CS 237: Probability in Computing

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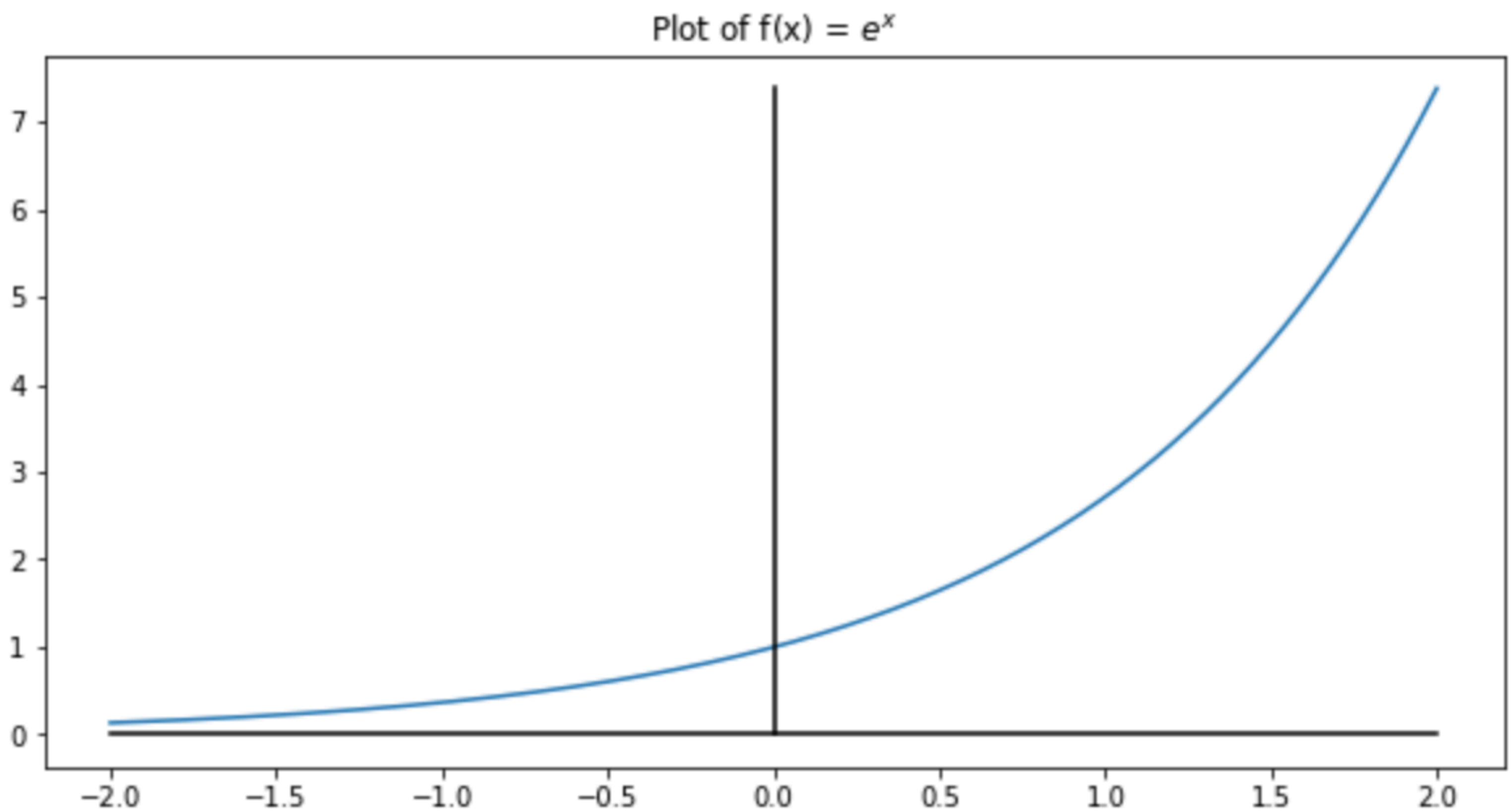
Lecture 13:

- Normal Distribution
- Exponential Distribution

Gaussian Exponential Function

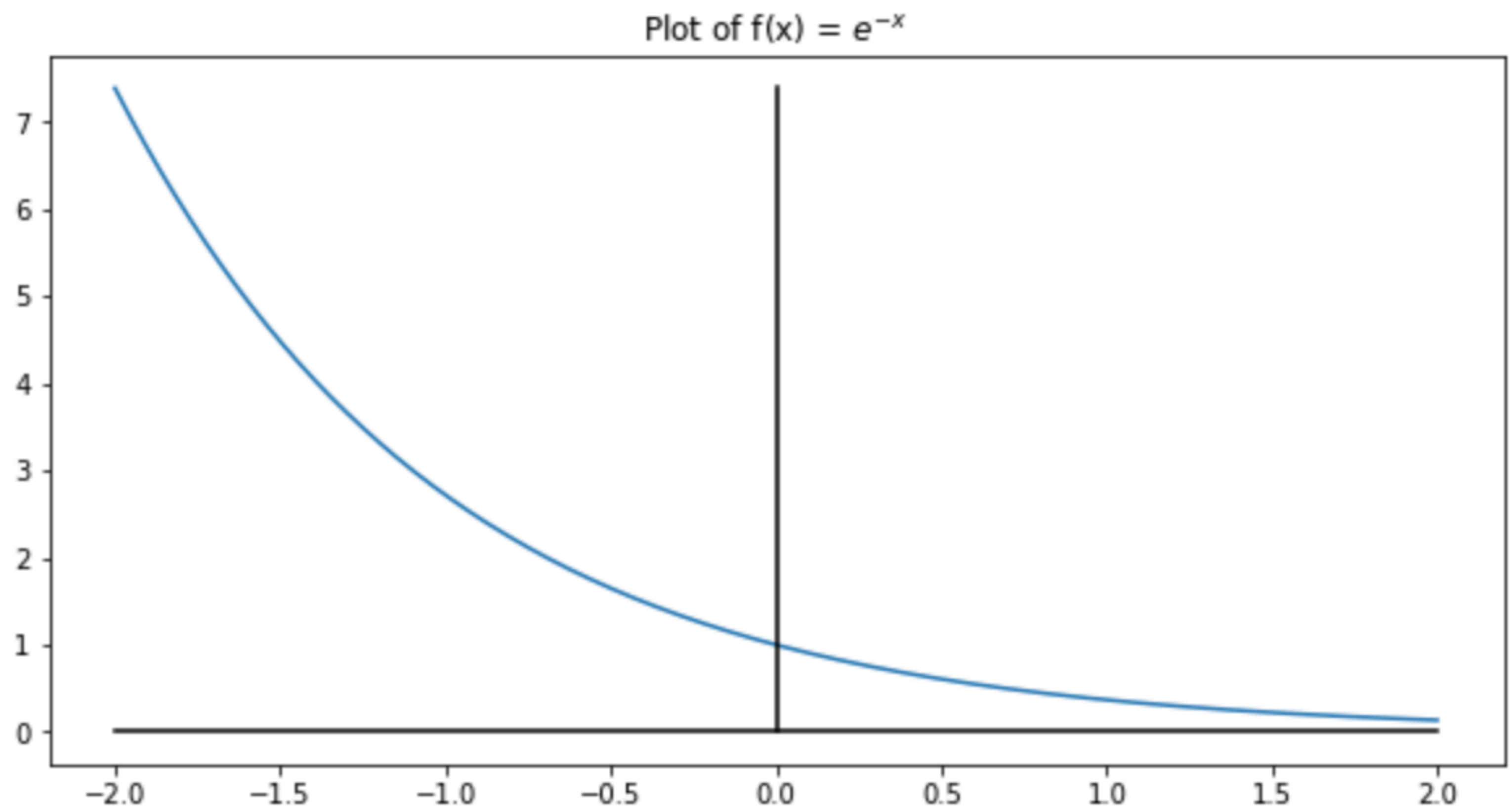
The normal distribution is one of a class of Gaussian exponential functions – what does that mean? Since both of the continuous distributions we study (Normal and Exponential) use exponentials, let's think about this a bit....

Here is a graph of the exponential function e^x, where $e = 2.71828...$ (Euler's Constant):



Gaussian Exponential Function

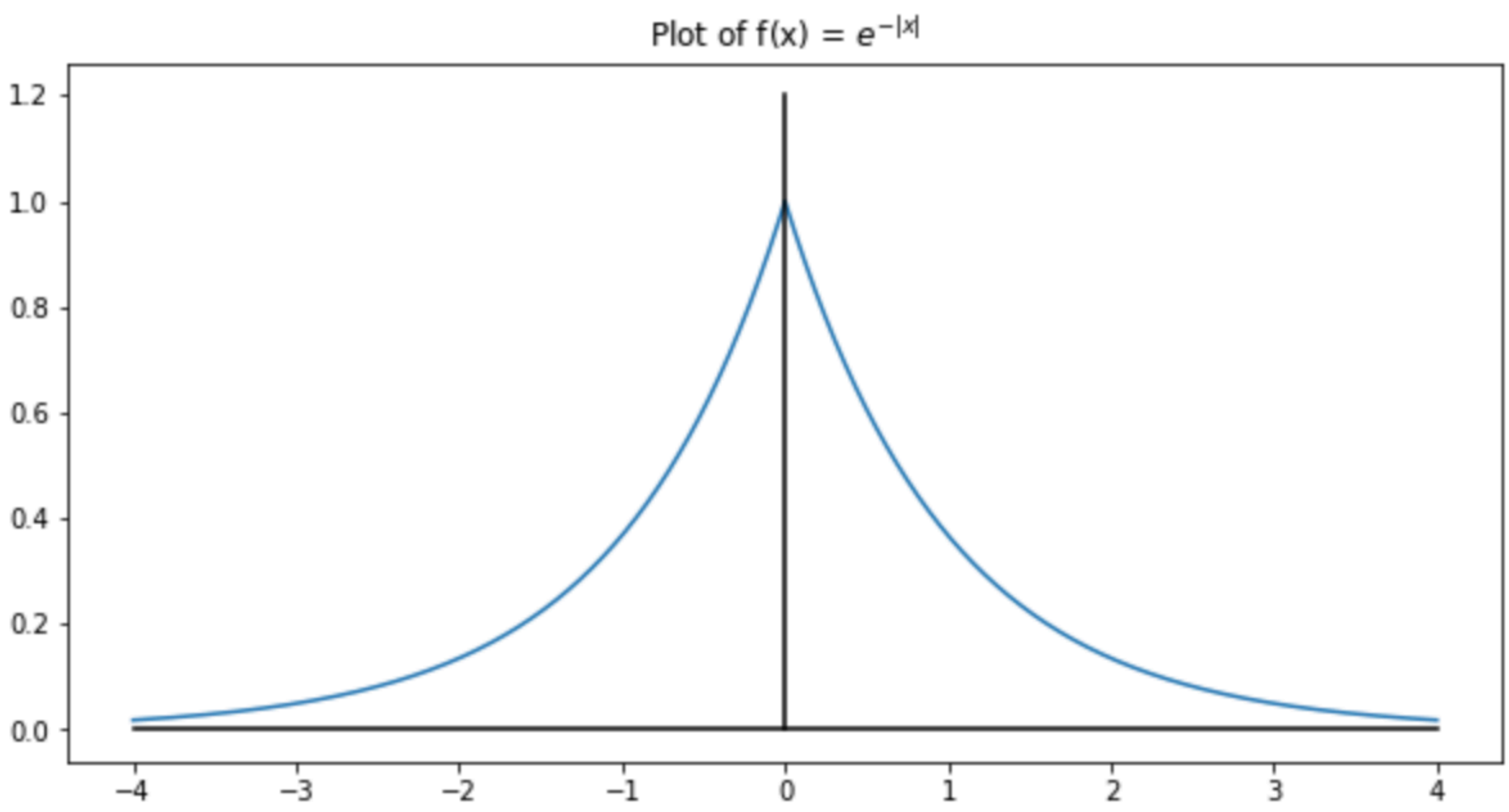
Here is a graph of e^{-x}, which flips the function around the Y axis:



Gaussian Exponential Function

But we want to make it look like the Binomial Distribution, so it has to be symmetric around the Y axis, so we'll reflect it around the y axis with the absolute value:

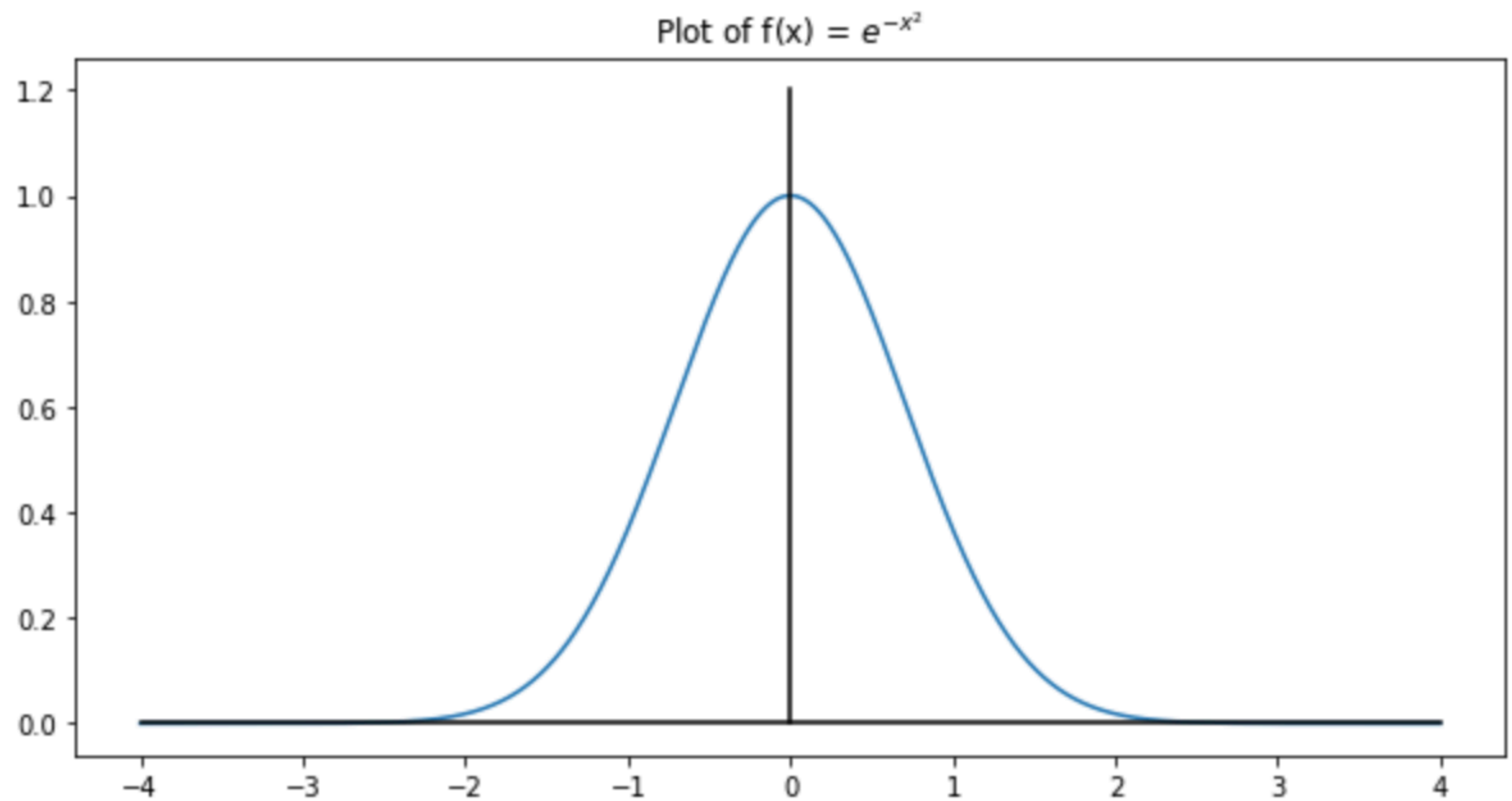
Graph of $e^{-|x|}$:



Gaussian Exponential Function

Whoops, forgot we hate absolute value, so let's use the square instead, which has the added advantage of eliminating the discontinuity at 0:

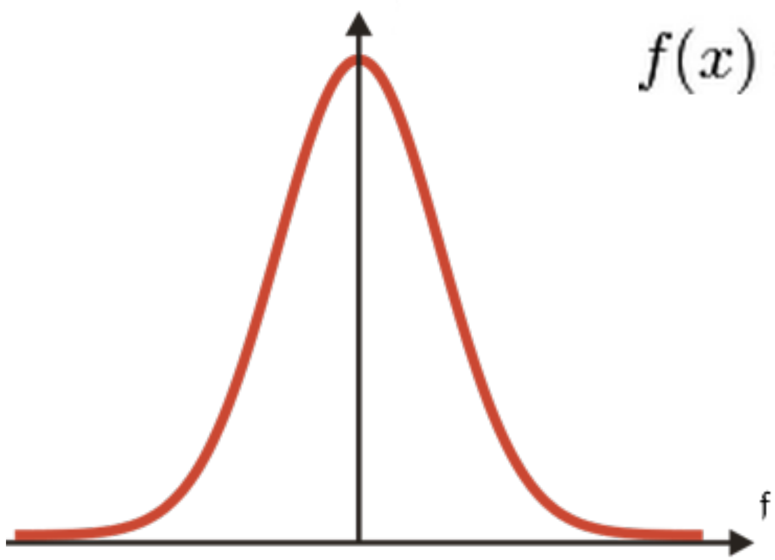
Graph of e^{-x^2}



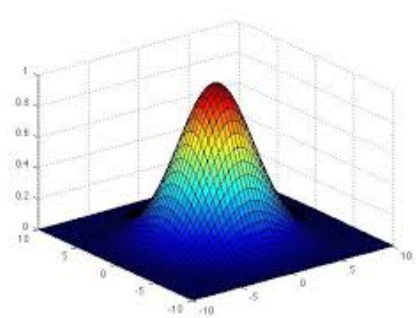
Gaussian Exponential Function

By adding parameters to adjust the height and the width, we have the Gaussian Exponential Function:

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}}$$



This function is widely used to describe phenomena (light, electric charge, gravity, quantum probabilities) that decrease in effect with distance, to create filters in signal processing (e.g., audio programming, graphics), etc., in addition to their wide use in statistics and probability…….



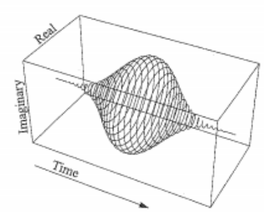


Figure 10.6
Gabor's elementary signal.

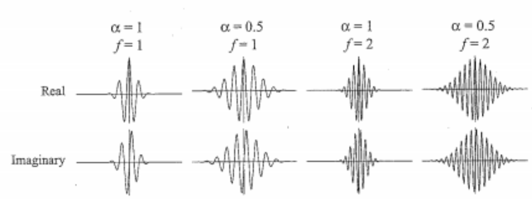
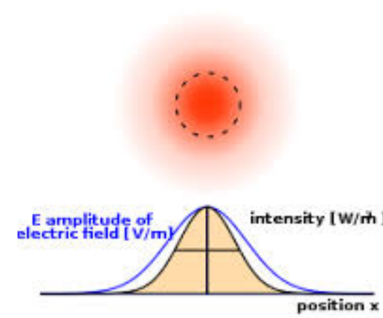


Figure 10.7
Examples of Gabor's Gaussian probability pulses.



Normal Distribution

By using parameters to fit the requirements of probability theory (e.g., that the area under the curve is 1.0), we have the formula for the Normal Distribution, which can be used to approximate the Binomial Distribution and which models a wide variety of random phenomena:

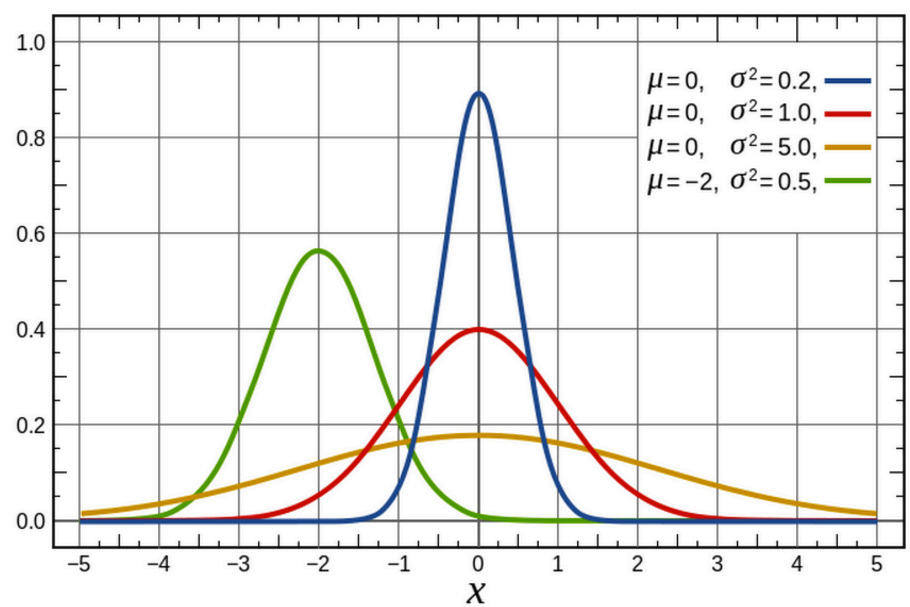
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where

μ = mean/expected value

σ = standard deviation

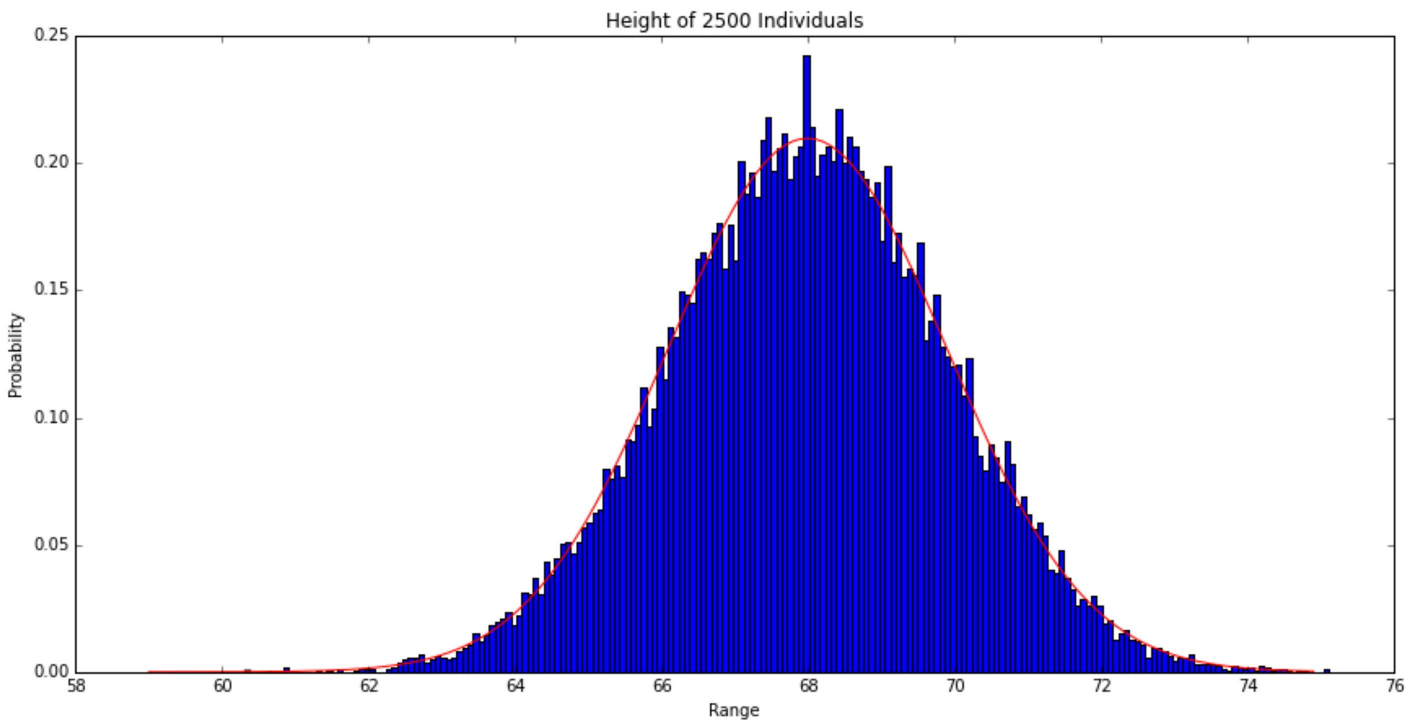
σ^2 = variance



Normal Distribution

The normal distribution, as the limit of B(N,0.5), occurs when a very large number of factors add together to create some random phenomenon.

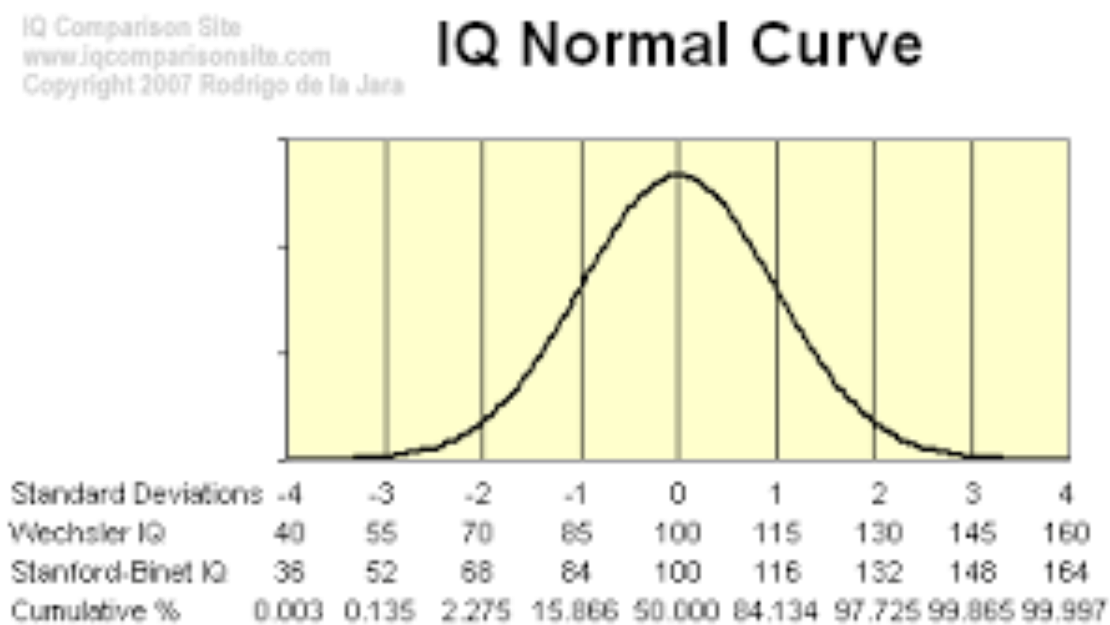
Example: What is the height of a human being?



Normal Distribution

The normal distribution, as the limit of B(N,0.5), occurs when a very large number of factors add together to create some random phenomenon.

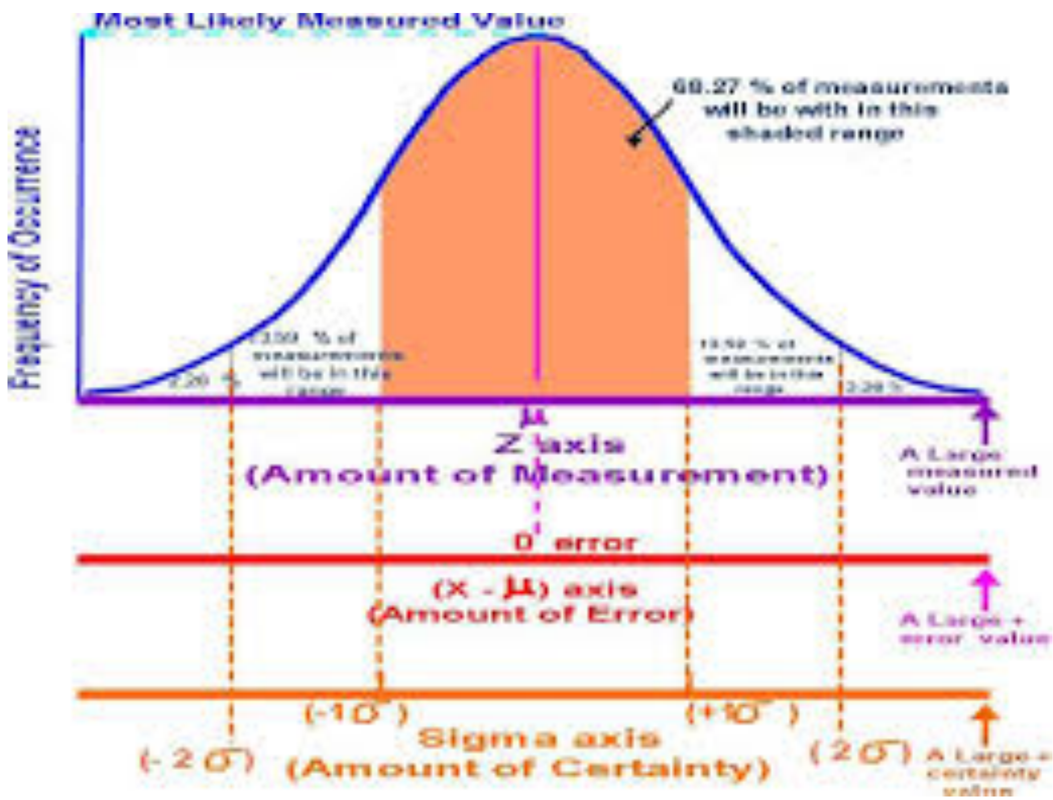
Example: What is the IQ of a human being?



Normal Distribution

The normal distribution, as the limit of B(N,0.5), occurs when a very large number of factors add together to create some random phenomenon.

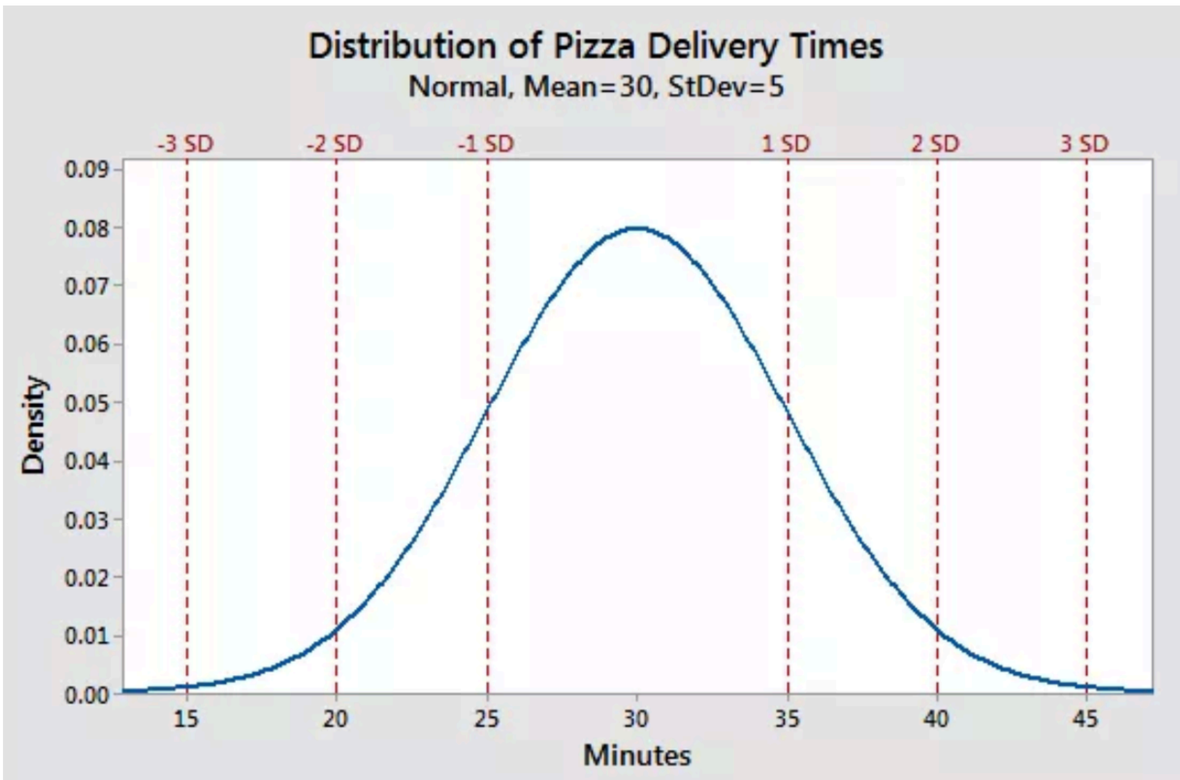
Example: What is the distribution of measurement errors?



Normal Distribution

The normal distribution, as the limit of B(N,0.5), occurs when a very large number of factors add together to create some random phenomenon.

Example: Even REALLY IMPORTANT things are normally distributed!



Normal Distribution

Recall that the only way we can analyze probabilities in the continuous case is with the CDF:

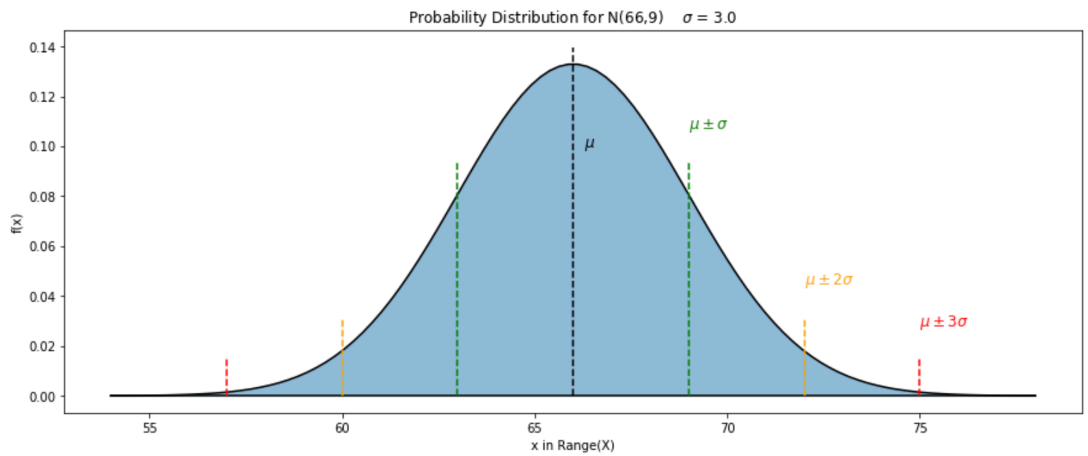
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

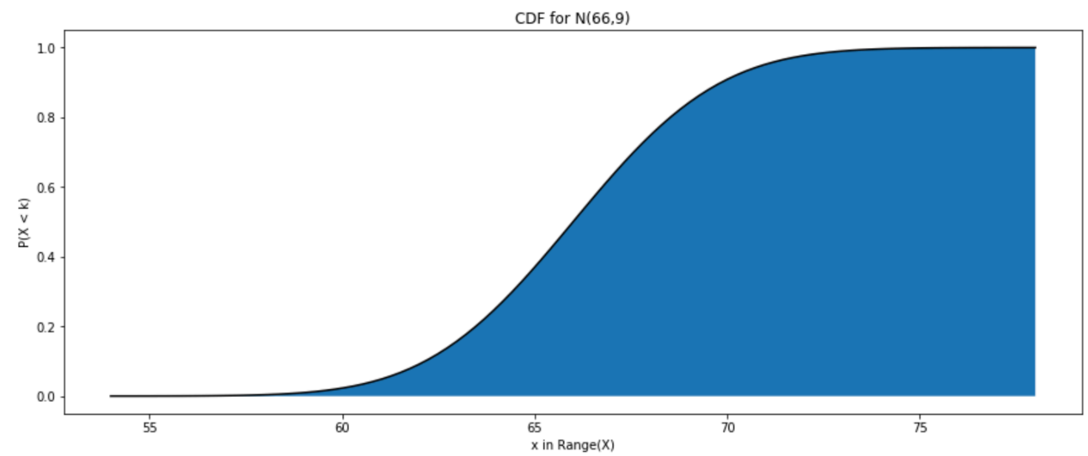
$$F(a) = \int_{-\infty}^{a} f(x)\, dx$$

P(X < a) = F(a)

P(X > a) = 1.0 – F(a)

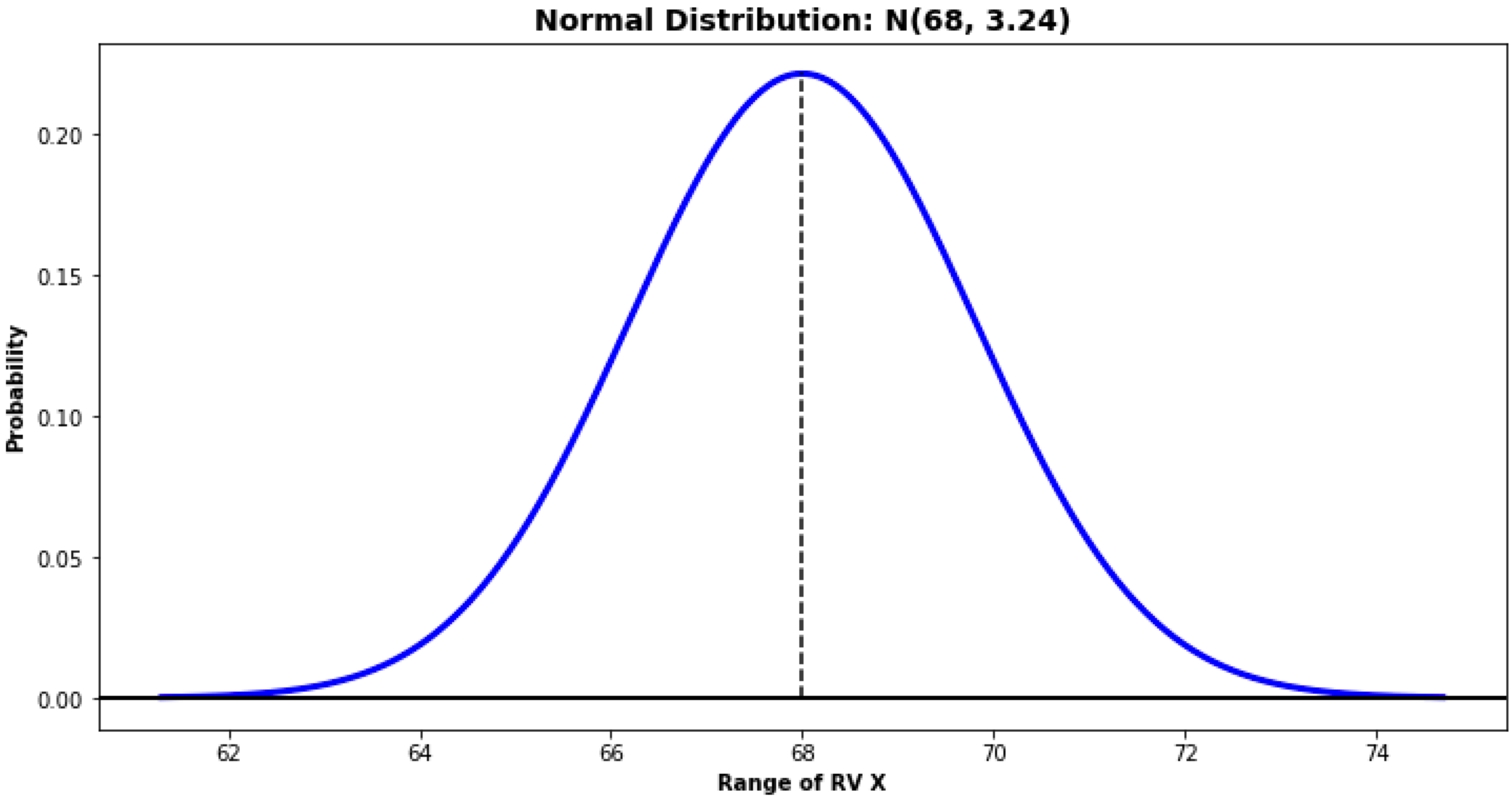
P(a < X < b) = F(b) – F(a)





Normal Distribution

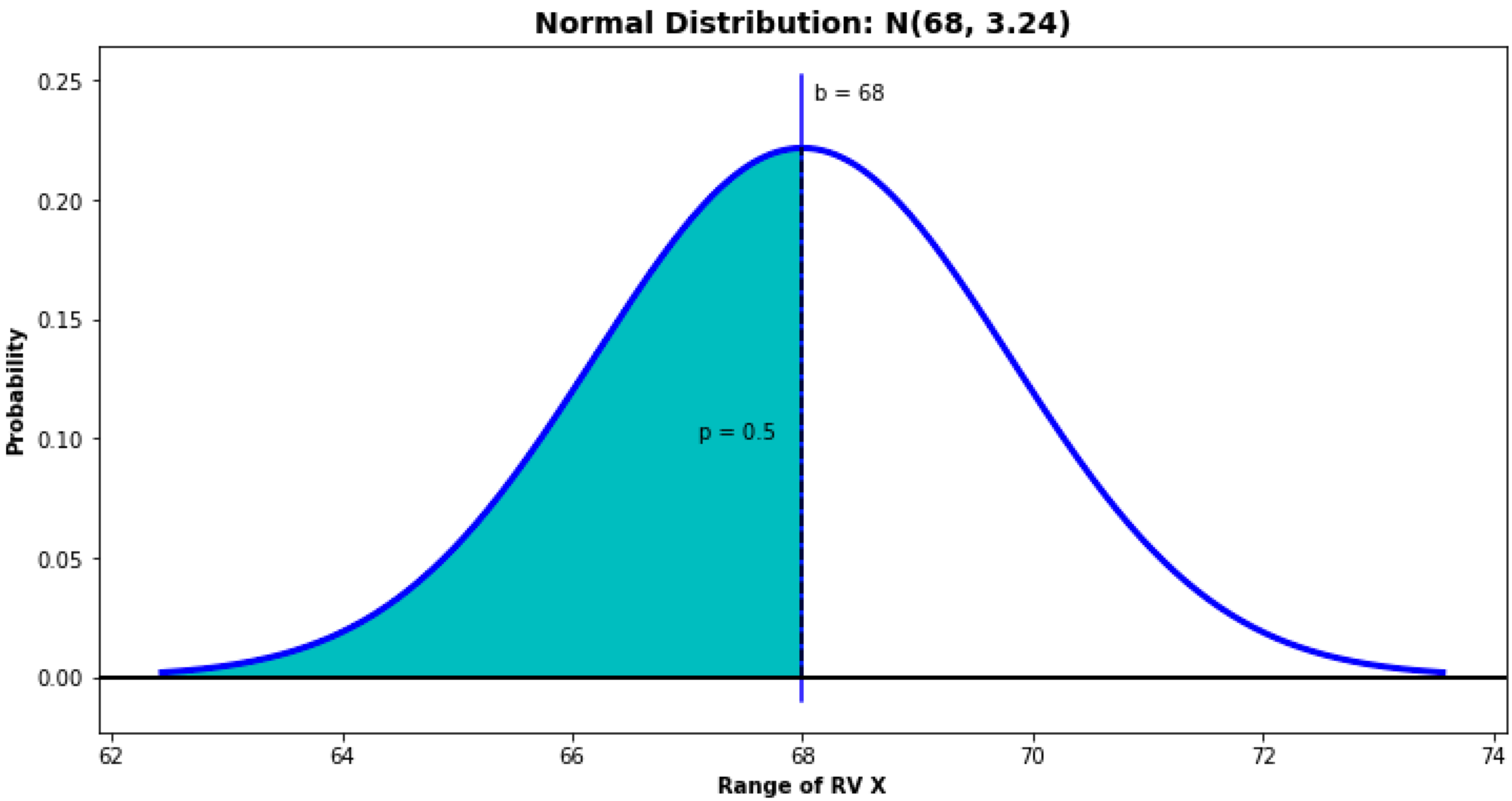
Suppose heights at BU are distributed normally with a mean of 68 inches and a standard deviation of 1.8 inches.




```
mean = 68
var = 3.24
stdev = 1.8
```

Normal Distribution

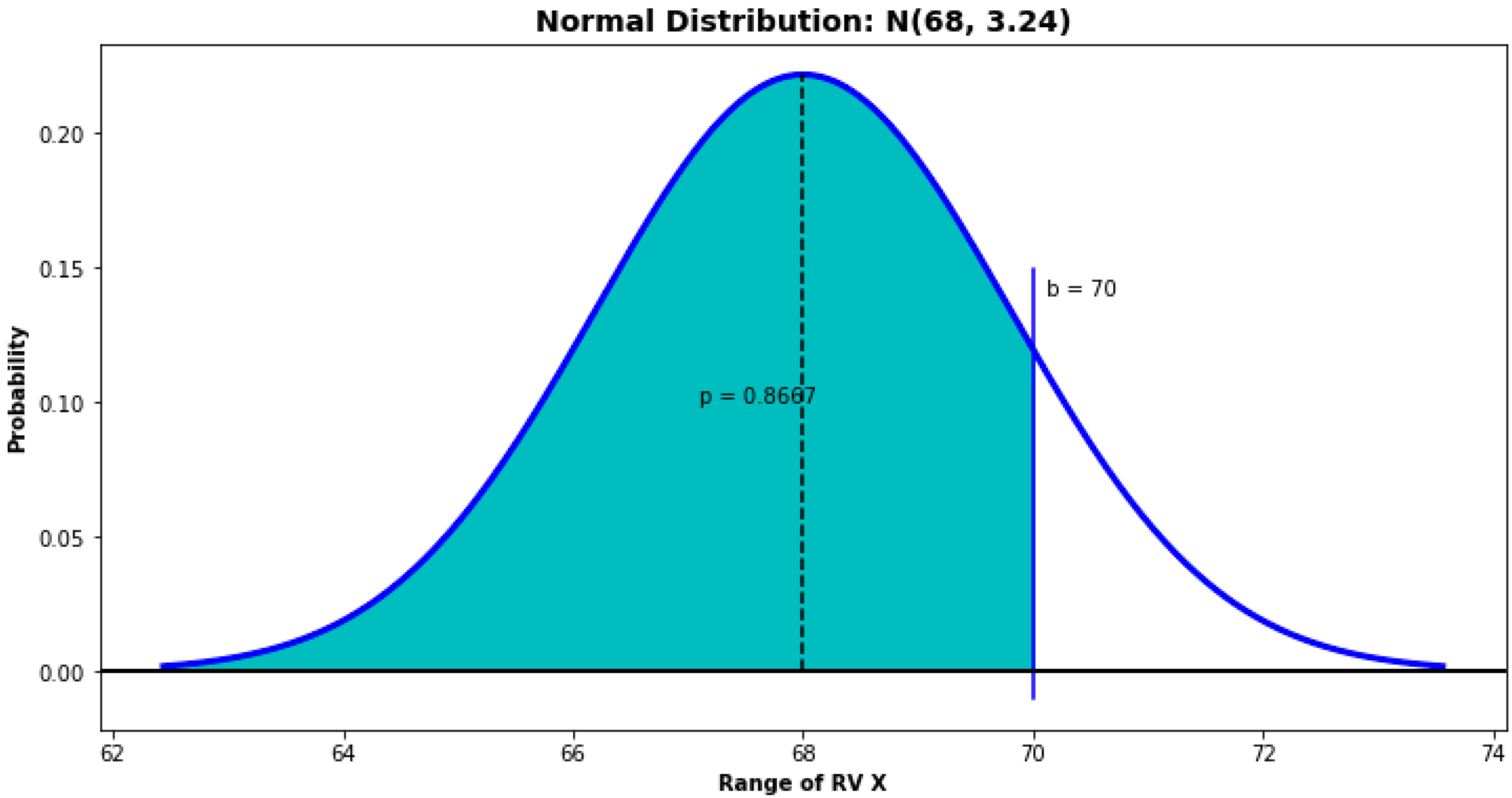
How many people are of less than average height?




```
mean = 68
var = 3.24
stdev = 1.8
P(X < 68) = 0.5
```

Normal Distribution

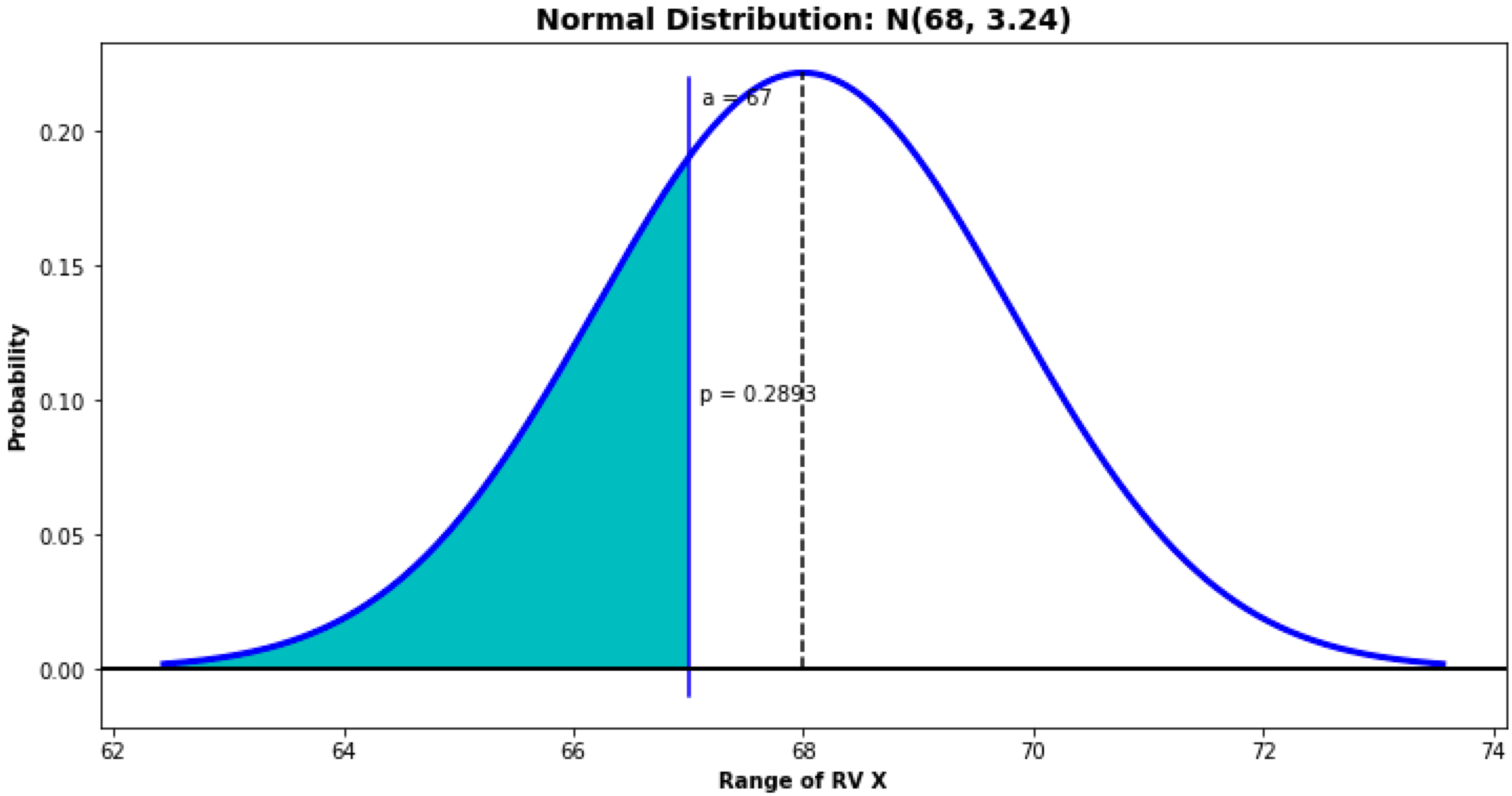
How many people are less than 70 inches?




```
mean = 68
var = 3.24
stdev = 1.8
P(X < 70) = 0.8667
```

Normal Distribution

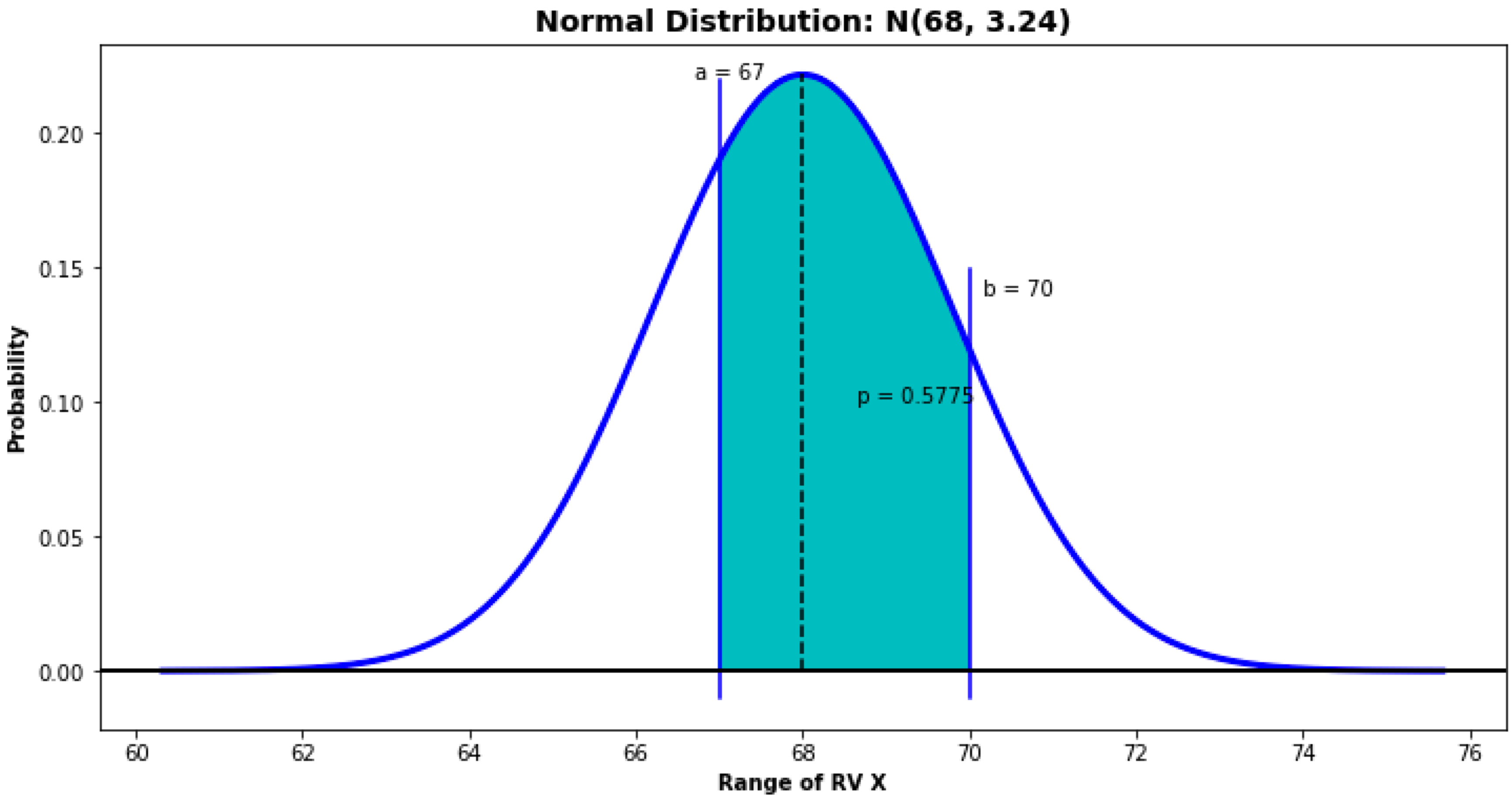
How many people are less than 67 inches?




```
mean = 68
var = 3.24
stdev = 1.8
P(X < 67) = 0.2893
```

Normal Distribution

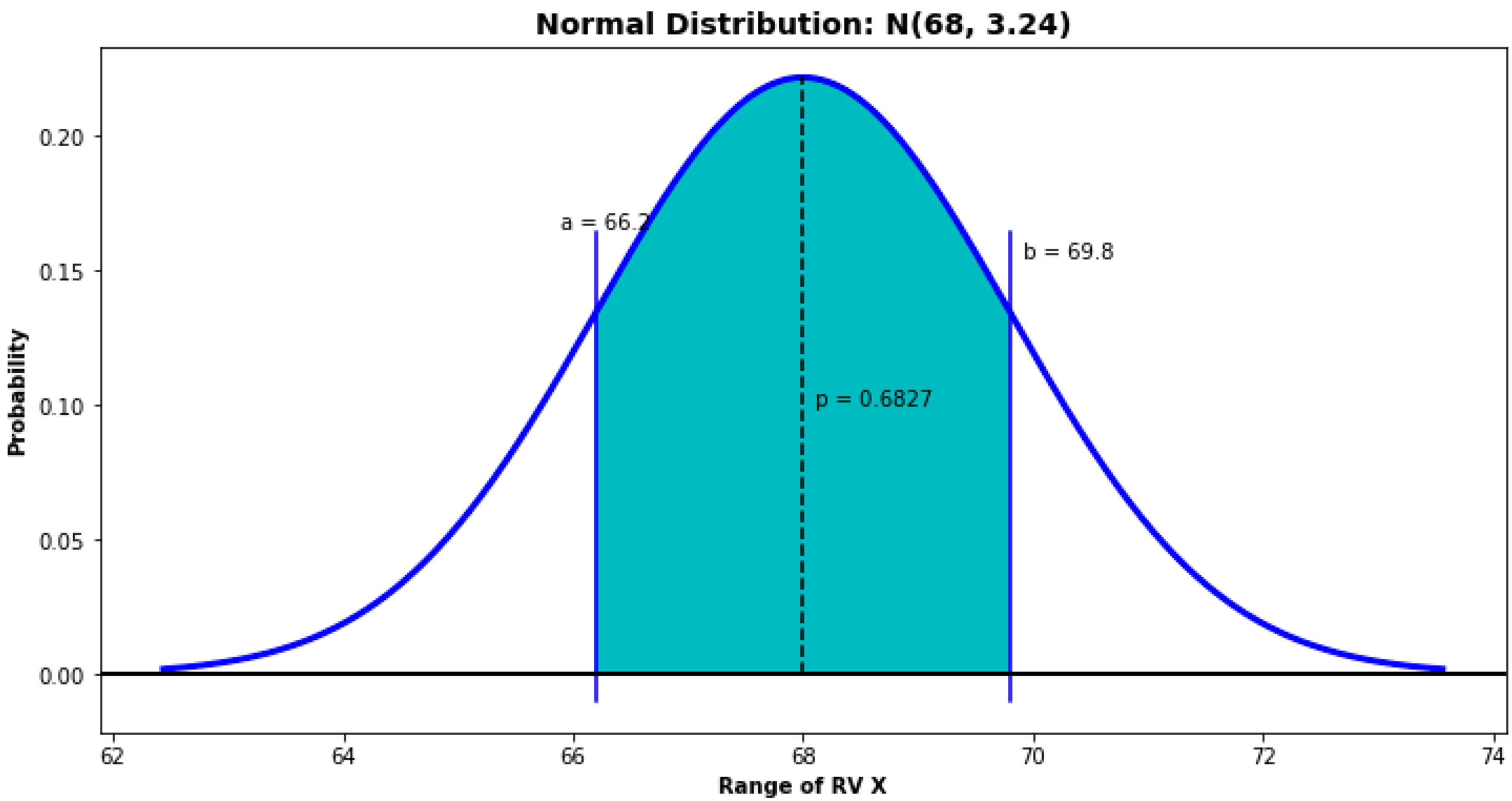
How many people are between 67 and 70 inches?




```
mean = 68
var = 3.24
stdev = 1.8
P(67 < X < 70) = P(X < 70) − P(X < 67) = 0.8667 − 0.2893 = 0.5775
```

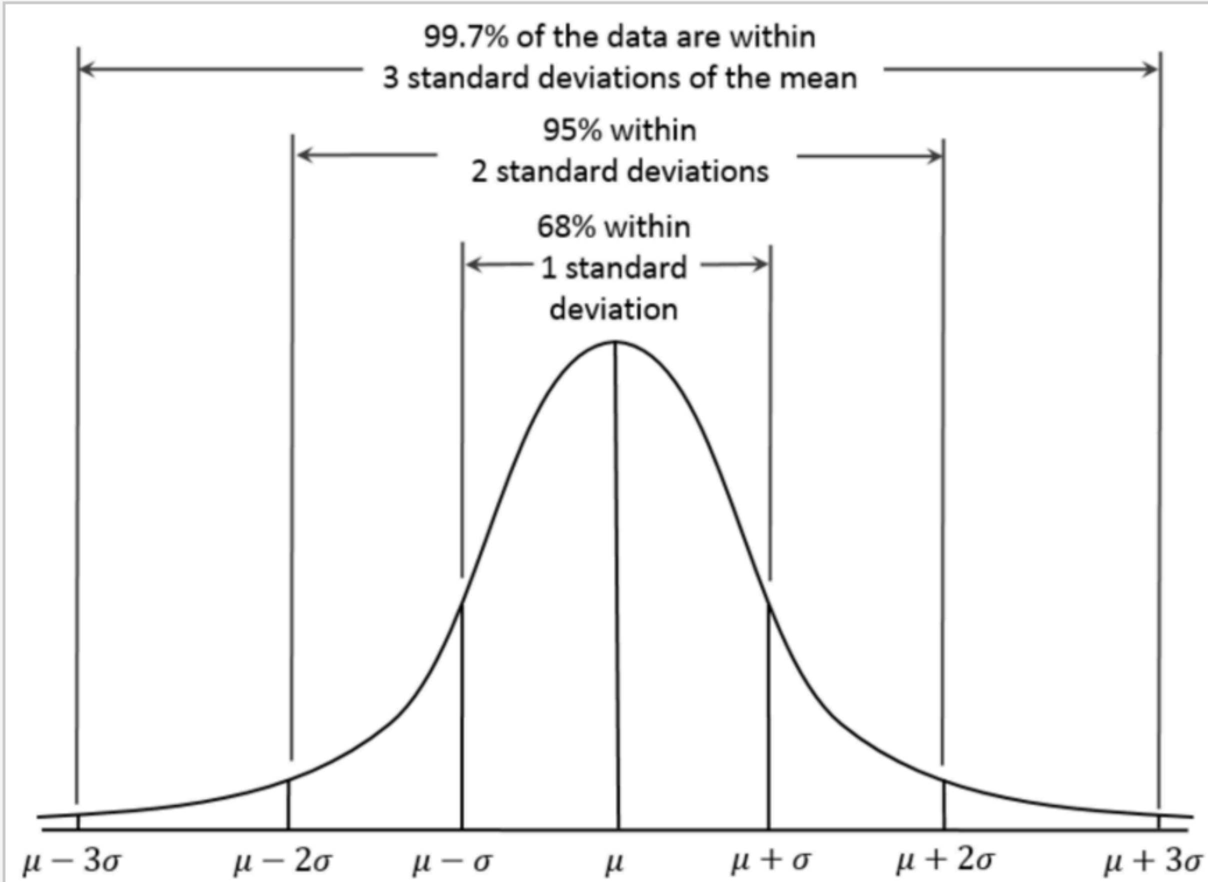
Normal Distribution

How many people are within one standard deviation of the mean height?




```
mean = 68
var = 3.24
stdev = 1.8
P(66.2 < X < 69.8) = P(X < 69.8) - P(X < 66.2) = 0.8413 - 0.1587 = 0.6827
```

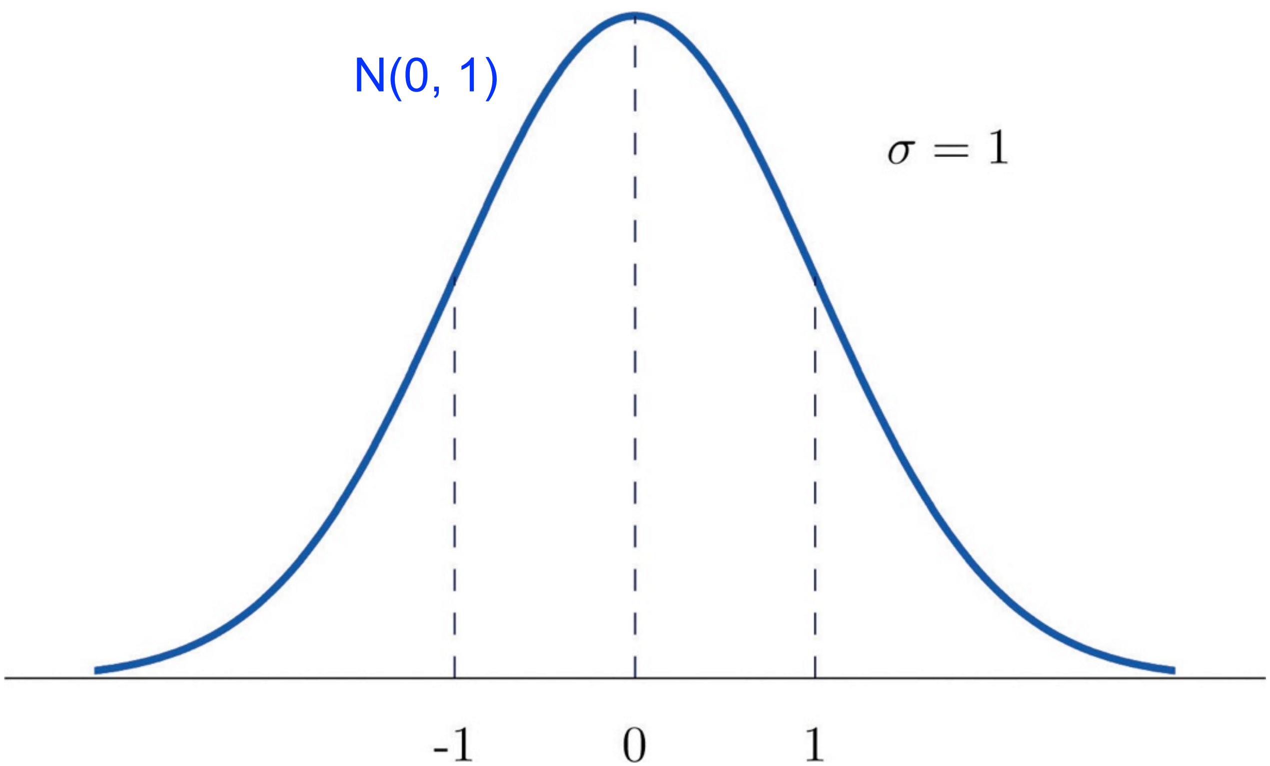
Normal Distribution



For the normal distribution, the values less than one standard deviation away from the mean account for 68.27% of the set; while two standard deviations from the mean account for 95.45%; and three standard deviations account for 99.73%.

Standard Normal Distribution

Since there are a potentially infinite number of Normal Distributions, usually we calculate using a normalized version, the Standard Normal Distribution with mean 0 and standard deviation (and variance) 1:

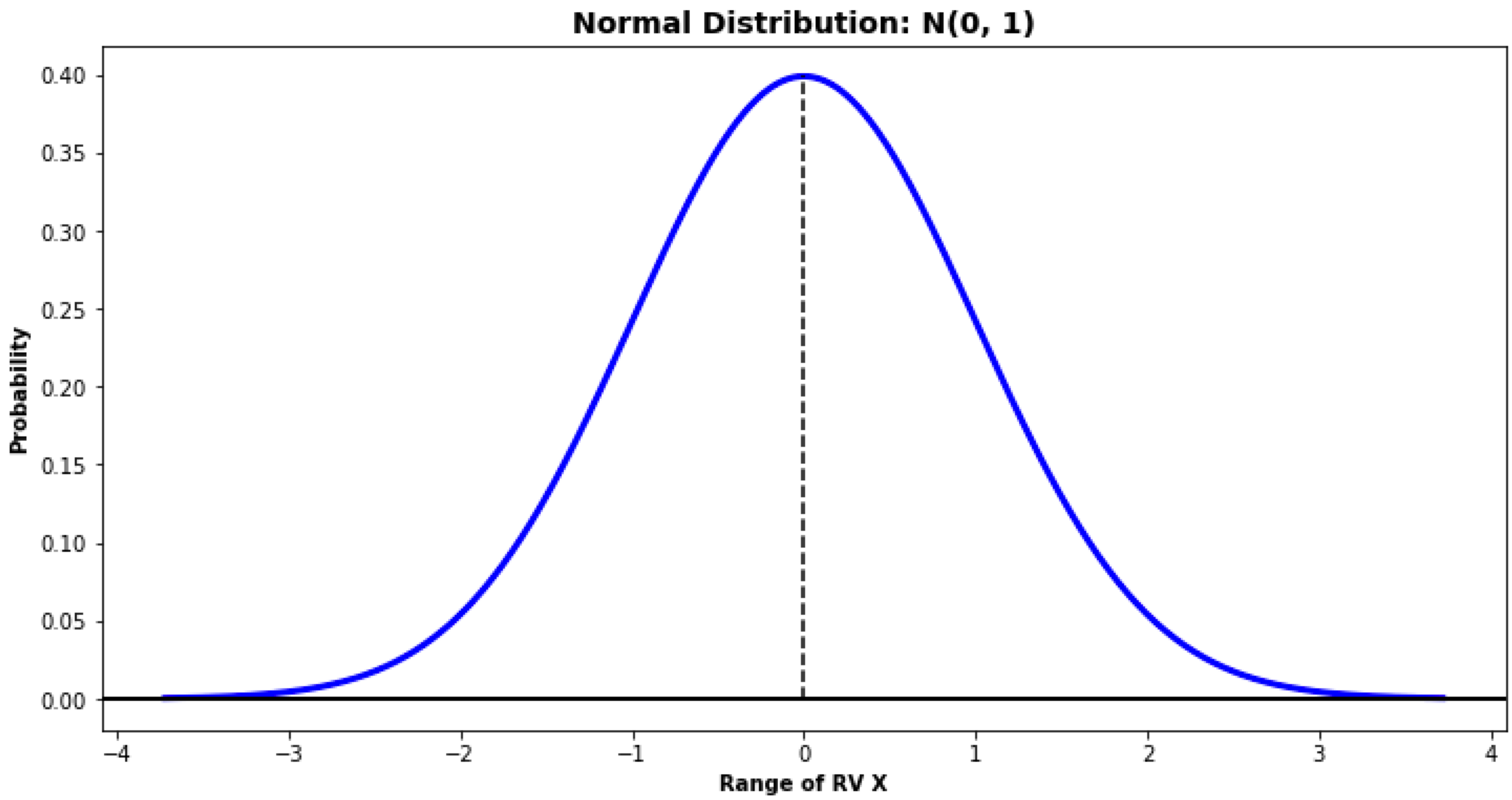


Any random variable X which has a normal distribution N(μ,σ^2) can be converted into a standardized random variable X* with distribution N(0,1) by the usual form. In the case of the normal distribution this is labelled as Z:

$$Z = \frac{X - \mu}{\sigma}$$

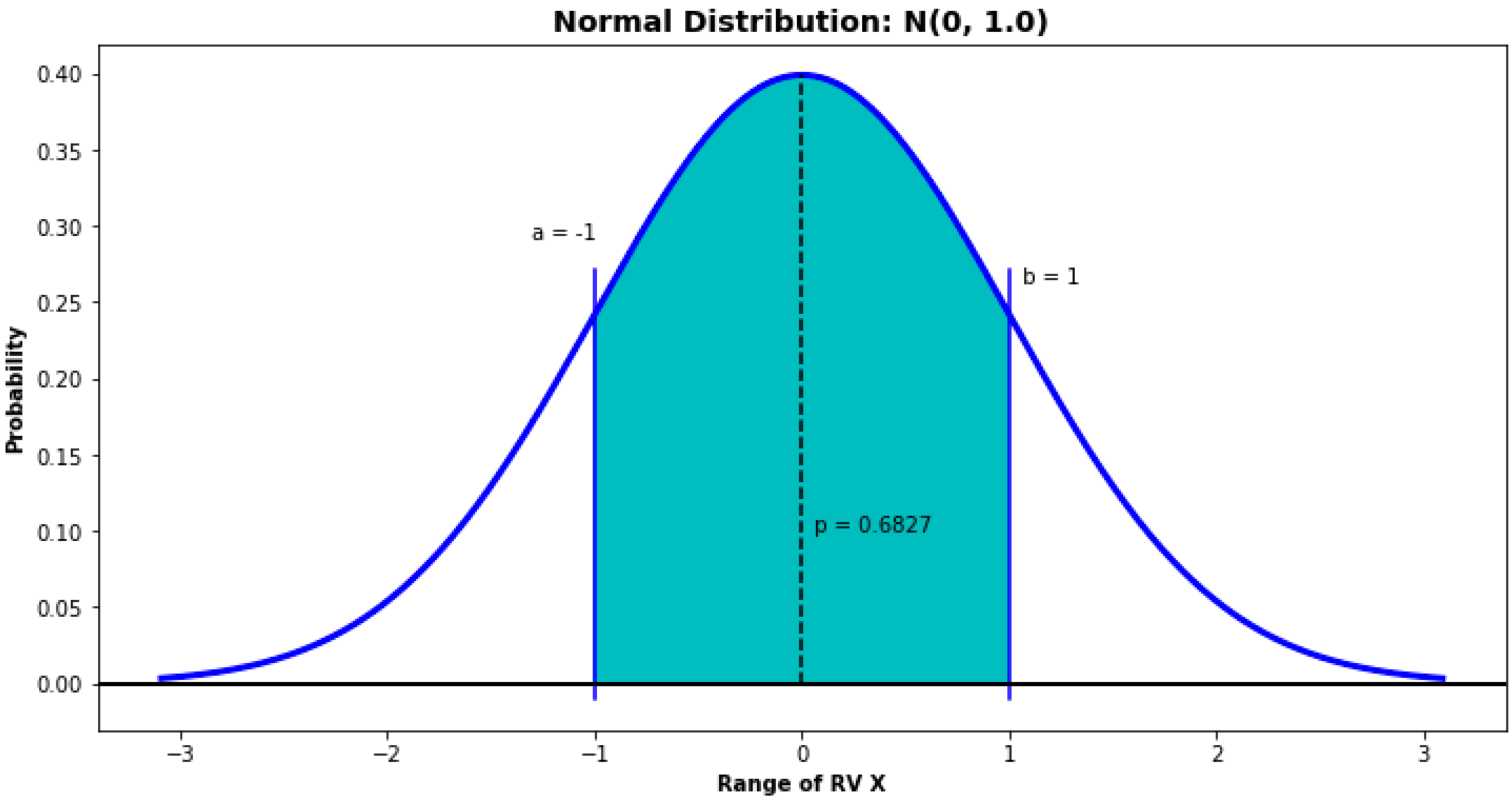
This is usually helpful in HAND calculations, since μ and σ have been factored out......

Normal Distribution



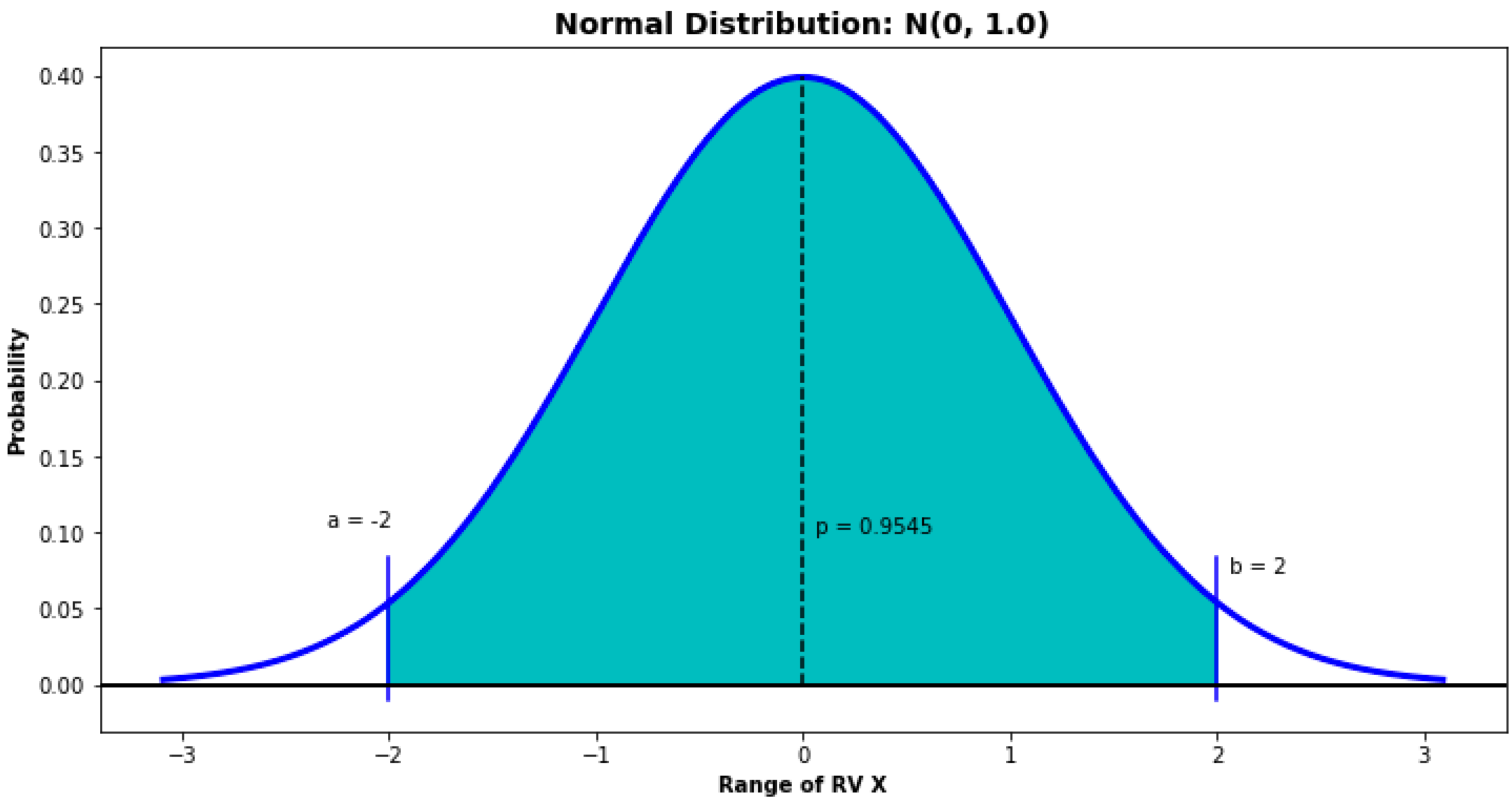

```
mean = 0
var = 1
stdev = 1
```

Normal Distribution



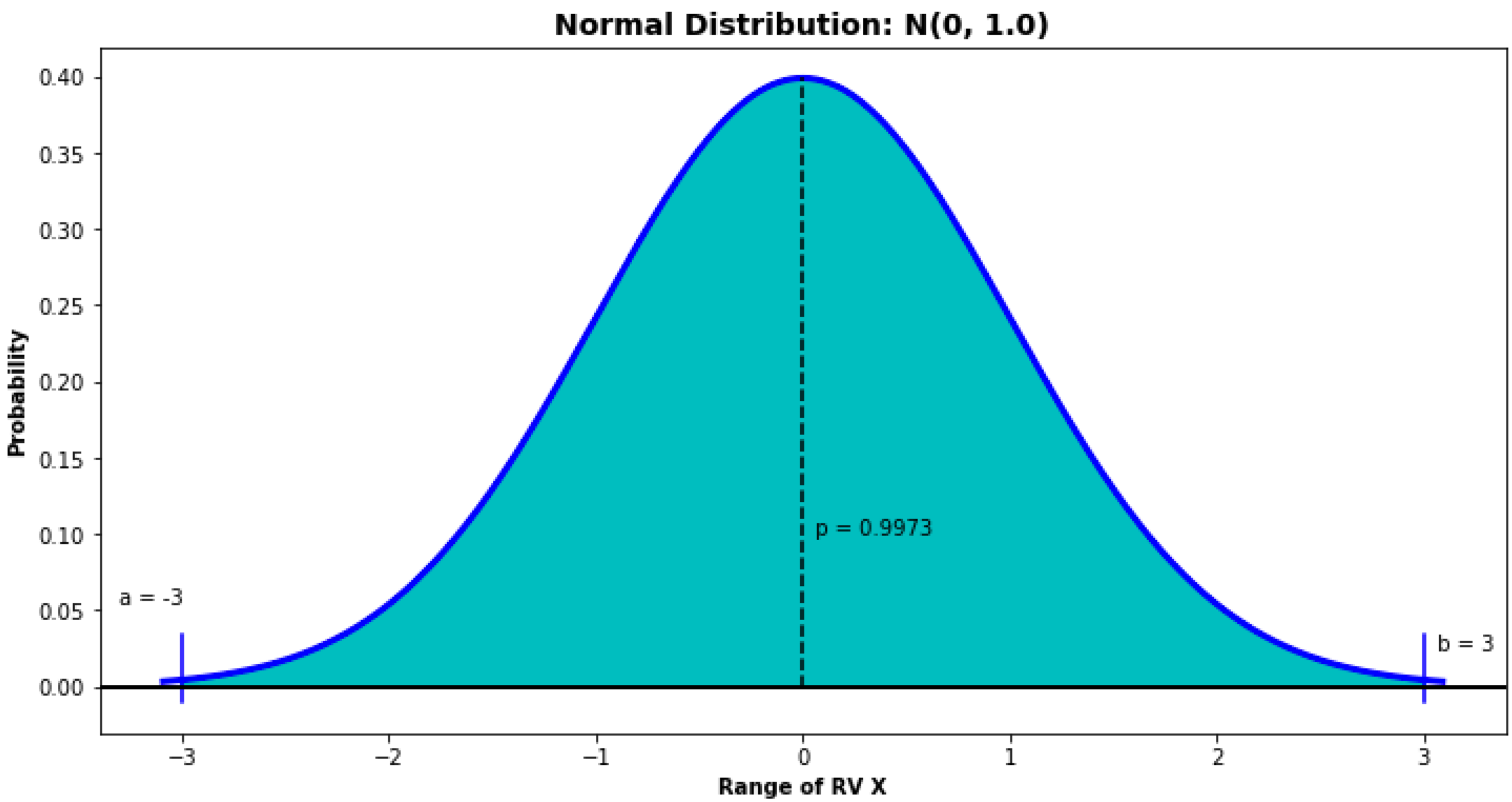

```
mean = 0
var = 1
stdev = 1.0
P(−1 < X < 1) = P(X < 1) − P(X < −1) = 0.8413 − 0.1587 = 0.6827
```

Normal Distribution




```
mean = 0
var = 1
stdev = 1.0
P(−2 < X < 2) = P(X < 2) − P(X < −2) = 0.9772 − 0.0228 = 0.9545
```

Normal Distribution



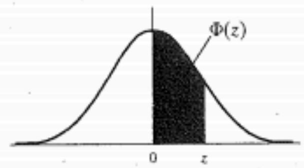

```
mean = 0
var = 1
stdev = 1.0
P(−3 < X < 3) = P(X < 3) − P(X < −3) = 0.9987 − 0.0013 = 0.9973
```

Standard Normal Distribution

If you were doing these calculations in 1900, or haven't heard of a computer, or if you were taking a test where you didn't have a calculator, here is how you would calculate the probability of a normally-distributed random variable:

Table 6-1 Standard Normal Curve Areas

This table gives areas $\Phi(z)$ under the standard normal distribution ϕ between 0 and $z \geq 0$ in steps of 0.01.



z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981

Normal Distribution

Modern people use the appropriate formulae:

```
def f_normal(mu,var,x):
    return (1/(math.sqrt(var*2*math.pi))) * math.exp(-(x-mu)*(x-mu)/(2*var))

def F_normal(mu,var,x):
    return (1 + math.erf((x-mu)/(var**0.5 * 2.0**0.5))) / 2

def normalRange(mu,var,low,high):
    return F_normal(mu,var,high) - F_normal(mu,var,low)


# OR use the scipy.stats.norm functions given at the top of the notebook:

# Loc = mean, scale = standard deviation

norm.pdf(x=50,loc=40,scale=5)

norm.cdf(x=50,loc=40,scale=5)

norm.rvs(loc=40,scale=5)      # random variates
```

Normal Distribution

Or a calculator or a web site:

- Enter a value in three of the four text boxes.
- Leave the fourth text box blank.
- Click the **Calculate** button to compute a value for the blank text box.

Standard score (z)	1.5
Cumulative probability: P(Z ≤ 1.5)	0.933
Mean	0
Standard deviation	1

Calculate

Normal Approximation to the Binomial

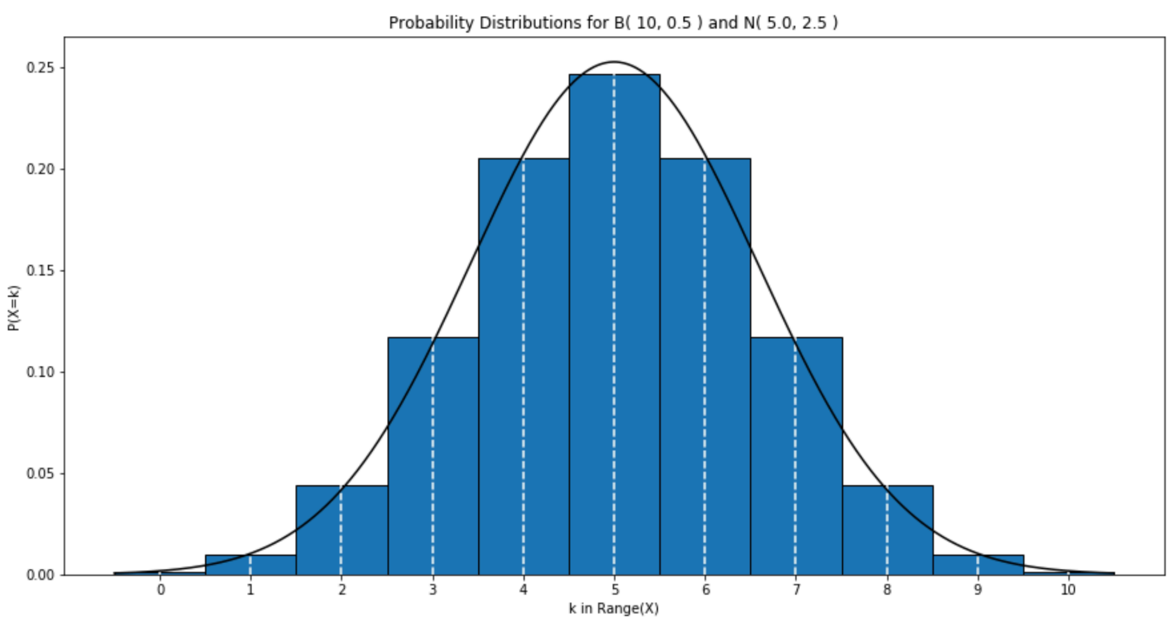
Since the binomial distribution with p close to 0.5 can be viewed as the limit of the binomial as N gets large, we can use the normal distribution to approximate the binomial.

To do the approximation, we have to have the same mean and standard deviation, so for a binomial RV

$$X \sim B(n,p)$$

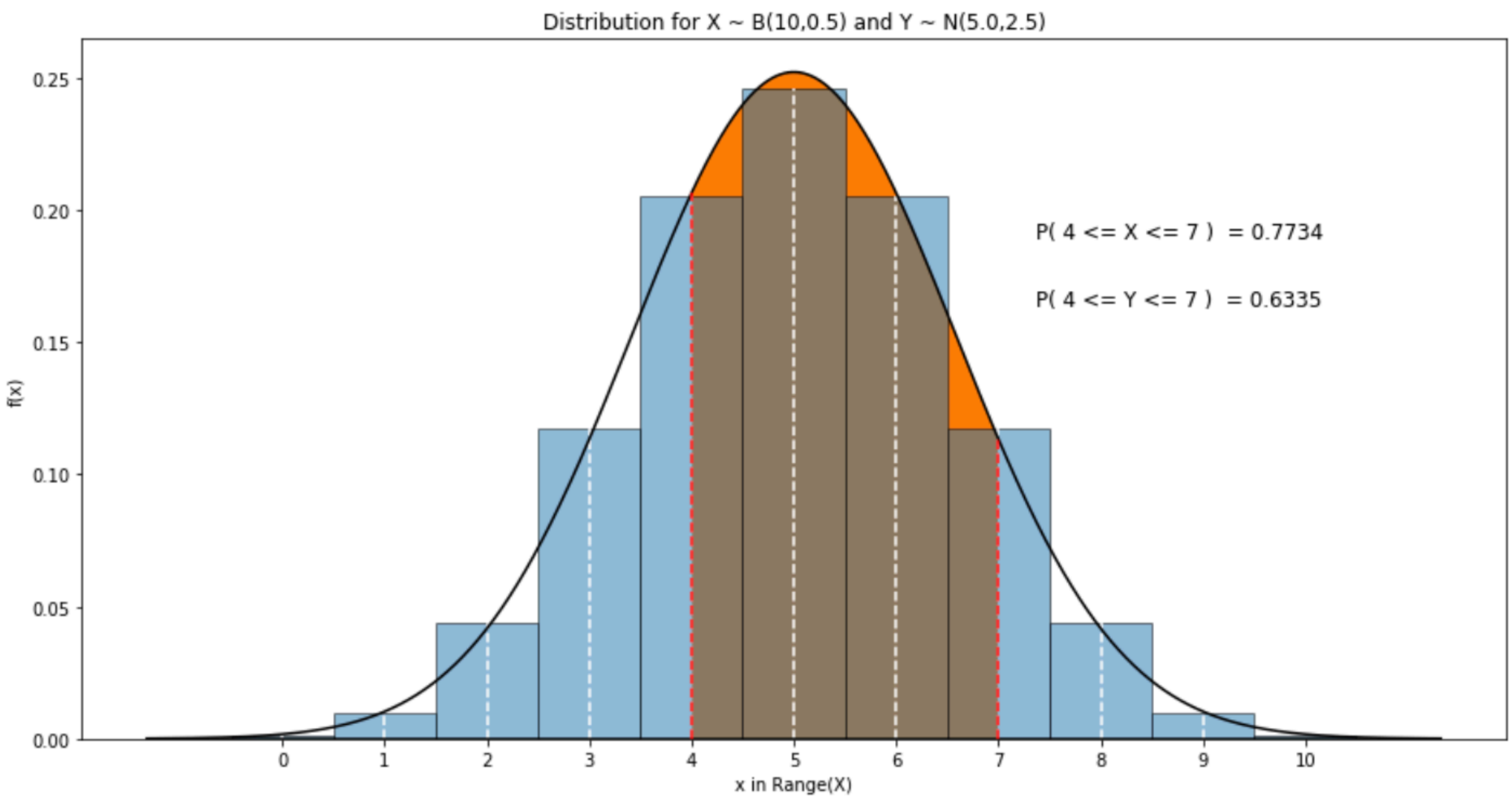
we consider a normal RV

$$Y \sim N(np, np(1-p))$$



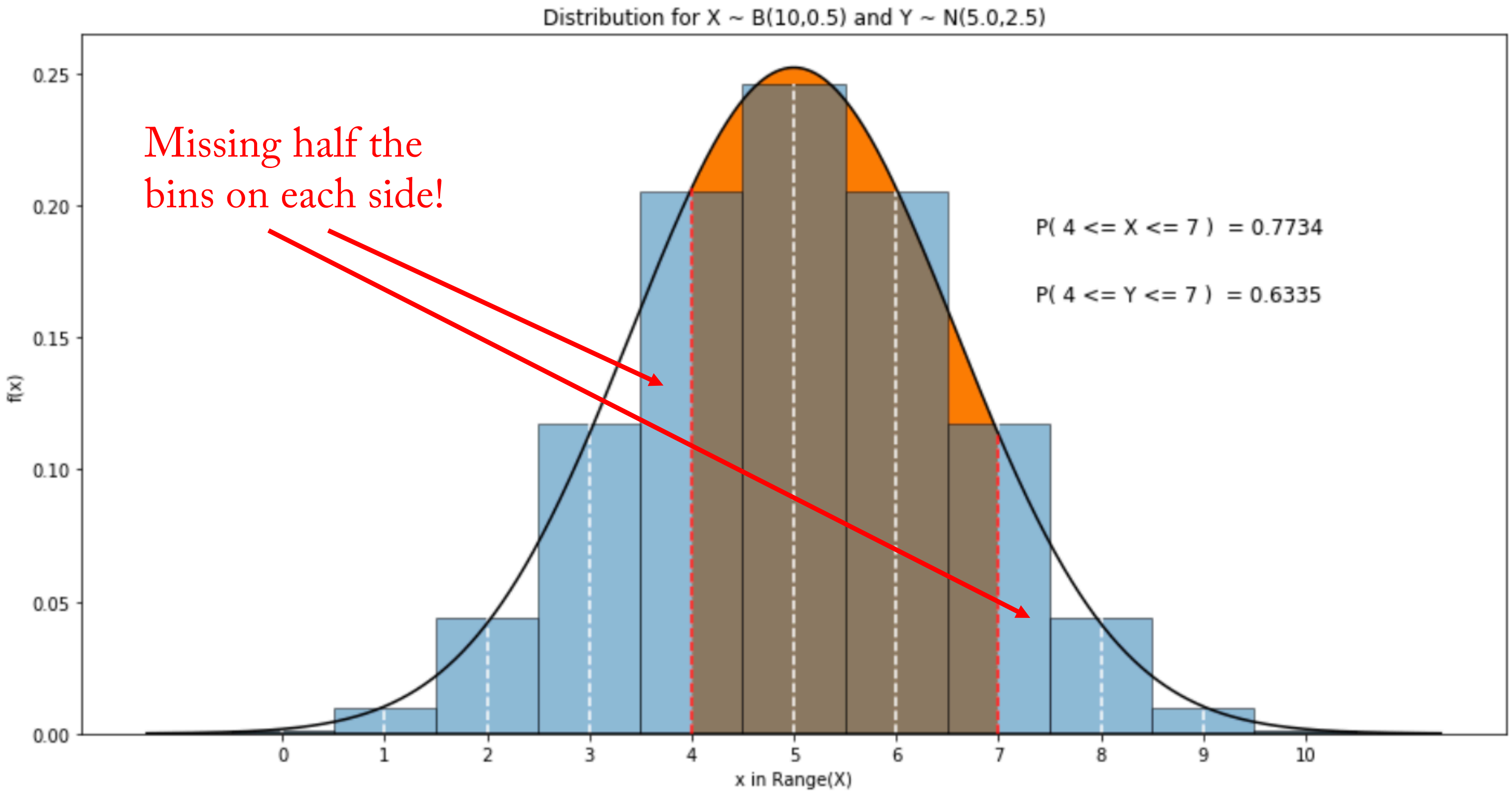
Normal Approximation to the Binomial

However, this is not quite right when we come to do actual calculations:



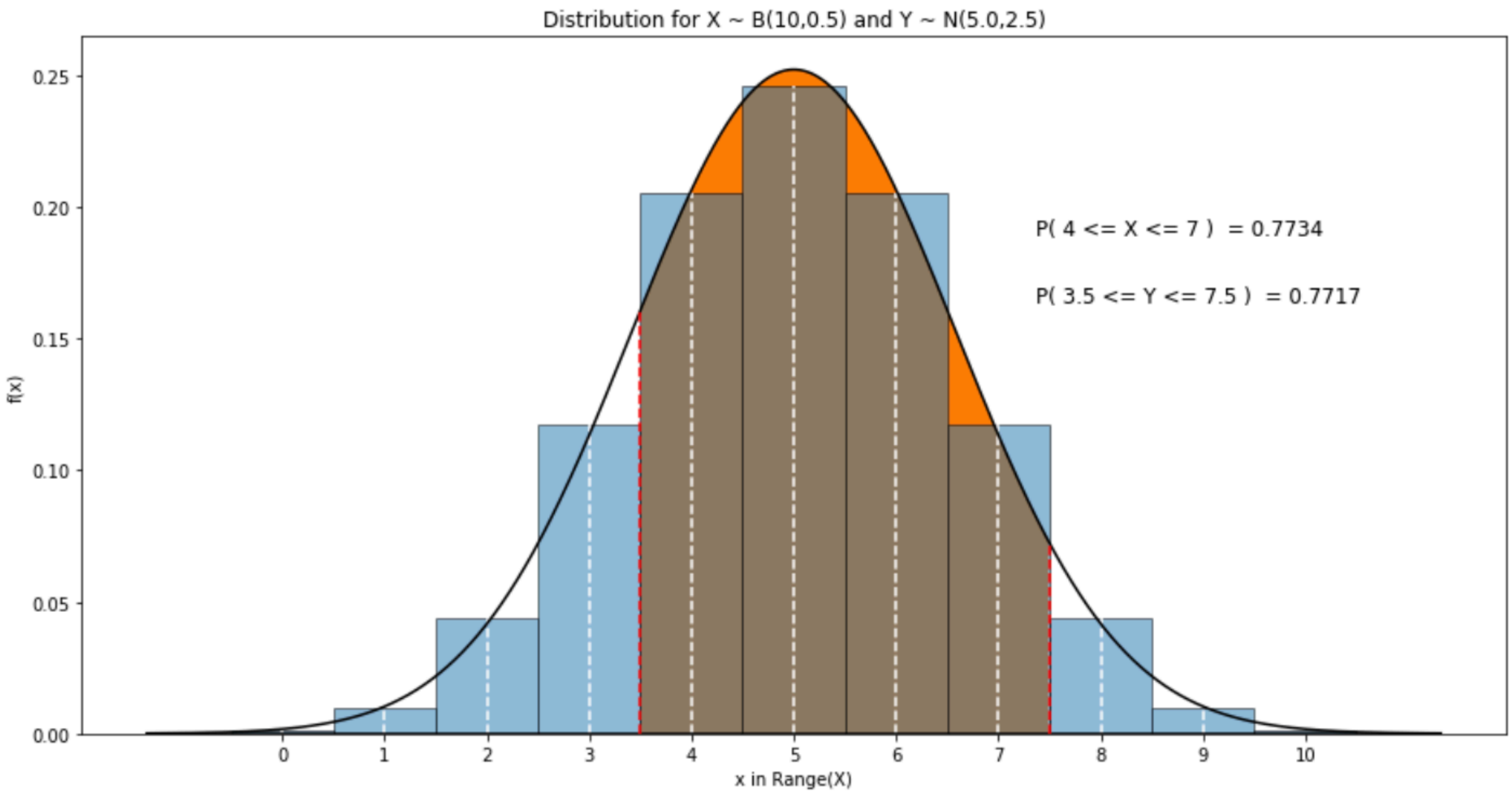
Normal Approximation to the Binomial

However, this is not quite right when we come to do actual calculations, as we underestimate by stopping half way through the lower and upper bins!



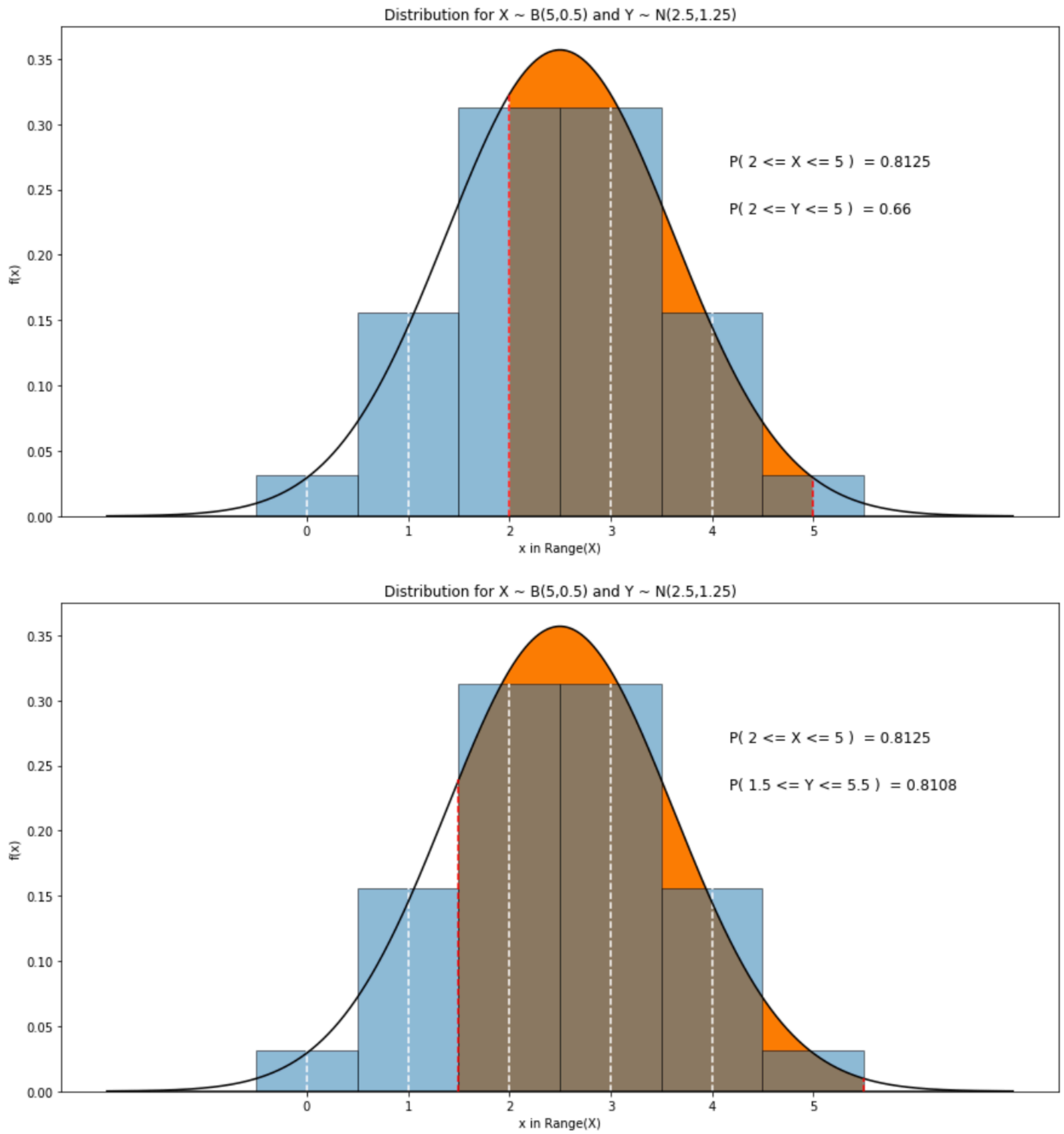
Normal Approximation to the Binomial

So, the continuity correction is to subtract 0.5 from any lower bound and add 0.5 to any upper bound:



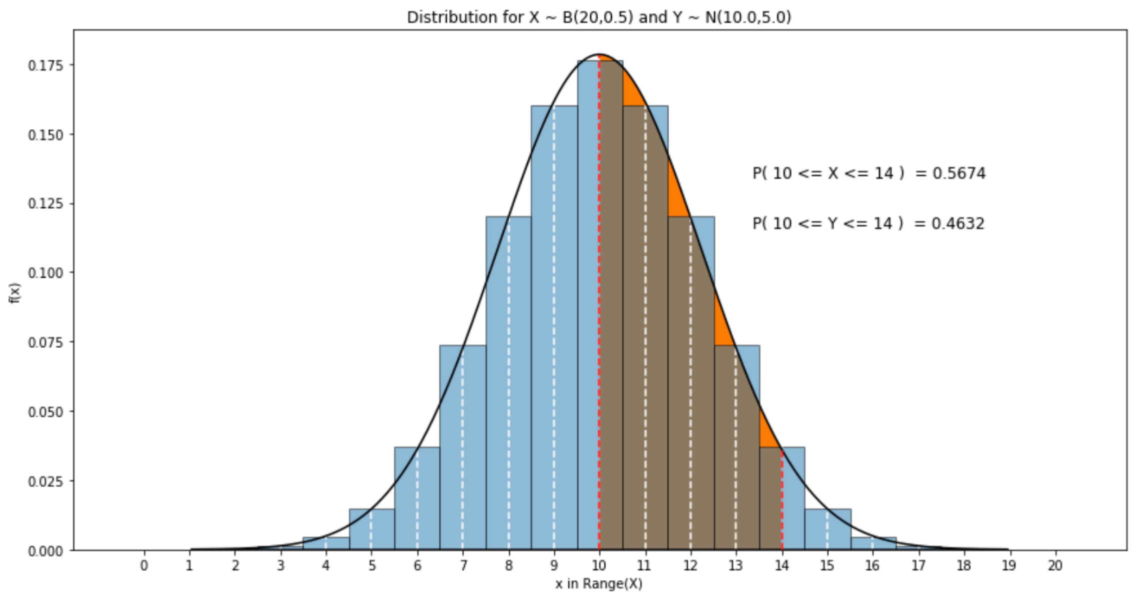
Normal Approximation to the Binomial

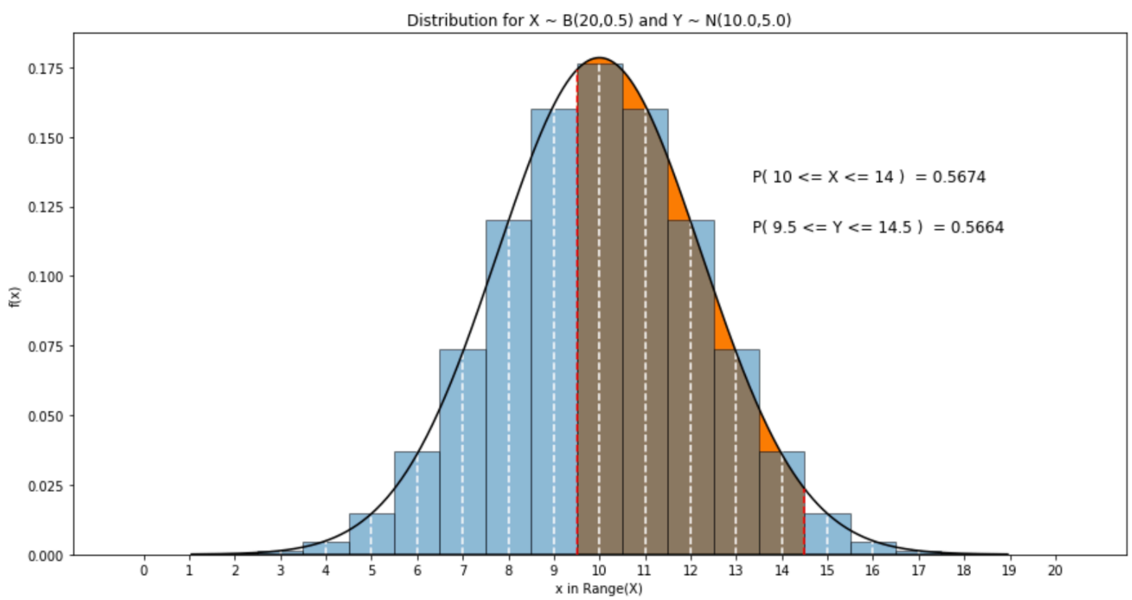
This actually works well even for small N:



Normal Approximation to the Binomial

For larger N the uncorrected value gets better, but the correction is **still** better:





Normal Approximation to the Binomial

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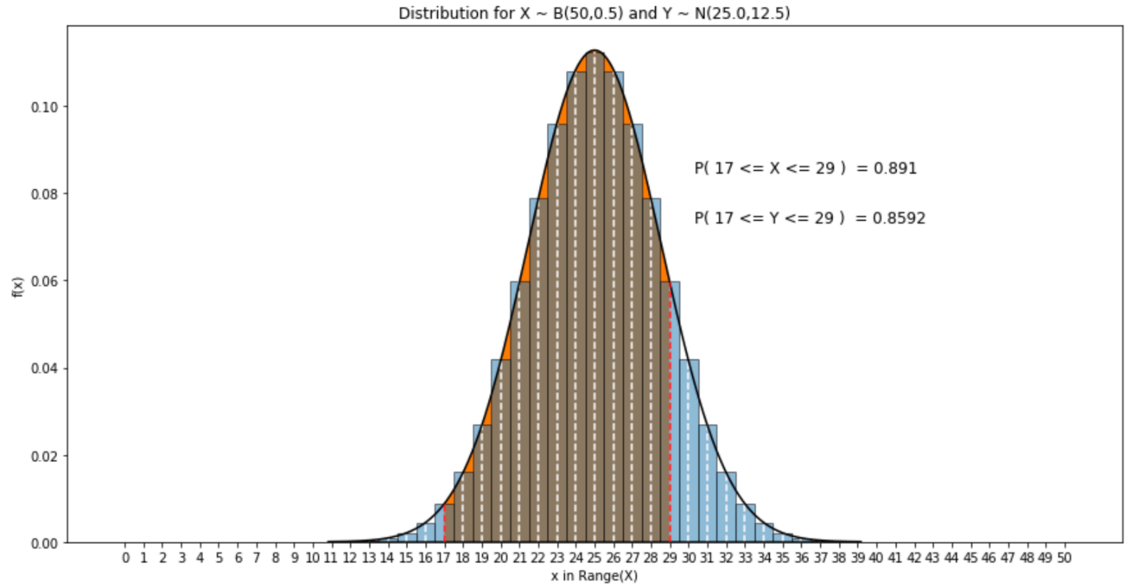

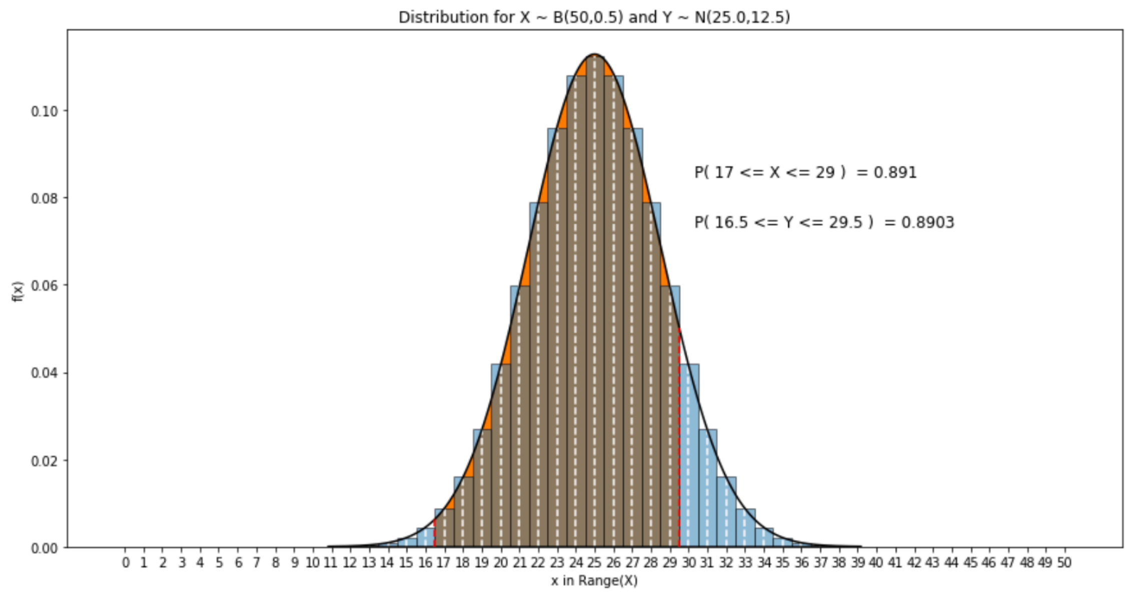

Normal Approximation to the Binomial

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