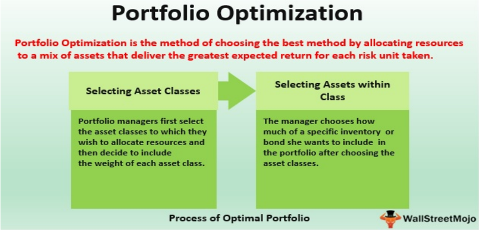
Overview

Introduction



- fundamental question to be asked:
  - In which of these would we invest? (bonds, options,security and commodity, mutual fund etc.)
  - How much would we invest in each of them?

Intro...

- Formally, financial portfolio optimization adheres to a formal approach in making investment decisions:
  - for selection of investment portfolios containing the financial instruments,
  - to allocate a specified capital over a number of available assets,
  - to meet certain pre-defined objectives,
  - to mitigate financial risks and ensure better preparedness for uncertainties,
  - to establish mathematical and computational methods on realistic constraints,
  - to manage profit and loss of the portfolio (across long and short positions), and,
  - to provide stability across inter and intraday market fluctuations etc.

Intro..

- Break through progress in porfolio optimization methods:

  -Markowitz's Modern Portfolio Theory [Markowitz, 1952]

  -Tobin James's work [Hester and James, 1967]

  - Sharpe's Capital Asset Pricing Model (CAPM) [Sharpe, 1964]

  -Fernholzs Stochastic Portfolio Theory [Fernholz, 2002]
  -Black-Litterman's model

  -Machine learning based methods (mechanism for regularizing the portfolio optimization, artificial neural networks and extreme value theory

Markowitz model

- Transforms asset allocation problem into optimization proble.
- There is an objective function to optimize and there are constraints to be respected.
- Objective is to find the values of decision variable (in our case the fraction of different assets classes to be allocated to maximize returns and averse the risk)
- The robustness of the model depends on selection of constraints and methods to decide over average returns and covariance structure of assets.

Markowitz model

We will look at following different variants of model:

1. Variance efficient model
2. Expected return efficient model
3. Parametric efficient model
4. Model with risk free return assets
5. Model with other constraints (extension of Markowitz model)

Markowitz Model

- Variance efficient Model

  $$\min\{x'\Sigma x|\mu' x = \mu_p, I'x = 1\}$$

  Here x is faction of total asset allocation to one particular class/type of asset
  Σ is covariance matrix (positive definite)
  Expected return of portfolio is μ_p and variance of portfolio is σ_p
  $\mu_p = \mu' x$ and $\sigma_p^2 = x'\Sigma x$

- Expected return-efficient Model

  $$\max\left\{\mu' x|x'\Sigma x = \sigma_p^2, \quad I'x = 1\right\}$$

Markowitz Model

- Parametric efficient model

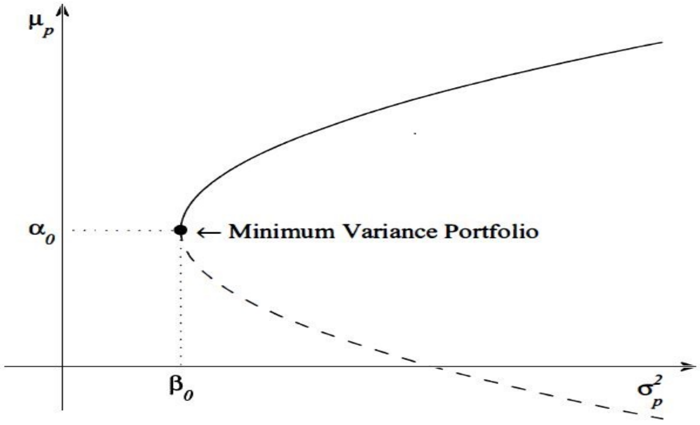
  $\min \left\{ -t\mu' x + \frac{1}{2} x' \Sigma x | l' x = 1 \right\}$
  this model is parameterized for the degree of risk aversion. the more the value of t the less risk averse the portfolio will be.

- solving this simple minimization we get

  $t = \frac{(\mu_p - \alpha_0)}{\alpha_1}$, and $t^2 = \frac{\left(\sigma_p^2 - \beta_0\right)}{\beta_2}$
  $\sigma_p^2 - \beta_0 = \left(\mu_p - \alpha_0\right)^2 / \alpha_1$

- Relation between variance and expected return is parabolic which is called efficient frontier.
- All portfolio strictly within the efficent frontier are inefficient.

Markowitz Model



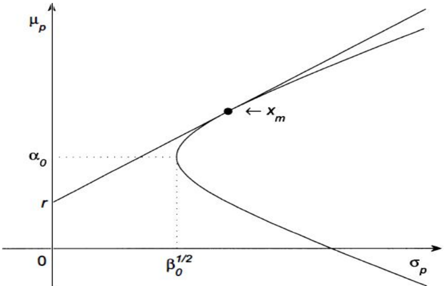
The plot is a parabola. Nose of the parabola is minimum variance portfolio which corresponds to $t = 0$

Efficient portfolio lies on the solid part of parabola curve.

- Now consider this problem

$$\begin{aligned}\text{minimize}: &\quad -t\,(\mu', r)\begin{bmatrix} x \\ x_{n+1}\end{bmatrix} + \frac{1}{2}\begin{bmatrix} x \\ x_{n+1}\end{bmatrix}'\begin{bmatrix} \Sigma & 0 \\ 0' & 0\end{bmatrix}\begin{bmatrix} x \\ x_{n+1}\end{bmatrix} \\ \text{subject to} &\quad : l'x + x_{n+1} = 1\end{aligned}$$

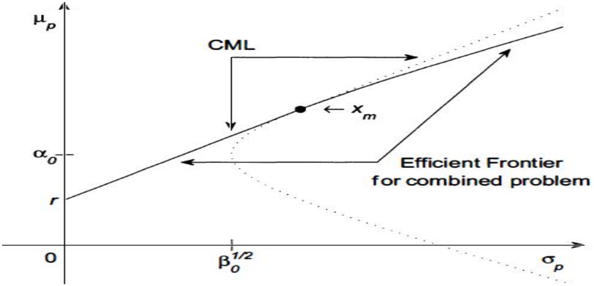
- Because $x_{n+1}(t)$ is a strictly decreasing function of t, as t is increased eventually $x_{n+1}(t)$ is reduced to zero. This occurs at x_m
- Point on efficient frontier where risk free asset fraction goes to zero is called market portfolio.



Markowitz Model

following results are worth mentioning (in the figure we assume no short sell):

1. $\mu_p - r = \sigma_p \left[(\mu - rI)' \Sigma^{-1} (\mu - rI)\right]^{\frac{1}{2}}$
2. $x(t) = \frac{t}{t_m} x_m$
3. Slope of the CML (capital market line) is $\frac{(\mu_m - r)}{\sigma_m}$



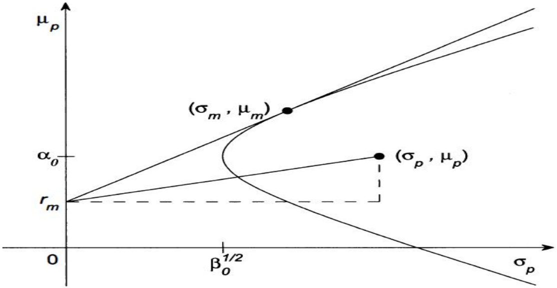
Sharpe Ratio

Following analysis is important because if analyst hypothesizes that one of μ_m, r_m, R changes what is the effect on other 2 parameters.

- Given the expected return on the market portfolio μ_m, what is the implied risk free return r_m?
- Given the implied risk free return r_m, what is the implied market portfolio x_m or equivalently, what is μ_m or t_m from which all other relevant quantities can be deduced?
- Given the slope R of the CML, what is the implied market portfolio x_m or equivalently, what is μ_m or t_m from which all other relevant quantities can be deduced?

Sharpe Ratio

- To answer these type of queries we will take the optimization route. Following figures help understand the sharpe ratio and objectives of optimization problem.



- $\max\left\{\frac{\mu' x - r_m}{(x'\Sigma x)^{\frac{1}{2}}} \middle| l'x = 1\right\}$

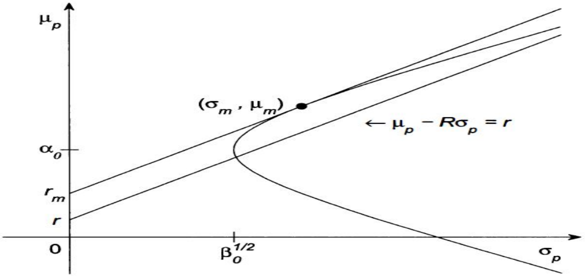
Sharpe ratio

Solving the above optimization problem is same as solving parametric efficient model. following results are found:

- For fixed r_m, let x_0 be optimal for Sharpe ration problem Then x_0 is also optimal for parametric efficient model with $t = x_0' \Sigma x_0 / (\mu' x_0 - r_m)$
- For fixed $t = t_1$, let $x_1 \equiv x(t_1)$ be optimal for (4.9). Then x_1 is optimal for (4.6) with $r_m = \mu' x_1 - x_1' \Sigma x_1 / t_1$

Sharpe ratio

If Sharpe ratio R is known, the question to be answered is to what market ratio does this R corresponds to and what is the corresponding implied risk free return.



$$\max\left\{\mu' x - R\left(x'\Sigma x\right)^{\frac{1}{2}} \middle| l' x = 1\right\}$$

Sharpe ratio

Solving the optimization problem defined as above gives following conclusion:

- For fixed Sharpe ratio R, let x_0 be optimal for Sharpe ratio problem, then x_0 is also optimal for parametric efficient model with $t = \frac{\left(x_0' \Sigma x_0\right)^{\frac{1}{2}}}{R}$.
- For fixed $t = t_1$, let $x_1 \equiv x\left(t_1\right)$ be optimal for parametric efficient then x_1 is optimal for Sharpe ratio model with $R = \frac{\left(x_1' \Sigma x_1\right)^{\frac{1}{2}}}{t_1}$
- Let $x\left(t_1\right) = x_1$ be a point on the efficient frontier for $t = t_1$. Then $\left(\mu' x_1 - r_m\right) / \left(x_1' \Sigma x_1\right)^{\frac{1}{2}}$ is the maximum Sharpe ratio for $r_m = \mu' x_1 - x_1' \Sigma x_1 / t_1$, and , $r_m = \mu' x_1 - R\left(x_1' \Sigma x_1\right)^{\frac{1}{2}}$ is the maximum risk free return with Sharpe ratio $R = \left(x_1' \Sigma x_1\right)^{\frac{1}{2}} / t_1$

Constrained Markowitz model

- Many extensions are possible. One standard extension, for example, is to allow short positions, *i.e.*, $x_i < 0$. To do this we introduce variables x_{long} and x_{short} with

$$x_{\text{long}} \succeq 0, \quad x_{\text{short}} \succeq 0, \quad x = x_{\text{long}} - x_{\text{short}}, \quad 1^T x_{\text{short}} \leq \eta 1^T x_{\text{long}}$$

- As another extension we can include linear transaction costs in the portfolio optimization problem. Starting from a given initial portfolio x_{init} we buy and sell assets to achieve the portfolio x, which we then hold over the period as described above.We have the constraints

$$x = x_{\text{init}} + u_{\text{buy}} - u_{\text{sell}}, \quad u_{\text{buy}} \succeq 0, \quad u_{\text{sell}} \succeq 0$$

  We replace the simple budget constraint $1^T x = 1$ with the condition that the initial buying and selling, including transaction fees, involves zero net cash:

$$(1 - f_{\text{sell}}) 1^T u_{\text{sell}} = (1 + f_{\text{buy}}) 1^T u_{\text{buy}}$$

Constrained Markowitz model

- Portfolio optimization with loss risk constraints. We assume here that the price change vector $p \in \mathbf{R}^n$ is a Gaussian random variable, with mean $\bar{p}$ and covariance Σ. Therefore the return r is a Gaussian random variable with mean $\bar{r} = \bar{p}^T x$ and variance $\sigma_r^2 = x^T \Sigma x$ Consider a loss risk constraint of the form

$$\mathrm{prob}(r \leq \alpha) \leq \beta$$

where α is a given unwanted return level (*e.g.*, a large loss) and β is a given maximum probability.

$$\begin{array}{ll} \text{maximize} & \bar{p}^T x \\ \text{subject to} & \bar{p}^T x + \Phi^{-1}(\beta) \left\| \Sigma^{1/2} x \right\|_2 \geq \alpha \\ & x \succeq 0, \quad 1^T x = 1 \end{array}$$

Constrained Markowitz model

- Bounding portfolio risk with incomplete covariance information. SDP to solve will be following:
  $\bar{p} = \mathbf{E}p, \quad \Sigma = \mathbf{E}(p - \bar{p})(p - \bar{p})^T$
  In the risk bounding problem considered here, we assume the portfolio x is known, but only partial information is available about the covariance matrix Σ. We might have, for example, an upper and lower bound on each entry:

$$L_{ij} \leq \Sigma_{ij} \leq U_{ij}, \quad i, j = 1, \ldots, n$$

  define the worst-case variance of the portfolio as

$$\sigma_{\mathrm{wc}}^2 = \sup \left\{ x^T \Sigma x | L_{ij} \leq \Sigma_{ij} \leq U_{ij}, i, j = 1, \ldots, n, \Sigma \succeq 0 \right\}$$

  We can find σ_{wc} by solving the SDP

$$\begin{array}{ll} \text{maximize} & x^T \Sigma x \\ \text{subject to} & L_{ij} \leq \Sigma_{ij} \leq U_{ij}, \quad i, j = 1, \ldots, n \\ & \Sigma \succeq 0 \end{array}$$

Constrained Markowitz model

from an optimal Σ for the SDP.we can take
$p = \bar{p} + \Sigma^{1/2} v$, where v is any random vector with $\mathbf{E}v = 0$ and $\mathbf{E}vv^T = I$
we can add other constraints as follows:

- Known variance of certain portfolios. We might have equality constraints such as

  $$u_k^T \Sigma u_k = \sigma_k^2$$

  where u_k and σ_k are given.
- Including effects of estimation error. If the covariance Σ is estimated from empirical data, the estimation method will give an estimate $\hat{\Sigma}$, and some information about the reliability of the estimate, such as a confidence ellipsoid. This can be expressed as

  $$C(\Sigma - \hat{\Sigma}) \leq \alpha$$

  where C is a positive definite quadratic form on $\mathbf{S}^n$, and the constant α determines the confidence level.

Constrained Markowitz model

- Factor models. The covariance might have the form

$$\Sigma = F\Sigma_{\text{factor}}\, F^T + D$$

where $F \in \mathbf{R}^{n\times k}, \Sigma_{\text{factor}} \in \mathbf{S}^k$, and D is diagonal. This corresponds to model of the price changes of the form

$$p = Fz + d$$

where z is a random variable (the underlying factors that affect the price changes) and d_i are independent (additional volatility of each asset price). We assume that the factors are known. since Σ is linearly related to Σ factor and D, we can impose any convex constraint on them (representing prior information) and still compute σ_{wc} using convex optimization.

- Information about correlation coefficients.

$$l_{ij} \leq \rho_{ij} = \frac{\Sigma_{ij}}{\Sigma_{ii}^{1/2}\Sigma_{jj}^{1/2}} \leq u_{ij}, \quad i, j = 1, \ldots, n$$

since Σ_{ii} are known, but Σ_{ij} for $i \neq j$ are not.

Summary

- In a mean-variance optimization framework, accurate estimation of the variance-covariance matrix is paramount. Quantitative techniques that use Monte-Carlo simulation with the Gaussian copula and well-specified marginal distributions are effective.
- All the above models are one of quadratic, second order cone, robust linear or semi-definite programming types and there are solvers to solve them (CVX packages with python and MATLAB). Going deep into solver algorithms is all another vast subject to cover.
- Any number of consistent constraints can be added to mean-variance model as required.

Reference

1. Best Micheal J. Portfolio Optimization, Chapman Hall/CRC Taylor Francis Group, 2010.

2. Boyd Stephe, Lieven Vandenberghe, Convex Optimization, Cambridge University Press 2004.

3. https://en.wikipedia.org/wiki/Portfolio$_o$*ptimization*, *Wikipedia*.

Thank you