Name

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CS 237—Midterm Exam

Fall 2018

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write "Do Not Grade," on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention—no exceptions! If answers are on the back of the page please tell me so. **Circle final answers**. No calculators allowed, and *you may leave complicated formulae uncomputed,* but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself.

In writing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

Problem One

There are 12 students in a class, and they need to be divided into 3 teams to play a contest.

You may leave these answers as formulae.

(A) Suppose that they need to be divided into 3 equal-sized teams named "A Team", "B Team", and "C Team." How many ways can this be done?

$$\binom{12}{4}\binom{8}{4}$$

(B) Now suppose they need to be divided into two teams of 5, and one team of 2 each. How many ways can this be done?

$$\frac{\binom{12}{5}\binom{7}{5}}{2}$$

(C) Now suppose you need to divide the class into 4 equal-sized teams, and for each team you need to select a captain. How many ways can this be done? (Realize that once you select the members of each team, there are multiple ways to select the captains of each team.)

$$\frac{3^4\binom{12}{3}\binom{9}{3}\binom{6}{3}}{4!}$$

Problem Two

Suppose X ~ Uniform(0,10), i.e., it produces a random real number between 0 and 10 according to the following graph of the Probability Density Function $f_X(a)$ (the y axis has been left off on purpose):

(a) Give of $f_X(a)$ as a mathematical formula.

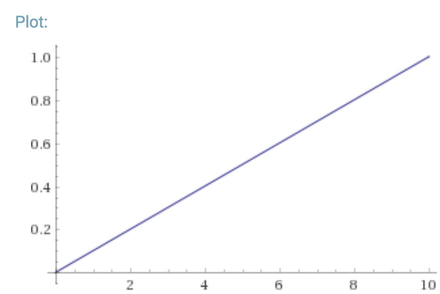
$$f_X(a) = \begin{cases} \frac{1}{10} & \text{FOR } 0 \le a \le 10 \\ 0 & \text{OTHERWISE} \end{cases}$$

(b) Calculate the Cumulative Distribution Function $F_X(a)$ using integrals and draw the graph.

$$F_X(a) = \int_{\cancel{-\infty}\ 0}^{a} \frac{1}{10}\, dx$$

$$= \frac{1}{10} x \Big|_0^a$$

$$= \begin{cases} 1 & \text{FOR } a > 10 \\ \frac{a}{10} & \text{FOR } 0 \le a \le 10 \\ 0 & \text{FOR } a < 0 \end{cases}$$



(c) Calculate E(X) using integrals (you can probably guess what it is, but I want you to derive it using integrals).

$$\text{(c)}\quad E(X) = \int_{\cancel{-\infty}\ 0}^{\cancel{\infty}\ 10} x \cdot \frac{1}{10}\, dx$$

CHECK:

$$\frac{\left(\frac{1}{20}x^2\right)}{dx} = \frac{2}{20}x = \frac{1}{10}x \checkmark$$

$$= \frac{1}{20}x^2 \Big|_0^{10} = \frac{100}{20} = 5.0$$

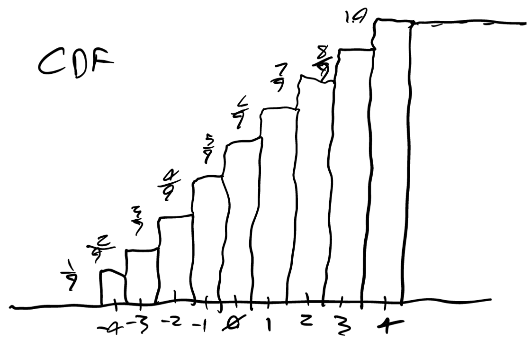
Problem Three

Suppose a discrete random variable **X** is distributed according to the uniform discrete distribution in the range [-4, -3, -2, -1, 0, 1, 2, 3, 4], that is,

$$f_X(k) = \begin{cases} 1/9 & \text{if } -4 \le k \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, let the random variable $\mathbf{Z} = \mathbf{X}^2$.

(a) Draw the CDF of **X**.



(b) Give the range R_Z and probability function f_X for **Z**.

$$R_Z = \{0, 1, 4, 9, 16\}$$

$$f_Z = \left\{\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right\}$$

(c) What is E[**Z**]? Show all work.

$$E[Z] = \frac{2}{9} + \frac{8}{9} + \frac{18}{9} + \frac{32}{9} = \frac{60}{9} = \frac{20}{3}$$

(d) What is Var(**X**)? Show all work.

$$Var(X) = E(X^2) - E(X)^2$$

$$= E(Z) - 0$$

$$= \frac{20}{3}$$

E(X) = 0 symmetric!

Problem Four

Wayne and Lenka are playing a game in which each has a fair coin, and they flip the coins at the same time, and keep doing so until they have the same face showing (both heads or both tails). A round is one simultaneous flip. For example, , they might have the following:

L: T T H
W: H H H (three rounds = three flips)

or they might have:

L: T
W: T (one round = one flip)

(A) If X = the number of rounds the game lasts, what is the distribution of X? Be absolutely precise.

THIS IS $X \sim$ GEOMETRIC$\left(\frac{1}{2}\right)$

$R_X = \{1, 2, \ldots\}$ $f_X = \{\frac{1}{2}, \frac{1}{4}, \ldots\}$

(B) What is the expected number of rounds when they play this game?

$$E(X) = \frac{1}{p} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

(C) What is the probability that the game lasts more than 2, but less than 7 rounds?

B $P(2 < X < 7) = P(X > 2) - P(X > 6)$

$= \left(1-\frac{1}{2}\right)^2 - \left(1-\frac{1}{2}\right)^6$

$= \frac{1}{4} - \frac{1}{64}$

$= \frac{16}{64} - \frac{1}{64} = \frac{15}{64}$

1 2	3 4 5 6	7
$\frac{1}{2}+\frac{1}{4}$	$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$	$\frac{1}{64}$
	$\frac{15}{64}$	

(D) What is the probability that if the game lasts exactly 9 rounds, that Wayne has landed heads more times than Lenka? You may calculate it or guess, but if you guess you must give a reason for your guess.

(D) GAME IS SYMMETRIC (COULD EXCHANGE HEADS & TAILS WITHOUT CHANGING PROBS)

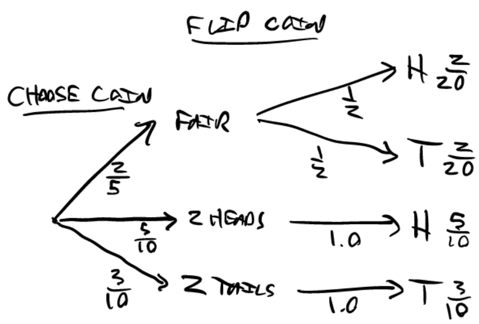
SO $\frac{1}{2}$

Problem Five

A box contains 10 coins where 5 coins have a head on each side, 3 coins have a tail on each side and 2 are fair coins (head and tail with 50% chance of each when tossed).

A tree diagram is the best way to start this problem.

(A) Suppose a coin is chosen at random and tossed. Find the probability that a head appears.



$P(H) = \frac{2}{20} + \frac{5}{10} = 0.6 \qquad P(T) = \frac{2}{20} + \frac{3}{10} = 0.4$

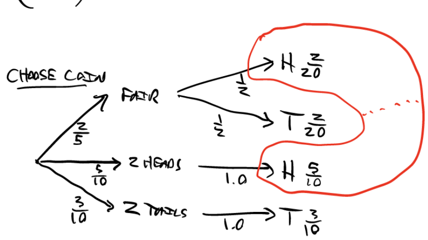
(B) Suppose we repeat the experiment in (A) 10 times. What is the probability that at least 7 heads appear? (You may just give the formula.)

THIS IS BINOMIAL $X \sim B(10, \frac{6}{10})$

$$\sum_{k=7}^{10} \binom{10}{k} (0.6)^k (1-0.6)^{10-k}$$

(C) Suppose a coin is selected at random and tossed. If a head appears, find the probability that the coin was fair, i.e., one with a head on one side and tail on the other.

(C)



$P(\text{FAIR} \mid H) = \frac{P(\text{BOTH})}{P(H)}$

$= \frac{\frac{2}{20}}{\frac{2}{20} + \frac{5}{10}} = \frac{2}{12} = \frac{1}{6}$

Problem Six Let X be a random variable defined as follows. Toss a die: if the number of dots showing is odd, let Y = 1, otherwise let Y = 4.

(A) Give the range R_X and probability distribution f_X for X.

This is Bernoulli(0.5): $$R_x = \{1,4\} \quad f_x = \{0.5, 0.5\}$$

(B) Calculate explicitly (not just by quoting a formula) the expected value E(X) and show every step. You may leave the result as a fraction.

$$E(X) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$$

(C) Calculate the variance Var(X) and standard deviation σ_X of X. Again, do not just quote a formula, but show explicitly all calculations. You may leave the result as a fraction.

$$\begin{aligned} VAR(X) &= E(X^2) - E(X)^2 \\ &= \left(\frac{1}{2} + \frac{16}{2}\right) - \left(\frac{5}{2}\right)^2 \\ &= \frac{17}{2} - \frac{25}{4} \\ &= \frac{34}{4} - \frac{25}{4} \\ &= \frac{9}{4} \end{aligned}$$

$$\sigma_x = \sqrt{\frac{9}{4}} = \frac{3}{2}$$