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CS 237 — Midterm Exam

Fall 2019

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write “Do Not Grade,” on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers.** No calculators allowed, and *you may leave complicated formulae uncomputed*, but please do multiply $1/2 * 1/2$ to get $1/4$ if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

Problem One

Suppose a standard set of 52 playing cards is randomly shuffled and you are dealt a hand of seven (7) cards, without replacement (as is usual in card games). For the following, just give the appropriate formula.

(A) What is the probability that you get 4 cards of one suit and 3 cards of a different suit, so there are two suits represented in all? [Example: 4 Spades and 3 Hearts]

Solution:

$$\frac{\binom{4}{1} \binom{13}{4} \binom{3}{1} \binom{13}{3}}{\binom{52}{7}}$$

(B) What is the probability that you get 3 of one denomination, 3 of a different denomination, and one more card of a denomination different from the other 6 cards (so there are three denominations represented in all, for example, 3 Kings, 3 Aces, and a 10)?

Solution:

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{3} \binom{44}{1}}{2 \cdot \binom{52}{7}} \quad \text{OR} \quad \frac{\binom{13}{2} \binom{4}{3} \binom{4}{3} \binom{44}{1}}{\binom{52}{7}}$$

A diagram of the deck of cards is given on the last page.

Problem Two

Consider the following events for a family with children:

A = There are children of both genders (girls and boys)

B = There is at most one boy

Show that A and B are independent if the family has exactly three children.

Show all work!

Solution: (This is Schaum's Problem 4.29.)

There are two ways to solve this, depending on whether you thought of a family as a sequence or a set.

Sequence:

Sample space = { BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG }

A = { BBG, BGB, BGG, GBB, GBG, GGB } B = { BGG, GBG, GGB, GGG }

A and B = { BGG, GBG, GGB }

Thus: $P(AB) = 3/8 = P(A) \cdot P(B) = 6/8 \cdot 4/8$

Set:

Sample space = { {B,B,B} }, {B,B,G} }, {B,G,G} }, {G,G,G} }

A = { {B,B,G} }, {B,G,G} } B = { {B,G,G} }, {G,G,G} }

A and B = { {B,G,G} }

Thus: $P(AB) = 1/4 = P(A) \cdot P(B) = 1/2 \cdot 1/2$

Problem Three

To play a game, the player tosses two fair coins.

The player wins \$2 if 2 heads occur and \$1 if 1 head occurs. On the other hand, the player loses \$3 if no heads occur.

(A) Give the range R_X and PDF f_X for the appropriate random variable X .

$$R_X = \{ -3, 1, 2 \}$$

$$f_X = \{ 1/4, 1/2, 1/4 \}$$

(B) Find the expected value $E(X)$ of the game.

$$E(X) = -3/4 + 1/2 + 2/4 = 1/4 = \$ 0.25$$

(C) Find the variance $\text{Var}(X)$ of the game.

$$\begin{aligned}\text{Var}(X) &= E(X^2) - E(X)^2 \\ &= [9/4 + 1/2 + 4/4] - (1/4)^2 \\ &= 15/4 - 1/16 = 60/16 - 1/16 = 59/16\end{aligned}$$

(D) Is the game fair? If not, how would you change it to make it fair?

No, it is not fair. The player should pay \$0.25 to play the game, or the values changed (e.g., the player loses \$4 if no heads occur).

Problem Four

You have a sack containing 3 red balls and 5 black balls. Consider each of these situations, which are independent (you start each case with a sack of the 8 red and black balls). In each case you are selecting 3 balls *without replacement*. Show all work.

(A) Suppose you select 2 balls from the sack and observe that they are the same color. What is the probability that the 3rd ball you select will be a different color from the first two?

S = same color D = third different

$$P(BB) = 5 \cdot 4 / (8 \cdot 7) = 20/56$$

$$P(RR) = 3 \cdot 2 / (8 \cdot 7) = 6/56$$

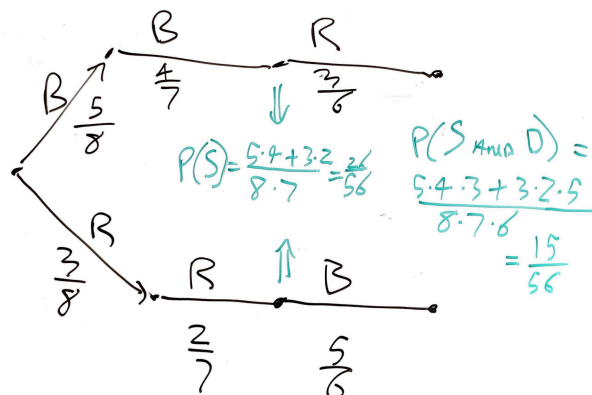
$$P(S) = 6/56 + 20/56 = 26/56$$

$$P(BBR) = 5 \cdot 4 \cdot 3 / (8 \cdot 7 \cdot 6) = 10/56$$

$$P(RRB) = 3 \cdot 2 \cdot 5 / (8 \cdot 7 \cdot 6) = 5/56$$

$$P(S \text{ and } D) = 5/56 + 10/56 = 15/56$$

$$P(D|S) = P(S \text{ and } D) / P(S) = (15/56) / (26/56) = 15/26$$



(B) Suppose you select 3 balls from the sack and observe that there are 2 red and 1 black. You did not notice which colors came out in which order (they are now just a set of 3 balls). What is the probability that the first ball you selected was in fact red?

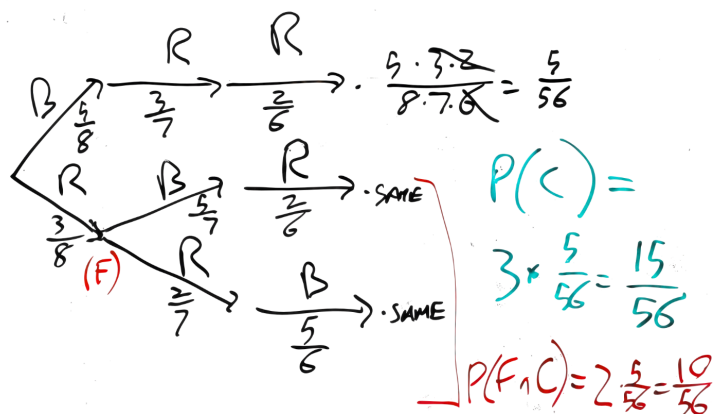
C = 2 red and 1 black

$$P(C) = P(BRR) + P(RBR) + P(RRB) \\ = 3 * (3 \cdot 2 \cdot 5 / (8 \cdot 7 \cdot 6)) = 15/56$$

F = first ball is red

$$P(F \text{ and } C) = P(RBR) + P(RRB) \\ = 2 * (3 \cdot 2 \cdot 5 / (8 \cdot 7 \cdot 6)) = 10/56$$

$$P(F | C) = P(F \text{ and } C) / P(C) = (10/56) / (15/56) = 10/15 = 2/3$$



Problem Five

In the following assume that we are dealing with Poisson processes and so potentially you could use the Poisson or the Exponential distribution. (You also could possibly use other distributions as needed.) For each of the following, give the distribution involved (including any parameters) and give the formula to solve the problem.

(A) Suppose that every 12 hours, on average, I get a spam call on my iPhone. What is the probability that I get exactly 2 spam calls tomorrow between 12 noon and 6pm?

Distribution: Assuming the time unit is 6 hours we are counting using the Poisson:

$$X \sim \text{Poi}(\lambda = 0.5)$$

since we can scale the rate parameter to fit the question.

Formula:
$$P(X = 2) = \frac{e^{-0.5} 0.5^2}{2!} = 0.0758$$

(only the formula is necessary)

(B) Suppose my iPhone is expected to last 5000 hours before the battery fails. What is the probability it lasts more than 4000 but less than 6000 hours?

Distribution: This is time measuring arrivals of failures, so we have an expected value of 5000 hours, which means the rate parameter is 1/5000 arrivals per hour:

$$X \sim \text{Exp}(\lambda = 1/5000)$$

Formula:

$$\begin{aligned} P(4000 < X < 6000) &= P(X > 4000) - P(X > 6000) \\ &= e^{\frac{-4000}{5000}} - e^{\frac{-6000}{5000}} = 0.1481 \end{aligned}$$

(C) I drop my iPhone, on average, twice a week. Suppose I do not drop it for 5 days in a row, what is the probability that I finish out the last two days of the week without dropping my iPhone?

Distribution: This is counting arrivals (dropped phone) at a rate of 2/week. Since arrivals are independent, the previous five days do not matter. Scaling to arrivals in a two day period, we have $X \sim \text{Poi}(4/7)$

Formula:

$$P(X = 0) = \frac{e^{-4/7} (4/7)^0}{0!} = e^{-4/7} = 0.5647$$

Problem Six

In this problem, various random experiments will be described and you must give the name and any parameters of the distribution involved and perhaps some other information. Be precise! You may simply give formulae for any calculations required.

Wayne and Trevor play a dart game in which they each throw a dart and try to hit a bulls-eye. Wayne's probability of hitting the bulls-eye is $1/4$ and Trevor's probability of hitting the bulls-eye is $1/3$. We may assume that these events are independent.

(A) Suppose Wayne and Trevor both shoot at the target. Let $X = "1 \text{ if they both hit the target and } 0 \text{ if they don't}."$

What is the distribution of X ?

$$X \sim \text{Bernoulli}(p = 1/12)$$

Give the name and any relevant parameters for each distribution.

What is $\text{Var}(X)$?

$$\text{Var}(X) = (11/12) \cdot (1/12) = 11/144$$

(B) Suppose Wayne and Trevor play a new game in which in each round they both shoot at the target, and they stop when they both hit the target on the same round. Let $Y = "the \text{ number of rounds until they stop}."$

What is the distribution of Y ?

$$Y \sim \text{Geometric}(p = 1/12)$$

What is $E(Y)$?

$$E(Y) = 1/(1/12) = 12$$

(C) Now suppose Wayne shoots at the target 10 times a day, for practice (but he never gets any better at it, and his probability of hitting the bulls-eye is always $1/4$). $Z = "the \text{ number of bulls-eyes he hits in a given week}."$

There are two possible reasonable interpretations of this, either as Poisson or Binomial (which illustrates why Poisson can approximate the Binomial):

What is the distribution of Z ?

$$Z \sim \text{Poi}(70/4)$$

What is the probability that he gets 5 bulls-eyes next weekend?

$$\frac{e^{-20/4} (20/4)^5}{5!} = 0.1755$$

What is the distribution of Z ?

$$Z \sim \text{Binomial}(N = 70, p = 1/4)$$

What is the probability that he gets 5 bulls-eyes next weekend?

$$P(Z' = 5) = \binom{20}{5} \cdot (1/4)^5 \cdot (3/4)^{15} = 0.2023$$

for $Z' \sim B(20, 1/4)$.