A Rational Geometric Series Approximation for $\pi/2$

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Abstract

We present a new geometric series with rational parameters that converges to a rational approximation of $\pi/2$. The series is

$$\frac{\pi}{2} \approx 6 \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{7}\right)^{n} \left(\frac{8}{25}\right)^{n-1},$$

whose closed form yields $\frac{300}{191} \approx 1.570680628$, with a relative error of 0.007366%. The rapid convergence (ratio $16/175 \approx 0.0914$) and simple structure make it an interesting pedagogical example.

1 Introduction

Approximations of π using infinite series have a long history, from the Leibniz–Madhava series to more modern fast-converging iterations. This note presents a previously undocumented geometric series that provides a good rational approximation to $\pi/2$ using only the rational numbers $\frac{2}{7}$ and $\frac{8}{25}$.

2 Derivation

Consider the series

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{7}\right)^{n} \left(\frac{8}{25}\right)^{n-1}.$$

Factor $\frac{2}{7}$:

$$S = \frac{2}{7} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{16}{175}\right)^{n-1}.$$

Let $k = n - 1$:

$$S = \frac{2}{7} \sum_{k=0}^{\infty} (-1)^{k} \left(\frac{16}{175}\right)^{k}.$$

This is a geometric series with ratio $r = -\frac{16}{175}$, $|r| < 1$.

3 Closed-form summation

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad r = -\frac{16}{175},$$

so

$$S = \frac{2}{7} \cdot \frac{1}{1+\frac{16}{175}} = \frac{2}{7} \cdot \frac{175}{191} = \frac{350}{1337}.$$

Multiplying by 6 gives the approximation for $\pi/2$:

$$6S = \frac{2100}{1337} = \frac{300}{191} \approx 1.57068062827.$$

4 Error analysis

Exact value:

$$\frac{\pi}{2} \approx 1.57079632679.$$

Absolute error:

$$E_a \approx 1.156985 \times 10^{-4}.$$

Relative error percentage:

$$E_r \approx \frac{1.156985 \times 10^{-4}}{1.57079632679} \times 100\% \approx 0.007366\%.$$

The series converges quickly; after only 4 terms the truncation error is smaller than the inherent approximation error.

5 Generalization

Let $a, b > 0$ and define

$$S(a,b) = \sum_{n=1}^{\infty} (-1)^{n+1} a^n b^{n-1} = \frac{a}{1+ab}.$$

Choosing M such that $M \cdot S(a,b) \approx \pi/2$ yields a family of approximations. For rational a, b, the limit is rational, so perfect accuracy requires irrational parameters.

6 Comparison with known approximations

- Archimedes' bound: $\pi/2 \approx 11/7 \approx 1.57142857$, error 0.040%.
- Zu Chongzhi's fraction: $\pi/2 \approx 355/226 \approx 1.57079646$, error $1.33 \times 10^{-7}\%$.
- This series: $\pi/2 \approx 300/191 \approx 1.57068063$, error 0.00737%.

While less accurate than 355/226, the series representation is novel and geometrically elegant.

7 Conclusion

We have presented a new geometric series with rational parameters that converges to a good rational approximation of $\pi/2$. Its simple structure and clear derivation make it suitable for educational purposes and further exploration in number theory.

References

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[2] Wallis, J. (1656). *Arithmetica Infinitorum*.

[3] OEIS Foundation Inc. (2024). The On-Line Encyclopedia of Integer Sequences. `https://oeis.org`