

Name:

CS 237—Midterm Exam Solution

Summer 2016

You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write “Do Not Grade,” on the two problems you are eliminating; I will grade the first 5 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers.** No calculators allowed, and you may leave *complicated* arithmetic expressions uncomputed, but please do multiply $1/2 * 1/2$ to get $1/4$ if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

Problem One. Out of 2000 families with 4 children, how many would you expect to find which have at least one girl and one boy (i.e., not all girls or all boys)? Assume the probability of any particular child being a boy is 0.5.

You must do this in two different ways: (a) using the Binomial Distribution, and (b) using the Inclusion/Exclusion Principle without referring to the Binomial Distribution.

Solution:

(a) If X = number of girls in a family with 4 children, we have $B(4,0.5)$ and are looking for

$$\begin{aligned} & P(X = 1) + P(X = 2) + P(X = 3) \\ &= C(4,1)*(1/2)^4 + C(4,2)*(1/2)^4 + C(4,3)*(1/2)^4 \\ &= 4 * 1/16 + 6 * 1/16 + 4 * 1/16 \\ &= 14/16 = 7/8 \end{aligned}$$

Then $7/8 * 2000 = 14000/8 = 1750$.

Or, you might use $1.0 - [C(4,0)*(1/2)^4 + C(4,4)*(1/2)^4] = 1.0 - 2*(1/2)^4 = 1 - 1/8 = 7/8$ etc.

(b)

The probability of all girls is $(1/2)^4 = 1/16$ and for all boys is the same, so $2/16 = 1/8$ of the families would have at least children of only one sex, or $2000/8 = 250$. Thus $2000 - 250 = 1750$ would have at least one girl and one boy.

Problem Two. A coin is tossed twice. Let A = the first toss is a head, and B = at least one of the tosses is a head. Let

P_1 = the probability of two heads given that we know the first toss is a head,
and

P_2 = the probability of two heads given that we know *at least one* of the tosses is a head.

(a) Suppose the coin is fair. Wayne claims that P_1 is at least as large as P_2 . Is he right?

Solution: Let A = first toss is a head and B = second toss is a head; $P(A)=P(B) = 1/2$ and $P(A \text{ and } B) = 1/4$ and $P(A \text{ or } B) = 3/4$

$$P_1 = P((A \text{ and } B) | A) = P(A \text{ and } B) / P(A) = (1/4)/(1/2) = 1/2$$

and

$$P_2 = P((A \text{ and } B) | (A \text{ or } B)) = P(A \text{ and } B)/P(A \text{ or } B) = (1/4)/(3/4) = 1/3.$$

So, YES, Wayne is right because $1/2 \geq 1/3$.

(b) Suppose the coin is not necessarily fair. Wayne makes the same claim as in (a). Is he still right?

Solution: We are asking if $P(A \text{ and } B) / P(A) \geq P(A \text{ and } B) / P(A \text{ or } B)$, which is the same as asking if $P(A) \leq P(A \text{ or } B)$. But this is obviously true, since A is a subset of $(A \text{ union } B)$.

For this one, I DO want you to do all the calculations, and not just show formula. The values are small fractions.

By the way, this is problem 1.15 from p.57 in the textbook:

Solution to Problem 1.15. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $P(A \cap B | A)$ and $P(A \cap B | A \cup B)$. We have

$$P(A \cap B | A) = \frac{P((A \cap B) \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)},$$

and

$$P(A \cap B | A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}.$$

Since $P(A \cup B) \geq P(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH , HT , TH , TT are equally likely, we have

$$\frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \quad \frac{P(A \cap B)}{P(A \cup B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Problem Three. You are dealt 5 cards from a shuffled deck of 52 cards. What is the probability that you get three different suits and no face cards? [Hint: these two events are independent and you will need to consider all the ways that you could get three different suits. You don't know, of course, what the suits are.]

Solution: Let E = you get three different suits. There are $C(4,3) = 4$ ways of choosing the three suits. Having chosen the three suits, there are two different possible patterns (note these are unordered):

$$E_1 = A A A B C \quad \text{and} \quad E_2 = A A B B C$$

$$|E_1| = C(13,3)*C(13,1)*C(13,1) \quad \text{and} \quad |E_2| = C(13,2)*C(13,2)*C(13,1)$$

$$P(E) = C(4,3)*[C(13,3)*C(13,1)*C(13,1) + C(13,2)*C(13,2)*C(13,1)] / C(52,5) = 0.1961.$$

Let F = no face cards.

$$P(F) = C(40,5) / C(52,5) = 0.2532$$

$$\text{Thus, } P(\text{three different suits and no face cards is}) = P(E) * P(F) = 0.0497.$$

Problem Four. A group of 20 freshmen, 20 sophomores, 20 juniors, and 20 seniors is randomly assigned to 4 different classrooms for a test, such that there are 20 students in each classroom. What is the probability that each classroom has exactly 5 seniors?

Solution: This can be solved at least two ways. First, we might use *multinomial coefficients*. There are $80!$ possible permutations of students, and

$$A = 80! / (20! * 20! * 20! * 20!)$$

ways of assigning groups of students to the four classrooms. This is the sample space.

For the event of interest, there are

$$B = 20! / (5! * 5! * 5! * 5!)$$

ways of assigning the seniors to the four rooms, and

$$C = 60! / (15! * 15! * 15! * 15!)$$

ways of assigning the non-seniors to the four rooms. Then the probability is

$$B * C / A = 0.0163,$$

We could also use combinations:

$$A = C(80,20)*C(60,20)*C(40,20)*C(20,20)$$

$$B = C(20,5)*C(15,5)*C(10,5)*C(5,5) = 11732745024$$

$$C = C(60,15)*C(45,15)*C(30,15)*C(15,15)$$

and the result is the same.

Problem Five. From an ordinary deck of 52 cards we draw cards at random, with replacement, and successively until a face card (Jack, Queen, or King) is drawn.

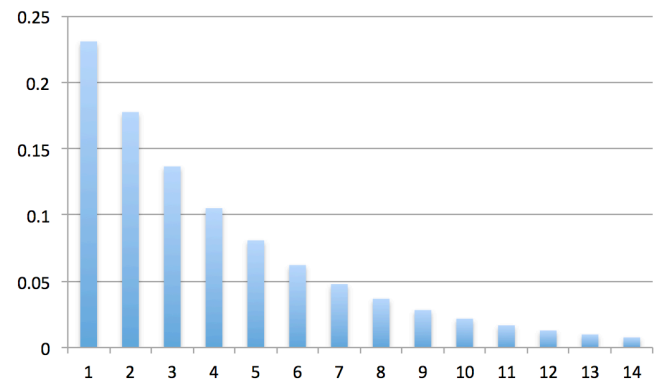
(a) Express this problem as a Random Variable X , giving (i) the $\text{Rng}(X)$, (ii) the formula for $P_X(y)$, and (iii) sketching the distribution.

Solution: The probability of a face card is $12/52 = 3/13$. So this is $\text{Geo}(3/13)$.

$$\text{Rng}(X) = \{ 1, 2, 3, \dots \}$$

$$P_X(y) = (10/13)^{y-1} * (3/13)$$

$$P_X = [0.2308, 0.1775, 0.1365, \dots]$$



(b) What is the probability that at least 3 draws are needed?

Solution: The simplest calculation is to find the probability of 2 non-face cards (since after that, any possibility fits our situation) or $(10/13)^2 = 0.5917$. Or, you could use the Exclusion Principle and count the probabilities of the cases to exclude:

$$\begin{aligned}
 & 1.0 - (P(X=1) + P(X=2)) \\
 = & 1.0 - (3/13 + (10/13)(3/13)) \quad \ll \text{this answer is fine} \\
 = & 1.0 - (0.2308 + 0.1775) \\
 = & 0.5917
 \end{aligned}$$

Problem Six. A fair die is rolled twice. Let A denote the event that the first roll shows at least as many dots as the second, and B denote the event that the sum of the dots is evenly divisible by 3. Are A and B independent? Explain carefully.

Solution: *The two events are independent.* $P(A) = 21/36$ and $P(B) = 12/36$ and $P(AB) = 7/36$. Since $P(AB) = P(A) * P(B)$, the two events are independent.

Outcomes from Two Dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12