Lecture 12
ROB-GY 7863 / CSCI-GA 3033 7863: Planning, Learning, and Control for Space Robotics

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Logistics

- Project 2 Deadlines:
  - Final presentations: December 8th
- Next two classes:
  - Guest Lecture
  - Recap / Questions

Agenda

- High Level Class Concept and Goals
- Autonomous System Diagram
- Translation Exercise
- Recap Contraction
- Recap Theoretical Results

High Level Class Concept I

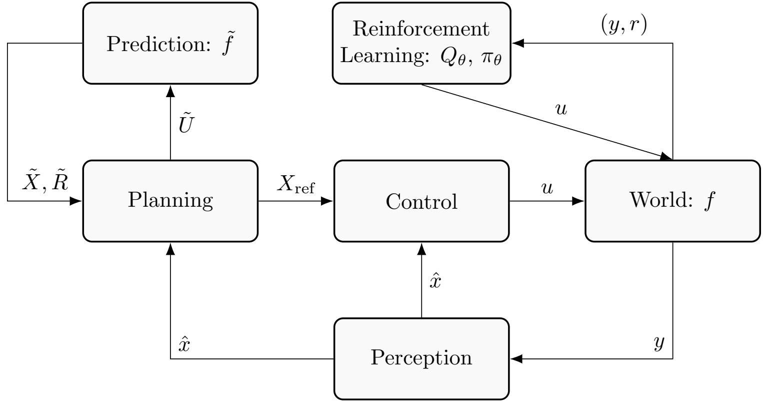
- Goals:
  - To introduce mathematical fundamentals of planning, learning, and control for robotics.
  - To get experience with modern software tools for robotics
  - To get excited about space robotic applications!
  - Deliverable: Cool demo video of your project that you can post / talk about / show people as part of your portfolio.
- Challenge: Autonomy for space robotics requires a lot of knowledge:
  - algorithms: dynamics, control, estimation, planning, and learning
  - notation: aerospace, robotics, computer science
  - culture: where the field has been and where it is going.
- Strategy:
  - Contraction-based convergence analysis is a widely-applicable tool:
    - fundamental understanding of algorithms.

High Level Class Concept II

- helps us design algorithms with good properties.
- predictable performance and understanding of which terms help/hurt convergence.

- Design, implement, and present a mission using modern software tools and fundamental understanding of autonomy concepts.

Autonomous Systems



Translation I

Concept	Computer Science	Optimal Control
State	s	x
Action/Control	a	u
Transition/Dynamics	$p(s' \vert s, a)$	$f(x, u, w)$
Reward/Stage Cost	$R(s, a)$	$c(x, u)$
Policy/Controller	$\pi(s) = a$	$k(x) = u$
Value/Cost-to-Go	$V^{\pi}(s_k)$	$J(x_k, U_k)$
Likelihood/Measurement	$p(o \vert s)$	$y = h(x, v)$
Belief	$p(s_k \vert a_{1:k}, o_{1:k})$	$p(x_k \vert u_{1:k}, y_{1:k})$

Translation II

LQG	GridWorld	Spacecraft
x	cell index	$[\mathbf{r}, \mathbf{v}, \Theta, \omega]$
u	N,E,S,W	$f_{\text{ext}}, \tau_{\text{ext}}$
$Ax + Bu + w$	$\begin{cases} s + a & \text{w.p.} \quad 1 - \epsilon \\ s & \text{w.p.} \quad \epsilon \end{cases}$	$\begin{cases} \mathbf{v} \\ \frac{1}{m} \sum_{i \in \text{forces}} f_i \\ \mathcal{B}(\Theta)^{-1} \omega \\ \mathcal{I}^{-1}(-\omega \times (\mathcal{I}\omega) + \tau_{\text{ext}}) \end{cases}$
$x_k^T Q_x x_k + u_k^T R u_k$	$\begin{cases} 1 & s = g \\ 0 & \text{else} \end{cases}$	$\lVert x_k - x_k^{\text{ref}} \rVert_{Q_x} + u_k^T R u_k$
$u_k = -K_k x_k$	$\pi(s) = \arg\max_a Q^*(s, a)$	$U_k^{\text{ref}} = \min_{U_k} J(\hat{x}, U_k)$ $u_k = u_k^{\text{ref}} - K_k(\hat{x} - x_k^{\text{ref}})$
$\sum_{t=k}^{k+K} c_t + \lVert x_{k+K} - x_{k+K}^{\text{ref}} \rVert_{Q_f}$	$E[\sum_{t=k}^{\infty} \gamma^{t-k} r_t]$	$\sum_{t=k}^{k+K} c_t + \lVert x_{k+K} - x_{k+K}^{\text{ref}} \rVert_{Q_f}$
$y_k = Cx_k + v_k$	$\begin{cases} s & \text{w.p } 1 - \epsilon \\ s + \delta & \text{w.p } \epsilon \end{cases}$	$y_k = Cx_k + v_k$
$b_k = \text{KF}(b_k, y_k, u_k)$	$b_k = \text{PF}(b_k, y_k, u_k)$	$b_k = \text{EKF}(b_k, y_k, u_k)$

Recap Contraction I

- Contraction:
  - Let (X, d) be a metric space where X is a set and d is a distance.
  - Let T be a mapping from X to X.
  - We say T is a contraction mapping if there exists a $\alpha \in [0, 1)$ such that:

    $$d(T(x), T(y)) \leq \alpha d(x, y), \quad \forall x, y \in X.$$

    If we already know x^* is a fixed point of T, then we only need to check that

    $$d(T(x), x^*) < \alpha d(x, x^*), \forall x \in X.$$

  - Consider a contraction mapping T and a sequence $x_{k+1} = T(x_k)$. Then, by induction, $d(x^*, x_k) \leq \alpha^k d(x^*, x_0)$.
  - Note: I originally introduced this concept as a comparison lemma in Lecture 2. This statement is a corollary of Banach Fixed Point Theorem or Contraction Mapping Theorem.

Recap Contraction II

- Standard Lyapunov as Contraction (Lecture 2):
  - Setup: Consider a nonlinear dynamical system $x_{k+1} = f(x_k)$ with fixed point $f(0) = 0$ where $f : R^n \to R^n$.
  - Metric space: $(R^n, d(x, y) = \|x - y\|_M)$
  - Lyapunov function: $V(x) = d(x, 0)^2$
  - Iterative Mapping: $x_{k+1} = f(x_k)$
  - Contraction Condition: $V(x_{k+1}) \leq \alpha V(x_k)$
  - Sequence Behavior: $V(x_k) \leq \alpha^k V(x_0) \implies \|x_k\|^2 \leq C\alpha^k \|x_0\|^2$
- Differential Lyapunov as Contraction (Lecture 4):
  - Setup: Consider a nonlinear dynamical system $x_{k+1} = f(x_k)$ where $f : R^n \to R^n$. Let $x_k(x_0) = f(...f(x_0))$ be the trajectory mapping, let $x_0^\epsilon = x_0 + \epsilon h$ be a finite perturbation on the initial condition, let $\delta_k = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} x_k(x_0^\epsilon)$ be the variation of the trajectory mapping with dynamics: $\delta_{k+1} = \frac{\partial f}{\partial x}\delta_k = F(x)\delta_k$.
  - Metric space: $(R^n, d(\delta_1, \delta_2) = \|\delta_1 - \delta_2\|_M)$
  - Lyapunov function: $V(x, \delta) = d(\delta, 0)^2$
  - Iterative Mapping: $\delta_{k+1} = F(x)\delta_k$

Recap Contraction III

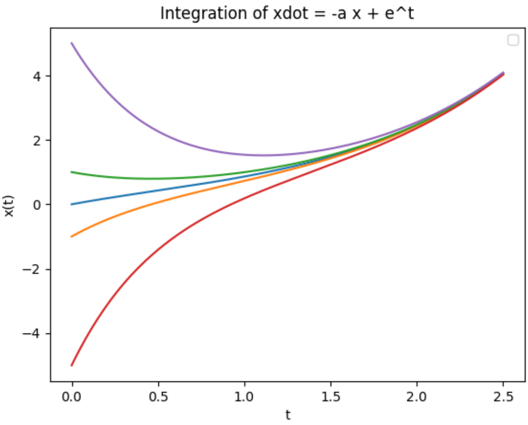
- Contraction Condition: $V(x, \delta_{k+1}) \leq \alpha V(x, \delta_k)$
- Sequence Behavior: $V(x, \delta_k) \leq \alpha^k V(x, \delta_0) \implies \|\delta_k\|^2 \leq C\alpha^k \|\delta_0\|^2 \implies \|x_k - y_k\|^2 \leq C\alpha^k \|x_0 - y_0\|^2$

- Benefits of Differential Stability:
  - Hierarchical: Consider a hierarchical system: $x_{k+1} = f(x_k)$ and $y_{k+1} = g(x_k, y_k)$. If f is contracting in x and g is contracting in y then $z = [x, y]$ is contracting (Lohmiller et al., 1998). We used this for LQG stability.
  - Robust: Consider a system $x_{k+1} = f(x_k) + d(x_k)$. If f is contracting in x with rate α, then $\|x_k - y_k\|_2^2 \leq \alpha^k \|x_0 - y_0\|_2^2 + \frac{1-\alpha^k}{1-\alpha} \max_x d(x)$ (Lohmiller et al., 1998).
  - Adaptive: Consider a system $x_{k+1} = f(x_k, \theta_k)$. If f is contracting in x then we can update θ asynchronously without losing stability (e.g. incremental least squares). We saw this in adaptive optimization in Lecture 4 (Davydov et al., 2025).

Recap Contraction IV

- Stochastic: Consider a system: $x_{k+1} = f(x_k) + w_k$ where $w_k \sim \mathcal{N}(0, \Sigma)$. If f is contracting with rate α, then $E[\|x_k - y_k\|^2] \leq C_1 \alpha^k E[\|x_0 - y_0\|^2] + C_2 \text{trace}(\Sigma)/(1-\alpha)$ (Pham, 2008). We saw this for Kalman Filter.

Recap Contraction V



Convergence results we went over I

- Control: $\pi(x) = u$
  - LQR Control (Lecture 2):
    - Setup: Consider a linear dynamical system $\dot{x} = Ax + Bu$, with cost $c(x, u) = x^T Qx + u^T Ru$ and error signal $e = x - x^{\text{ref}}$. Choose a controller $u = -Ke$ where $K = R^{-1}B^T P$ and P solves CARE equation: $A^T P + PA - 2PBR^{-1}B^T P + Q = 0$.
    - Metric space $(R^n, d(x, y) = \|x - y\|_P)$.
    - Lyapunov function: $V(x) = d(x, 0)^2$
    - Iterative Mapping: $\dot{x} = Ax + Bu$
    - Contraction Condition: $\dot{V} \leq -\alpha V$
    - Sequence Behavior: $\dot{V}(x) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(x) \implies \|e(t)\|^2 \leq C \exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} t\right) \|e(0)\|^2$
  - PD Spacecraft Control (Lecture 2):
    - Setup: Consider state $x = [q, \dot{q}]$, Lagrange robot equations: $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u$, setpoint reference $q^{\text{ref}}(t) = q^{\text{ref}}$, error signal $e = q - q^{\text{ref}}$ and gravity-compensated PD controller $u = G(q) - K_p e - K_d \dot{e}$.
    - Metric Space: $(R^n, d(x, y) = \|x - y\|_{K_p, M}$.

Convergence results we went over II

- Lyapunov function:
  $V(x) = d(x, x^{\text{ref}})^2 = \frac{1}{2}e^T K_p e + \frac{1}{2}\dot{e}^T M(q)\dot{e})$
- Iterative Mapping: $\dot{x} = [\dot{q}, M^{-1}(-C(q,\dot{q})\dot{e} - K_p e - K_d \dot{e})$
- Contraction Condition: $\dot{V} \leq 0$
- Sequence Behavior: $\lim_{t\to\infty} V(t) = 0 \implies \lim_t \|e(t)\| = 0$

- Sliding Mode Control (Lecture 3):
  - Setup: Consider state $x = [q, \dot{q}]$, Lagrange robot equations: $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$, tracking reference $q^{\text{ref}}(t)$, error signal $e = q - q^{\text{ref}}$ and sliding-mode controller $u = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - K(\dot{q} - \dot{q}_r)$ where $\dot{q}_r = \dot{q}^{\text{ref}} - \Lambda(q - q^{\text{ref}})$.
  - Metric Space: $(R^n, d(\delta_1, \delta_2) = \|\delta_1 - \delta_2\|_M)$
  - Lyapunov function: $V(x, \delta) = d(\delta, 0)^2$.
  - Iterative Mapping: $\dot{x} = f(x)$
  - Contraction Condition: $\dot{V} \leq -\alpha V$
  - Sequence Behavior:
    $V(t) \leq \exp(-\alpha t)V(0) \implies \|e(t)\|^2 \leq C\exp(-\alpha t)\|e(0)\|^2$.

- Estimation:
  - Kalman Filter (Lecture 7/8):

Convergence results we went over III

- Setup: Consider a linear dynamical system: $x_{k+1} = Ax_k + Bu_k + w_k$ where $w_k \sim \mathcal{N}(0, \Sigma_x)$ and linear measurement equation $y_k = Cx_k + v$ where $v_k \sim \mathcal{N}(0, \Sigma_y)$. Construct an estimate: $\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_k(y_{k+1} - C(A\hat{x}_k + Bu_k))$.
- Metric Space: $(R^n, d(y_k, x_k) = \|y_k - x_k\|_{P_k^{-1}})$
- Lyapunov function: $V(x, \hat{x}) = d(x, \hat{x})^2$.
- Iterative Mapping: $\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_k(y_{k+1} - C(A\hat{x}_k + Bu_k))$.
- Contraction Condition: $V(\hat{x}_{k+1}, x_{k+1}) \leq \rho V(\hat{x}_k, x_k)$.
- Sequence Behavior: $E[\|\hat{x}_k - x_k\|]_2^2 \leq C\rho^k E[\|x_0 - y_0\|_2^2] + C\frac{1-\rho^k}{1-\rho}$.

- LQG (Lecture 8):

Convergence results we went over IV

- Setup: Same equations as Kalman filter. Add estimation error system: $e_k = x_k - \hat{x}_k$ and create hierarchical system: $e_{k+1} = (A - LC)e_k = g(e_k)$ and $x_{k+1} = (A - BK)x_k + K_k e_k = f(x_k, e_k)$. From previous results, we know g is contracting in e and f is contracting in x. Let $z_k = [e_k, x_k]$ and $z_{k+1} = h(z_k)$, direct application of hierarchical contraction result results in h contracting in z.
- Metric Space: $(R^{2n}, d(z_1, z_2) = \|z_1 - z_2\|^2_{P_{k-1},M})$.
- Lyapunov function: $V(z) = d(z, 0)^2$.
- Iterative Mapping: $z_{k+1} = h(z_k)$
- Contraction Condition: $V(z_{k+1}) \leq \alpha V(z_k)$
- Sequence Behavior: $\|z_k\|_2^2 \leq \alpha^k \|z_0\|_2^2$.

- Gradient and Hessian-Based Optimization:
  - Continuous Gradient Descent (Lecture 4):
    - Setup: Consider the m-strongly convex function $f : R^n \to R$ and the optimization problem $\min_x f(x)$. We have access to ∇f and we run the gradient-flow dynamics $\dot{x} = -\nabla f(x)$.
    - Metric Space: $(R^n, d(x, y) = |f(x) - f(y)|)$
    - Lyapunov function: $V(x, x^*) = d(x, x^*)$.

Convergence results we went over V

  - Iterative Mapping: $\dot{x} = -\nabla f(x)$
  - Contraction Condition: $\dot{V} \leq -mV$
  - Sequence Behavior: $V(x(t)) \leq \exp(-mt) V(x(0))$
- Discrete Gradient Descent (Lecture 4):
  - Setup: Consider the m-strongly convex and M-smooth function $f : R^n \to R$ and the optimization problem $\min_x f(x)$. We have access to ∇f and we run the gradient-descent dynamics $x_{k+1} = x_k - \alpha \nabla f(x_k)$.
  - Metric Space: $(R^n, d(x, y) = |f(x) - f(y)|)$
  - Lyapunov function: $V(x, x^*) = d(x, x^*)$.
  - Iterative Mapping: $x_{k+1} = x_k - \alpha \nabla f(x_k)$
  - Contraction Condition: $V(x_{k+1}) \leq \frac{\kappa-1}{\kappa+1} V(x_k)$ where $\kappa = M/m$.
  - Sequence Behavior: $V(x_k) \leq \left(\frac{\kappa-1}{\kappa+1}\right)^k V(x_0) \implies f(x_k) - f(x^*) \leq \left(\frac{\kappa-1}{\kappa+1}\right)^k (f(x_0) - f(x^*))$.
- Newton Descent (Lecture 4, reference (Desoer et al., 1972)):
- Primal Dual Descent (Lecture 4):
- Interior Point Method (Lecture 4):

Convergence results we went over VI

- Sampling-Based Trajectory Optimization (Lecture 7):

$$U_k = U_{k-1} - \alpha \widehat{\nabla_U J(U)}$$

- Search-Based Optimization:
  - Monte Carlo Tree Search:

$$\mathbb{E}[V^*(x_0) - \widehat{V}_n(x_0)] \leq c\ n^{-1/2}$$

    Active area of research!

- Reinforcement Learning:
  - Value Iteration (Lecture 10):
    - Setup: Consider an MDP $\langle S, A, T, R, \gamma \rangle$.
    - Metric Space: $(\mathcal{B}, \|.\|_\infty = \sup_x |.|)$
    - Iterative Mapping: $(\mathcal{T}V)(s) = \max_a [R(s,a) + \gamma E[V(s')]]$
    - Contraction Condition: $\|\mathcal{T}V - \mathcal{T}W\|_\infty \leq \gamma \|V - W\|_\infty$
    - Sequence Behavior: $\|V_k - V^*\|_\infty \leq \gamma^k \|V_0 - V^*\|_\infty$.

Convergence results we went over VII

- Q-Learning (Lecture 10):

$$Q_{k+1} = (1-\alpha)Q_k + \alpha\widehat{\mathcal{T}Q_k}$$

- Policy Gradient (Lecture 11):

$$\theta_{k+1} = \theta_k - \alpha\widehat{\nabla_\theta J(\theta)}$$

- Dynamics: $f(x,u)$
  - Lagrange Equation from Minimum Action and Calculus of Variations (Lecture 1):
    - Setup: Consider state $x = [q, \dot{q}]$ and Lagrangian $L = T - V$ where T is kinetic energy and V is potential energy. Principal of least energy says trajectories are the extrema of action integral: $S[q] = \int_{t=t_1}^{t_2} L(q,\dot{q})dt$. Functionals require variational derivatives (similar to differential Lyapunov contraction), and setting that derivative to zero gives us:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad (1)$$

Convergence results we went over VIII

- Picard Iteration: (Lecture -1 = we did not cover this, but I wish we had):
  - **TODO:**

Path Integral Analysis: Convergence of Solutions I

- Let $x_{k+1} = f(x_k)$. Prove that $\|\delta_k\|^2 \leq Cq^k \|\delta_0\|^2$ implies $\|x_k - y_k\|^2 \leq Cq^k \|x_0 - y_0\|^2$
- Define path z:

$$z_0(s) = x_0 + s(y_0 - x_0) \tag{2}$$
$$z_{k+1}(s) = f(z_k(s)) \tag{3}$$
$$z_k(1) = y_k \tag{4}$$
$$z_k(0) = x_k \tag{5}$$

- Define variation along path, which obeys same dynamics and convergence result as δ_k, because we proved contraction uniformly across x:

$$\delta_k(s) = \frac{\partial z_k(s)}{\partial s} \tag{6}$$
$$\delta_{k+1}(s) = F_k(s)\delta_k(s) \tag{7}$$
$$\delta_0(s) = y_0 - x_0 \tag{8}$$

Path Integral Analysis: Convergence of Solutions II

- Use fundamental theorem of calculus and then apply triangle inequality and contraction inequality:

$$y_k - x_k = z_k(1) - z_k(0) = \int_s \frac{\partial z_k(s)}{\partial s} ds = \int_s \delta_k(s) ds \quad (9)$$

$$\begin{aligned} \|y_k - x_k\|^2 &= \| \int_s \delta_k(s) ds \|^2 && (10) \\ &\leq \int_s \|\delta_k(s)\|^2 ds && (11) \\ &\leq \int_s Cq^k \|x_0 - y_0\|^2 ds && (12) \\ &= Cq^k \|x_0 - y_0\|^2 && (13) \end{aligned}$$

Path Integral Analysis: Robustness

- Let $x_{k+1} = f(x_k) + w(x_k)$. Prove that, if f is contracting with rate α, then
$\|x_k - y_k\|^2 \leq \alpha^k \|x_0 - y_0\|^2 + \sup_x \|w(x)\|^2/(1-\alpha)$.

Davydov, Alexander, Veronica Centorrino, Anand Gokhale, Giovanni Russo, and Francesco Bullo (2025). "Time-varying convex optimization: A contraction and equilibrium tracking approach". In: IEEE Transactions on Automatic Control (cit. on p. 11).

Desoer, C. and H. Haneda (1972). "The measure of a matrix as a tool to analyze computer algorithms for circuit analysis". In: IEEE Transactions on Circuit Theory 19.5, pp. 480–486. DOI: 10.1109/TCT.1972.1083507 (cit. on p. 18).

Lohmiller, Winfried and Jean-Jacques E Slotine (1998). "On contraction analysis for non-linear systems". In: Automatica 34.6, pp. 683–696 (cit. on p. 11).

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