Lecture 3
ROB-GY 7863 / CSCI-GA 3033 7863: Planning, Learning, and Control for Space Robotics

Benjamin Riviere

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Logistics

- A note on class philosophy: Graduate level special topics is a survey of active areas of research, rather than building fundamentals. You do not have to understand every topic 100%. Then, you dive into something you want to learn more about for the project.
- Reminder of deadlines:
  - Project 1 and 2 Proposal: September 29th
  - Project 1 Report and Presentation: October 13th
  - Project 2 Report and Presentation: December 8th

Recap Last Week

- Dynamics: matched Lagrange $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$, Newton $F = ma$, and Mujoco models and discussed physics vs learning-based models.
- Controls: comparison lemma $\dot{z} \leq az(t) + d \implies z(t) \leq e^{at}z(0) + \frac{d}{a}(e^{at} - 1)$, Lyapunov $V(x) > 0$, $\dot{V} \leq -aV$, linear regulator, gravity compensator PD control.

Space Culture I

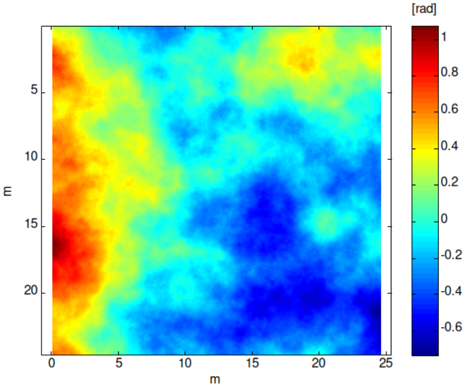


Figure: My summer internship.

Space Culture II

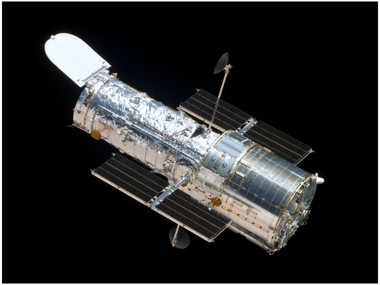


Figure: Hubble Space Telescope, Operational: 1990 - today.

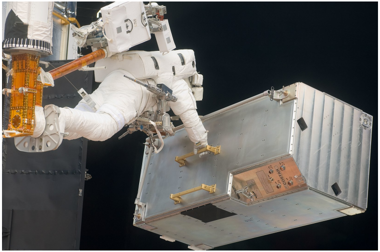


Figure: Servicing Mission 4 (but let's talk about Servicing Mission 1 in 1993).

Space Culture III

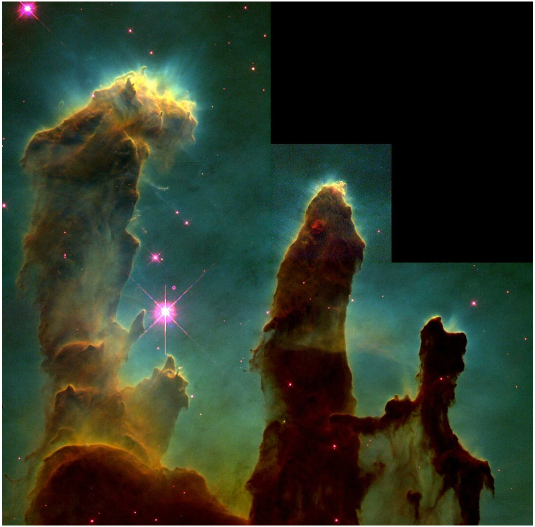


Figure: Pillars of Creation in 1995.

Agenda

- Contraction Theory
- Sliding Mode Controller
- Break
- Formation Flying

Differential Lyapunov is Contraction Theory I

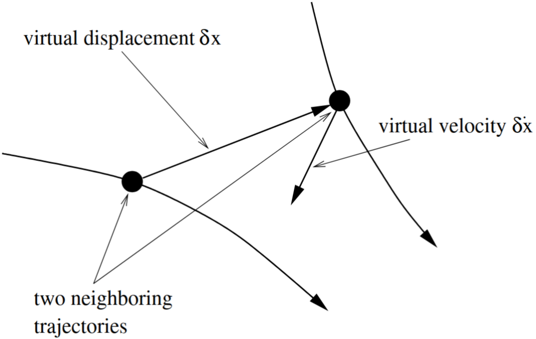


Figure: From (Lohmiller et al., 1998). I would call δx a variation, not a virtual displacement.

- Consider a non-autonomous system: $\dot{x} = f(x, t)$.

Differential Lyapunov is Contraction Theory II

- Define solution:

$$x(t, x_0) = x_0 + \int_{s=t_0}^{t} f(x(s, x_0), s) ds \tag{1}$$

- Define perturbed initial condition where $h \in \mathbb{R}^n$ and $\epsilon \geq 0$:

$$x_0^\epsilon = x_0 + \epsilon h \tag{2}$$

- Define the variation:

$$\delta x(t) = \frac{\partial}{\partial \epsilon} x(t, x_0^\epsilon)|_{\epsilon=0} \tag{3}$$

$$\delta x(t_0) = h \tag{4}$$

Differential Lyapunov is Contraction Theory III

- Derive the dynamics of the variation: For notational simplicity: $x = x(t, x_0^\epsilon)$

$$\frac{d}{dt}x = f(x,t) \tag{5}$$
$$\frac{\partial}{\partial \epsilon}\frac{d}{dt}x = \frac{\partial}{\partial \epsilon}f(x,t) \tag{6}$$
$$\frac{d}{dt}\frac{\partial x}{\partial \epsilon} = \frac{\partial}{\partial x}f(x,t)\frac{\partial x}{\partial \epsilon} \tag{7}$$
$$\dot{\delta x} = \frac{\partial f}{\partial x}\delta x \tag{8}$$

- Check for stability with Lyapunov: $V = \delta x^T \delta x$:

$$V(\delta x) = \delta x^T \delta x \tag{9}$$
$$\dot{V}(\delta x) = \dot{\delta x}^T \delta x + \delta_x^T \dot{\delta x} = \delta_x^T (\frac{\partial f}{\partial x}^T + \frac{\partial f}{\partial x})\delta x \tag{10}$$

Differential Lyapunov is Contraction Theory IV

- Contraction condition is comparison lemma on differential Lyapunov:

$$\dot{V} \leq -\alpha V \iff \frac{\partial f}{\partial x}^T + \frac{\partial f}{\partial x} \preceq -\alpha I \tag{11}$$

where $\alpha > 0$.

- What does it mean for the variation to go to zero?:

$$\delta x(t) = \frac{\partial}{\partial \epsilon} x(t, x_0^\epsilon)|_{\epsilon=0} = 0 \tag{12}$$

$\rightarrow$ All solutions of the flow contract to each other and initial condition is "forgotten".

Differential Lyapunov is Contraction Theory V

- We can consider a larger class of differential lyapunov functions: $V = \delta x^T M \delta x$ where M is a **contraction metric**. This gives generalized contraction condition:

$$\frac{\partial f}{\partial x}^T M + M\frac{\partial f}{\partial x} + \dot{M} + \alpha M \preceq 0 \tag{13}$$

- Sanity check for LTI systems: $\frac{\partial f}{\partial x} = A$ and $\dot{M} = 0$. Then contraction condition is very similar to CARE where $M = P$.

Types of Stability I

Classic Lyapunov: $V = x^T P x$

- Intuition: Trajectory energy goes down
- Good at "autonomous" systems: $f(x)$
- Applications: Regulation, tracking is possible but requires more complicated lemmas.
- Takeaway: Good starting point

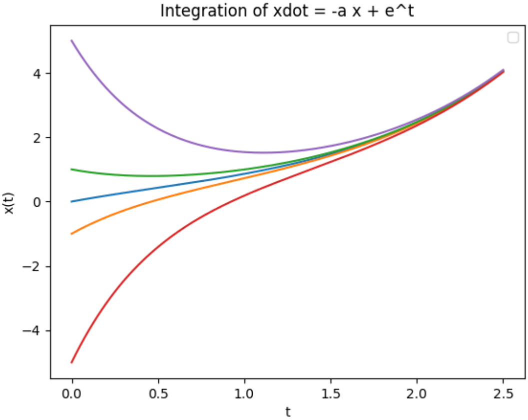
Differential Lyapunov: $V = \delta x^T M \delta x$

- Intuition: Distance between trajectories goes down
- Better at "non-autonomous" systems $f(x, t)$.
- Applications: tracking, adaptive control, hierarchical systems
- Both approaches are equivalent for LTI systems.

Types of Stability II

Example of unstable, contracting system:

$$\dot{x} = -ax + e^t \tag{14}$$



Sliding Mode Contraction Controller I

- Dynamics and reference $x^{\text{ref}}(t)$:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u$$

- Pick controller:

$$u = M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + G(q) - K(\dot{q} - \dot{q}_r) \tag{15}$$

where $\dot{q}_r = \dot{q}^{\text{ref}} - \Lambda(q - q^{\text{ref}})$

- Consider virtual system for y with solutions $y = \dot{q}_r$ and $y = \dot{q}$

$$M(q)\dot{y} + C(q, \dot{q})y + G(q) - K(\dot{q} - y) = u \tag{16}$$

- If we can show y is contracting, then we can say $\dot{q}$ goes to $\dot{q}_r$.
- Define the "sliding mode surface" $S = \{(q, \dot{q}) \mid \dot{q}_r = \dot{q}\}$. On this surface, we have good error dynamics, letting $e(t) = q - q^{\text{ref}}$: $\dot{e} = -\Lambda e$ where we chose $\Lambda \succ 0$.

Sliding Mode Contraction Controller II

- **TODO:** picture.
- Remains to show y is contracting.
- Variation dynamics:

$$M(q)\dot{\delta y} + C(q, \dot{q})\delta y + K\delta y = 0 \tag{17}$$

- Lyapunov analysis

$$\frac{d}{dt}\delta y^T M(q)\delta y = 2\delta y^T M(q)\dot{\delta}_y + \delta y^T \dot{M}(q)\delta y \tag{18}$$

$$= -2\delta y^T K\delta y \tag{19}$$

- We typically have two timescales: $\lambda_{\min}(\Lambda) < \lambda_{\min}(K)$.

Formation Flying

- Motivating Applications of Formation Flying in Space
- Basics of Spectral Graph Theory
- Consensus
- Formation Flying

Formation Flying and Optics I

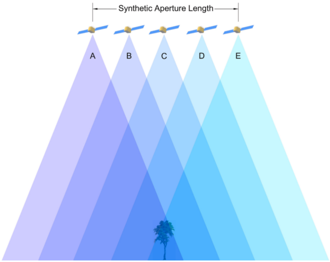


Figure: Synthetic Apertures: Picture from (Shennan et al., 2024). Other work (Lavalle et al., 2021)

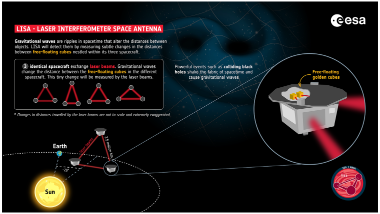


Figure: Interferometry: European Space Agency's space-based gravitational wave observatory, LISA (Wikipedia contributors, 2025a) Other work (Wikipedia contributors, 2025b)

Basics of Spectral Graph Theory I

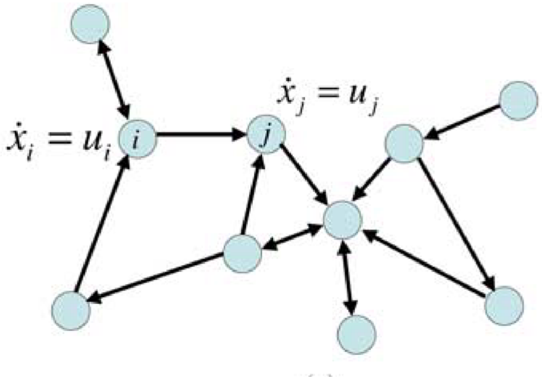


Figure: "Seminal" paper (Olfati-Saber et al., 2007)

Basics of Spectral Graph Theory II

- A graph G is a set of nodes $V = \{i = 1, ..., N\}$ and edges $E = \{(i,j)\}$, $G = (V, E)$
- $A \in \mathbb{R}^{N \times N}$ is the Adjacency Matrix, a_{ij} is greater than zero if there exist an edge from i to j.
- A graph is **balanced** if the total weight of edges entering a node and leaving the same node are equal for all nodes: $\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}, \forall i$.
- A graph is **directed** if edges have directionality.
- Two nodes i and j are **connected** if there exists a path of edges from i to j. A graph is **connected** if every pair of vertices is **connected**.
- The Laplacian of a graph, $L = D - A$ where D is the degree matrix: $d_{ii} = \sum_{j \in N(i)} a_{ij}$.

Basics of Spectral Graph Theory III

- Spectral graph theory refers to properties of L:
- L is positive semi-definite: $x^T L x = \sum_{i,j \in E} a_{ij}(x_i - x_j)^2$.
- By definition, $\mathbb{1}$ is always an eigenvector of L: $L\mathbb{1} = 0\mathbb{1}$
- The number of connected components is the number of zero eigenvalues of L (eigenvectors are indicator vectors on the components)
- The Fiedler eigenvalue is the second smallest eigenvalue of the Laplacian $\lambda_2(L)$.

Basic Consensus I

- Let G be a balanced, (strongly-)connected (di-)graph.
- Simple system for ith node:

$$\dot{x}_i = u_i \tag{20}$$

- Desired convergence to "agreement space":

$$x_1 = x_2 = \ldots = x_N \tag{21}$$
$$x = \alpha \mathbb{1} \tag{22}$$

where $\alpha = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$.
- Basic consensus dynamics:

$$\dot{x}_i = \sum_{j \in N(i)} a_{ij}(x_j - x_i) \tag{23}$$
$$\dot{x} = -Lx \tag{24}$$

and pick $a_{ij} = 1$ if $(i, j) \in E$.

Basic Consensus II

- Define the disagreement vector:

$$e = x - \alpha \mathbb{1} \tag{25}$$

- Try Lyapunov $V = e^T e$.

$$\dot{V} = 2e^T \dot{e} = -2e^T L e \leq 0 \tag{26}$$

where we use $\dot{e} = -Lx = -L(e + \alpha \mathbb{1}) = -Le$ by the definition of L.

- Wait, that is not exponential!
- We can get a **tighter bound** by using additional information. First, note that $e \in \{\mathbb{1}\}^{\perp}$:

$$\mathbb{1}^T e = \mathbb{1}^T (x(t) - \alpha \mathbb{1}) = 0 \tag{27}$$

Basic Consensus III

- Brief Tangent: The classical interpretation of the spectrum is algebraic: For matrix A, its eigenvalues Λ and eigenvectors V are:

$$(A - \Lambda I)V = 0 \tag{28}$$

In the 20th century, spectrum was interpretted in a variational sense: Let M be a symmetric, positive semidefinite matrix and let eigenvalues be sorted in ascending order $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then, the kth eigenvalue is the solution of an optimization problem (corallary to the Courant-Fisher theorem):

$$\lambda_k, v_k = \min_{\substack{x \in \{v_1, \ldots, v_{k-1}\}^\perp \\ \|x\|=1}} x^T M x \tag{29}$$

where $v_1, \ldots, v_{k-1}$ are previous eigenvectors.

Basic Consensus IV

- In our context, we know the first eigenvector is $\mathbb{1}$, and we know that $e \in \{\mathbb{1}\}^\perp$, and we know that $\lambda_2(L) > 0$ (by connectedness). Then:

$$\dot{V} = -2e^T L e = -2e^T L_s e \underbrace{\leq}_{\text{Courant-Fischer}} -2\lambda_2(L_s) e^T e = -2\lambda_2(L_s) V \tag{30}$$

- Above line implies $V(t) \leq e^{-2\lambda_2(L_s)t} V(0)$ implies $\|e(t)\| \leq e^{-\lambda_2(L_s)t} \|e(0)\|$
- Another approach to same result: first diagonalize L_s, $L = UDU^T$ and consider $y = Ux$

$$\dot{y} = -Dy \implies y_i(t) = e^{-\lambda_i t} y_i(0), \quad \forall i \geq 2 \tag{31}$$

where we recall that real symmetric positive semi-definite matrices are always diagonalizable, and their eigenvectors are an orthogonal basis.

There are many extensions!

- Leader follower dynamics:

$$\dot{x}_i = \sum_{j \in N(i)} a_{ij}(x_j - x_i) - \kappa_i(x_i - x^*) \tag{32}$$

where x^* is a leader node.

- Formation control:

$$\dot{x}_i = \sum_{j \in N(i)} a_{ij}[(x_j - x_i) + (r_j - r_i)] \tag{33}$$

where $r_j - r_i$ are desired relative position vectors of the formation.

- Time varying graphs:

$$G = (V, E) \to G(t) = (V(t), E(t)) \tag{34}$$

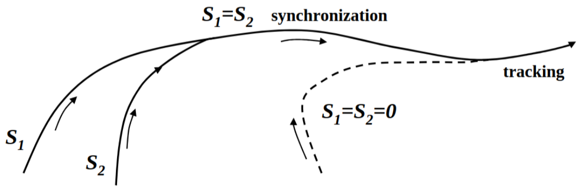
- Distributed sensor networks:
- Discrete time consensus:

$$x_{k+1} = (I - \epsilon L)x_k \tag{35}$$

Formation Flying

- Basic extensions of consensus are enough for **position control** formation flying
- Large slew maneuvers in induce nonlinearities that make basic consensus performance suffer.
- Seminal work (Chung et al., 2009b) in contraction theory for networks of Euler-Lagrangian systems, application for formation flying (Chung et al., 2009a), and extension to adaptive network topologies Chung et al., 2013

Key Ideas of (Chung et al., 2009b) I



- Main point: fast synchronization, slow tracking enables concurrent synchronization (synchronization with irregular graphs)
- We are using Euler Lagrange manipulator dynamics for ith robot:

$$M_i(q_i)\ddot{q}_i + C(q_i, \dot{q}_i)\dot{q}_i + g(q_i) = \tau_i \tag{36}$$

Key Ideas of (Chung et al., 2009b) II

- Pick the previously derived controller:

$$\tau_i = M_i(q_i)\ddot{q}_{i,r} + C(q_i, \dot{q}_i)\dot{q}_{i,r} + g(q_i) \tag{37}$$

$$- K_1(\dot{q}_i - \dot{q}_{i,r}) + \sum_{j \in N_i(t)} \frac{2}{m} K_2(\dot{q}_j - \dot{q}_{j,r}) \tag{38}$$

- Write stacked closed-loop equation:

$$[M]\dot{x} + [C]x + [L]x = 0 \tag{39}$$

where $x = [\dot{q}_1 - \dot{q}_{1,r}, \ldots, \dot{q}_n - \dot{q}_{n,r}]$. Note: this is a different Laplacian than classical, but still has ones eigenvector.

- Spectral decomposition of Laplacian:

$$[L] = [U]^T[D][U] \tag{40}$$

where $[U] = [\mathbb{1}, U_{\text{sync}}]$

Key Ideas of (Chung et al., 2009b) III

- Project system into eigenspace of Laplacian:

$$[U]^T[M][U]\dot{y} + [U]^T[C][U]y + [D]y = 0 \tag{41}$$

where $y = U^T x$.

- Differential Lyapunov: $V = \delta y^T [M] \delta y$ with $y = [y_t, y_s]$:

$$\frac{d}{dt}\begin{bmatrix}\delta y_t \\ \delta y_s\end{bmatrix}^T \begin{bmatrix}[\mathbb{1}]^T[M][\mathbb{1}] & [\mathbb{1}]^T[M][U_{\text{sync}}] \\ [U_{\text{sync}}]^T[M][\mathbb{1}] & [U_{\text{sync}}]^T[M][U_{\text{sync}}]\end{bmatrix}\begin{bmatrix}\delta y_t \\ \delta y_s\end{bmatrix} \tag{42}$$

$$= -2\begin{bmatrix}\delta y_t \\ \delta y_s\end{bmatrix}^T \begin{bmatrix}D_s & 0 \\ 0 & D_t\end{bmatrix}\begin{bmatrix}\delta y_t \\ \delta y_s\end{bmatrix} \tag{43}$$

where we can choose K_1 and K_2 such that $D_s > D_t$, synchronization is faster than tracking.

- Then, you can treat irregular graph edges as references, rather than synchronization, and maintain convergence guarantees.

Key Ideas of (Chung et al., 2009b) IV

- Takeaway: Timescale separation is useful here to relax assumptions on graph structure. Previously, timescale separation was useful for sliding mode control to get exponential stability. Timescale separation is a general tool, we will see it again in actor-critic algorithms in reinforcement learning.

Example Project Ideas

- Formation Flying (either with classic consensus or contraction or guidance (will discuss soon)) should be a pretty straightforward project
- "Dimming the Sun (DimSun) using Controllable Swarm of Smallbody Regolith Particles" (Bandyopadhyay et al., 2025)
- Rocket Landing
- Rovers
- Mimic an existing mission

Zooming Out

- Last time, we finished dynamics
- This time, we finished control / spacecraft
- Next time, moving to optimization / planning / rockets
- Connections between control and planning:
  - Stability analysis is convergence of physical state
  - Planning/Optimization analysis is convergence of optimization variables
  - Solution of linear quadratic regulator problem and of Lyapunov stability is the CARE equation.

References I

- Contraction Theory (Lohmiller et al., 1998)
- Consensus (Olfati-Saber et al., 2007)
- Formation Flying (Chung et al., 2009a)

Bandyopadhyay, Saptarshi, Sriramya Bhamidipati, Maira Saboia da Silva, Mark T Richardson, Maria Z. Hakuba, Matthew D. Lebsock, Aditya A. Paranjape, Angadh Nanjangud, Tushar Jadhav, Carl J. Percival, Evan F. Fishbein, John T. Reager, and Amir Rahmani (2025). "Dimming the Sun (DimSun) using Controllable Swarm of Smallbody Regolith Particles". In: *2025 IEEE Aerospace Conference*, pp. 1–15. DOI: 10.1109/AERO63441.2025.11068752 (cit. on p. 32).

Chung, Soon-Jo, Umair Ahsun, and Jean-Jacques E Slotine (2009a). "Application of synchronization to formation flying spacecraft: Lagrangian approach". In: *Journal of Guidance, Control, and Dynamics* 32.2, pp. 512–526 (cit. on pp. 27, 34).

References II

Chung, Soon-Jo, Saptarshi Bandyopadhyay, Insu Chang, and Fred Y Hadaegh (2013). "Phase synchronization control of complex networks of Lagrangian systems on adaptive digraphs". In: *Automatica* 49.5, pp. 1148–1161 (cit. on p. 27).

Chung, Soon-Jo and Jean-Jacques E. Slotine (2009b). "Cooperative Robot Control and Concurrent Synchronization of Lagrangian Systems". In: *IEEE Transactions on Robotics* 25.3, pp. 686–700. DOI: 10.1109/TRO.2009.2014125 (cit. on pp. 27–31).

References III

Lavalle, Marco, Ilgin Seker, James Ragan, Eric Loria, Razi Ahmed, Brian P. Hawkins, Samuel Prager, Duane Clark, Robert Beauchamp, Mark Haynes, Paolo Focardi, Nacer Chahat, Matthew Anderson, Kai Matsuka, Vincenzo Capuano, and Soon-Jo Chung (2021). "Distributed Aperture Radar Tomographic Sensors (DARTS) to Map Surface Topography and Vegetation Structure". In: *2021 IEEE International Geoscience and Remote Sensing Symposium IGARSS*, pp. 1090–1093. DOI: 10.1109/IGARSS47720.2021.9553170 (cit. on p. 18).

Lohmiller, Winfried and Jean-Jacques E Slotine (1998). "On contraction analysis for non-linear systems". In: *Automatica* 34.6, pp. 683–696 (cit. on pp. 8, 34).

References IV

Olfati-Saber, Reza, J Alex Fax, and Richard M Murray (2007). "Consensus and cooperation in networked multi-agent systems". In: *Proceedings of the IEEE* 95.1, pp. 215–233 (cit. on pp. 19, 34).

Shennan, Glen and Richard Crabbe (2024). "A review of spaceborne synthetic aperture radar for invasive alien plant research". In: *Remote Sensing Applications: Society and Environment* 36, p. 101358. ISSN: 2352-9385. DOI: https://doi.org/10.1016/j.rsase.2024.101358. URL: https://www.sciencedirect.com/science/article/pii/S2352938524002222 (cit. on p. 18).

Wikipedia contributors (2025a). *Laser Interferometer Space Antenna*. Accessed: 2025-09-20. URL: https://en.wikipedia.org/wiki/Laser_Interferometer_Space_Antenna (cit. on p. 18).

References V

Wikipedia contributors (2025b). *Terrestrial Planet Finder*. Accessed: 2025-09-20. URL: https://en.wikipedia.org/wiki/Terrestrial_Planet_Finder (cit. on p. 18).