Consider the following linear elliptic problem

$$\left.\begin{aligned} -\nabla\cdot(K\nabla p) &= f, && \text{in } \Omega \\ p &= 0, && \text{on } \partial\Omega \end{aligned}\right\} \quad (S)$$

Assume that we have q, an approximation to p.

$$e := |\underbrace{p}_{\text{true}} - \underbrace{q}_{\text{approx.}}|_{\ast} \quad \text{some suitable norm}$$

(i) If we have p, we have e.

(ii) If we don't have p. How do we compute e?

↳ Error estimation.

$$\underbrace{\mu^{\ominus}}_{\text{lower bound / minorant}} \leq e \leq \underbrace{\mu^{\oplus}}_{\text{upper bound / majorant}}$$

$$\mu^{\oplus} = \mu^{\oplus}(q, \dots)$$

Today's aim: Derive an upper bound $\mu^{\oplus}$ for e, measured in the energy norm.

- Assumption for (S)
  (i) $\Omega \subset \mathbb{R}^2$, is open bounded with Lipschitz bound $\partial\Omega$.
  (ii) K is symmetric, positive-definite :

$$c_1^2 |\xi|^2 \le K\xi\cdot\xi \le c_2^2 |\xi|^2 \quad \forall\ \xi \in \mathbb{R}^2.$$

(iii) $f \in L^2(\Omega)$.

- *Function spaces*

(i) We say $f \in L^2(\Omega)$ if $\int_\Omega |f|^2\,dx < +\infty$.

$$\|f\| = \left(\int_\Omega |f|^2\,dx\right)^{1/2}$$

$$(f,g) = \int_\Omega f\cdot g\,dx$$

(ii) $H^1(\Omega)$: $\{q \in L^2(\Omega) : \nabla q \in [L^2(\Omega)]^2\}$.

(iii) $H_0^1(\Omega)$: $\{q \in H^1(\Omega) : q = 0 \text{ on } \partial\Omega\}$.

(iv) $H(\mathrm{div},\Omega)$: $\{\vec{v} \in [L^2(\Omega)]^2 : \nabla\cdot\vec{v} \in L^2(\Omega)\}$.

- Energy norm

$$B(p,q) := (K\nabla p, \nabla q) \quad \forall\ p,q \in H_0^1(\Omega)$$

$$|||q||| = B(q,q)^{1/2} = \|K^{1/2}\nabla q\| \quad \forall\ q \in H_0^1(\Omega).$$

- Tools

(i) Green's thm. There holds

$$(\vec{v}, \nabla q) + (\nabla\cdot\vec{v}, q) = 0, \quad \forall\ \vec{v} \in H(\mathrm{div},\Omega),\ q \in H_0^1(\Omega).$$

(ii) Poincaré_Friedrich inequality. There holds

$$\|q\| \le C_{F,\Omega} \|\nabla q\|, \quad \forall q \in H_0^1(\Omega).$$

(iii) Def: Weak primal form of (S). Find $p \in H_0^1(\Omega)$ s.t.

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$$(K\nabla p, \nabla q) = (f, q), \quad \forall q \in H_0^1(\Omega). \quad (W)$$

■ A posteriori bound.

Thm (1). Let $p \in H_0^1(\Omega)$ be a solution to (W), and let $q \in H_0^1(\Omega)$ be arbitrary. Then,

$$|||p-q||| \le \underbrace{\|K^{-1/2}(\vec{v}+K\nabla q)\|}_{\text{diffusive error}} + \underbrace{\frac{C_{F,\Omega}}{c_1} \|f-\nabla\cdot\vec{v}\|}_{\text{residual error}}, \quad \forall \vec{v} \in H(\mathrm{div},\Omega)$$

$$:= \mathcal{M}^{\oplus}(q, \vec{v}; f).$$

Proof:

$$|||p-q|||^2 = B(p-q, p-q)$$

$$= (K\nabla(p-q), \nabla(p-q))$$

$$= (K\nabla p, \nabla(p-q)) + (-K\nabla q, \nabla(p-q))$$

$$= (f, p-q) + \quad ''$$

Add the following identity:

$$-(\vec{v}, \nabla(p-q)) - (\nabla\cdot\vec{v}, p-q) = 0, \quad \forall \vec{v} \in H(div, \Omega)$$

$$= (f-\nabla\cdot\vec{v}, p-q) + (-[\vec{v}+K\nabla q], \nabla(p-q))$$

$$|||p-q|||^2 = \text{''} + (-K^{-1/2}[\vec{v}+K\nabla q], K^{1/2}\nabla(p-q))$$

$$\le \|f-\nabla\cdot\vec{v}\| \, \underbrace{\|p-q\|} + \underbrace{\|K^{-1/2}(\vec{v}+K\nabla q)\|} \, \underbrace{\|K^{1/2}\nabla(p-q)\|}_{|||p-q|||}$$

$$\le C_{F,\Omega}\|f-\nabla\cdot\vec{v}\| \, \underbrace{\|\nabla(p-q)\|} + \text{''}$$

$$\le \frac{C_{F,\Omega}}{C_1}\|f-\nabla\cdot\vec{v}\| \, |||p-q||| + \text{''}$$

$$|||p-q|||^{\cancel{2}} \le \cancel{|||p-q|||}\left(\frac{C_{F,\Omega}}{C_1}\|f-\nabla\cdot\vec{v}\| + \|K^{-1/2}(\vec{v}+K\nabla q)\|\right)$$

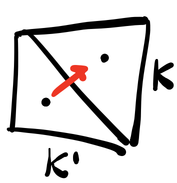
$$:= \mathcal{M}^{\oplus}(q, \vec{v}, f) \qquad \blacksquare$$

- Cell-centered Finite volume method (CCFVM).

$$\vec{u} = -K\nabla p$$

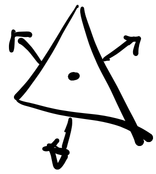
$$\int_K \nabla\cdot\vec{u}\,dx = \int_{\partial K} \vec{u}\cdot\vec{n}\,ds = \sum_{e\in\mathcal{E}_K} \overbrace{\vec{u}_{K,e}\cdot\vec{n}_{K,e} A_{K,e}}^{F_{K,e}} = \sum_{e\in\mathcal{E}_K} F_{K,e} = \int_K f\,dx$$

$$F_{K,e} \simeq T_{K,e}\left(p_h|_K - p_h|_{K'}\right)$$



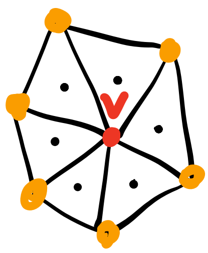
The result of applying TPFA:

(i) $p_h|_K \in \mathbb{P}_0(K) \quad \forall\ K \in \mathcal{T}_h$



(ii) $F_{K,e} \in \mathbb{P}_0(e) \quad \forall\ e \in \mathcal{E}_K \ \forall\ K \in \mathcal{T}_h.$

(i) Potential reconstruction $(\tilde{p}_h)$.



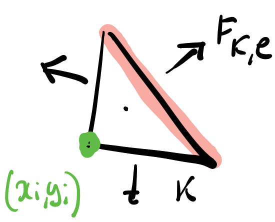
$\mathcal{T}_v$: set of K associated with v.

$$\tilde{p}_h|_v = \begin{cases} \dfrac{\sum_{K \in \mathcal{T}_v} |K|\, p_h|_K}{\sum_{K \in \mathcal{T}_v} |K|}, & v \text{ is internal} \\ 0, & v \text{ is external} \end{cases}$$

Locally: $\tilde{p}_h|_K \in \mathbb{P}_1(K)$

Globally: $\tilde{p}_h \in H^1(\mathcal{T}_h) \cap H^1_0(\Omega) \subset H^1_0(\Omega)$ ✓

(ii) Flux reconstruction. ↱ $\mathbb{RT}_0$ basis functions



$$\vec{u}_h|_K = \sum_{e \in \mathcal{E}_K} F_{K,e}\, \vec{\varphi}_{K,e}$$

$$\vec{\varphi}_{K,e} = \frac{\operatorname{sign}(\vec{n}_e)}{2|K|} \begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix}$$

Locally $\vec{u}_h|_K \in \mathbb{RT}_0(K)$

Globally: $\vec{u}_h \in \mathbb{RT}_0(\mathcal{T}_h) \subset H(\mathrm{div}, \Omega)$ ✓

In summary: In Thm (1).

(i) Set $q = \tilde{p}_h \in H_0^1(\Omega)$

(ii) Set $\vec{v} = \vec{u}_h \in H(\mathrm{div}, \Omega)$.