Lecture 8
ROB-GY 7863 / CSCI-GA 3033 7863: Planning, Learning, and Control for Space Robotics

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Logistics I

- Midterm grades are posted with comments
- Project 2 Deadlines:
  - Final proposals: ~~November 24th~~ $\rightarrow$ November 3rd
  - Final presentations: December 8th
- Midterm feedback report (lets take 5 minutes to do this together in class)

(a) CS

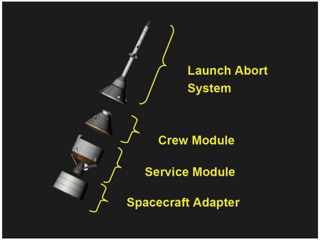
(b) Robotics

Space Culture I

- Artemis Program Goals:
  - reestablish a human presence on the Moon
  - establish a permanent base on the Moon
  - facilitate human missions to Mars
- Funding status: Artemis budget passed in 2025 One Big Beautiful Bill
- Important Components:

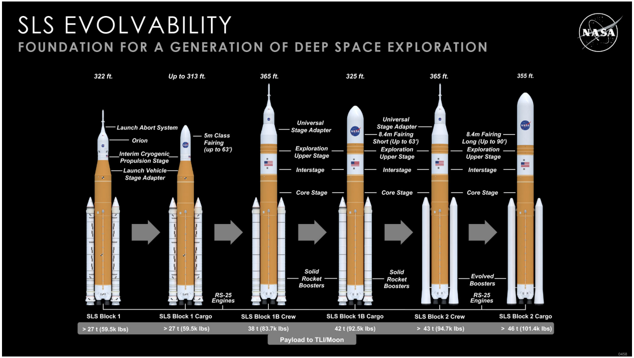
Space Culture II

- Orion Spacecraft:



Space Culture III

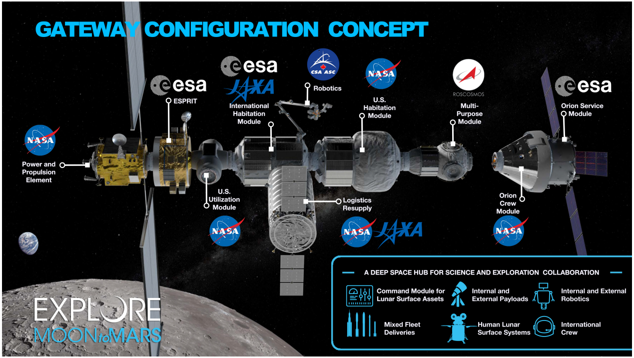
- Space Launch System (SLS) rocket:



Space Culture IV

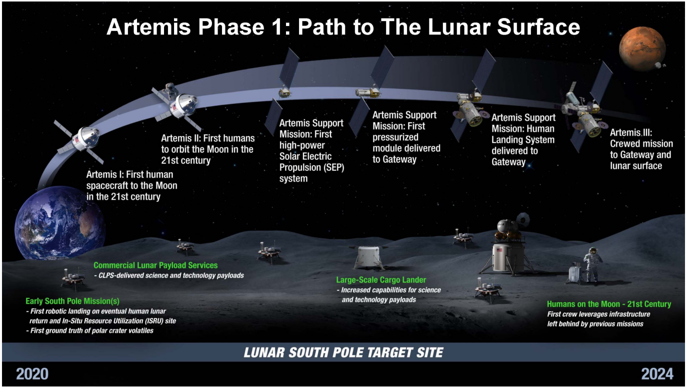
Space Culture V

- Lunar Gateway:



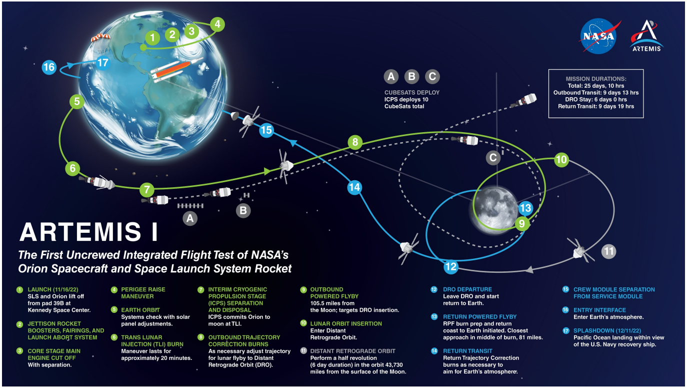
Space Culture VI

- Existing and Planned Missions:



Space Culture VII

- Artemis I Mission (2022):



- Artemis II: first crewed flights, (Planned Febuary 5, 2026)

Recap

- Connections
- Sampling-Based Optimization: Algorithm, Analysis, Implementation, and Examples. Main point: Sampling-based optimization $\approx$ gradient descent:

$$U_{k+1} = U_k + \frac{\sum_i w^i \epsilon^i}{\sum_j w^j} = U_k - \alpha \nabla_U J_{\lambda,\Sigma}(U) \qquad (1)$$

- Probability Review: Axioms of Probability, Bayes Rule
- Kalman Filter derivation from Bayes Filter with Markov assumption:

$$b_k(x_k) = p(x_k | u_{1:k}, y_{1:k}) \qquad (2)$$

$$b_{k+1}(x_{k+1}) = \eta p(y_{k+1}|x_{k+1}) \int_{x_k} p(x_{k+1}|u_k, x_k) b_k(x_k) \qquad (3)$$

Agenda

- Contraction of Kalman Filter
- Separation Principle and Linear Quadratic Gaussian
- Break
- Coding Exercise

Kalman Filter Contraction Analysis I

- Discrete Stochastic Contraction Theorem **pham**. Consider a stochastic difference equation:

$$x_{k+1} = f(x_k, k) + \sigma(x_k, k) w_k$$

where $w_k \sim \mathcal{N}(0, I)$. If: (i) f is contracting in state-independent metric M_k with rate $\alpha \in (0,1)$ and (ii) the impact of noise is bounded in expectation: $\mathbb{E}\|\sigma(x_k, k) w_k\|^2_{M_{k+1}} = \text{trace}(M_{k+1}\sigma(x_k, k)) \leq C$, $\forall k$. Then:

$$\mathbb{E}\|x_k - y_k\|^2_{M_k} \leq \alpha^k \mathbb{E}\|x_0 - y_0\|^2_{M_0} + \frac{2C}{1-\alpha}$$

Takeaway: For additive gaussian noise, compute contraction of deterministic system then add noise contribution later.

Kalman Filter Contraction Analysis II

- Linear Gaussian system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k + v_k, \end{aligned}$$

  with $w_k \sim \mathcal{N}(0, \Sigma_x)$, $v_k \sim \mathcal{N}(0, \Sigma_y)$, $\Sigma_x, \Sigma_y \succ 0$.
- Kalman filter (predict/update) for $\hat{x}_k$:

$$\begin{aligned} \bar{x}_{k+1} &= A\hat{x}_k + Bu_k \\ \bar{P}_{k+1} &= AP_kA^T + \Sigma_x \\ K_{k+1} &= \bar{P}_{k+1}C^T(C\bar{P}_{k+1}C^T + \Sigma_y)^{-1}, \\ \hat{x}_{k+1} &= \bar{x}_{k+1} + K_{k+1}(y_{k+1} - C\bar{x}_{k+1}), \\ P_{k+1} &= (I - K_{k+1}C)\overline{P}_{k+1} \end{aligned}$$

Kalman Filter Contraction Analysis III

- Assume controllable $(A, \Sigma_x^{1/2})$ and observable (A, C), which implies solutions of Ricatti equation, uniformly bounded covariance:

$$\underline{p}\, I \preceq P_k \preceq \overline{p}\, I, \quad \forall k.$$

- Virtual **deterministic** system (noise process is $K_{k+1} v_{t+1} + w_t$, we will compute its contribution later):

$$z_{k+1} = Az_k + Bu_k + K_{k+1} C\big(x_{k+1} - (Az_k + Bu_k)\big)$$

  Identify two solutions of this discrete system: $z_k = \hat{x}_k$ and $z_k = x_k$, so contraction gives us desired behavior.

- Differential System: Let $\delta_k = \lim_{\epsilon \to 0} \frac{\partial z_k^\epsilon}{\partial \epsilon}$. Then

$$\delta_{k+1} = (I - K_{k+1} C) A \delta_k$$

Kalman Filter Contraction Analysis IV

- Select **precision matrix** metric: $M_k = P_k^{-1}$ and define differential discrete Lyapunov function: $V_k = \delta_k^T M_k \delta_k$.
- Key identity for the measurement update (with $\bar{M}_{k+1} = \bar{P}_{k+1}^{-1}$):

$$
\begin{aligned}
&(I - K_{k+1}C)^T M_{k+1} (I - K_{k+1}C) \\
&\quad = (I - K_{k+1}C)^T \bar{M}_{k+1} \\
&\quad = \bar{M}_{k+1} - C^T K_{k+1}^T \bar{M}_{k+1} \\
&\quad = \bar{M}_{k+1} - C^T (C\overline{P}_{k+1}C^T + \Sigma_y)^{-1} C\overline{P}_{k+1}\bar{M}_{k+1} \\
&\quad = \bar{M}_{k+1} - C^T (C\overline{P}_{k+1}C^T + \Sigma_y)^{-1} C \\
&\quad = \bar{M}_{k+1} - C^T (S_{k+1})^{-1} C
\end{aligned}
$$

Kalman Filter Contraction Analysis V

- Go "backwards": apply innovation step, apply identity, then apply propagation:

$$
\begin{aligned}
V_{k+1} &= \delta_{k+1}^T M_{k+1} \delta_{k+1} \\
&= \delta_k^T A^T (I - K_{k+1} C)^T M_{k+1} (I - K_{k+1} C) A \delta_k \\
&= \delta_k^T A^T (\bar{M}_{k+1} - C^T S_{k+1}^{-1} C) A \delta_k \\
&= \delta_k^T \left[A^T \bar{M}_{k+1} A \; - \; A^T C^T S_{k+1}^{-1} C A \right] \delta_k \\
&= \delta_k^T \left[A^T (A P_k A^T + \Sigma_x)^{-1} A - A^T C^T S_{k+1}^{-1} C A \right] \delta_k
\end{aligned}
$$

- Look at first term. By $\Sigma_x \succ 0$,

$$
\begin{aligned}
\delta_k^T (A^T (A P_k A^T + \Sigma_x)^{-1} A) \delta_k &\leq \delta_k^T (A^T (A P_k A^T)^{-1} A) \delta_k \\
&= \delta_k^T P_k^{-1} \delta_k = \delta_k^T M_k \delta_k
\end{aligned}
$$

Kalman Filter Contraction Analysis VI

- Look at second term $A^T C^T S_{k+1}^{-1} CA$.

$$\overline{P}_{\max} = \overline{p} A A^T + \Sigma_x$$

$$S_{\max} = C \overline{P}_{\max} C^T + \Sigma_y$$

$$A^T C^T S_{k+1}^{-1} CA \succeq A^T C^T (S_{\max})^{-1} CA \succeq \mu I \succeq \mu \underline{p} M_k = \sigma M_k$$

  where $\mu = \lambda_{\min}(A^T C^T (S_{\max})^{-1} CA)$ and $\sigma = \mu \underline{p}$. Note that $\mu > 0$ when CA has no null space (this is a one-step observability requirement, we can extend this argument to m-steps to recover observability result from linear theory).

- Contraction inequality:

$$V_{k+1} \leq \delta_k^T (M_k - \sigma M_k) \delta_k = (1 - \sigma) V_k.$$

  We already know that $\sigma > 0$ (if observable), and we also know that $V_k \geq 0 \; \forall k$, therefore, $\rho = (1 - \sigma) \in (0, 1)$ and we have contraction at rate ρ.

Important Variations I

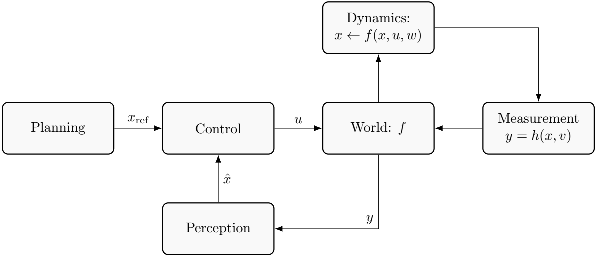
- Extended Kalman filter (local linear approximation)
- Unscented Kalman filter (pick sigma points, push through nonlinear functions, compute weighted average update using spectrum of prior)
- Ensemble Kalman filter (sample points, push through nonlinear functions, fit normal)
- Particle filter (predict, update, resample)

Putting it all together for Linear Systems I

- A discrete-time linear-time-invariant control system with additive gaussian noise is a tuple: $(A, B, C, \Sigma_x, \Sigma_y)$:

$$x_{k+1} = Ax_k + Bu_{k+1} + w_k \tag{4}$$
$$y_k = Cx_k + v_k \tag{5}$$

  where $w_k \sim \Sigma_x$ and $v_k \sim \Sigma_y$.
- We get to pick u_k and we have access to y_k (and u_k).



Putting it all together for Linear Systems II

- LQG Control = LQR control and Kalman filter estimator:

$$u_{k+1} = -K_{k+1}\hat{x}_k \tag{6}$$
$$\hat{x}_{k+1} = A\hat{x}_k + Bu_{k+1} + L_{k+1}(y_{k+1} - C(A\hat{x}_k + Bu_k)) \tag{7}$$

- Hierarchical System $e_k = x_k - \hat{x}_k$

$$e_{k+1} = f(e_k) = (A - L_{k+1}C)e_k \tag{8}$$
$$x_{k+1} = g(x_k, e_k) = (A - BK_{k+1})x_k + K_{k+1}e_{k+1} \tag{9}$$

  we know f is contracting, and we know $g(x_k, 0)$ is contracting.
- Desired behavior:

$$\mathbb{E}\|\hat{x}_k - x_k\| \leq \rho_e^k \mathbb{E}\|\hat{x}_0 - x_0\| + c_e \tag{10}$$
$$\|x_k - x_k^{\text{des}}\| \leq \rho_c^k \|x_0 - x_0^{\text{des}}\| + c_c \tag{11}$$

  Generally we want $\rho_e < \rho_c$.

More General Form

- General (nonlinear, unstructured uncertainty) system form:

$$x_{k+1} \sim p(x_{k+1}|x_k, u_k) \tag{12}$$
$$y_k \sim p(y_k|x_k) \tag{13}$$

- If you have a ton of GPU compute: Sampling-Based MPC Controller and Ensemble Kalman Filter.

Break

- Break

Coding Exercise I

Implement an LQG Controller

- Create a general system class with likelihood and dynamics functions.
- Create and test a LTI system class with A, B, C, Q_x, Q_y, Σ_x, Σ_y, P_0.
- Create and test a controller class and an instance of LQR controller.
- Create and test an estimator class and an instance of Kalman filter.
- Challenge: Create MJX system, create MPPI controller, create ensemble kalman filter

Test Plan I

- Generate system, empty controls, plot visualization
- LQR on real state -¿ check convergence
- Kalman filter -¿ check convergence
- LQR on estimated state -¿ check convergence

References I