3 Diamonds?

Solution: You need to separate this problem into cases, and might as well choose 3, 4, or 5 Diamonds, and for each find the probability and sum:

$$P(3 \text{ Diamonds}) = \frac{\binom{13}{3} \binom{39}{2}}{\binom{52}{5}} = 0.0815$$

$$P(4 \text{ Diamonds}) = \frac{\binom{13}{4} \binom{39}{1}}{\binom{52}{5}} = 0.0107$$

$$P(5 \text{ Diamonds}) = \frac{\binom{13}{5}}{\binom{52}{5}} = 0.0005$$

These sum to 0.0928.

Poker Probability

Problem: What is the probability of a Flush (all the same suit)?

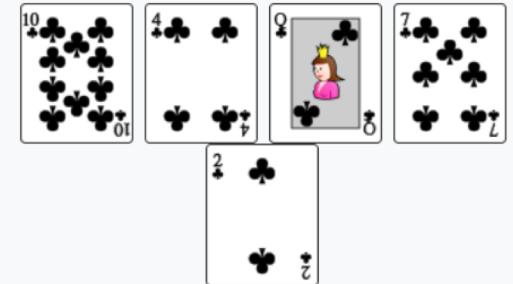
Solution: Choose a suit and then choose 5 cards from that suit:

$$\frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} = 0.00198079$$

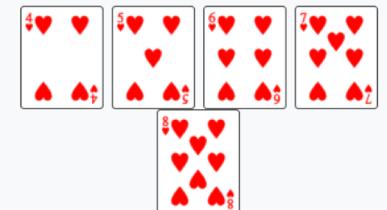
Note: This is the **cumulative** probability, in that it is a flush, but includes the straight and royal flushes. If we wish to exclude them, we must subtract all 40 of them:

$$\frac{\binom{4}{1} \binom{13}{5} - \binom{10}{1} \binom{4}{1}}{\binom{52}{5}} = 0.00196540$$

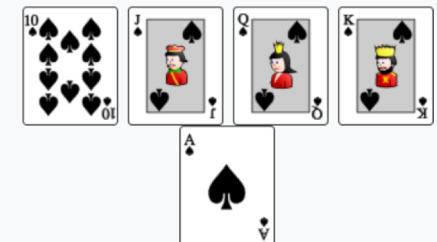
Flush (excluding royal flush and straight flush)



Straight flush (excluding royal flush)



Royal flush



Poker Probability

Problem: What is the probability of a Straight? Assume that Ace can be below 2 or above King.

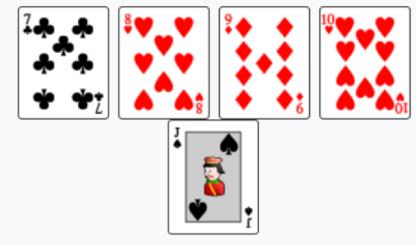
Solution: There are 10 sequences which form a straight, so just choose one of the 10 and then suits for each of the 5 cards:

A 2 3 4 5
2 3 4 5 6
3 4 5 6 7
4 5 6 7 8
5 6 7 8 9
6 7 8 9 10
7 8 9 10 J
8 9 10 J Q
9 10 J Q K
10 J Q K A

$$\frac{\binom{10}{1} * \binom{4}{1}^5}{\binom{52}{5}} = 0.00394$$

$$\frac{\binom{10}{1} * \binom{4}{1}^5 - \binom{10}{1} \binom{4}{1}}{\binom{52}{5}} = 0.003925$$

Straight (excluding royal flush
and straight flush)



Poker Probability

Example set of 52 poker playing cards													
Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs	♣ A	♣ 2	♣ 3	♣ 4	♣ 5	♣ 6	♣ 7	♣ 8	♣ 9	♣ 10	♣ J	♣ Q	♣ K
Diamonds	♦ A	♦ 2	♦ 3	♦ 4	♦ 5	♦ 6	♦ 7	♦ 8	♦ 9	♦ 10	♦ J	♦ Q	♦ K
Hearts	♥ A	♥ 2	♥ 3	♥ 4	♥ 5	♥ 6	♥ 7	♥ 8	♥ 9	♥ 10	♥ J	♥ Q	♥ K
Spades	♠ A	♠ 2	♠ 3	♠ 4	♠ 5	♠ 6	♠ 7	♠ 8	♠ 9	♠ 10	♠ J	♠ Q	♠ K

Problem: What is the probability of a Pair, 3-of-a-Kind, and 4-of-a-Kind?

Solution: First choose the denomination of the 2, 3 or 4 of a kind, then the suits of those cards, then the remaining cards of different denominations, again choosing the denomination, then the suits::

Pair:

$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}} = 0.4226$$

3-of-a-Kind:

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2}{\binom{52}{5}} = 0.0211$$

4-of-a-Kind:

$$\frac{\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}} = 0.00024$$

