Optimality conditions

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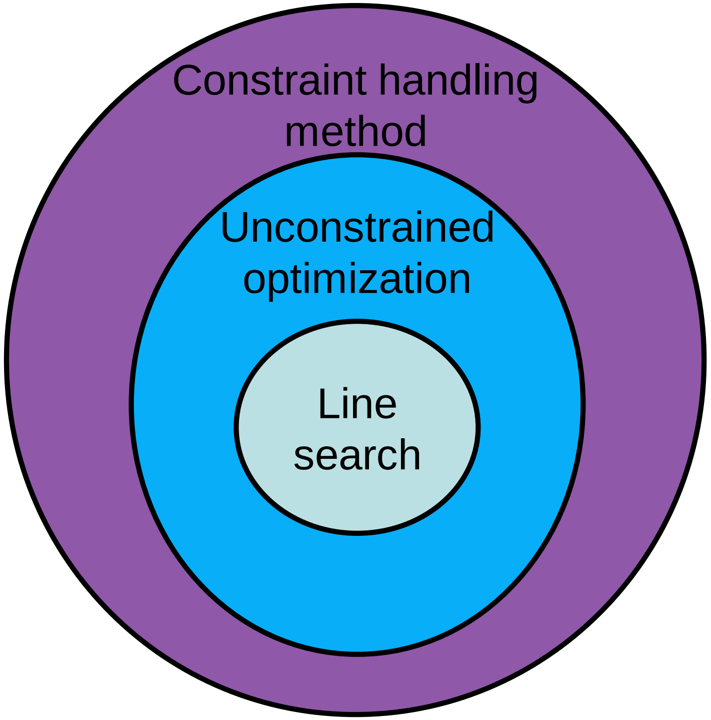
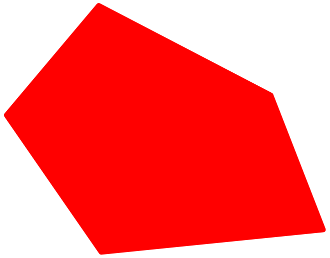
Structure of optimization methods

- Typically
  - Constraint handling converts the problem to (a series of) unconstrained problems
  - In unconstrained optimization a search direction is determined at each iteration
  - The best solution in the search direction is found with line search



Constrained problem

- $\min f(x),\ s.t.\ x \in S \subset R^n$
- S is the *feasible region*, consists of all the points that satisfy the constraints
- Constraints can be e.g.
  – Box constraints: $x_i^l \le x_i \le x_i^u,\ i = 1, \dots, n$
  – Inequality constraints: $g_j(x) \le 0,\ j = 1, \dots, m$
  – Equality constraints: $h_j(x) = 0,\ j = 1, \dots, l$
- Special case: linear constraints
  – $Ax \le b$ (i.e., $g(x) = Ax - b$)
  – $Ax = b$ (i.e., $h(x) = Ax - b$)



Reminder: Descent direction

- **Definition**: Let $f: R^n \to R$. A vector $d \in R^n$ is a descent direction for f in $x^* \in R^n$ if

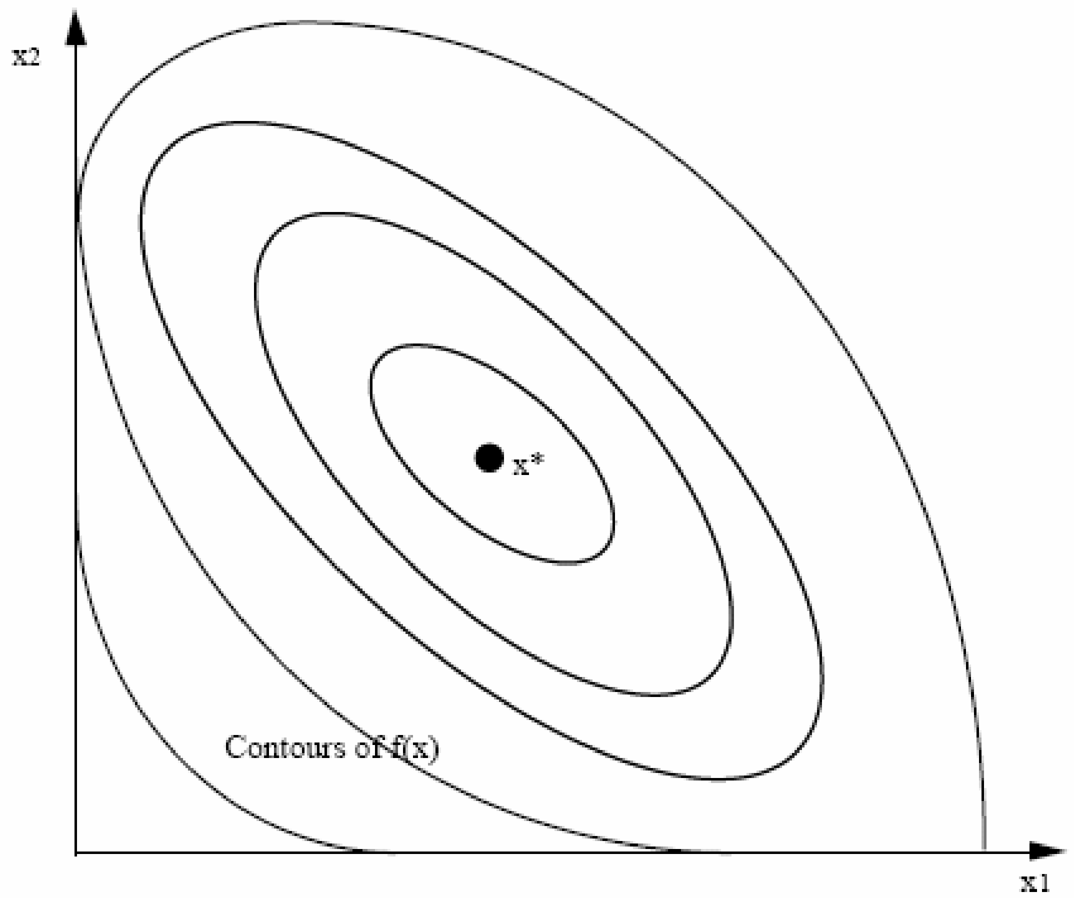
$$\exists\, \delta > 0 \text{ s.t. } \quad f(x^* + \lambda d) < f(x^*)\ \forall\, \lambda \in (0, \delta].$$

- **Result**: Let $f: R^n \to R$ be differentiable in x^*. If

  $\exists d \in R^n$ s.t. $\nabla f(x^*)^T d < 0$,

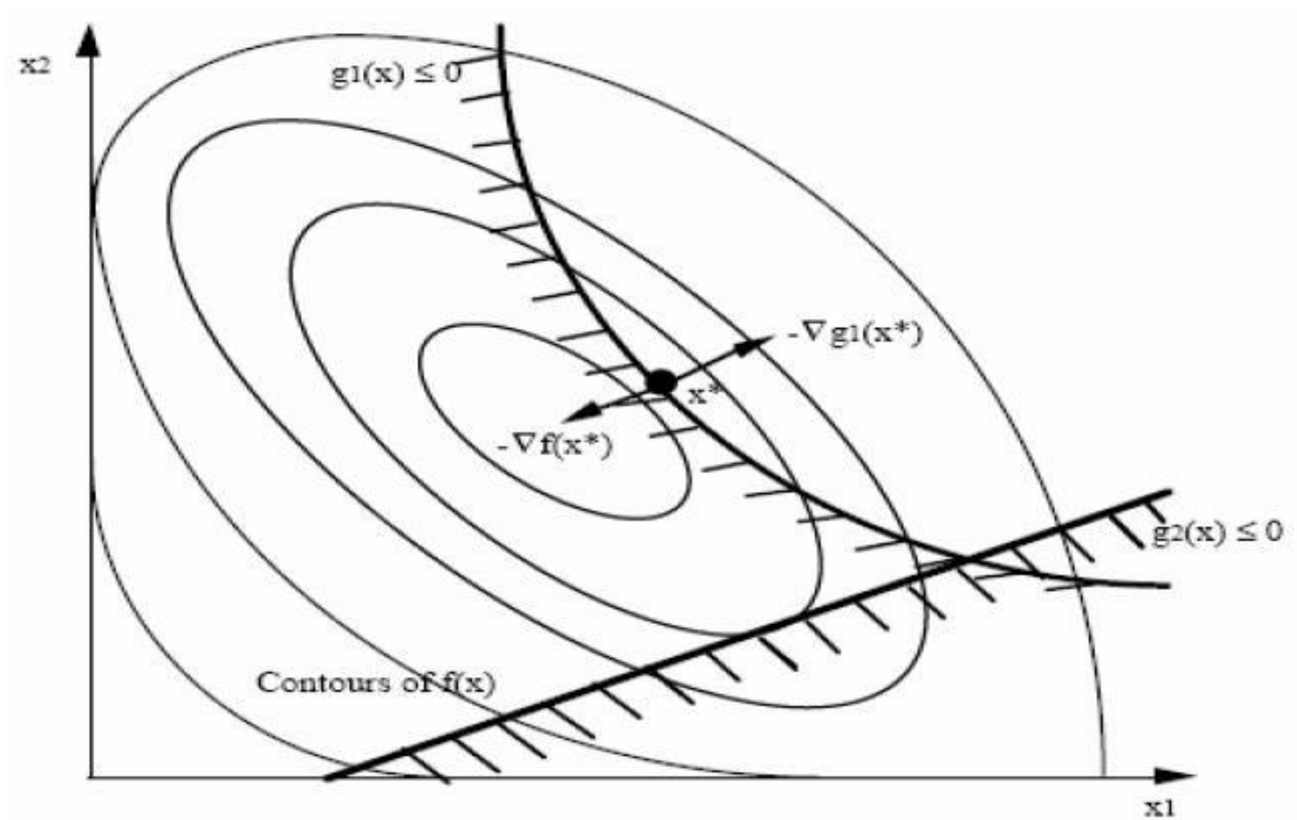
  then d is a descent direction for f in x^*.

Unconstrained problem



Adopted from Prof. L.T. Biegler (Carnegie Mellon University)

Inequality constraints



Min $f(x)$

s.t. $g(x) \leq 0$

Analogy: Ball rolling down valley pinned by fence

Note: Balance of forces (∇f, ∇g_1)

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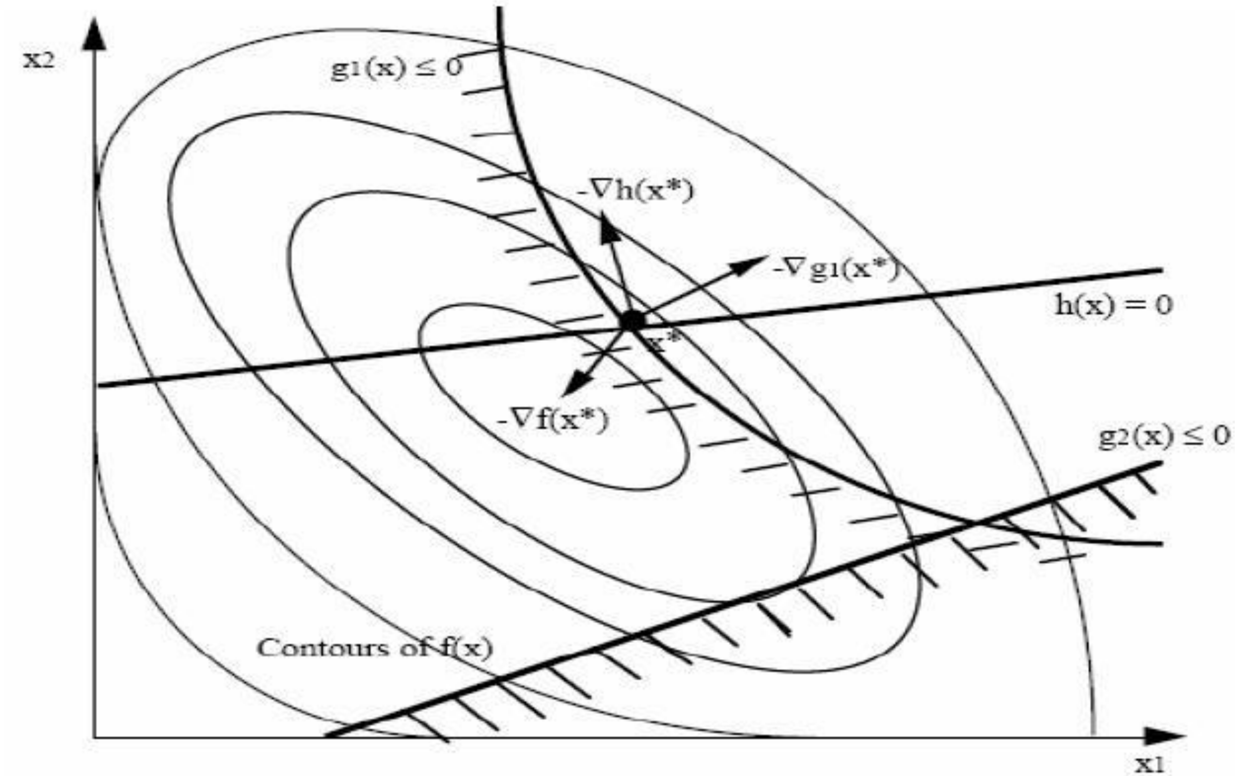
Inequality and equality constraints



Problem: Min $\quad f(x)$

s.t. $\quad g(x) \leq 0$

$h(x) = 0$

<u>Analogy</u>: Ball rolling on rail pinned by fences

Balance of forces: ∇f, ∇g_1, ∇h

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Feasible descent directions

- **Definition**: Let $S \subset R^n, S \neq \emptyset$ and $x^* \in cl\, S$. Then $$D = \{d \in R^n \mid d \neq 0 \;\&\; x^* + \alpha d \in S\ \forall \alpha \in (0, \delta)\}$$ for some $\delta > 0$ is the cone of feasible directions of S in x^*.
  - Each $d \in D, d \neq 0$ is a feasible direction
- **Definition**:

  $$F = \{d \in R^n \mid f(x^* + \alpha d) < f(x^*),\ \ \forall \alpha \in (0, \delta)\}$$

  for some $\delta > 0$ is the cone of descent directions in x^*.
- **Note:** Set

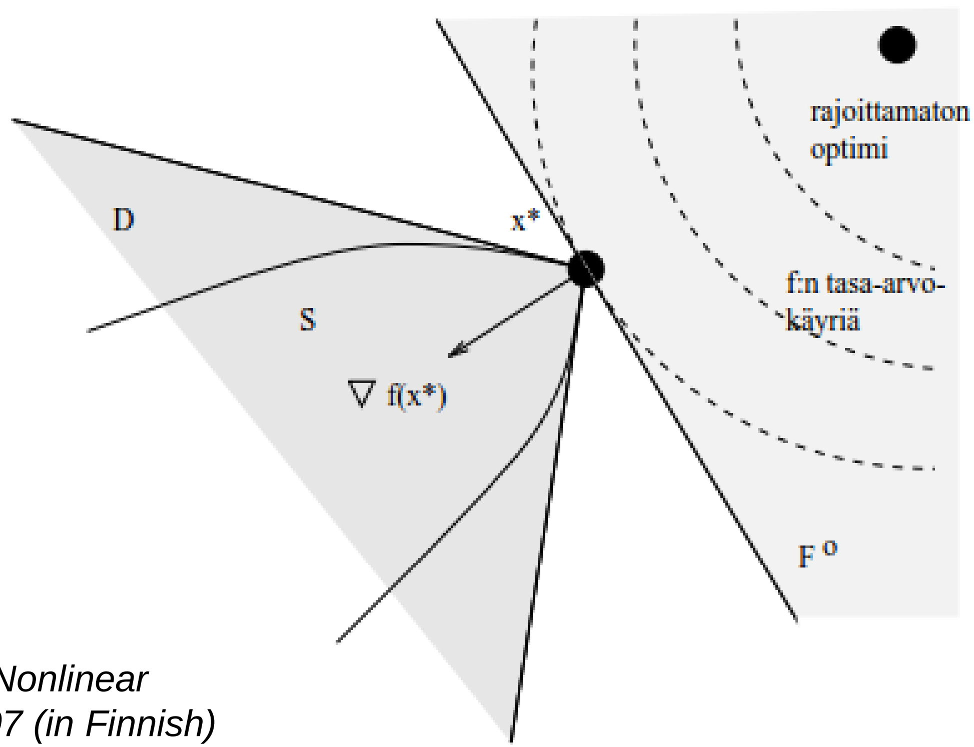
  $F \cap D = \{d \in R^n \mid d \in F \;\&\; d \in D\}$ is the cone of feasible descent directions

A necessary condition

- Let us consider $\min f(x),\ s.t.\ x \in S \subset R^n$ where $S \neq \emptyset$
- **Theorem:** Let f be differentiable in $x^* \in S$. If x^* is a local minimizer, then $F^\circ \cap D = \emptyset$ where

$$F^\circ = \{d \in R^n \,|\nabla f(x^*)^T d < 0\}.$$

  – *That is, no descent direction in x^* is feasible*



From *Miettinen: Nonlinear optimization, 2007 (in Finnish)*

Problem with only inequality constraints

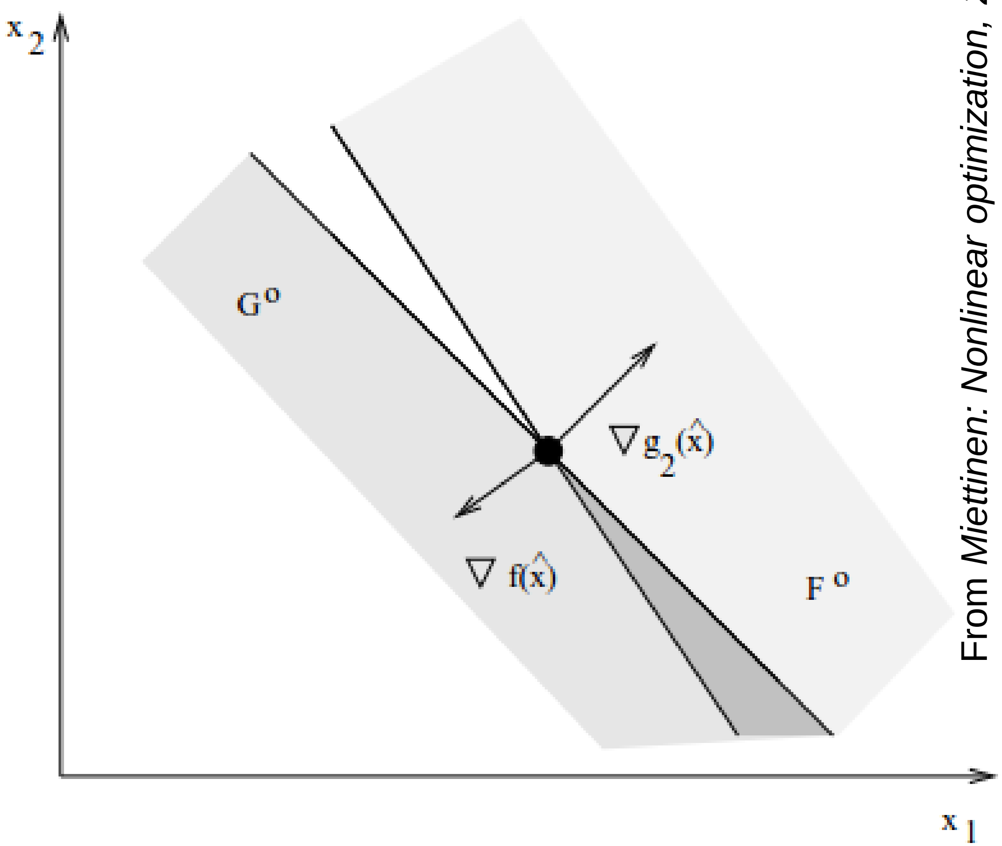
- Let us consider
  $\min f(x),\ s.t.\ g_i(x) \leq 0, i = 1, \dots, m$
- $S = \{x \in R^n \mid g_i(x) \leq 0\ \forall\ i = 1, \dots, m\}$ where $g_i: R^n \to R\ \forall i$
- Let I denote the set of active constraints in x^* i.e. $I = \{i \mid g_i(x^*) = 0\}$.

A necessary condition (for inequality constraints only)

- Define $G^{\circ} = \{d \in R^n \,|\nabla g_i(x^*)^T d < 0 \;\forall i \in I\}$
- **Theorem:** Let f and g_i be continuously differentiable in $x^* \in S$. If x^* is a local minimizer, then $F^{\circ} \cap G^{\circ} = \emptyset$.

Example

- $\min f(x) = (x_1 - 3)^2 + (x_2 - 2)^2$
- $$s.t.\ \begin{aligned} x_1^2 + x_2^2 &\leq 5, \\ x_1 + x_2 &\leq 3, \\ x_1, x_2 &\geq 0 \end{aligned}$$
- $\hat{x} = \left(\frac{9}{5}, \frac{6}{5}\right)^T, I = \{2\}$
- $F^\circ \cap G^\circ \neq \emptyset$
- $\hat{x}$ is not a local minimizer!



From Miettinen: Nonlinear optimization, 2007 (in Finnish)

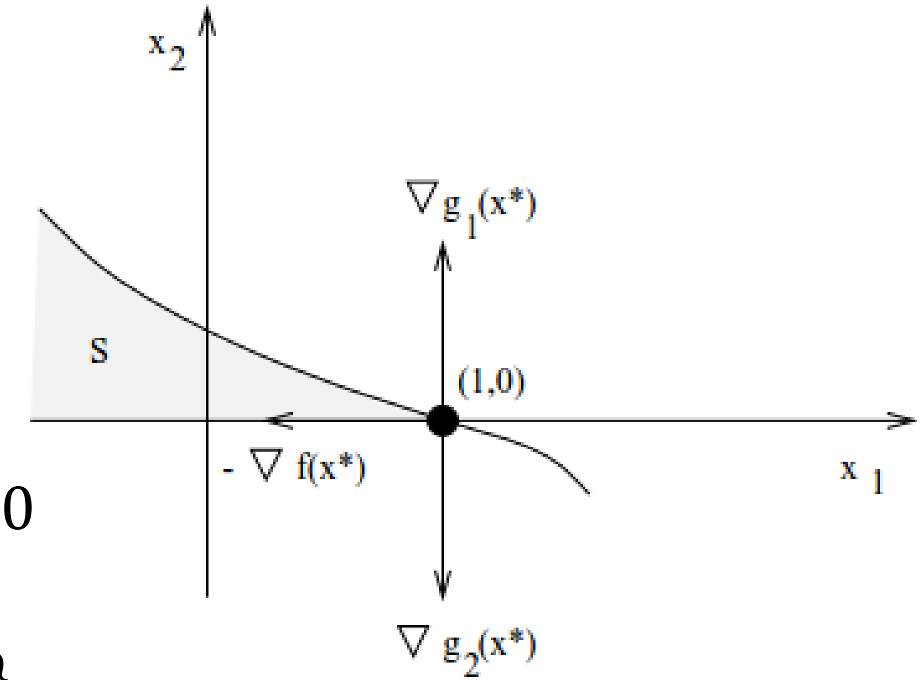
Notes

- If $\nabla f(x^*) = 0$, then $F^\circ = \emptyset$ and $F^\circ \cap G^\circ = \emptyset$
  – All critical points satisfy necessary conditions
- Any $x^* \in S$ that satisfies $g_i(x^*) = 0$ for some $i \in I$ satisfies also the necessary conditions

Fritz John conditions (for inequality constraints only)

- **Necessary conditions:** Let f and g_i be continuously differentiable in $x^* \in S$. If x^* is a local minimizer, then there exist multipliers $\lambda \geq 0$ and $\mu_i \geq 0,\ i = 1, \ldots, m$ such that $(\lambda, \mu) \neq 0$ and
  1) $\lambda \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0$
  2) $\mu_i g_i(x^*) = 0$ for all $i = 1, \ldots, m$.
- Multipliers are usually called as *Lagrange multipliers*
  - $\min L(x, \mu) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x), s.t. x \in R^n$
- *Conditions 2) are called complementarity conditions*

Example



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- $\min f(x) = x_1$
- $s.t.\ \begin{matrix} x_2 - (1 - x_1)^3 \leq 0 \\ -x_2 \leq 0 \end{matrix}$
- $x^* = (1{,}0)^T, I = \{1{,}2\}$
- $\nabla f(x^*) = (1{,}0)^T, \nabla g_1(x^*) = (0{,}1)^T, \nabla g_2(x^*) = (0, -1)^T$
- $\lambda \binom{1}{0} + \mu_1 \binom{0}{1} + \mu_2 \binom{0}{-1} = \binom{0}{0}$
  - $\lambda = 0\ \ and\ \mu_1 = \mu_2 = \alpha\ (> 0)$ will do → necessary conditions are satisfied even though x^* is not a local minimizer

KKT conditions

- Assume that $\lambda > 0$ i.e. the gradient of the objective function is taken into account every time.
- Since $\lambda > 0$ we can divide by it and can assume that $\lambda = 1$
- In order to satisfy KKT conditions, some regularity must be assumed from the constraints
  - Constraint qualifications
- **Definition:** A point $x^* \in S$ is *regular* if the set of gradients of the active constraints $\nabla g_i(x^*)$ $(i \in I)$ is linearly independent.

KKT conditions (for inequality constraints)

- **Necessary conditions:** Let f and g_i be continuously differentiable in a regular $x^* \in S$. If x^* is a local minimizer, then there exist multipliers $\mu_i \geq 0,\ i = 1, \ldots, m$ such that
  1) $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0$
  2) $\mu_i g_i(x^*) = 0$ for all $i = 1, \ldots, m$.
- Note: In problems including $g_i(x) \geq 0$, *we have* $\mu_i \leq 0$

KKT conditions
(for inequality constraints)

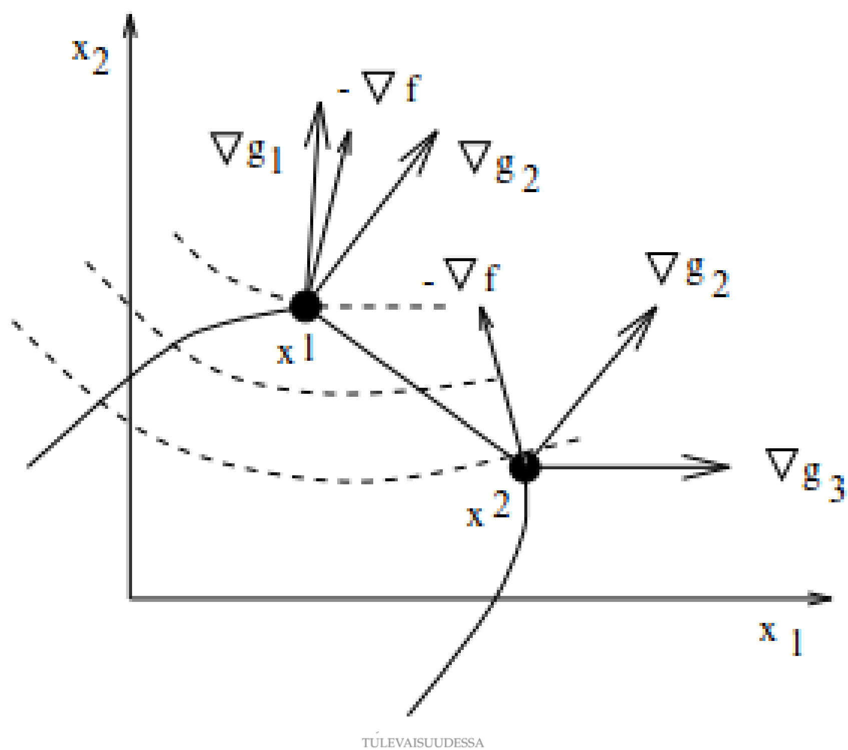
- **Sufficient conditions:** Let f and g_i be continuously differentiable and convex. Consider $x^* \in S$. If there exist multipliers $\mu_i \geq 0,\ i = 1, \dots, m$ such that
  1) $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0$
  2) $\mu_i g_i(x^*) = 0$ for all $i = 1, \dots, m$,

  then x^* is a global minimizer.

Note

- Alternatively, one can write
  $$-\nabla f(x^*) = \sum_{i=1}^{m} \mu_i \nabla g_i(x^*)$$
- That is, $-\nabla f(x^*)$ belongs to the cone defined by the gradients of

  the active constraints



Problem with both inequality and equality constraints

- Let us consider
  $\min f(x),\ s.t.\ g_i(x) \leq 0, i = 1, \dots, m$ and $h_j(x) = 0, j = 1, \dots, l$
- $S = \{x \in R^n \mid g_i(x) \leq 0\ \forall\ i = 1, \dots, m\ \&\ h_j(x) = 0\ \forall\ j = 1, \dots, l\}$ where $g_i: R^n \to R\ \forall i$ and $h_j: R^n \to R\ \forall j$

KKT conditions

- **Definition:** A point $x^* \in S$ is *regular* if the set of gradients of the active inequality constraints $\nabla g_i(x^*)$ $(i \in I)$ and equality constraints $\nabla h_i(x^*)$ $(i = 1, \dots, l)$ are linearly independent.
- **Necessary conditions:** Let f, g_i and h_i be continuously differentiable in a regular $x^* \in S$. If x^* is a local minimizer, then there exist multipliers $\mu_i \geq 0,\ i = 1, \dots, m$ and $\nu_i, i = 1, \dots, l$ such that
  1) $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{i=1}^{l} \nu_i \nabla h_i(x^*) = 0$
  2) $\mu_i g_i(x^*) = 0$ for all $i = 1, \dots, m$.

KKT conditions

- **Sufficient conditions:** Let f, g_i and h_i be continuously differentiable and convex. Consider $x^* \in S$. If there exist multipliers $\mu_i \geq 0,\ i = 1, \ldots, m$ and $\nu_i, i = 1, \ldots, l$ such that
  1) $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{i=1}^{l} \nu_i \nabla h_i(x^*) = 0$
  2) $\mu_i g_i(x^*) = 0$ for all $i = 1, \ldots, m$,

  then, x^* is a global minimizer.

Example

- $\min f(x) = (x_1 - 3)^2 + (x_2 - 2)^2$
  $x_1^2 + x_2^2 \leq 5,$
- s.t. $x_1 + 2x_2 = 4,$
  $x_1, x_2 \geq 0$
- **Formulate the KKT conditions**
- $$\begin{gathered} \boldsymbol{2x_1 - 6 + 2\mu_1 x_1 - \mu_2 + \nu_1 = 0} \\ \boldsymbol{2x_2 - 4 + 2\mu_1 x_2 - \mu_3 + 2\nu_1 = 0} \\ \boldsymbol{\mu_1\left(x_1^2 + x_2^2 - 5\right) = 0} \\ \boldsymbol{-\mu_2 x_1 = 0} \\ \boldsymbol{-\mu_3 x_2 = 0} \\ \boldsymbol{x_1 + 2x_2 - 4 = 0} \end{gathered}$$
- $\boldsymbol{\mu_1, \mu_2, \mu_3 \geq 0}$