Clifford-algebraic formulation of nonlinear conformal transformations in electrodynamics

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We derive a set of Clifford-algebraic formulas for two major nonlinear conformal transformations of the physical quantities related to Maxwell's equations. The superiority of these formulas over their vector-tensorial counterparts are demonstrated through three illustrative examples.

KEYWORDS: Clifford algebra, geometric algebra, special conformal transformation, conformal inversion, classical electrodynamics.

I. INTRODUCTION

Maxwell's equations are renowned for being form-invariant under Lorentz transformation, which is a linear transformation of spacetime. As shown by Cunningham and Bateman back in 1909 [1, 2], this invariance can be extended to conformal transformation which is nonlinear in general.

While the manifestations of conformal transformation in classical electrodynamics have been extensively explored [3], the inherent complexity of the mathematical formulation continues to pose challenges for both novices and experienced researchers in this field. This complexity, in our opinion, arises mainly from the prevalent use of vectors and tensors in electrodynamics as well as special relativity. Although the advantage of invoking Clifford algebra (also known as geometric algebra) as a substitute is recognized by some authors [4, 5, 6], there has been little research on applying Clifford algebra to conformal transformation strictly in four-dimensional spacetime.

Inspired by the Clifford-algebraic versions of the Lorentz transformations for four-vectors and electromagnetic tensor, we set a goal to derive the analogues of conformal inversion, which is the simplest nonlinear conformal transformation. Through a series of attempts, we achieved a set of concise formulas in covariant as well as non-covariant Clifford-algebraic formulations. Building on this foundation, we further generalized the formulas to apply to the most important nonlinear conformal transformation, the special conformal transformation (SCT). As expected, these formulas turned out to exhibit superior simplicity and elegance compared to their vector-tensorial counterparts.

The organization of this paper is as follows: In the next section we introduce the necessary background knowledge, then we derive the said formulas step by step in Sec. III. In order to demonstrate the superiority of these formulas, we work on three practical examples in Sec. IV. Finally in Sec. V we enumerate the advantages of these newly-found formulas. Two appendices are provided at the end of this paper to enhance readers' comprehension of this research.

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II. PRELIMINARIES

A. Conventions and notations

Throughout this paper c = 1, and all quantities are nondimensionalized.

Euclidean space is defined as a three-dimensional dimensionless space with the coordinates (x, y, z) and metric diag(1, 1, 1). The corresponding orthonormal basis is denoted by $\{\hat{x}, \hat{y}, \hat{z}\}$, and the position vector by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}.\tag{1}$$

A Euclidean vector is denoted by

$$\vec{U} = U_x \hat{x} + U_y \hat{y} + U_z \hat{z},\tag{2}$$

and the scalar product of two Euclidean vectors by

$$\vec{U} \cdot \vec{V} = U_x V_x + U_y V_y + U_z V_z. \tag{3}$$

Minkowski space is defined as a (1+3)-dimensional dimensionless space with the coordinates $\mathbf{x}^{\alpha}=(t,x,y,z)$ and metric diag $(1,-1,-1,-1)=:\eta_{\alpha\beta}$, where the Greek indices run from 0 to 3. The corresponding orthonormal basis is denoted by $\{\mathbf{e}_{\alpha}\}$, and the position four-vector by

$$\mathbf{x} = \mathbf{x}^{\alpha} \mathbf{e}_{\alpha} \tag{4}$$

under the summation convention. Other four-vectors follow the same convention, for example,

$$A = A^{\alpha} \mathbf{e}_{\alpha} \tag{5}$$

means the contravariant components of A are

$$A^{\alpha} = (A^{0}, A^{1}, A^{2}, A^{3}) = (A_{0}, -A_{1}, -A_{2}, -A_{3}).$$
(6)

The scalar product of two four-vectors is denoted by

$$A \cdot J = \eta_{\alpha\beta} A^{\alpha} J^{\beta} = A^{\alpha} J_{\alpha}. \tag{7}$$

We always use $\mathbf{x}'^{\mu}=(t',x',y',z')$ for the new coordinates after conformal inversion, and $\mathbf{x}''^{\mu}=(t'',x'',y'',z'')$ for that after special conformal transformation. For a generic conformal transformation, the new coordinates are denoted by $\tilde{\mathbf{x}}^{\mu}=(\tilde{t},\tilde{x},\tilde{y},\tilde{z})$, hence both x'^{μ} and x''^{μ} are special cases of \tilde{x}^{μ} . Functions of these coordinates follow the same convention, for example, $\vec{E}'(t',x',y',z')$ denotes the transformed electric field after conformal inversion.

B. Conformal transformation

The conformal transformation

$$\tilde{\mathbf{x}}^{\mu} = \mathcal{C}^{\mu}(\mathbf{x}) \tag{8}$$

is defined as a one-to-one relation in Minkowski space that satisfies

$$\Lambda^2 \frac{\partial \tilde{\mathbf{x}}^{\mu}}{\partial \mathbf{x}^{\alpha}} \frac{\partial \tilde{\mathbf{x}}^{\nu}}{\partial \mathbf{x}^{\beta}} \eta_{\mu\nu} = \eta_{\alpha\beta} \text{ with } \Lambda > 0.$$
 (9)

It is obvious that Λ is related to the Jacobian determinant of this transformation, i.e.,

$$\Lambda = \left| \det \left[\frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{x}} \right] \right|^{-\frac{1}{4}}.$$
 (10)

There exist two interpretations for Eq. (8) that correspond to the active and the passive transformations. In this paper we adopt the former, i.e., interpreting Eq. (8) as mapping each spacetime point and its contents to another point. The results we obtain are also valid for the so-called conformal coordinate transformation [7] which is the passive conformal transformation plus a scale transformation, a detailed discussion can be found in [8].

It can be proved that every conformal transformation is a composition of four fundamental ones [9]:(1) Four-dimensional dilation

$$\tilde{\mathbf{x}} = \lambda^{-1} \mathbf{x} \iff \tilde{\mathbf{x}}^{\mu} = \lambda^{-1} \mathbf{x}^{\mu},\tag{11}$$

where λ is a positive constant; (2) Four-dimensional translation

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{b} \iff \tilde{\mathbf{x}}^{\mu} = \mathbf{x}^{\mu} + \mathbf{b}^{\mu},\tag{12}$$

where $b = b^{\mu}e_{\mu}$ is a constant four-vector; (3) Lorentz transformation

$$\tilde{\mathbf{x}}^{\mu} = \mathcal{L}^{\mu}_{\alpha} \mathbf{x}^{\alpha} \text{ with } \mathcal{L}^{\mu}_{\beta} \mathcal{L}^{\nu}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}, \tag{13}$$

where \mathcal{L} is a constant matrix, and (4) Special conformal transformation (abbreviated as SCT henceforward) which is defined in Eq. (19). The degrees of freedom of these transformations are one, four, six, and four respectively.

C. Conformal inversion

Examples of linear conformal transformation include Eqs. (11)–(13). As for the nonlinear ones, the simplest example is the conformal inversion

$$x' = \varepsilon x/x^2 \iff x'^{\mu} = \varepsilon x^{\mu}/x^2,$$
 (14)

where $\varepsilon = 1$ or -1, and

$$\mathbf{x}^2 := t^2 - r^2 \neq 0,\tag{15}$$

i.e., the light cone $t^2 = r^2$ is excluded from the domain of Eq. (14). This transformation will be referred to as Inversion from now on.

From the Jacobian matrix element

$$\frac{\partial \mathbf{x}'^{\mu}}{\partial \mathbf{x}^{\alpha}} = \varepsilon \left(\mathbf{x}^{2} \delta^{\mu}_{\alpha} - 2 \mathbf{x}^{\mu} \mathbf{x}_{\alpha} \right) / (\mathbf{x}^{2})^{2}, \tag{16}$$

we can prove that Inversion is a conformal transformation with

$$\Lambda = |\mathbf{x}^2| = |\mathbf{x}'^2|^{-1}.\tag{17}$$

D. Special conformal transformation (SCT)

As the only nonlinear fundamental conformal transformation, SCT is made of three consecutive conformal transformations (Inversion \longrightarrow translation \longrightarrow Inversion):

$$x' = \varepsilon x/x^2 \text{ with } x^2 \neq 0,$$

$$\tilde{x} = x' + \varepsilon a \text{ with } a^2 \neq 0,$$

$$x'' = \varepsilon \tilde{x}/\tilde{x}^2 \text{ with } \tilde{x}^2 \neq 0.$$
(18)

The composition of Eq. (18) takes the form

$$x'' = \Sigma^{-1}(x + x^2 a) \iff x''^{\mu} = \Sigma^{-1}(x^{\mu} + x^2 a^{\mu}), \tag{19}$$

where

$$\Sigma := x^{2} \tilde{x}^{2} = (x'^{2} x''^{2})^{-1}$$

$$= 1 + 2a \cdot x + a^{2} x^{2} = (1 - 2a \cdot x'' + a^{2} x''^{2})^{-1}.$$
(20)

It is easy to prove that Eqs. (18) and (19) are conformal transformations with $\Lambda = |\Sigma|$, and $\Sigma = 0$ is a light cone whose vertex lies at $-a^{\mu}/a^2$ [10].

E. Conformal transformations of electrodynamic quantities

Under the active conformal transformation where the metric is fixed, the transformation laws for the components of potential four-vector, current density four-vector ("current four-vector" for short henceforward), and electromagnetic tensor are as follows [8].

$$\tilde{A}^{\mu}(\tilde{\mathbf{x}}) = \vartheta \Lambda^2 \frac{\partial \tilde{\mathbf{x}}^{\mu}}{\partial \mathbf{x}^{\alpha}} A^{\alpha}(\mathbf{x}), \tag{21}$$

$$\tilde{A}_{\mu}(\tilde{\mathbf{x}}) = \eta_{\mu\nu}\tilde{A}^{\nu}(\tilde{\mathbf{x}}) = \vartheta \frac{\partial \mathbf{x}^{\alpha}}{\partial \tilde{\mathbf{x}}^{\mu}} A_{\alpha}(\mathbf{x}), \tag{22}$$

$$\tilde{J}^{\mu}(\tilde{\mathbf{x}}) = \vartheta \Lambda^4 \frac{\partial \tilde{\mathbf{x}}^{\mu}}{\partial \mathbf{x}^{\alpha}} J^{\alpha}(\mathbf{x}), \tag{23}$$

$$\tilde{J}_{\mu}(\tilde{\mathbf{x}}) = \eta_{\mu\nu} \tilde{J}^{\nu}(\tilde{\mathbf{x}}) = \vartheta \Lambda^{2} \frac{\partial \mathbf{x}^{\alpha}}{\partial \tilde{\mathbf{x}}^{\mu}} J_{\alpha}(\mathbf{x}), \tag{24}$$

$$\tilde{F}^{\mu\nu}(\tilde{\mathbf{x}}) = \vartheta \Lambda^4 \frac{\partial \tilde{\mathbf{x}}^{\mu}}{\partial \mathbf{x}^{\alpha}} \frac{\partial \tilde{\mathbf{x}}^{\nu}}{\partial \mathbf{x}^{\beta}} F^{\alpha\beta}(\mathbf{x}), \tag{25}$$

$$\tilde{F}_{\mu\nu}(\tilde{\mathbf{x}}) = \eta_{\mu\rho}\eta_{\nu\sigma}\tilde{F}^{\rho\sigma}(\tilde{\mathbf{x}}) = \vartheta \frac{\partial \mathbf{x}^{\alpha}}{\partial \tilde{\mathbf{x}}^{\mu}} \frac{\partial \mathbf{x}^{\beta}}{\partial \tilde{\mathbf{x}}^{\nu}} F_{\alpha\beta}(\mathbf{x}), \tag{26}$$

where

$$\vartheta := \operatorname{sgn}\left(\frac{\partial \tilde{t}}{\partial t}\right). \tag{27}$$

For Inversion, we can derive

$$\vartheta = -\varepsilon \tag{28}$$

from Eq. (16), and it follows that $\vartheta = (-\varepsilon)^2 = 1$ for SCT.

III. DERIVATIONS OF THE FORMULAS

A. Review of Clifford algebra $C\ell_{1,3}$

Clifford algebra $Cl_{1,3}$ of Minkowski space is defined as a 16-dimensional vector space with the basis

$$\{1, \mathbf{e}_{\alpha}, \mathbf{e}_{\alpha} \mathbf{e}_{\beta}, \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\gamma}, \mathbf{e}_{0} \mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{3}\},\tag{29}$$

where the associative product between e_{α} 's is called geometric product and is defined by

$$\mathsf{e}_{\alpha}\mathsf{e}_{\beta} + \mathsf{e}_{\beta}\mathsf{e}_{\alpha} = 2\eta_{\alpha\beta}.\tag{30}$$

Once the geometric product is introduced, x^2 in Eq. (15) is no longer a symbol. It is straightforward to derive $xy + yx = 2x \cdot y$ from Eqs. (4) and (30), and it follows that

$$x^{2} = xx = x \cdot x = t^{2} - r^{2}.$$
 (31)

Accordingly, the left hand side of Eq. (19) can be written as

$$x'' = \Sigma^{-1}(1 + ax)x = \Sigma^{-1}x(1 + xa), \tag{32}$$

where

$$\Sigma = 1 + 2a \cdot x + a^2 x^2 = (1 + ax)(1 + xa) = (1 + xa)(1 + ax)$$
$$= (1 - 2a \cdot x'' + a^2 x''^2)^{-1} = [(1 - ax'')(1 - x''a)]^{-1} = [(1 - x''a)(1 - ax'')]^{-1}.$$
(33)

When we use $\mathcal{C}\ell_{1,3}$ as the mathematical tool for electrodynamics [4, 5], as long as magnetic monopole is not taken into account, only two kinds of basis vectors of this algebra are needed, i.e., $\{e_{\alpha}\}$ for four-vectors, and $\{e_{\alpha}e_{\beta}\}$ for bivectors that correspond to anti-symmetric tensors of rank two. We will discuss them separately in the following paragraphs.

(i) $\{e_{\alpha}\}$: Both of the potential and current four-vectors take the form of Eq. (5). Under four-dimensional dilation Eq. (11), the four-vector W(x) transforms as

$$\tilde{W}(\tilde{\mathbf{x}}) = \lambda^n W(\mathbf{x}),\tag{34}$$

where n = 1 for potential four-vector and n = 3 for current four-vector according to scaling law; while under four-dimensional translation Eq. (12), the corresponding transformation is simply

$$\tilde{W}(\tilde{\mathbf{x}}) = W(\mathbf{x}). \tag{35}$$

Involving no geometric product, Eqs. (34) and (35) are the same as those in vector-tensorial formulation. On the other hand, formulas of Lorentz transformations usually contain geometric products. For example, parity transformation

$$\tilde{\mathbf{x}}^{\mu} = (t, -x, -y, -z),$$
(36)

which is one of the simplest Lorentz transformations, leads to the transformation law

$$\tilde{W}(\tilde{\mathbf{x}}) = \mathbf{e}_0 W(\mathbf{x}) \mathbf{e}_0, \tag{37}$$

which includes the Clifford-algebraic version of Eq. (36), $\tilde{x} = e_0 x e_0$, as a special case.

Note that Eq. (34) is compatible with Eqs. (21)-(24) with $\vartheta = 1$ and $\Lambda = \lambda$, while both Eqs. (35) and (37) are compatible with Eqs. (21)-(24) with $\vartheta = 1$ and $\Lambda = 1$.

For completeness, we list the Clifford-algebraic formulas for four kinds of Lorentz transformations in Appendix II.

(ii) $\{e_{\alpha}e_{\beta}\}$: In electrodynamics, the most important bivector is

$$F = \frac{1}{2}F^{\alpha\beta}\mathbf{e}_{\alpha}\mathbf{e}_{\beta},\tag{38}$$

where

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$
(39)

is the contravariant electromagnetic tensor. F in Eq. (38) is called Faraday bivector, and its explicit expression is

$$F = E_x e_1 e_0 + E_y e_2 e_0 + E_z e_3 e_0 + B_x e_3 e_2 + B_y e_1 e_3 + B_z e_2 e_1. \tag{40}$$

Under four-dimensional dilation Eq. (11), four-dimensional translation Eq. (12), and parity transformation Eq. (36), Faraday bivector transforms respectively as

$$\tilde{F}(\tilde{\mathbf{x}}) = \lambda^2 F(\mathbf{x}),\tag{41}$$

$$\tilde{F}(\tilde{\mathbf{x}}) = F(\mathbf{x}),\tag{42}$$

$$\tilde{F}(\tilde{\mathbf{x}}) = \mathbf{e}_0 F(\mathbf{x}) \mathbf{e}_0,\tag{43}$$

where Eq. (41) is compatible with Eqs. (25) and (26) with $\vartheta = 1$ and $\Lambda = \lambda$, while both Eqs. (42) and (43) are compatible with Eqs. (25) and (26) with $\vartheta = 1$ and $\Lambda = 1$.

It is worthwhile to point out the similarity between Eqs. (37) and (43), which finds no analogue in vector-tensorial formulation.

B. Formulas for Inversion

In order to derive the formulas associated with Inversion that imitate Eqs. (37) and (43), we first introduce an identity

$$x^{4} \frac{\partial x^{\prime \mu}}{\partial x^{\alpha}} \mathbf{e}_{\mu} = -\varepsilon \mathbf{x} \mathbf{e}_{\alpha} \mathbf{x},\tag{44}$$

which can be proved via Eq. (16) and

$$\mathbf{e}_{\alpha}\mathbf{x} + \mathbf{x}\mathbf{e}_{\alpha} = 2\mathbf{x}_{\alpha},\tag{45}$$

which is a derivative of Eq. (30).

Using this identity in company with Eqs. (17), (21), and (28), we obtain the sought-for formula for potential four-vector,

$$A' = A'^{\mu} \mathbf{e}_{\mu} = (-\varepsilon) \mathbf{x}^4 \frac{\partial \mathbf{x}'^{\mu}}{\partial \mathbf{x}^{\alpha}} A^{\alpha} \mathbf{e}_{\mu} = \mathbf{x} A \mathbf{x}, \tag{46}$$

where the arguments of A' = A'(x') and A = A(x) are related by Eq. (14).

The Inversion formula of Faraday bivector can be derived as follows: According to Eqs. (17), (25), and (28),

$$F' = \frac{1}{2}F'^{\mu\nu}\mathsf{e}_{\mu}\mathsf{e}_{\nu} = \frac{1}{2}(-\varepsilon)\mathsf{x}^{8}\frac{\partial\mathsf{x}'^{\mu}}{\partial\mathsf{x}^{\alpha}}\frac{\partial\mathsf{x}'^{\nu}}{\partial\mathsf{x}^{\beta}}F^{\alpha\beta}\mathsf{e}_{\mu}\mathsf{e}_{\nu}. \tag{47}$$

Employing Eq. (44) twice, we convert the above relation to

$$F' = \frac{1}{2}(-\varepsilon)^3(\mathbf{x}\mathbf{e}_{\alpha}\mathbf{x})(\mathbf{x}\mathbf{e}_{\beta}\mathbf{x})F^{\alpha\beta} = -\varepsilon\mathbf{x}^2\mathbf{x}F\mathbf{x} = -\varepsilon\Omega\mathbf{x}F\mathbf{x},\tag{48}$$

where

$$\Omega := x^2 = x'^{-2}. \tag{49}$$

Note that the results in Eqs. (46) and (48) are also quite similar.

Then, using the inverse of Eq. (14),

$$\mathbf{x} = \varepsilon \mathbf{x}'^{-2} \mathbf{x}' \iff \mathbf{x}^{\mu} = \varepsilon \mathbf{x}'^{-2} \mathbf{x}'^{\mu},\tag{50}$$

we can re-express the results of Eqs. (46) and (48) in terms of the new coordinates,

$$A' = x'^{-4}x'Ax' = \Omega^2 x'Ax', (51)$$

$$F' = -\varepsilon x'^{-6} x' F x' = -\varepsilon \Omega^3 x' F x'. \tag{52}$$

C. Formulas for SCT

According to Eqs. (35) and (46), the transformation of potential four-vector with respect to Eq. (18) can be expressed as

$$A' = xAx,$$

$$\tilde{A} = A',$$

$$A'' = \tilde{x}\tilde{A}\tilde{x}.$$
(53)

It is straightforward to write down the composition

$$A'' = \tilde{\mathbf{x}} \mathbf{x} A \mathbf{x} \tilde{\mathbf{x}} = (1 + \mathbf{a} \mathbf{x}) A (1 + \mathbf{x} \mathbf{a}), \tag{54}$$

where $\tilde{\mathbf{x}}\mathbf{x} = \varepsilon(1+\mathbf{a}\mathbf{x})$ and $\mathbf{x}\tilde{\mathbf{x}} = \varepsilon(1+\mathbf{x}\mathbf{a})$ have been used, and the arguments of $A'' = A''(\mathbf{x}'')$ and $A = A(\mathbf{x})$ are related by Eq. (19).

In light of Eq. (51), we can rewrite Eq. (54) in terms of the new coordinates,

$$A'' = x'^{-4}x''^{-4}x''x'Ax'x'' = \Sigma^{2}(1 - x''a)A(1 - ax''),$$
(55)

where Σ is given in Eq. (33).

Similarly, the SCT formulas of Faraday bivector can be derived from Eqs. (42), (48), and (52),

$$F'' = x^2 \tilde{x}^2 \tilde{x} x F x \tilde{x} = \Sigma (1 + ax) F (1 + xa)$$

$$(56)$$

$$= x'^{-6}x''^{-6}x''x'Fx'x'' = \Sigma^{3}(1 - x''a)F(1 - ax'').$$
(57)

The Inversion and SCT formulas of current four-vector, which resemble those of potential four-vector, are listed in Appendix II.

D. Review of Clifford algebra $C\ell_3$

Serving as an alternative tool for electrodynamics [6], Clifford algebra $\mathcal{C}\ell_3$ of Euclidean space is an eight-dimensional vector space with the basis

$$\{1; \hat{x}, \hat{y}, \hat{z}; \hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}; \hat{x}\hat{y}\hat{z}\},\tag{58}$$

where the geometric product is defined by

$$\hat{x}^2 = \hat{y}^2 = \hat{z}^2 = 1 \text{ and } \hat{x}\hat{y} = -\hat{y}\hat{x}, \text{ etc.}$$
 (59)

Hence for a Euclidean vector

$$\vec{U}^2 = \vec{U} \cdot \vec{U} = U_x^2 + U_y^2 + U_z^2 =: U^2.$$
 (60)

In $\mathcal{C}\ell_3$ formulation, the correspondents of position four-vector, potential four-vector, and Faraday bivector of $\mathcal{C}\ell_{1,3}$ formulation are constructed as follows.

$$\mathbf{x} = t + \vec{r},\tag{61}$$

$$A = A_0 + \vec{A},\tag{62}$$

$$F = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} + B_x \hat{y} \hat{z} + B_y \hat{z} \hat{x} + B_z \hat{x} \hat{y}. \tag{63}$$

Since the basis vector $\hat{x}\hat{y}\hat{z}$ satisfies

$$(\hat{x}\hat{y}\hat{z})^2 = -1 \text{ and } (\hat{x}\hat{y}\hat{z})\hat{x} = \hat{x}(\hat{x}\hat{y}\hat{z}), \text{ etc.}, \tag{64}$$

it can be formally identified with $\sqrt{-1} =: i$. With this notation, we rewrite Eqs. (58) and (63) respectively as

$$\{1; \hat{x}, \hat{y}, \hat{z}; i\hat{z}, i\hat{x}, i\hat{y}; i\},\tag{65}$$

$$\vec{F} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} + i(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) = \vec{E} + i\vec{B}. \tag{66}$$

Although the symbols x, A, and F in Eqs. (61)–(63) are the same as those in $\mathcal{C}\ell_{1,3}$ formulation, we use \vec{F} instead of F in Eq. (66) to emphasize it may be taken as a complex-valued Euclidean vector, which will henceforward be called Faraday vector.

Moreover, this i notation enables us to express the geometric product of two Euclidean vectors as

$$\vec{U}\vec{V} = \vec{U}\cdot\vec{V} + i\vec{U}\times\vec{V},\tag{67}$$

and it follows that

$$\vec{U}\vec{V}\vec{U} = 2(\vec{U}\cdot\vec{V})\vec{U} - U^2\vec{V}. \tag{68}$$

E. Correspondence between $Cl_{1,3}$ and Cl_3 formulations

In comparison, the covariant $\mathcal{C}\ell_{1,3}$ formulation is suitable for deriving elegant formulas, while the non-covariant $\mathcal{C}\ell_3$ formulation is easier to manipulate in practical calculations. The correspondence between the formulas of these two formulations are listed here, the conversion rules are provided in Appendix I.

(i) Inversion:

Eqs. (46) and (51)
$$\iff A' = x\bar{A}x = \omega^2 x'\bar{A}x',$$
 (69)

Eqs. (48) and (52)
$$\iff \vec{F}' = \varepsilon \omega x \vec{F}^* \bar{x} = \varepsilon \omega^3 x' \vec{F}^* \bar{x}',$$
 (70)

where

$$\omega := x\bar{x} = \bar{x}x = (x'\bar{x}')^{-1} = (\bar{x}'x')^{-1}, \tag{71}$$

and the bar and asterisk symbols are defined via the following relations:

$$\bar{\mathbf{x}} = t - \vec{r},
\bar{A} = A_0 - \vec{A},
\vec{F}^* = \vec{E} - i\vec{B}$$
(72)

(ii) SCT:

Eq. (32)
$$\iff x'' = \sigma^{-1}(1 + a\bar{x})x = \sigma^{-1}x(1 + \bar{x}a),$$
 (73)

Eqs. (54) and (55)
$$\iff A'' = (1 + a\bar{x})A(1 + \bar{x}a) = \sigma^2(1 - x''\bar{a})A(1 - \bar{a}x''),$$
 (74)

Eqs. (56) and (57)
$$\iff \vec{F}'' = \sigma(1 + a\bar{x})\vec{F}(1 + x\bar{a}) = \sigma^3(1 - x''\bar{a})\vec{F}(1 - a\bar{x}''),$$
 (75)

where

$$\sigma := (1 + a\bar{\mathbf{x}})(1 + x\bar{\mathbf{a}}) = (1 + \bar{\mathbf{a}}\mathbf{x})(1 + \bar{\mathbf{x}}\mathbf{a})$$

$$= \left[(1 - \mathbf{x}''\bar{\mathbf{a}})(1 - a\bar{\mathbf{x}}'') \right]^{-1} = \left[(1 - \bar{\mathbf{x}}''\mathbf{a})(1 - \bar{\mathbf{a}}\mathbf{x}'') \right]^{-1}. \tag{76}$$

Note that $\omega = \Omega$ and $\sigma = \Sigma$ when they are expressed in terms of coordinates, i.e.,

$$\omega = t^2 - r^2 = (t'^2 - r'^2)^{-1},\tag{77}$$

$$\sigma = 1 + 2(\mathbf{a}_0 t - \vec{\mathbf{a}} \cdot \vec{r}) + (\mathbf{a}_0^2 - \vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(t^2 - r^2)$$

$$= \left[1 - 2(\mathbf{a}_0 t'' - \vec{\mathbf{a}} \cdot \vec{r}'') + (\mathbf{a}_0^2 - \vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(t''^2 - r''^2)\right]^{-1}.$$
(78)

(iii) For completeness, we include the correspondence for parity transformation formulas.

Eq. (37)
$$\iff \tilde{W} = \bar{W},$$
 (79)

Eq. (43)
$$\iff \vec{\tilde{F}} = -\vec{F}^*$$
. (80)

IV. APPLICATIONS OF THE FORMULAS

A. Inversion and SCT of Lorentz invariants

In electrodynamics, there exist two renowned Lorentz invariants

$$I_1 = \vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B} \text{ and } I_2 = 2\vec{E} \cdot \vec{B}$$
 (81)

that can be generated by Faraday vector via Eq. (60), i.e.,

$$\vec{F}^2 = \vec{F} \cdot \vec{F} = I_1 + iI_2. \tag{82}$$

After Inversion, we can still define

$$I_1' + iI_2' := \vec{F}^{\prime 2} = \vec{E}^{\prime} \cdot \vec{E}^{\prime} - \vec{B}^{\prime} \cdot \vec{B}^{\prime} + 2i\vec{E}^{\prime} \cdot \vec{B}^{\prime}, \tag{83}$$

although I'_1 and I'_2 may not be invariant under Lorentz transformation. On the other hand, we can use Eqs. (70) and (82) to obtain

$$\vec{F}^{\prime 2} = \omega^2 \mathbf{x} \vec{F}^* \bar{\mathbf{x}} \mathbf{x} \vec{F}^* \bar{\mathbf{x}} = \omega^4 (\vec{F}^*)^2 = \omega^4 (I_1 - iI_2). \tag{84}$$

Comparing Eqs. (83) and (84), we find

$$I_1' = \omega^4 I_1 \text{ and } I_2' = -\omega^4 I_2.$$
 (85)

Similarly, the SCT formula Eq. (75) leads to

$$\vec{F}^{"2} = \sigma^2 (1 + a\bar{x}) \vec{F} (1 + x\bar{a}) (1 + a\bar{x}) \vec{F} (1 + x\bar{a}) = \sigma^4 \vec{F}^2, \tag{86}$$

which yields

$$I_1'' = \sigma^4 I_1 \text{ and } I_2'' = \sigma^4 I_2.$$
 (87)

As a comparison, we sketch the derivations of I'_1 and I'_2 in tensorial formulation [11]:

$$I_1 = -\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} \Longrightarrow I_1' = -\frac{1}{2}F'^{\mu\nu}F_{\mu\nu}'; \tag{88}$$

$$I_2 = -\frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \Longrightarrow I_2' = -\frac{1}{4} \epsilon'^{\mu\nu\rho\sigma} F_{\mu\nu}' F_{\rho\sigma}', \tag{89}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol that transforms as

$$\epsilon^{\prime\mu\nu\rho\sigma} = \det \left[\frac{\partial \mathbf{x}}{\partial \mathbf{x}'} \right] \frac{\partial \mathbf{x}'^{\mu}}{\partial \mathbf{x}^{\alpha}} \frac{\partial \mathbf{x}'^{\nu}}{\partial \mathbf{x}^{\beta}} \frac{\partial \mathbf{x}'^{\rho}}{\partial \mathbf{x}^{\gamma}} \frac{\partial \mathbf{x}'^{\sigma}}{\partial \mathbf{x}^{\delta}} \epsilon^{\alpha\beta\gamma\delta}. \tag{90}$$

Invoking Eqs. (25) and (26), it is easy to obtain

$$I_1' = \Lambda^4 I_1 = \omega^4 I_1 \tag{91}$$

from Eq. (88). Nevertheless, Eq. (89) leads to

$$I_2' = \det \left[\frac{\partial \mathbf{x}}{\partial \mathbf{x}'} \right] I_2,$$
 (92)

and it is laborious to show the Jacobian determinant equals $-\omega^4$.

B. Inversion of an electromagnetic field

In this example, we demonstrate how to expand \vec{F}' in Eq. (70) to obtain the expression for Inversion of a given electromagnetic field.

To express the result in terms of the original coordinates, we use the formula

$$\vec{F}' = \varepsilon \omega x \vec{F}^* \bar{x} = \varepsilon \omega (t + \vec{r}) \vec{F}^* (t - \vec{r})$$
(93)

$$= \varepsilon \omega [(t^2 + r^2)\vec{F}^* - 2(\vec{r} \cdot \vec{F}^*)\vec{r} + 2it(\vec{r} \times \vec{F}^*)], \tag{94}$$

where Eqs. (67) and (68) have been used in the derivation of Eq. (94). Alternatively, we can express the result in terms of the new coordinates,

$$\vec{F}' = \varepsilon \omega^3 \mathbf{x}' \vec{F}^* \bar{\mathbf{x}}' = \varepsilon \omega^3 (t' + \vec{r}') \vec{F}^* (t' - \vec{r}')$$
(95)

$$= \varepsilon \omega^{3} [(t'^{2} + r'^{2})\vec{F}^{*} - 2(\vec{r}' \cdot \vec{F}^{*})\vec{r}' + 2it'(\vec{r}' \times \vec{F}^{*})]. \tag{96}$$

Note that Eqs. (94) and (96) are isomorphic except for the powers of ω .

It is straightforward to separate Eq. (94) or Eq. (96) into electric and magnetic parts. In order to compare with existing studies [1, 12], we list the result of Eq. (94) as follows.

$$\vec{E}' = \varepsilon(t^2 - r^2)[(t^2 + r^2)\vec{E} - 2(\vec{r} \cdot \vec{E})\vec{r} + 2t(\vec{r} \times \vec{B})]; \tag{97}$$

$$\vec{B}' = \varepsilon(t^2 - r^2)[-(t^2 + r^2)\vec{B} + 2(\vec{r} \cdot \vec{B})\vec{r} + 2t(\vec{r} \times \vec{E})], \tag{98}$$

or, equivalently,

$$\vec{E}' = \varepsilon(t^2 - r^2)[(t^2 - r^2)\vec{E} - 2\vec{r} \times (\vec{r} \times \vec{E}) + 2t(\vec{r} \times \vec{B})]; \tag{99}$$

$$\vec{B}' = \varepsilon(t^2 - r^2)[-(t^2 - r^2)\vec{B} + 2\vec{r} \times (\vec{r} \times \vec{B}) + 2t(\vec{r} \times \vec{E})]. \tag{100}$$

It is apparent that, since $\vec{F}^* = \vec{E} - i\vec{B}$ and $i\vec{F}^* = \vec{B} + i\vec{E}$, the inverse duality transformation $(\vec{E}, \vec{B}) \longrightarrow (-\vec{B}, \vec{E})$ leads to Eq. (97) \longrightarrow Eq. (98) and Eq. (99) \longrightarrow Eq. (100).

The tensorial counterpart of Eq. (94) takes the form [8]

$$F'^{\mu\nu} = -\varepsilon \left[x^4 F^{\mu\nu} + 2x^2 (x^{\mu} F^{\nu\rho} - x^{\nu} F^{\mu\rho}) x_{\rho} \right], \tag{101}$$

which can be obtained by substituting Eqs. (16), (17), and (28) into Eq. (25). Although Eq. (101) is equivalent to Eq. (94), there is no simple way to separate it into electric and magnetic parts, nor can we see the electro-magnetic duality directly from this expression.

Note that if we employ the covariant formula in Eq. (48) instead of the non-covariant one in Eq. (70), the expansion result will be essentially Eq. (101), i.e., $F' = \frac{1}{2}F'^{\mu\nu}\mathbf{e}_{\mu}\mathbf{e}_{\nu}$.

C. SCT of an electromagnetic field

The goal of this example is to expand $\vec{F}'' = \vec{E}'' + i\vec{B}''$ in terms of the original coordinates, and the strategy is to imitate the previous example as much as possible. Hence we start with expressing \vec{F}'' in Eq. (75) as

$$\vec{F}'' = \sigma(u + \vec{v})\vec{F}(u - \vec{v}),\tag{102}$$

where

$$u = 1 + \frac{1}{2}(a\bar{x} + x\bar{a}) = 1 + a_0t - \vec{a} \cdot \vec{r}$$
 (103)

is a scalar, and

$$\vec{v} = \frac{1}{2}(a\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{a}}) = t\vec{\mathbf{a}} - a_0\vec{r} - i\vec{\mathbf{a}} \times \vec{r}$$
(104)

is a complex-valued Euclidean vector.

From Eqs. (93) and (94), the expansion of Eq. (102) takes the form

$$\vec{F}'' = \sigma[(u^2 + v^2)\vec{F} - 2(\vec{v} \cdot \vec{F})\vec{v} + 2iu(\vec{v} \times \vec{F})], \tag{105}$$

where

$$v^{2} = (t\vec{a} - a_{0}\vec{r})^{2} - (\vec{a} \times \vec{r})^{2} = (t\vec{a} - a_{0}\vec{r}) \cdot (t\vec{a} - a_{0}\vec{r}) - (\vec{a} \times \vec{r}) \cdot (\vec{a} \times \vec{r}).$$
(106)

For the sake of clarity, we separate Eq. (105) into three parts,

$$\vec{F}'' = \vec{F}_1'' + \vec{F}_2'' + \vec{F}_3'', \tag{107}$$

where

$$\vec{F}_1'' = \vec{E}_1'' + i\vec{B}_1'' = \sigma(u^2 + v^2)\vec{F},\tag{108}$$

$$\vec{F}_2'' = \vec{E}_2'' + i\vec{B}_2'' = -2\sigma(\vec{v} \cdot \vec{F})\vec{v}, \tag{109}$$

$$\vec{F}_3'' = \vec{E}_3'' + i\vec{B}_3'' = 2i\sigma u(\vec{v} \times \vec{F}). \tag{110}$$

For \vec{F}_1'' in Eq. (108), we can directly write down the result

$$\vec{E}_1'' = \sigma \left[(1 + a_0 t - \vec{a} \cdot \vec{r})^2 + (t\vec{a} - a_0 \vec{r})^2 - (\vec{a} \times \vec{r})^2 \right] \vec{E}; \tag{111}$$

$$\vec{B}_1'' = \sigma \left[(1 + a_0 t - \vec{a} \cdot \vec{r})^2 + (t\vec{a} - a_0 \vec{r})^2 - (\vec{a} \times \vec{r})^2 \right] \vec{B}.$$
 (112)

For $\vec{F}_2^{"}$ in Eq. (109), we first calculate

$$(\vec{v} \cdot \vec{F})\vec{v} = \{t(\vec{\mathbf{a}} \cdot \vec{E}) - \mathbf{a}_0(\vec{r} \cdot \vec{E}) + (\vec{\mathbf{a}} \times \vec{r}) \cdot \vec{B} + i[t(\vec{\mathbf{a}} \cdot \vec{B}) - \mathbf{a}_0(\vec{r} \cdot \vec{B}) - (\vec{\mathbf{a}} \times \vec{r}) \cdot \vec{E}]\}[t\vec{\mathbf{a}} - \mathbf{a}_0\vec{r} - i\vec{\mathbf{a}} \times \vec{r}],$$

$$(113)$$

it then follows that

$$\vec{E}_{2}^{"} = -2\sigma[t(\vec{\mathbf{a}} \cdot \vec{E}) - \mathbf{a}_{0}(\vec{r} \cdot \vec{E}) + (\vec{\mathbf{a}} \times \vec{r}) \cdot \vec{B}](t\vec{\mathbf{a}} - \mathbf{a}_{0}\vec{r})
- 2\sigma[t(\vec{\mathbf{a}} \cdot \vec{B}) - \mathbf{a}_{0}(\vec{r} \cdot \vec{B}) - (\vec{\mathbf{a}} \times \vec{r}) \cdot \vec{E}]\vec{\mathbf{a}} \times \vec{r};$$

$$\vec{B}_{2}^{"} = -2\sigma[t(\vec{\mathbf{a}} \cdot \vec{B}) - \mathbf{a}_{0}(\vec{r} \cdot \vec{B}) - (\vec{\mathbf{a}} \times \vec{r}) \cdot \vec{E}](t\vec{\mathbf{a}} - \mathbf{a}_{0}\vec{r})$$
(114)

$$+2\sigma[t(\vec{\mathbf{a}}\cdot\vec{E}) - \mathbf{a}_0(\vec{r}\cdot\vec{E}) + (\vec{\mathbf{a}}\times\vec{r})\cdot\vec{B}]\vec{\mathbf{a}}\times\vec{r}. \tag{115}$$

Finally for \vec{F}_3'' in Eq. (110), by applying the formula

$$(\vec{\mathbf{a}} \times \vec{r}) \times \vec{U} = (\vec{\mathbf{a}} \cdot \vec{U})\vec{r} - (\vec{r} \cdot \vec{U})\vec{\mathbf{a}}$$
(116)

to

$$\vec{v} \times \vec{F} = [(t\vec{a} - a_0\vec{r}) \times \vec{E} + (\vec{a} \times \vec{r}) \times \vec{B}] + i[(t\vec{a} - a_0\vec{r}) \times \vec{B} - (\vec{a} \times \vec{r}) \times \vec{E}], \tag{117}$$

we obtain

$$\vec{E}_3'' = 2\sigma(1 + \mathbf{a}_0 t - \vec{\mathbf{a}} \cdot \vec{r})[(\vec{\mathbf{a}} \cdot \vec{E})\vec{r} - (\vec{r} \cdot \vec{E})\vec{\mathbf{a}} - t\vec{\mathbf{a}} \times \vec{B} + \mathbf{a}_0 \vec{r} \times \vec{B}]; \tag{118}$$

$$\vec{B}_{3}'' = 2\sigma(1 + a_{0}t - \vec{a} \cdot \vec{r})[(\vec{a} \cdot \vec{B})\vec{r} - (\vec{r} \cdot \vec{B})\vec{a} + t\vec{a} \times \vec{E} - a_{0}\vec{r} \times \vec{E}].$$
 (119)

In summary, the expressions for the transformed electric and magnetic fields are respectively [13, 14]

$$\vec{E}'' = \vec{E}_1'' + \vec{E}_2'' + \vec{E}_3''; \tag{120}$$

$$\vec{B}'' = \vec{B}_1'' + \vec{B}_2'' + \vec{B}_3''. \tag{121}$$

Since $\vec{F} = \vec{E} + i\vec{B}$ and $i\vec{F} = -\vec{B} + i\vec{E}$, the duality transformation $(\vec{E}, \vec{B}) \longrightarrow (\vec{B}, -\vec{E})$ leads to Eq. $(120) \longrightarrow Eq. (121)$.

Comparing the two expressions in Eq. (75), we find that to express \vec{E}'' and \vec{B}'' in terms of the new coordinates, one only has to replace each explicit t in Eqs. (111), (112), (114), (115), (118) and (119) by t'', and each explicit \vec{r} therein by \vec{r}'' , then make the following changes:

$$\forall \ \sigma \longrightarrow \sigma^3, \tag{122}$$

$$\forall 1 + \mathbf{a}_0 t'' - \vec{\mathbf{a}} \cdot \vec{r}'' \longrightarrow 1 - \mathbf{a}_0 t'' + \vec{\mathbf{a}} \cdot \vec{r}''. \tag{123}$$

In tensorial formulation as well as $\mathcal{C}\ell_{1,3}$ formulation, the corresponding calculations are much lengthier, and the result [7]

$$F''^{\mu\nu} = \sigma^2 F^{\mu\nu} - 2\sigma (a^{\mu} F^{\nu\rho} - a^{\nu} F^{\mu\rho}) [x_{\rho} + 2(a \cdot x) x_{\rho} - x^2 a_{\rho}] + 2\sigma (x^{\mu} F^{\nu\rho} - x^{\nu} F^{\mu\rho}) (a^2 x_{\rho} + a_{\rho}) + 4\sigma (a^{\mu} x^{\nu} - a^{\nu} x^{\mu}) a_{\rho} F^{\rho\sigma} x_{\sigma}$$
(124)

is not easy to be separated into electric and magnetic parts. Moreover, even if we accomplish the separation [3], a lot of further work is needed to rearrange the result into the forms of Eqs. (111), (112), (114), (115), (118) and (119).

V. DISCUSSION AND CONCLUSION

In this research we derived a set of Clifford-algebraic formulas for Inversion and SCT of the quantities related to Maxwell's equations. The advantages of these formulas are threefold:

- (i) As evidenced by the three illustrative examples in Sec. IV, these formulas enjoy superiority in conciseness and convenience compared to the vector-tensorial formulas. This strengthens our confidence in our results in the face of possible discrepancies with existing studies [1, 12, 13, 14].
- (ii) These formulas neatly integrate the covariant and contravariant formulas in vectortensorial formulation. For example, owing to $A'^{\mu}e_{\mu}=A'_{\mu}e^{\mu}$, the $\mathcal{C}\ell_{1,3}$ formula A'=xAx is mathematically equivalent to both

$$A^{\prime\mu} = \vartheta x^4 \frac{\partial x^{\prime\mu}}{\partial x^{\alpha}} A^{\alpha} = -x^2 A^{\mu} + 2(x \cdot A) x^{\mu}$$
 (125)

and

$$A'_{\mu} = \eta_{\mu\nu}A'^{\nu} = \vartheta \frac{\partial \mathbf{x}^{\alpha}}{\partial \mathbf{x}'^{\mu}} A_{\alpha} = -\mathbf{x}^2 A_{\mu} + 2(\mathbf{x} \cdot A)\mathbf{x}_{\mu}. \tag{126}$$

Similarly, A'' = (1 + ax)A(1 + xa) is equivalent to

$$A''^{\mu} = \left[\sigma \delta^{\mu}_{\alpha} - 2(\mathbf{a}_{\alpha} \mathbf{x}^{\mu} - \mathbf{x}_{\alpha} \mathbf{a}^{\mu}) + 4(\mathbf{a} \cdot \mathbf{x}) \mathbf{x}_{\alpha} \mathbf{a}^{\mu} - 2(\mathbf{a}^{2} \mathbf{x}_{\alpha} \mathbf{x}^{\mu} + \mathbf{x}^{2} \mathbf{a}_{\alpha} \mathbf{a}^{\mu}) \right] A^{\alpha}$$

$$= \sigma A^{\mu} - 2 \left[\mathbf{a} \cdot A + \mathbf{a}^{2} (\mathbf{x} \cdot A) \right] \mathbf{x}^{\mu} + 2 \left[\mathbf{x} \cdot A - \mathbf{x}^{2} (\mathbf{a} \cdot A) + 2(\mathbf{a} \cdot \mathbf{x}) (\mathbf{x} \cdot A) \right] \mathbf{a}^{\mu}$$
(127)

and its covariant counterpart $A''_{\mu} = \eta_{\mu\nu}A''^{\nu}$. For the same reason $F' = -\varepsilon\Omega xFx$ is equivalent to $F'^{\mu\nu}$ in Eq. (101) and $F'_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F'^{\rho\sigma}$, while $F'' = \Sigma(1+ax)F(1+xa)$ to $F''^{\mu\nu}$ in Eq. (124) and its covariant counterpart.

(iii) In the literature, there is no lack of introducing higher-dimensional spacetime to realize conformal transformation, either in vector-tensorial formulation [3, 7, 11] or Clifford-algebraic formulation [15]. In contrast, our formulas are constructed and operated entirely in four dimensions. Therefore, as listed in Appendix II, all of the fundamental conformal transformations of the quantities related to Maxwell's equations can be expressed as Clifford-algebraic formulas in ordinary spacetime.

In conclusion, the formulas we derived offer a streamlined mathematical framework for the major nonlinear conformal transformations in electrodynamics. This not only paves the way for future research, but also holds the potential for bringing new insight into existing studies.

Appendix I: $C\ell_{1,3} \longrightarrow C\ell_3$ CONVERSION

(i) Basic rules:

$$e_1e_0 \longrightarrow \hat{x}, \ e_2e_0 \longrightarrow \hat{y}, \ e_3e_0 \longrightarrow \hat{z}, \ \text{and} \ e_0^2 = 1.$$

(ii) Four-vectors:

$$\mathbf{xe}_0 \longrightarrow \mathbf{x} = t + \vec{r},$$
 $\mathbf{e}_0 \mathbf{x} \longrightarrow \bar{\mathbf{x}} = t - \vec{r},$
 $A\mathbf{e}_0 \longrightarrow A = A_0 + \vec{A},$
 $\mathbf{e}_0 A \longrightarrow \bar{A} = A_0 - \vec{A}.$

(iii) Faraday (bi-)vector:

$$\begin{split} F &= E_x \mathsf{e}_1 \mathsf{e}_0 + E_y \mathsf{e}_2 \mathsf{e}_0 + E_z \mathsf{e}_3 \mathsf{e}_0 + B_x \mathsf{e}_3 \mathsf{e}_2 + B_y \mathsf{e}_1 \mathsf{e}_3 + B_z \mathsf{e}_2 \mathsf{e}_1 \\ &= E_x \mathsf{e}_1 \mathsf{e}_0 + E_y \mathsf{e}_2 \mathsf{e}_0 + E_z \mathsf{e}_3 \mathsf{e}_0 + B_x \mathsf{e}_3 \mathsf{e}_0 \mathsf{e}_0 \mathsf{e}_2 + B_y \mathsf{e}_1 \mathsf{e}_0 \mathsf{e}_0 \mathsf{e}_3 + B_z \mathsf{e}_2 \mathsf{e}_0 \mathsf{e}_0 \mathsf{e}_1 \\ &\longrightarrow E_x \hat{x} + E_y \hat{y} + E_z \hat{z} + B_x \hat{y} \hat{z} + B_y \hat{z} \hat{x} + B_z \hat{x} \hat{y} = \vec{E} + i \vec{B} = \vec{F}. \end{split}$$

(iv) Geometric products of two four-vectors:

$$\begin{split} xy &= x e_0 e_0 y \longrightarrow x \bar{y}, \\ x \cdot x &= x^2 = x e_0 e_0 x = e_0 x x e_0 \longrightarrow x \bar{x} = \bar{x} x, \\ 2x \cdot y &= xy + y x \longrightarrow x \bar{y} + y \bar{x} = \bar{x} y + \bar{y} x. \end{split}$$

(v) Faraday (bi-)vector sandwiched by two four-vectors:

$$xFy = xe_0(e_0Fe_0)e_0y \longrightarrow x(-\vec{E} + i\vec{B})\bar{y} = -x\vec{F}^*\bar{y}.$$

Appendix II: SUMMARY OF FORMULAS

A. $C\ell_{1,3}$ formulation:

Position four-vector:
$$\mathbf{x} = \mathbf{x}^{\alpha} \mathbf{e}_{\alpha}$$
,
Potential four-vector: $A(\mathbf{x}) = A^{\alpha} \mathbf{e}_{\alpha}$,
Current four-vector: $J(\mathbf{x}) = J^{\alpha} \mathbf{e}_{\alpha}$,
Faraday bivector: $F(\mathbf{x}) = \frac{1}{2} F^{\alpha\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta}$.

(1) Four-dimensional dilation:

$$\tilde{\mathbf{x}} = \lambda^{-1} \mathbf{x}, \ \tilde{A} = \lambda A, \ \tilde{J} = \lambda^3 J, \ \tilde{F} = \lambda^2 F.$$

(2) Four-dimensional translation:

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{b}, \ \tilde{A} = A, \ \tilde{J} = J, \ \tilde{F} = F.$$

(3) Lorentz transformation=Four-dimensional orthogonal transformation:

$$L := \exp(\alpha_x \mathsf{e}_1 \mathsf{e}_0 + \alpha_y \mathsf{e}_2 \mathsf{e}_0 + \alpha_z \mathsf{e}_3 \mathsf{e}_0 + \theta_x \mathsf{e}_3 \mathsf{e}_2 + \theta_y \mathsf{e}_1 \mathsf{e}_3 + \theta_z \mathsf{e}_2 \mathsf{e}_1).$$

(3.1) Proper orthochronous $(\vartheta = 1)$:

$$\tilde{\mathbf{x}} = L\mathbf{x}L^{-1}, \ \tilde{A} = LAL^{-1}, \ \tilde{J} = LJL^{-1}, \ \tilde{F} = LFL^{-1}.$$

(3.2) Improper orthochronous ($\vartheta = 1$):

$$\tilde{\mathbf{x}} = \mathbf{e}_0 L \mathbf{x} L^{-1} \mathbf{e}_0, \ \tilde{A} = \mathbf{e}_0 L A L^{-1} \mathbf{e}_0, \ \tilde{J} = \mathbf{e}_0 L J L^{-1} \mathbf{e}_0, \ \tilde{F} = \mathbf{e}_0 L F L^{-1} \mathbf{e}_0.$$

(3.3) Improper antichronous ($\vartheta = -1$):

$$\tilde{\mathbf{x}} = -\mathbf{e}_0 L \mathbf{x} L^{-1} \mathbf{e}_0, \ \tilde{A} = \mathbf{e}_0 L A L^{-1} \mathbf{e}_0, \ \tilde{J} = \mathbf{e}_0 L J L^{-1} \mathbf{e}_0, \ \tilde{F} = -\mathbf{e}_0 L F L^{-1} \mathbf{e}_0.$$

(3.4) Proper antichronous $(\vartheta = -1)$:

$$\tilde{\mathbf{x}} = -L\mathbf{x}L^{-1}, \ \tilde{A} = LAL^{-1}, \ \tilde{J} = LJL^{-1}, \ \tilde{F} = -LFL^{-1}.$$

(4') Conformal inversion:

$$\Omega = \mathbf{x}^2 = \mathbf{x}'^{-2};$$

$$\mathbf{x}' = \varepsilon \Omega^{-1} \mathbf{x},$$

$$A' = \mathbf{x} A \mathbf{x} = \Omega^2 \mathbf{x}' A \mathbf{x}',$$

$$J' = \Omega^2 \mathbf{x} J \mathbf{x} = \Omega^4 \mathbf{x}' J \mathbf{x}',$$

$$F' = -\varepsilon \Omega \mathbf{x} F \mathbf{x} = -\varepsilon \Omega^3 \mathbf{x}' F \mathbf{x}'.$$

(4) Special conformal transformation (SCT):

$$\Sigma = (1 + ax)(1 + xa) = [(1 - x''a)(1 - ax'')]^{-1};$$

$$x'' = \Sigma^{-1}(1 + ax)x = \Sigma^{-1}x(1 + xa),$$

$$A'' = (1 + ax)A(1 + xa) = \Sigma^{2}(1 - x''a)A(1 - ax''),$$

$$J'' = \Sigma^{2}(1 + ax)J(1 + xa) = \Sigma^{4}(1 - x''a)J(1 - ax''),$$

$$F'' = \Sigma(1 + ax)F(1 + xa) = \Sigma^{3}(1 - x''a)F(1 - ax'').$$

B. $C\ell_3$ formulation:

Position four-vector:
$$\mathbf{x} = t + \vec{r}$$
,

Potential four-vector: $A(\mathbf{x}) = A_0 + \vec{A}$,

Current four-vector: $J(\mathbf{x}) = J_0 + \vec{J}$,

Faraday vector: $\vec{F}(\mathbf{x}) = \vec{E} + i\vec{B}$.

(1) Four-dimensional dilation: $\tilde{\mathbf{x}} = \lambda^{-1}\mathbf{x}, \ \tilde{A} = \lambda A, \ \tilde{J} = \lambda^3 J, \ \tilde{\tilde{F}} = \lambda^2 \vec{F}.$

(2) Four-dimensional translation: $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{b}, \ \tilde{A} = A, \ \tilde{J} = J, \ \tilde{\vec{F}} = \vec{F}.$

- (3) Lorentz transformation: $L = \exp(\vec{\alpha} + i\vec{\theta})$.
- (3.1) Proper orthochronous: $\tilde{\mathbf{x}} = L\mathbf{x}L^*$, $\tilde{A} = LAL^*$, $\tilde{J} = LJL^*$, $\tilde{\tilde{F}} = L\tilde{F}\bar{L}$.
- (3.2) Improper orthochronous: $\tilde{\mathbf{x}} = \bar{L}^* \bar{\mathbf{x}} \bar{L}, \ \tilde{\mathbf{A}} = \bar{L}^* \bar{\mathbf{A}} \bar{L}, \ \tilde{J} = \bar{L}^* \bar{\mathbf{J}} \bar{L}, \ \tilde{\vec{F}} = -\bar{L}^* \vec{F}^* L^*.$
- (3.3) Improper antichronous: $\tilde{\mathbf{x}} = -\bar{L}^*\bar{\mathbf{x}}\bar{L}$, $\tilde{\mathbf{A}} = \bar{L}^*\bar{\mathbf{A}}\bar{L}$, $\tilde{\mathbf{J}} = \bar{L}^*\bar{\mathbf{J}}\bar{L}$, $\vec{\tilde{F}} = \bar{L}^*\vec{F}^*L^*$.
- (3.4) Proper antichronous: $\tilde{\mathbf{x}} = -L\mathbf{x}L^*$, $\tilde{A} = LAL^*$, $\tilde{J} = LJL^*$, $\tilde{\tilde{F}} = -L\tilde{F}\bar{L}$.
- (4') Conformal inversion:

$$\omega = x\bar{x} = \bar{x}x = (x'\bar{x}')^{-1} = (\bar{x}'x')^{-1};$$

$$x' = \varepsilon\omega^{-1}x,$$

$$A' = x\bar{A}x = \omega^{2}x'\bar{A}x',$$

$$J' = \omega^{2}x\bar{J}x = \omega^{4}x'\bar{J}x',$$

$$\vec{F}' = \varepsilon\omega x\vec{F}^{*}\bar{x} = \varepsilon\omega^{3}x'\vec{F}^{*}\bar{x}'.$$

(4) Special conformal transformation (SCT):

$$\begin{split} \sigma &= (1+a\bar{\mathbf{x}})(1+x\bar{\mathbf{a}}) = (1+\bar{\mathbf{a}}\mathbf{x})(1+\bar{\mathbf{x}}\mathbf{a}) \\ &= \left[(1-\mathbf{x}''\bar{\mathbf{a}})(1-a\bar{\mathbf{x}}'') \right]^{-1} = \left[(1-\bar{\mathbf{x}}''\mathbf{a})(1-\bar{\mathbf{a}}\mathbf{x}'') \right]^{-1}; \\ \mathbf{x}'' &= \sigma^{-1}(1+a\bar{\mathbf{x}})\mathbf{x} = \sigma^{-1}\mathbf{x}(1+\bar{\mathbf{x}}\mathbf{a}), \\ A'' &= (1+a\bar{\mathbf{x}})A(1+\bar{\mathbf{x}}\mathbf{a}) = \sigma^2(1-\mathbf{x}''\bar{\mathbf{a}})A(1-\bar{\mathbf{a}}\mathbf{x}''), \\ J'' &= \sigma^2(1+a\bar{\mathbf{x}})J(1+\bar{\mathbf{x}}\mathbf{a}) = \sigma^4(1-\mathbf{x}''\bar{\mathbf{a}})J(1-\bar{\mathbf{a}}\mathbf{x}''), \\ \vec{F}'' &= \sigma(1+a\bar{\mathbf{x}})\vec{F}(1+\mathbf{x}\bar{\mathbf{a}}) = \sigma^3(1-\mathbf{x}''\bar{\mathbf{a}})\vec{F}(1-a\bar{\mathbf{x}}''). \end{split}$$

In both formulations, the similarity between the expressions of A' and J' can be deduced by comparing Eq. (21) with Eq. (23), or Eq. (22) with Eq. (24). The same argument applies to the similarity between the expressions of A'' and J''.

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