Name: Time turned in:

CS 237—Midterm One

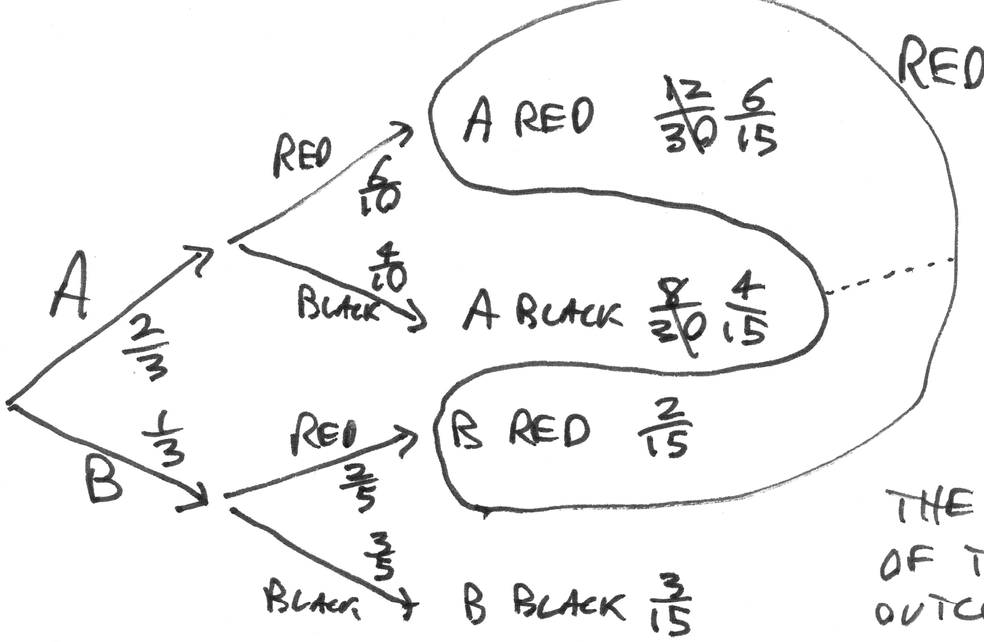
Fall 2017

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. You *must* leave blank, or draw an X through, or write "Do Not Grade," on one of the problems; we will grade the first 5 we get to if we can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers.** No calculators allowed, and you may leave *complicated* arithmetic expressions uncomputed, but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; show all work, and if you can not completely solve the problem, show me as much as you know and we will attempt to give you partial credit.

ALMOST SAME AS HW2 P8

Problem One. You have two sacks, A and B. In A there are 6 red and 4 black balls. In B there are 2 red and 3 black balls. Suppose you have a coin with "A" written on one side and "B" written on the other, and it is not fair: the probability when you flip the coin is 2/3 that it lands "A" and 1/3 that it lands "B". You flip the coin to choose a sack and draw a ball from that sack and find it to be red. What is the probability that this ball came from sack A? (Draw a tree diagram to solve this, and show all work.)



$+$

1.0

$$P(A \mid RED) = \frac{P(A\ RED)}{P(RED)} = \frac{\frac{6}{15}}{\frac{6}{15}+\frac{2}{15}} = \frac{\frac{6}{15}}{\frac{8}{15}} = \frac{6}{8} = \frac{3}{4}$$

THE PERCENTAGE OF THE RED OUTCOME DUE TO THE PATH STARTING WITH A IS 75% $= \frac{3}{4}$

Problem Three. There are 9 students in a class, and they need to be divided into 3 teams to play a contest.

(a) Suppose that they need to be divided into 3 teams named "Reds", "Blues", and "Blacks." How many ways can this be done? Show all work.

$$\binom{9}{3}\underbrace{\binom{6}{3}\binom{3}{3}}_{\text{NOT NECESSARY}} = 1680$$

(b) Now suppose they need to be divided into 3 equal-sized teams. How many ways can this be done? Show all work.

$$\frac{\binom{9}{3}\binom{6}{3}}{3!} = 280$$

(c) Now suppose they need to be divided into one team of 5, and two teams of 2 each. How many ways can this be done? Show all work.

$$\frac{\binom{9}{5}\binom{4}{2}\binom{2}{2}}{2!} = 378$$

THE FIRST STEP IN THIS PROBLEM WAS TO ADJUST THE RATE λ TO UNITS OF THE PROBLEMS.

Problem Four. Suppose that in the Pacific Northwest, tsunamis caused by mid-ocean earthquakes occur at the average rate of 1 every 250 years.

You do not have to evaluate complicated formula, but simplify them as much as you can.

ALMOST SAME AS HW6 P.5

(a) You buy a house in the Pacific Northwest and live in it for 50 years. What is the probability that your house will be destroyed by at least one tsunami during the time you live in it?

$$\lambda = \frac{50}{250} = \frac{1}{5} \qquad \text{POISSON!} \quad X \sim \text{POI}(\tfrac{1}{5})$$

$$1 - P(X=0) = 1 - \frac{e^{-\frac{1}{5}}\frac{1}{5}^0}{0!} = 1 - e^{-.5} = .1813$$

(b) The archeological record shows that the last time there was a tsunami in the Pacific Northwest was in 1650. What is the probability that in the next millennium there will be at least 5 tsunamis? (Assume you are asking this in the year 2000, so the millennium is years 2000 – 2999.)

$$\lambda = \frac{1000}{250} = 4 \qquad X \sim \text{POI}(4)$$

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{k=0}^{4} \frac{e^{-4}4^k}{k!}$$

$$= .3712$$

(c) What is the probability that in exactly 6 of the next 10 centuries, no tsunamis occur (assuming you are asking this question in the year 2000, so the 10 centuries are years 2000 – 2099, 2100 – 2199, ..., and 2900 – 2999).

POISSON:

$$\lambda = \frac{100}{250} = 0.4 \qquad X \sim \text{POI}(0.4)$$

"HOW MANY QUAKES IN A GIVEN CENTURY"

$$P(X=0) = \frac{e^{-.4}.4^0}{0!} = e^{-0.4} \quad (= .6703)$$

BINOMIAL:

$$Y \sim B(10, e^{-0.4})$$

"HOW MANY ~~EARTHQUAKES~~ CENTURIES IN A MILLENIUM HAVE NO EARTHQUAKES"

$$P(Y=6) = \binom{10}{6}(e^{-0.4})^6(1-e^{-0.4})^4$$

$$= .2251$$

Problem Three (Poisson and Exponential Distributions) The probability that a meteorite lands anywhere in the Sahara desert is modeled as a Poisson random variable with a mean of 1 every 10 days.

(a) Using the Poisson Distribution, answer the following question: It is currently midnight; what is the probability that at least one meteorite lands by noon?

A rate of 1 meteorite every 10 days, or $\lambda = 0.1$ per day, has to be modified to account for the period mentioned (half a day) to get $\lambda = 0.05$ per 12 hours. Then we have simply

$$1.0 - (0.05)^0 e^{-0.05} / 0! = 1 - e^{-0.05} \quad [= 0.0488]$$

(b) Using the Poisson Distribution, answer the following question: It is currently midnight; supposing no meteorite lands by 6am, what is the probability that at least one will land by noon?

Because of the "memoryless property," this is fundamentally the same problem as the last, but with a period of 6 hours, or 1/4 of a day, so we have $\lambda = 0.025$ per 6 hours. Then we have

$$1.0 - (0.025)^0 e^{-0.025} / 0! = 1 - e^{-0.025} \quad [= 0.0247]$$

(c) Recast problem (a) using the Exponential Distribution: explain what the corresponding random variable represents and calculate in this framework the answer to the question posed in (a).

Now the random variable gives the time between two arrivals of meteorites. The mean interarrival time $1/\lambda = 10$ days, so $\lambda = 0.1$ days and the CDF $F(x) = P(X < x) = 1 - e^{-\lambda x}$, so we have the same formula:

$$1 - P(X < 0.5) = 1 - e^{-0.5/10} = 1 - e^{-0.05} \quad [= 0.0488]$$

(d) Using the Exponential Distribution, answer the following question: It is currently midnight; what is the probability that the first meteorite lands some time between 6am and 6pm?

Let us use $\lambda = 0.025$ per 6 hours. Now we have (in terms of the unit of 6 hours):

$$P(X < 3) – P(X < 1) = 1 – e^{-3*0.025} - (1 – e^{-0.025}) = 0.0476.$$

Problems 4 and 5 are not relevant in Fall 2017 so I deleted them.

Problem Six (Wildcard Problem). Let X ~ Bernoulli(p) be a random variable distributed according to the Bernoulli distribution.

(a) Give the sample space and probability function for X.

$$Rng(X) = \{0, 1\}$$
$$f(X) = \{(1-p), p\}$$

Prove mathematically (as rigorously as you can) the following.

(b) Prove that X satisfies Axiom 2: "For the certain event S, we have P(S) = 1."

$$\sum_{k \in Rng(X)} f(k) = (1-p) + p = 1.0$$

(c) Prove that E(X) = p.

$$E(X) = \sum_{k \in Rng(X)} k \cdot f(k) = 0 \cdot (1-p) + 1 \cdot p = p$$

(d) Suppose I flip N coins, where the probability of heads is p. Let Y = "the number of heads which appear." Give the expression for E(Y) and *prove* that this is in fact the formula.

$$Y = B(N, p) \quad \text{so} \quad E(X) = N * p$$

$$Y = X_1 + \dots + X_n \quad \text{(SUM OF N INDEPENDENT BERNOULLIS)}$$

$$E(Y) = E(X_1 + \dots + X_n)$$
$$= E(X_1) + \dots + E(X_n) \quad \text{(BY LINEARITY OF EXPECTATION AS SHOWN IN CLASS)}$$
$$= p + \dots + p$$
$$= N * p$$