CS 237: Probability in Computing

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Lecture 2: Conclusion on Probability Spaces; Finite Spaces

- Probability Spaces and the Axiomatic Method
- Classification of Probability Spaces:
  - Discrete (Finite, Countably Infinite)
  - Continuous (Uncountably Infinite)
  - Equiprobable vs Not-Equiprobable
- Finite Probability
  - Equiprobable Case
  - Non-Equiprobable Case
- Problem Solving Strategies: Decision trees and the "Four-Step Method"

Review: Random Experiments, Sample Spaces and Sample Points

The Sample Points can be just about anything (numbers, letters, words, people, etc.) and the Sample Space (= **any** set of sample points) can be

Discrete

Finite

Example: Flip three coins, and output head if there are at least two heads showing, and tails otherwise (as if the coins "vote" for the outcome!)

S= { head, tail }

Countably Infinite

Example: Flip a coin until heads appears, and report the number of flips

S = { 1, 2, 3, 4, ... }

Continuous

Uncountably Infinite

Example: Spin a pointer on a circle labelled with real numbers [0..1) and report the number that the pointer stops on.

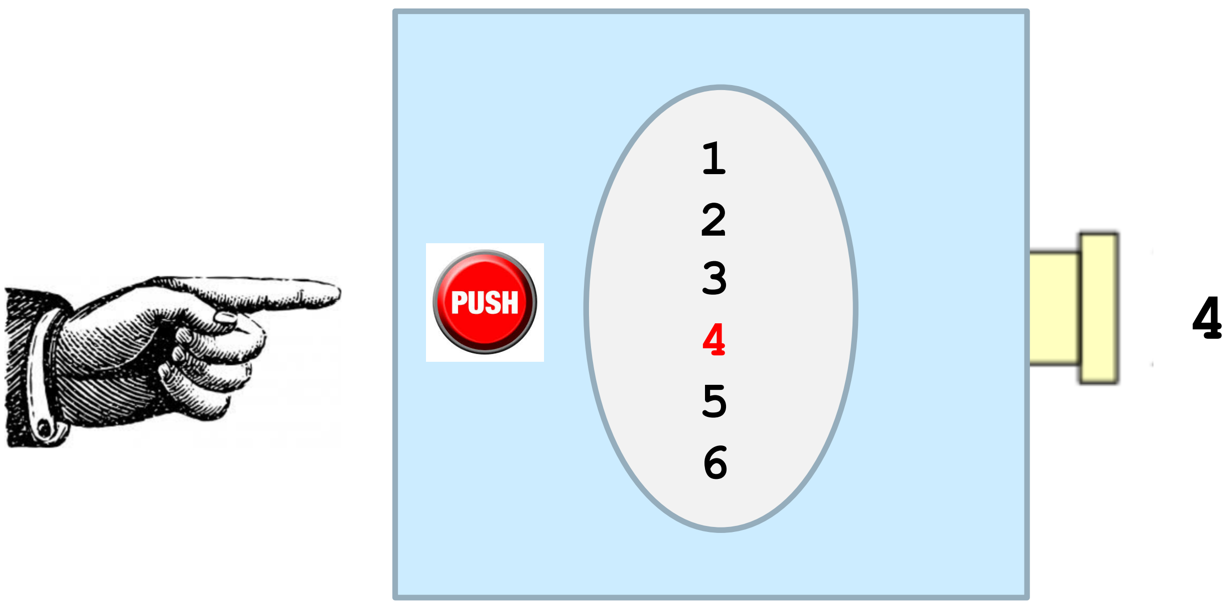
S = [0 .. 1)

Review: Sample Spaces, Sample Points, and Events

An Event is any **subset** of the Sample Space. An event A is said to have occurred if the outcome of the random experiment is a member of A. We will be mostly interested specifying a set by some characteristic, and then calculating the probability of that event occurring.

Example: Toss a die and output the number of dots showing.
Let A = "there are an even number of dots showing" and
B = "there are at least 5 dots showing."

$S = \{ 1, 2, 3, 4, 5, 6 \}$ $A = \{ 2, 4, 6 \}$ $B = \{ 5, 6 \}$



The event A occurred, since

$$4 \in \{2, 4, 6\}$$

The event B did not occur:

$$4 \notin \{ 5, 6 \}$$

Review: Sample Spaces, Sample Points, and Events

We will be mostly interested in **questions involving the probability of particular events occurring**, so let us pay particular attention to the notion of an event.

Example: Toss a die and output the number of dots showing. Let A = "there are an even number of dots showing."

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

The set of possible events is the power-set of S, the set of all subsets,

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}$$

So for this example we have $2^6 = 64$ possible events, including

- The empty or "impossible event" $\emptyset$. ("What is the probability of rolling a 9?")
- The "certain event" S. ("What is the probability of less than 10 dots?")
- All "elementary events" of one outcome: { 1 }, { 2 } , { 3 }, ..., { 6 }.

etc..... This gives you the most flexible way of discussing the results of an experiment....

Probability Spaces and Probability Axioms

To model a random experiment, we specify a Probability Space = a pair (S, P), where S is a sample space, and P is a probability function

$$P : \mathcal{P}(S) \mapsto \mathcal{R}$$

which assigns a probability (a real number) to each possible event and such that P satisfies the following three Axioms of Probability:

P_1: For any event A, we have $\mathbf{P(A) \geq 0}$.

P_2: For the certain event S, we have $\mathbf{P(S) = 1.0}$.

P_3: For any two disjoint events A and B ($A \cap B = \varnothing$), we have

$$P(A \cup B) = P(A) + P(B)$$

Also, we have an alternate version of the third axiom, for countable unions of events:

P'_3: For any **countably** infinite sequence of events A_1, A_2, A_3, ... which are pairwise disjoint (for any i and j, i ≠ j implies ($A_i \cap A_j = \varnothing$)) we have

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Probability Spaces and Probability Axioms

These axioms make perfect sense if we consider Venn Diagrams where we use area as indicating probability, so the area of an event in the diagram = probability of that event.

P_1: For any event A, we have $\mathbf{P(A) \geq 0}$.

"The area of each event is non-negative."

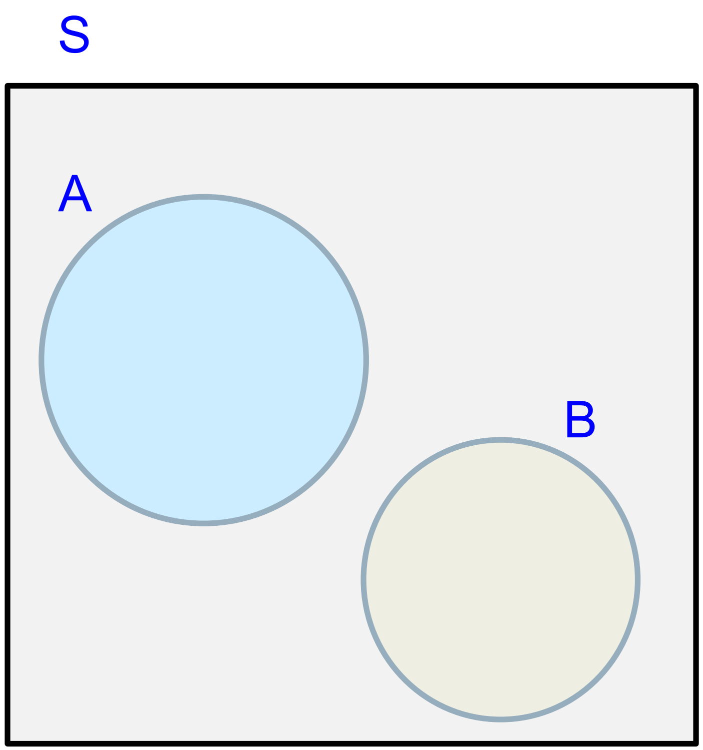
P_2: For the certain event S, we have $\mathbf{P(S) = 1.0}$.

"The area of the whole sample space is 1.0."

P_3: For any two disjoint events A and B we have

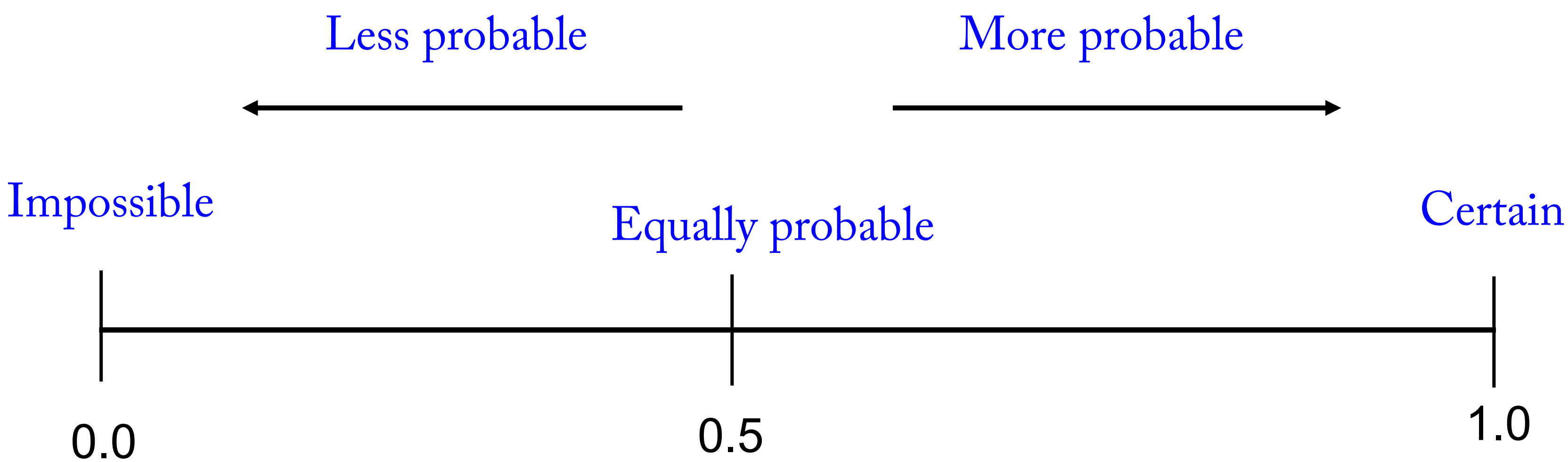
$$P(A \cup B) = P(A) + P(B)$$

"If two regions of S do not overlap, then the area of the two regions combined is the sum of the area of each region."



Probability Spaces and Probability Axioms

So we measure the probability of events on a real-number scale from 0 to 1:



Probability Functions: Equiprobable vs Not Equiprobable

Recall that probability spaces can be characterized by the characteristics of their sample space: discrete (finite or countably infinite) or continueous (uncountable).

Furthermore, we may characterize a probability function as being:

Equiprobable: All sample points (= elementary events) have the same probability.

Not Equiprobable: All sample points do NOT have the same probability.

When the sample space is finite, it is easy to see how this might happen:

Finite and Equiprobable:

Example: Flip a coin, report how many heads are showing.

$S = \{ 0, 1 \}$ $P(0) = 0.5$ $P(1) = 0.5$

Finite and NOT Equiprobable:

Example: Flip two coins, report how many heads are showing.

$S = \{ 0, 1, 2 \}$ $P(0) = 0.25$ $P(1) = 0.5$ $P(2) = 0.25$

How to Specify a Probability Space

In order to specify a probability space for a particular problem, it suffices to give

- The Probability Space (a set)
- The Probability Function (a function from the set to the interval [0..1])

In order to check that you indeed have a correct probability space, it generally suffices to check axiom **P_2: P(S) = 1.0.**

Example: Flip a coin, report how many heads are showing.

$S = \{ 0, \quad 1 \quad \}$

$P = \{ 0.5, 0.5 \}$ <= Just give the probability of each sample point.

Check: 0.5 + 0.5 = 1.0

Example: Flip two coins, report how many heads are showing.

$S = \{ 0, \quad 1, \quad 2 \quad \}$

$P = \{ 0.25, 0.5, 0.25 \}$

Check: 0.25 + 0.5 + 0.25 = 1.0

Probability Functions: Equiprobable vs Not Equiprobable

But when the sample space is countably infinite, the probability function can NOT be equiprobable!

Countably Infinite and Not Equiprobable:

Example: Flip a coin until a heads appears, and return the number of flips.

$S = \{ 1, \quad 2, \quad 3, \quad \ldots \}$

$P = \{ 1/2, \; 1/4, \; 1/8, \; \ldots \}$ Check: $1/2 + 1/4 + 1/8 + \ldots = 1.0$

But suppose a space were countably infinite and equiprobable:

$S = \{ 1, 2, 3, \ldots. \}$

$P = \{ a, a, a, \ldots \}$ for some $a > 0$.

Then $a + a + a + \ldots. = \infty$ not 1.0

Conclusion: When the sample space is countably infinite, the probability function can NOT be equiprobable.

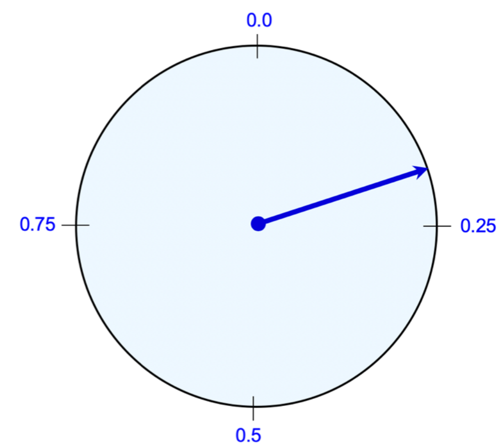
Probability Functions: Equiprobable vs Not Equiprobable

When the sample space is uncountable, say with the spinner, it is possible for the probability function to be equiprobable or non-equiprobable.

Uncountable and Equiprobable:

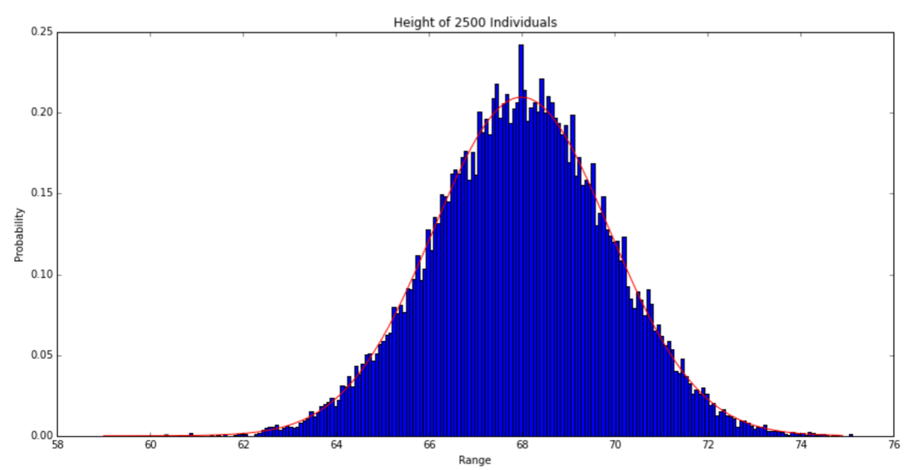
Example: Spin the spinner and report the real number showing.

S = [0..1) **Any point is equally likely**



Uncountable and NOT Equiprobable:

Example: Heights of Human Beings:

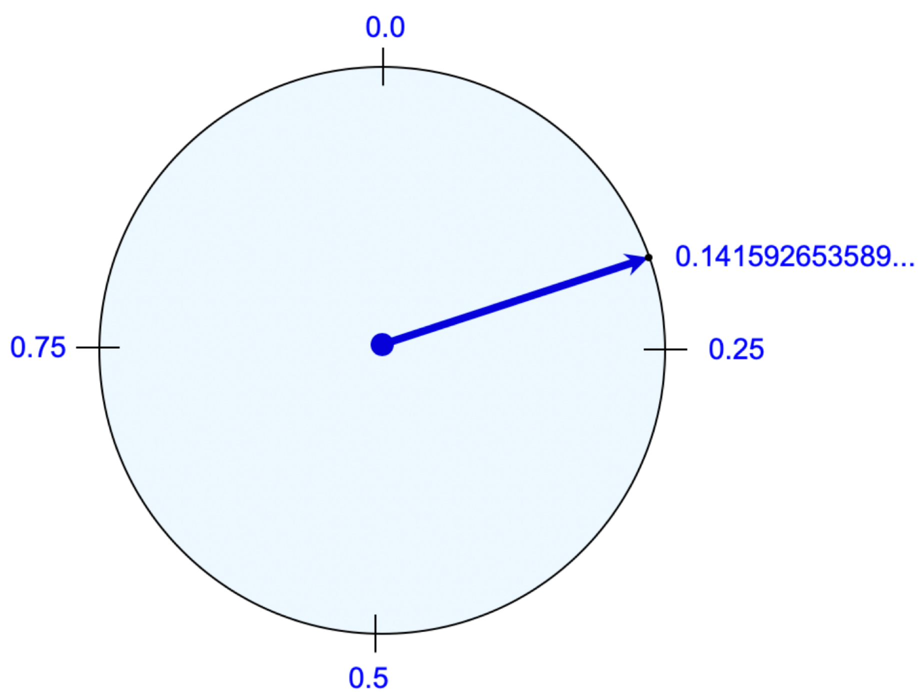


People are more likely to be close to the average height than at the extremes!

Anomolies with Continuous Probability Spaces

When the sample space is uncountable, as with our spinner, things can get a bit complicated.....

Question: Suppose you spin a spinner. What is the probability that the pointer lands EXACTLY on 0.141592... (the decimal part of π)?



Hint: There are two possibilities:

- The probability is 0.
- The probability is NOT 0.

Can you come up with an argument for or against either of these?

Anomolies with Continuous Probability Spaces

When the sample space is uncountable, as with our spinner, things can get a bit complicated.....

Question: Suppose you spin a spinner. What is the probability that the pointer lands EXACTLY on 0.5?

Answer: $\mathbf{0}$

Why? **Proof by contradiction:** Suppose the probability is **a** > 0. Then this must also be true for ANY real number in the range [0..1). But then we have the same problem as with countable non-equiprobable spaces: $a + a + a + \ldots = \infty$, violating P2.

As a consequence, when we discuss events in continuous probability, it only makes sense to talk about countable numbers of unions and intersections of all possible intervals [a..b] , [a..b), (a..b], (a..b), etc.

We will explore this further in the next homework......

There is actually a whole field of study in mathematics called "Measure Theory" that deals with this problem!

Finite Probability Spaces

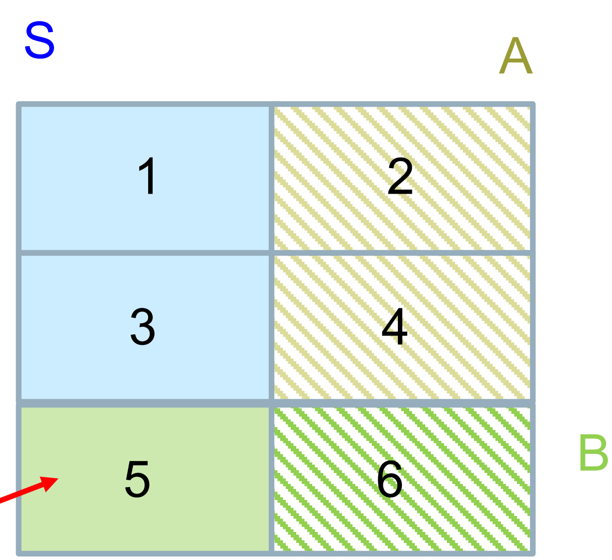
For finite probability spaces, it is easy to calculate the probability of an event; we just have to apply axiom P_3:

If event $\mathbf{A = \{ a_1, a_2, ..., a_n \}}$, then

$$\mathbf{P(A) = P(\{ a_1, a_2, ..., a_n \}) = P(a_1) + P(a_2) + ... + P(a_n)}$$

Example: Toss a die and output the number of dots showing. Let A = "there are an even number of dots showing" and B = "there are at least 5 dots showing."

We can illustrate simple problems by using the "area" = "probability" analogy:



Equiprobable: area of each elementary event is $1/6 = 0.16666...$

$$P(A) = P(2) + P(4) + P(6) = 1/6 + 1/6 + 1/6 = ½$$

$$P(B) = P(5) + P(6) = 1/6 + 1/6 = 1/3$$

Finite Probability Spaces

Example: Flip three fair coins and count the number of heads. Let A = "2 heads are showing" and B = "at most 2 heads are showing."

The equiprobable "pre-sample space" is

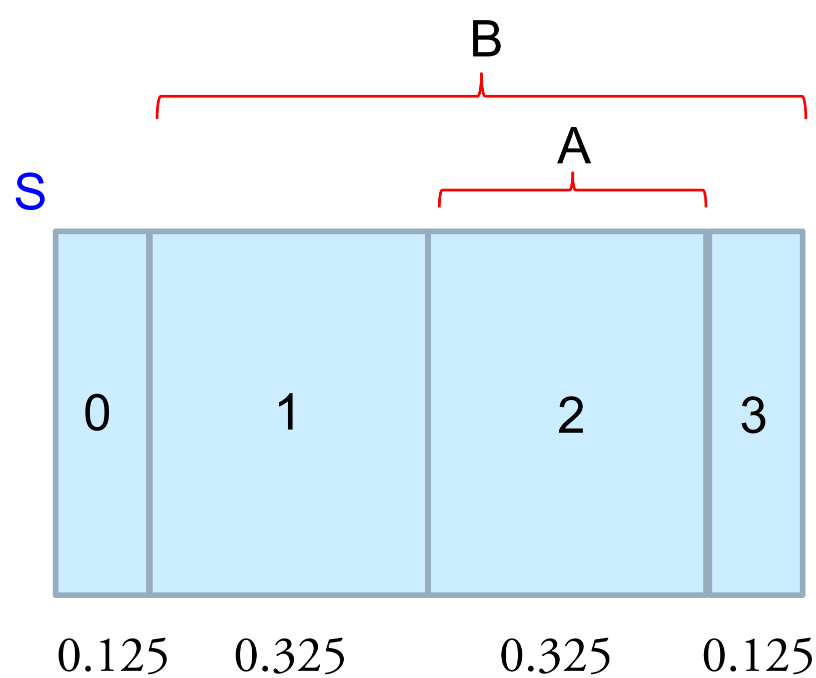
configuration:	**{ TTT,**	**TTH,**	**THT,**	**THH,**	**HTT,**	**HTH,**	**HHT,**	**HHH }**
# heads:	**0**	**1**	**1**	**2**	**1**	**2**	**2**	**3**

S = { 0, 1, 2, 3 }
P = { 1/8, 3/8, 3/8, 1/8 }

P(A) = P(2)
= 3/8

P(B) = P(0) + P(1) + P(2)
= 1/8 + 3/8 + 3/8
= 7/8

Not Equiprobable: area of each elementary event is different:



Finite Equiprobable Probability Spaces

For finite and equiprobable probability spaces,

it is easy to calculate the probability:

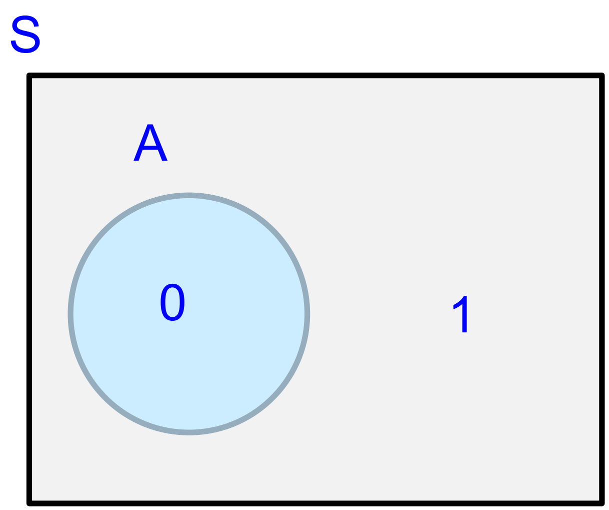
$$P(A) = \frac{|A|}{|S|}$$

Here, instead of area, we use the the number of elements.

Example: Flip a coin, report how many heads are showing? Let A = "the coin lands with tails showing"

S = { 0, 1 }

P = { ½, ½ }



Finite Equiprobable Probability Spaces

For finite and equiprobable probability spaces,

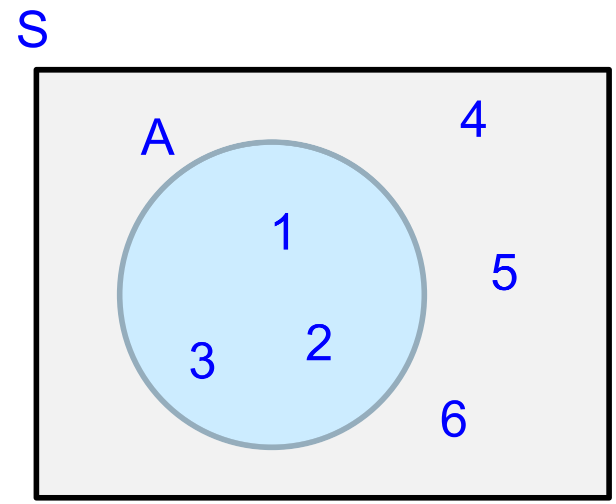
it is easy to calculate the probability:

$$P(A) = \frac{|A|}{|S|}$$

Example: Roll a die, how many dots showing on the top face? Let A = "less than 4 dots are showing."

S = { 1, 2, ... , 6 }

P = { 1/6, 1/6, ... , 1/6 }



Solving Probability Problems using Decision Trees

To the board......