Lecture 2
ROB-GY 7863 / CSCI-GA 3033 7863: Planning, Learning, and Control for Space Robotics

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Recap Last Week - Logistics I

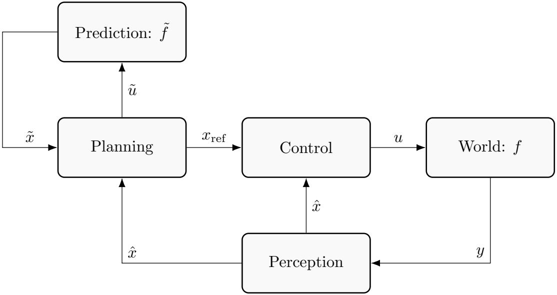
- Check that you have access to Brightspace.
- Check out the pdf notes from last time online.
- Check out class github, setup environment, and run sample scripts.
- Read/listen to something about space robots and be ready to talk about it, e.g. Space Policy Podcast.
- Start thinking about what you want your project to be.
- If you want to use an external simulator for Project 1, try doing the software installation / compatibility / unit tests as early as possible.

New Logistics I

- Project Details: Teams of 2 allowed
- Project Deadlines:
  - Project 1 and 2 Proposal: September 29th
  - Project 1 Report and Presentation: October 13th
  - Project 2 Report and Presentation: December 8th

Recap Last Week - Technical I

- Introduced Autonomy "Stack" and Stacks with Learned Models:



Recap Last Week - Technical II

- Spacecraft Dynamics via Newton Method:

$$
\begin{aligned}
\dot{\mathbf{r}} &= \mathbf{v} \\
\dot{\mathbf{v}} &= \sum_{i \in \text{bodies}} \frac{G m_i (\mathbf{r}_i - \mathbf{r})}{\|(\mathbf{r}_i - \mathbf{r})\|^3} + \frac{1}{m}\mathbf{f}_{\text{ext}} \\
\dot{\Theta} &= \mathcal{B}(\Theta)^{-1}\omega \\
\dot{\omega} &= \mathcal{I}^{-1}(-\omega \times (\mathcal{I}\omega) + \tau_{\text{ext}})
\end{aligned}
$$

- Lagrange Method for Robot Manipulator Equations:

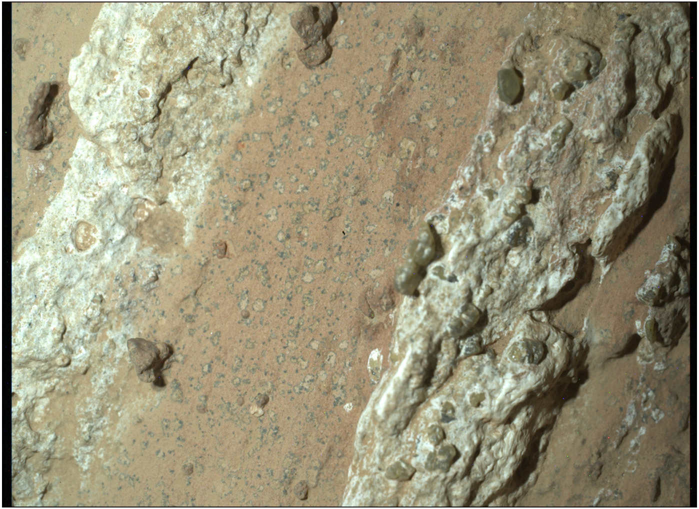
$$Bu = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q)$$

- Control u with Comparison Lemmas

Agenda Today

- Space Culture
- Dynamics Examples: Pendulum, Spacecraft, Interplanetary Super Highway, Mars Launch Times.
- Break
- Control Theory: Comparison Lemmas, Lyapunov and Differential Lyapuynov, Controller Synthesis, Estimator Synthesis
- Control Examples: Attitude Control for Spacecraft

Space Culture: Chevaya Falls



Chevaya Falls

- Nickname for region on Mars in Jezero Crater
- "The rock is covered in patterns of “leopard spots” that may have formed through chemical reactions known to fuel life. On Earth, spots like these are often associated with the fossilized record of microbes. Though other explanations still remain, Perseverance scientists have now published a peer-reviewed paper that finds no particularly strong evidence for any of the alternatives studied.”
- Planetary Society Article

Pattern for Deriving Dynamics I

- Pick Coordinates: q, $\dot{q}$
- Determine Mass Model: $M(q)$
- Determine Generalized Forces: $\delta W = (Bu)^T \delta q$
- Write Lagrangian and Derivatives
- Plug into Euler Langrange Equation

Examples of Dynamical Systems - Pendulum I

- Coordinates: $q = \theta$, $\dot{q} = \omega$
- Mass Model: Point mass and massless rod, $\mathcal{I} = (1/2)ml^2)$
- Generalized forces: $Bu = \tau_{\text{ext}}$, where $\delta W = (Bu)^T \delta q$
- Lagrangian:

$$
\begin{aligned}
L &= T - V \\
T &= \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}ml^2\omega^2 \\
V &= -mgl\cos\theta
\end{aligned}
$$

Examples of Dynamical Systems - Pendulum II

- Derivatives:

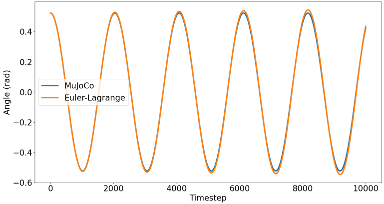
$$
\begin{aligned}
\frac{\partial L}{\partial \dot{q}} &= ml^2\omega \\
\frac{\partial L}{\partial q} &= -mgl\sin\theta \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} &= ml^2\dot{\omega}
\end{aligned}
$$

- Plug into Euler Lagrange Equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$:

$$
\begin{aligned}
ml^2\dot{\omega} + mgl\sin\theta &= \tau_{\text{ext}} \\
\dot{\omega} &= \frac{1}{ml^2}\tau_{\text{ext}} - \frac{g}{l}\sin\theta
\end{aligned}
$$

Examples of Dynamical Systems - Pendulum

```xml
<mujoco model="simple_pendulum_point_mass">
  <compiler angle="radian" inertiafromgeom="true"/>
  <option gravity="0 0 -9.81" integrator="RK4" timestep="0.001"/>
  <worldbody>
    <light pos="2 2 3" dir="-1 -1 -2"/>
    <body name="pendulum" pos="0 0 0">
      <joint name="hinge" type="hinge" axis="0 1 0" limited="false"
      damping="0"/>
      <geom name="rod" fromto="0 0 0   0 0 -1.0" mass="1e-6" type="
      capsule" size="0.01" rgba="0.3 0.6 0.9 1" contype="0"
      conaffinity="0"/>
      <geom name="bob" type="sphere" pos="0 0 -1.0" size="0.02" mass
      ="1.0" rgba="0.3 0.6 0.9 1" contype="0" conaffinity="0"/>
      <site name="tip" pos="0 0 -1.0" size="0.02" rgba="1 0.3 0.3 1"
      />
    </body>
  </worldbody>
  <actuator>
    <motor name="u" joint="hinge" gear="1" ctrlrange="-10 10"/>
  </actuator>
  <sensor>
    <jointpos   name="theta"    joint="hinge"/>
    <jointvel   name="thetadot" joint="hinge"/>
    <actuatorfrc name="tau"      actuator="u"/>
  </sensor>
</mujoco>
```



Examples of Dynamical Systems - Spacecraft I

- Coordinates: $q = [\mathbf{r}, \Theta]$, $\dot{q} = [\dot{\mathbf{r}}, \dot{\Theta}]$ and $\omega = \mathcal{B}(\Theta)\dot{\Theta}$
- Mass Model is Cylinder (same as last week):
  $\mathcal{I}_x = \mathcal{I}_y = (1/12)m(3r^2 + h^2)$ and $\mathcal{I}_z = (1/2)mr^2$.
- Generalized forces through Virtual work:

$$\delta W = \begin{bmatrix} f_{\text{ext}}^T \delta r \\ \tau_{\text{ext}}^T \delta \varphi \end{bmatrix} = \begin{bmatrix} f_{\text{ext}}^T \delta r \\ \tau_{\text{ext}}^T \mathcal{B}(\Theta)\delta\Theta \end{bmatrix} \tag{1}$$

$$B(q)u = \begin{bmatrix} I_3 & 0 \\ 0 & \mathcal{B}(\Theta)^T \end{bmatrix} \begin{bmatrix} f_{\text{ext}} \\ \tau_{\text{ext}} \end{bmatrix}$$

where f_{ext} in world frame and τ_{ext} in body frame.

Examples of Dynamical Systems - Spacecraft II

- Lagrangian:

$$
\begin{aligned}
L &= T - V \\
T &= \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}m\dot{\mathbf{r}}^T\dot{\mathbf{r}} + \frac{1}{2}\omega^T\mathcal{I}\omega = \frac{1}{2}m\dot{\mathbf{r}}^T\dot{\mathbf{r}} + \frac{1}{2}\dot{\Theta}^T J(\Theta)\dot{\Theta} \\
V &= -\sum_{i\in\text{bodies}} \frac{Gmm_i}{\|\mathbf{r}-\mathbf{r}_i\|}
\end{aligned}
$$

where $J(\Theta) = \mathcal{B}^T\mathcal{I}\mathcal{B}$

Examples of Dynamical Systems - Spacecraft III

- Derivatives:

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m\dot{\mathbf{r}} \\ J(\Theta)\dot{\Theta} \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m\ddot{\mathbf{r}} \\ \dot{J}(\Theta)\dot{\Theta} + J(\Theta)\ddot{\Theta} \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \sum_{i\in\text{bodies}} \frac{GMm_i(\mathbf{r}_i - \mathbf{r})}{\|\mathbf{r}-\mathbf{r}_i\|^3} \\ \frac{1}{2}\frac{\partial J}{\partial \Theta} : \Theta\Theta^T \end{bmatrix}$$

  where : is a tensor contraction: $(C : D)_{ij} = \sum_k \sum_l C_{i,j,k,l} D_{k,l}$.
- Plug into Euler Lagrange Equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$:

$$\begin{bmatrix} mI_3 & 0 \\ 0 & J(\Theta) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\Theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{J}(\Theta)\dot{\Theta} + \frac{1}{2}\frac{\partial J}{\partial \Theta} : \Theta\Theta^T \end{bmatrix}$$
$$+ \begin{bmatrix} \sum_{i\in\text{bodies}} \frac{GMm_i(\mathbf{r} - \mathbf{r}_i)}{\|\mathbf{r}-\mathbf{r}_i\|^3} \\ 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & \mathcal{B}(\Theta)^T \end{bmatrix} \begin{bmatrix} f_{\text{ext}} \\ \tau_{\text{ext}} \end{bmatrix}$$

Examples of Dynamical Systems - Spacecraft IV

- Complexity is computing C such that the following two things are true:

$$x^T(\dot{M} - 2C)x = 0, \quad \forall x \tag{2}$$

$$C(q, \dot{q})\dot{q} = \begin{bmatrix} 0 \\ \dot{J}(\Theta)\dot{\Theta} + \frac{1}{2}\frac{\partial J}{\partial \Theta} : \Theta\Theta^T \end{bmatrix} \tag{3}$$

- In general, computed with **Christoffel** symbols:

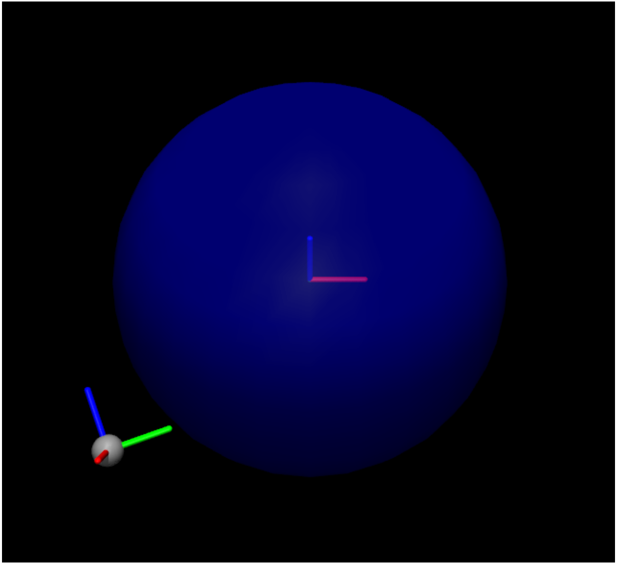
$$[C(q, \dot{q})\dot{q}]_i = \sum_{j,k} \Gamma_{ijk}\dot{q}_j\dot{q}_k \tag{4}$$

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ik}}{\partial q_j} + \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ij}}{\partial q_k}\right) \tag{5}$$

but that has a lot of derivatives, so there are special transforms someone else worked out for rigid body systems:

$$C(q, \dot{q}) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}^T\mathcal{I}\dot{\mathcal{B}} + \mathcal{B}^T S(\omega)\mathcal{I}\mathcal{B} \end{bmatrix} \tag{6}$$

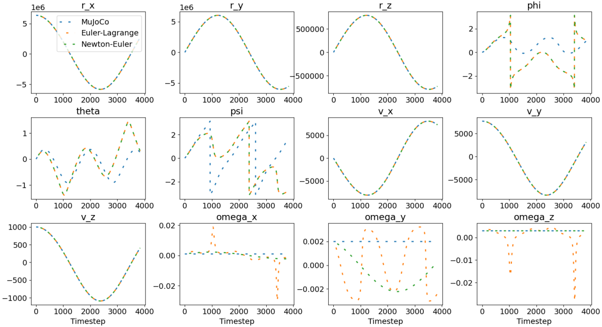
Examples of Dynamical Systems - Spacecraft V



Examples of Dynamical Systems - Spacecraft VI

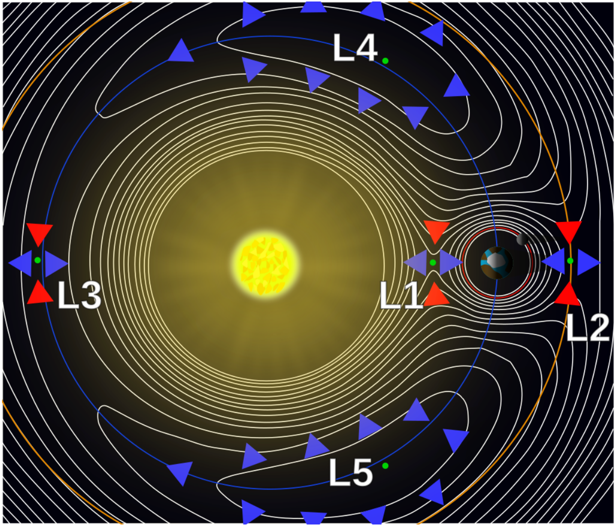
```xml
<mujoco model="spacecraft_nd">
  <compiler angle="radian" inertiafromgeom="false"/>
  <option timestep="1e-3" gravity="0 0 0" integrator="implicit"/>
  <default>
    <joint armature="0.0" damping="0.0" stiffness="0"/>
    <geom contype="0" conaffinity="0" friction="0 0 0" rgba="0.8 0.6 0.4 1"/>
    <motor ctrllimited="true" ctrlrange="-100 100"/>
  </default>
  <worldbody>
    <body name="spacecraft">
      <freejoint name="base6dof"/>
      <geom type="cylinder" size="0.001 0.0015" rgba="0.5 0.5 0.5 1"/>
      <inertial pos="0 0 0" mass="1" diaginertia="1 1 1"/>
      <!-- <site name="com" pos="0 0 0" size="0.2" rgba="1 0 0 1"/> -->
      <site name="spacecraft_com" pos="0 0 0" size="0.05" xyaxes="1 0 0 0 1 0"/>
      <geom name="spacecraft_axis_x" type="capsule" fromto="0 0 0  0.2 0 0" size="0.01" rgba="1 0 0 1"/>
      <geom name="spacecraft_axis_y" type="capsule" fromto="0 0 0  0 0.2 0" size="0.01" rgba="0 1 0 1"/>
      <geom name="spacecraft_axis_z" type="capsule" fromto="0 0 0  0 0 0.2" size="0.01" rgba="0 0 1 1"/>
    </body>
    <body name="earth" pos="0 0 0">
      <geom type="sphere" size="0.7" rgba="0 0 1 0.25" contype="0" conaffinity="0" friction="0 0 0"
      density="0"/>
      <geom name="earth_axis_x" type="capsule" fromto="0 0 0  0.2 0 0" size="0.01" rgba="1 0 0 1"/>
      <geom name="earth_axis_y" type="capsule" fromto="0 0 0  0 0.2 0" size="0.01" rgba="0 1 0 1"/>
      <geom name="earth_axis_z" type="capsule" fromto="0 0 0  0 0 0.2" size="0.01" rgba="0 0 1 1"/>
    </body>
    <light name="light1" pos="0 0 5" dir="0 0 -1" diffuse="1 1 1" specular="0.3 0.3 0.3" ambient="0.1
    0.1 0.1"/>
    <camera name="fixed_camera" pos="-150 -150 -150" quat="1 0 0 0" fovy="45"/>
  </worldbody>
  <actuator>
    <motor name="F_world_x" joint="base6dof" gear="1 0 0 0 0 0"/>
    <motor name="F_world_y" joint="base6dof" gear="0 1 0 0 0 0"/>
    <motor name="F_world_z" joint="base6dof" gear="0 0 1 0 0 0"/>
    <motor name="M_body_x" joint="base6dof" gear="0 0 0 1 0 0"/>
    <motor name="M_body_y" joint="base6dof" gear="0 0 0 0 1 0"/>
    <motor name="M_body_z" joint="base6dof" gear="0 0 0 0 0 1"/>
  </actuator>
</mujoco>
```

Examples of Dynamical Systems - Spacecraft VII



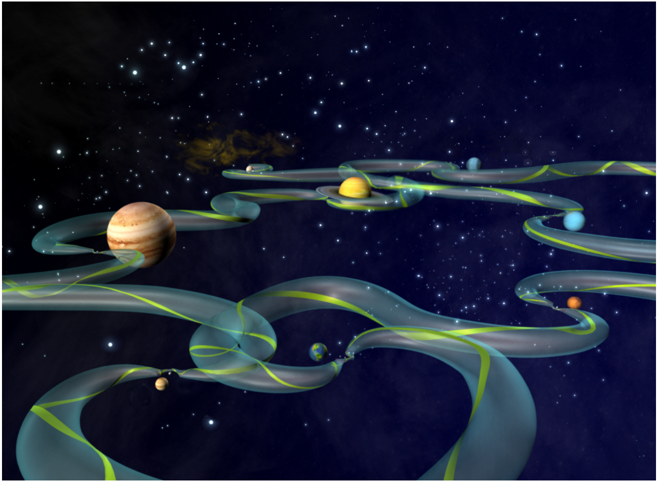
Other Example of Astrodynamics I

Lagrange Points:



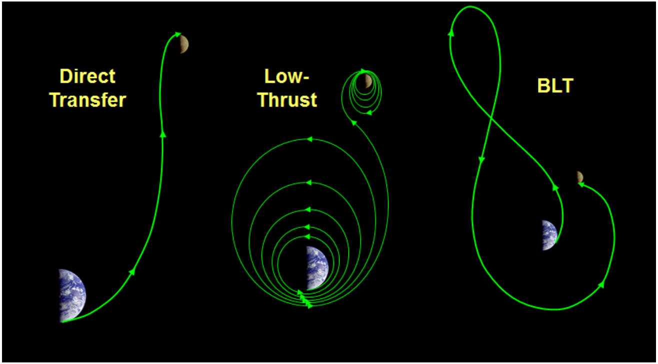
Other Example of Astrodynamics II

Interplanetary Transport Network:



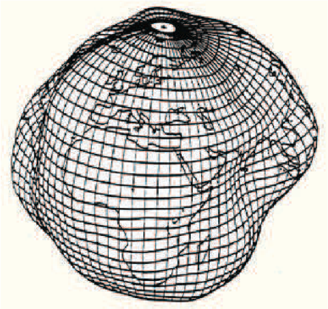
Other Example of Astrodynamics III

Moon (monthly) vs Mars (every two years) Launch Times mars animation



Other Example of Astrodynamics IV

J2 Perturbation:



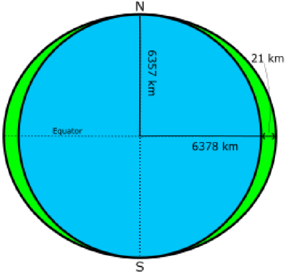
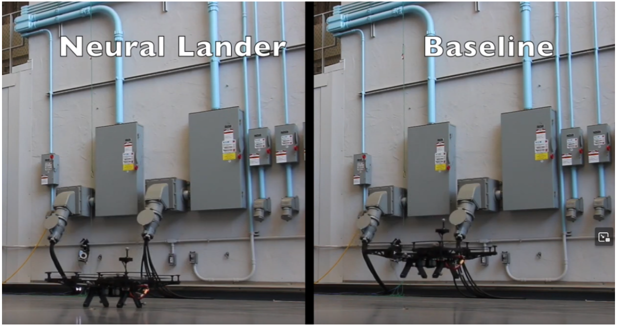


Figure: from asu lecture slides

Learned Dynamic Models I

Neural Lander - residual forces: $\dot{x} = \underbrace{f(x, u)}_{\text{Physics-Based}} + \underbrace{g(x, u)}_{\text{learned}}$



Learned Dynamic Models II

World Models - learn dynamics and observation model:
$y = h(f(x, u))$

Prompt: The video shows a first person perspective of someone navigating diffic...

Figure: Google/Deepmind's Genie Model

Learned Dynamic Models III

Physics-Based Dynamics

- Structured $\rightarrow$ small number of parameters $\rightarrow$ small amount of data needed to train.
- Limited representation: there exist things we cannot model well, like contact and fluid dynamics
- Good generalization : zero shot transfer

Learning-Based Dynamics Models

- Unstructured $\rightarrow$ large number of parameters $\rightarrow$ large amount of data needed to train
- Universal representation: good for effects you cannot model
- Weaker generalization - you should expect model errors out of training domain

Recap

- Goals of last two lectures-ish:
  - Fundamental understanding of dynamics through Newton and Lagrange. The point is not to memorize or study spacecraft dynamics derivation, but to be convince you that the fundamentals we covered are the same as the internal of the solvers.
  - Practical implementation of dynamics through Physics-Based Simulators
  - Pros and cons of physics-based vs learned dynamics
- Now, we move to control
- Break

Agenda for Control

- Comparison Lemma
- Lyapunov
- Example 1: Linear Regulation Control
- Example 2: Gravity Compensated PD Control
- Example 3: Sliding Mode and Contraction Control

Comparison Lemmas - Continuous Time

- **Continuous Time Gronwall Bellman Inequality**: Let $z : [0, T] \to \mathbb{R}_{\geq 0}$ be absolutely continuous. Suppose there exists constants a and d such that:

  $$\dot{z}(t) \leq az(t) + d, \quad z \geq 0$$

  Then:

  $$z(t) \leq e^{at} z(0) + \frac{d}{a}(e^{at} - 1)$$

- Proof: By integrating factors
- Possible to generalize to the inhomogeneous case: a_n, d_n
- For what values of a do we get stability?

Lyapunov Stability

- Given a nonlinear dynamical function $\dot{x} = f(x)$ with an equilibrium point, $f(0) = 0$.
- A Lyapunov candidate is a radially unbounded, positive definite function: $\lim_{\|x\| \to \infty} V(x) = \infty$, $V(x) > 0$, $\forall x \neq 0$ and $V(x) = 0$ for $x = 0$.
- Main Theorem: If: $\dot{V} \leq aV$ then $\exists B$ such that $\|x(t)\| \leq Be^{at}\|x(0)\|$.
  - Note: This is just the comparison lemma, and we will use this frequently
- Lasalle's Theorem: If: $\dot{V} \leq 0$, then stability to largest invariant set $\mathcal{E} = \{x \mid \dot{V} = 0\}$

Pattern for Deriving Stability I

- Given dynamics and reference
- Pick controller
- Write closed loop equations for error
- Pick Lyapunov function
- Apply Comparison Lemma

Linear Regulation Problem I

- Dynamics and Reference: $x^{\text{ref}}(t) = x^{\text{ref}}$, $u^{\text{ref}}(t) = 0$, $f(x^{\text{ref}}, 0) = 0$, $f(x, u) = Ax + Bu$,
- Pick feedback controller: $u = -Ke$, gain matrix $K = B^T P R^{-1}$ where $R = R^T \succ 0$ and $e = x - x^{\text{ref}}$
- Closed loop error dynamics: $\dot{e} = (A - BK)e$
- Pick quadratic Lyapunov function $V = e^T P e$ where $P = P^T \succ 0$.
- Compute $\dot{V}$:

$$
\begin{aligned}
\dot{V} &= \dot{e}^T P e + e^T P \dot{e} \\
&= e^T (A^T - PBR^{-1}B^T) P e + e^T P (A - BR^{-1}B^T P) e \\
&= e^T (A^T P + PA - 2PBR^{-1}B^T P) e
\end{aligned}
$$

Linear Regulation Problem II

- Choose P such that

$$A^T P + PA - 2PBR^{-1}B^T P = -Q$$

  where $Q = Q^T \succ 0$. This is called the continuous time algebraic Riccati equation (CARE).
- Then, we can plug in, manipulate, bound and apply comparison lemma:

$$V = e^T Pe \leq \lambda_{\max}(P)\|e\|^2$$

$$\dot{V} = -e^T Qe = -\|e\|_Q^2 \leq -\lambda_{\min}(Q)\|e\|^2 \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V$$

$$V(t) \leq e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} t} V(0)$$

Linear Regulation Problem III

- Then we apply spectrum bounds and manipulate for desired stability result:

$$\lambda_{\min}(P)\|e(t)\|^2 \leq \|e(t)\|_P^2 \leq \lambda_{\max}(P)\|e(t)\|^2$$

$$\|e(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t}\|e(0)\|$$

- Intuition: Lyapunov is a measure of "energy" in a system and a way of going from high dimensional state space to a positive scalar value. Comparison lemmas are a way of telling how that positive scalar value will evolve over time.

Gravity Compensated PD Control for Spacecraft I

- Dynamics and reference $x^{\text{ref}}(t) = x^{\text{ref}}$:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu$$

- Use Gravity Compensated PD Controller:

$$u = G(q) - K_p e - K_d \dot{e}$$

with error coordinates: $e = q - q^{\text{ref}}$, $\dot{e} = \dot{q}$, $\ddot{e} = \ddot{q}$.

- Closed-loop error system:

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_d \dot{e} + K_p e = 0$$

Gravity Compensated PD Control for Spacecraft II

- Use Lyapunov:

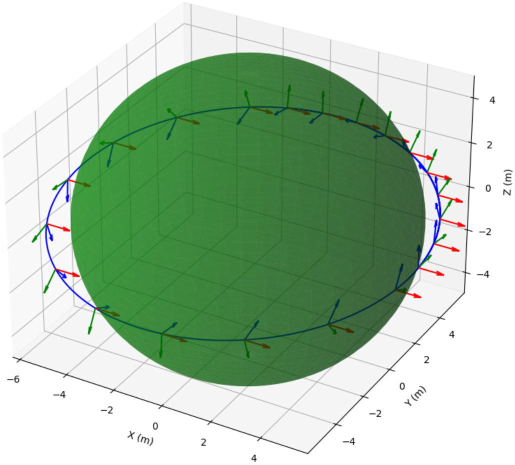
$$
\begin{aligned}
V(x) &= \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e \\
\dot{V}(x) &= \frac{1}{2}\dot{e}^T \dot{M}(q)\dot{e} + \dot{e}^T M(q)\ddot{e} + e^T K_p \dot{e} \\
&= \frac{1}{2}\dot{e}^T \dot{M}(q)\dot{e} + \dot{e}^T \left[-C(q,\dot{q})\dot{e} - K_d\dot{e} - K_p e\right] + e^T K_p \dot{e} \\
&= \frac{1}{2}\dot{e}^T \left[\dot{M}(q) - 2C(q,\dot{q})\right]\dot{e} - \dot{e}^T K_d \dot{e} \\
&= -\dot{e}^T K_d \dot{e} \leq 0
\end{aligned}
$$

- Apply Lasalle:

$$
\mathcal{E} = \{x | \dot{V} = 0\} = \{(0,0)\}
$$

Updated Simulation

TODO: see github script



Good References

- Spacecraft Engineering Larson et al., 1999
- Variational Dynamics Lanczos, 2012
- Nonlinear Systems Khalil et al., 2002