Reinforcement learning and Optimal control for Robotics (ROB-GY 6323)

New York University, Fall 2024

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Exercise series 1

For questions requesting a written answer, please provide a detailed explanation and typeset answers (e.g. using LaTeX1). Include plots where requested in the answers (or in a Jupyter notebook where relevant). For questions requesting a software implementation, please provide your code in runnable Jupyter Notebook. Include comments explaining how the functions work and how the code should be run if necessary. Code that does not run out of the box will be considered invalid.).

1 Exercise 1 [20 points]

Find all the minimum(s), if they exist, of the functions below. Characterize the type of minimum (global, local, strict, etc) and justify your answers (hint: you can plot the functions in Python to get an intuition of their form).

1.1 Function: $f_1(x) = -e^{-(x-1)^2}$

The function is:

$$f(x) = -e^{-(x-1)^2}$$

Minimum Value: -1
Value: $x = 1$
Type of Minimum: Global minimum

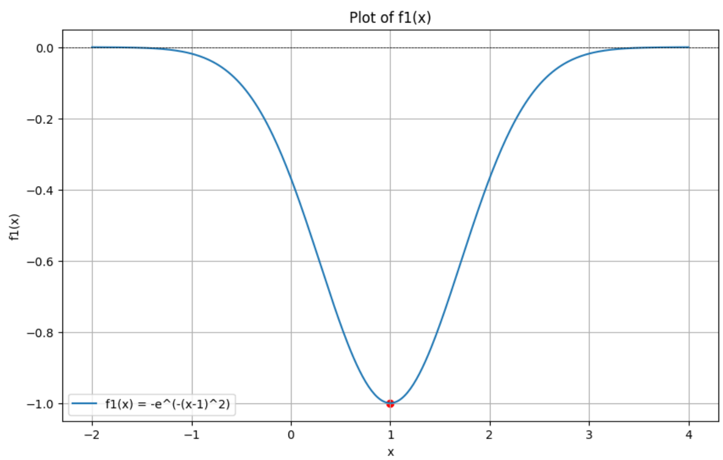


Figure 1: Plot of $f_1(x) = -e^{-(x-1)^2}$

1.2 Function: $f_2(x, y) = (1 - x)^2 + 100(y - x^2)^2$

The function is:

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2$$

Minimum Value: Approximately 0
Value: $(x, y) = (1, 1)$
Type of Minimum: Global minimum

1.3 Function: $f_3(x, y) = 20x + 2x^2 + 4y - 2y^2$

The function is:

$$f(x, y) = 20x + 2x^2 + 4y - 2y^2$$

Minimum Value: No minimum found.
Type: No minimum exist.

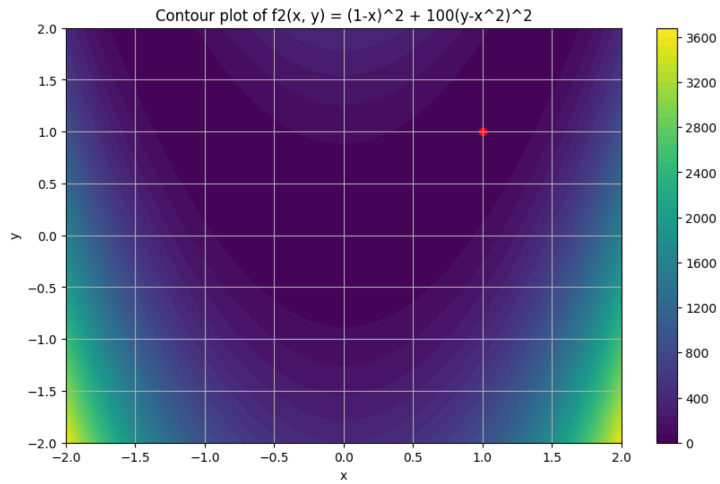


Figure 2: Contour plot of $f_2(x, y)$

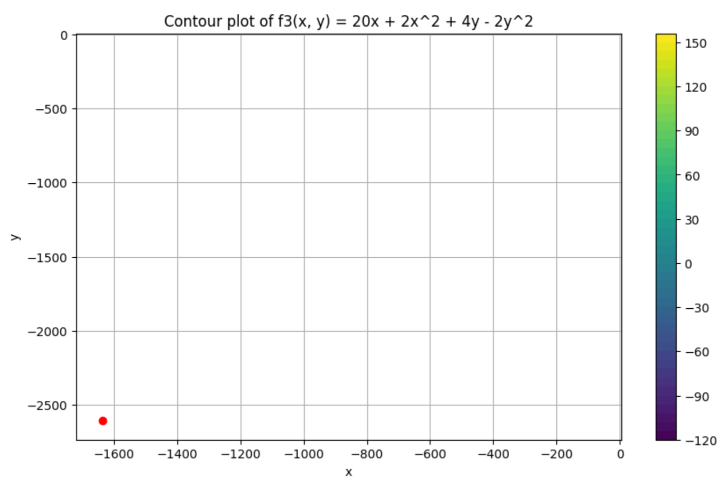


Figure 3: Contour plot of $f_3(x, y)$

1.4 Function: $f_4(x) = x^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T x$

The function is:

$$f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Minimum Value: Approximately -0.25
Value: $x = [0.25, -0.25]$
Type of Minimum: Global minimum

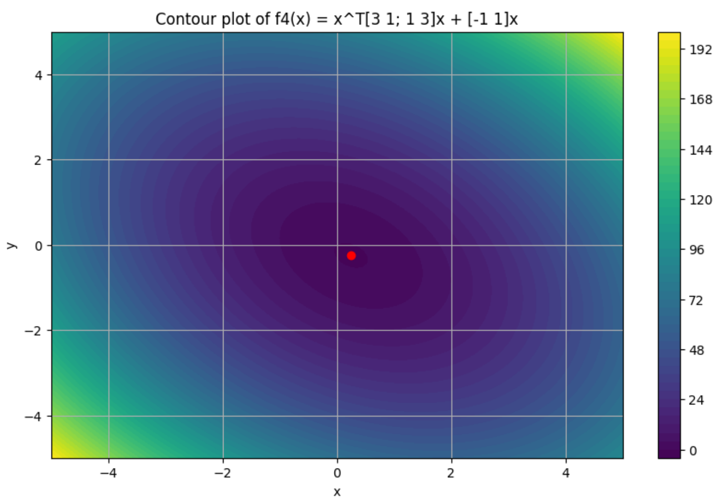


Figure 4: Contour plot of $f_4(x)$

1.5 Function: $f_5(x) = x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 10 \end{bmatrix}^T x$

The function is:

$$f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Minimum Value: No minimum found.
Type of Minimum: None.

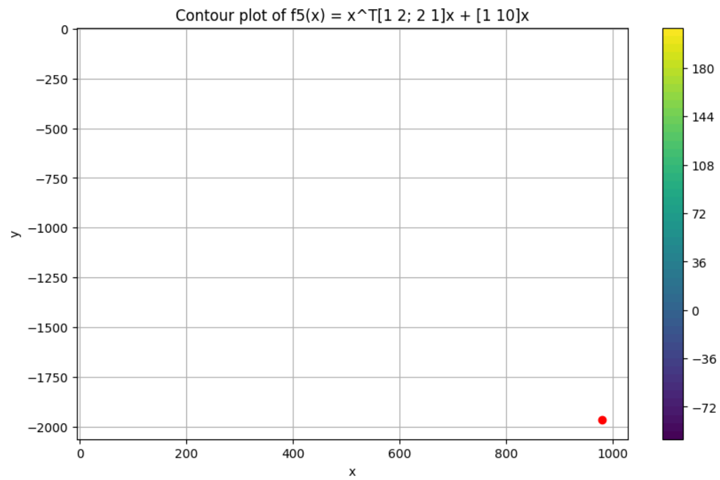


Figure 5: Contour plot of $f_5(x)$

1.6 Function: $f_6(x) = \frac{1}{2}x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x$

The function is:

$$f(x_1, x_2, x_3) = \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Minimum Value: 0
Value: $x = [0, 0, 0]$
Type of Minimum: Global minimum

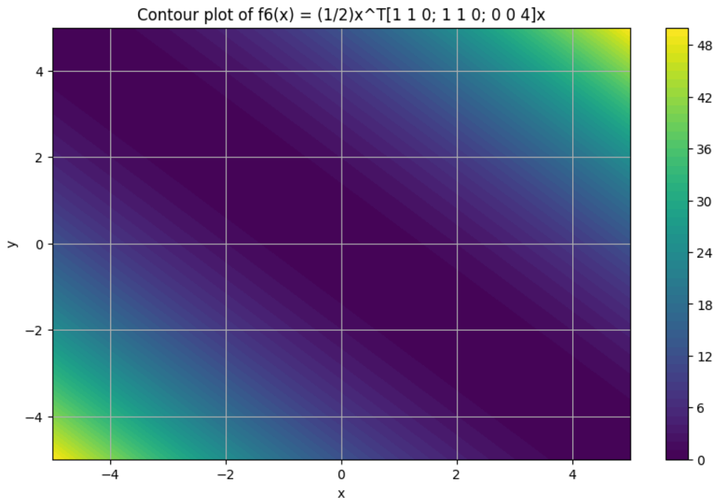


Figure 6: Contour plot of $f_6(x)$

2 Exercise 2 [20 points]

We would like to find a 2D point (x, y) as close as possible to the point $(1, 1)$ under the constraints that the sum $x + y$ is lower than 1 and that the differences $y - x$, $x - y$, and $-y - x$ are lower than 1.

- Write the problem above as a minimization problem with constraints (hint: use a quadratic cost)
- Draw a geometric sketch of the problem showing the level sets of the function to minimize and the constraints
- Write the Lagrangian of the optimization problem
- Write the KKT necessary conditions for a point x^* to be optimal
- Find the minimum and the values of x and y that reach this minimum
- At the minimum, which constraints are active (if any) and what are their associated Lagrange multipliers?

1. Minimization Problem:

Minimizing problem for this question can be:

$$f(x, y) = (x - 1)^2 + (y - 1)^2$$

Considering the constraints:

$$
\begin{aligned}
x + y &\leq 1 \\
y - x &\leq 1 \\
x - y &\leq 1 \\
-y - x &\leq 1
\end{aligned}
$$

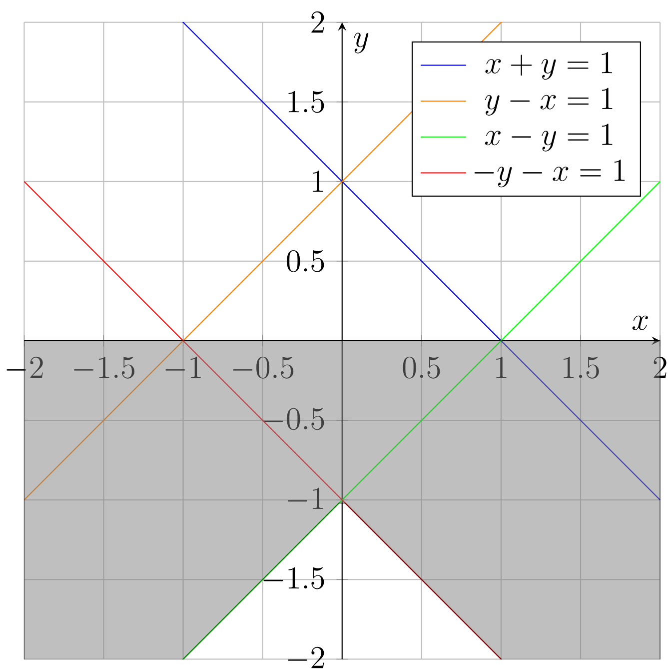
2. Geometric Sketch:

To visualize the constraints and the feasible region, we have the constraints as:

$$
\begin{aligned}
x + y &\leq 1 \\
y - x &\leq 1 \\
x - y &\leq 1 \\
-y - x &\leq 1
\end{aligned}
$$

These inequalities define a polygonal region in the plane.

Feasible Region for the Constraints



The shaded area shows the feasible region where all constraints are satisfied.

3. Lagrangian:
The general Lagrangian for an optimization problem can be written as:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

The Lagrangian of the optimization problem here is:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x-1)^2+(y-1)^2+\lambda_1(x+y-1)+\lambda_2(y-x-1)+\lambda_3(x-y-1)+\lambda_4(-y-x-1)$$

4. KKT Conditions:

The KKT conditions are:

$$\begin{aligned}
&2(x-1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0 \\
&2(y-1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \\
&\lambda_i \geq 0, \quad i = 1, 2, 3, 4 \\
&\lambda_i g_i(x, y) = 0, \quad i = 1, 2, 3, 4
\end{aligned}$$

5. Finding the Minimum:
If we solve for x and y, we have to assume that some of the constraints are active.
So, if $x + y = 1$ is active:
Substituting into the KKT conditions: $x^* = y^* = \frac{1}{2}$.
Checking with all constraints:
$x^* + y^* = 1$ (active)
$y^* - x^* = 0 \leq 1$
$x^* - y^* = 0 \leq 1$
$-y^* - x^* = -1 \leq 1$

This solution satisfies all constraints.

6. Active Constraints and Lagrange Multipliers:

At the minimum $(x^*, y^*) = (\frac{1}{2}, \frac{1}{2})$, the active constraint is $x + y = 1$. The associated Lagrange multiplier λ_1 is non-zero while λ_2, λ_3, and λ_4 are zero.

Exercise 3 [20 points]

Consider the following optimization problem

$$\min_x \frac{1}{2} x^T Q x$$

subject to

$$Ax = b$$

where $Q \in \mathbb{R}^{n \times n} > 0$ and $A \in \mathbb{R}^{m \times n}$ is full rank with $m < n$ and $b \in \mathbb{R}^m$ is an arbitrary vector.

- Write the Lagrangian of the optimization problem as well as the KKT conditions for optimality.
- Solve the KKT system and find the optimal Lagrange multipliers as a function of Q, A, and b.
- Use the above results to compute the minimum of the function below (and the value of x and of the Lagrange multipliers)

$$\frac{1}{2} x^T \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} x$$

under the constraint that the sum of the components of the vector $x \in \mathbb{R}^3$ should be equal to 1. Verify that the constraint is indeed satisfied for your result. (Hint: use python for all your numerical computation.)

The optimization problem is:

$$\min_x \frac{1}{2} x^T Q x$$

subject to:

$$Ax = b$$

where Q is a positive definite matrix, A is full rank, and b is a vector.

1. Lagrangian and KKT Conditions

Lagrangian: The Lagrangian $L(x, \lambda)$ for this problem is given by:

$$L(x, \lambda) = \frac{1}{2}x^T Q x + \lambda^T (Ax - b)$$

KKT Conditions: The KKT conditions for optimality are:
Stationarity: $Qx + A^T\lambda = 0$
Primal feasibility: $Ax = b$

2. Solution of the KKT System

For the specific problem with the matrix

$$Q = \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

and the constraint that the sum of components of x equals 1:

Equations for specific conditions:

From stationarity:
$100x_1 + 2x_2 + x_3 + \lambda = 0$
$2x_1 + 10x_2 + 3x_3 + \lambda = 0$
$x_1 + 3x_2 + x_3 + \lambda = 0$

From primal feasibility:
$x_1 + x_2 + x_3 = 1$

After solving these equations, we get:
$x_1 = \frac{1}{111}$, $x_2 = \frac{10}{111}$, $x_3 = \frac{100}{111}$, Lagrange multiplier $\lambda = -\frac{100}{111}$

3. Minimum value of the function

Substituting these values back into the objective function:

$$f(x) = \frac{1}{2}x^T Q x$$

Substituting x_1, x_2, x_3 into the expression.
The minimum value is approximately 0.405.

4. Verification

To verify that the constraint is satisfied, we will check the sum of the components:

Sum of components:
$x_1 + x_2 + x_3 = \frac{1}{111} + \frac{10}{111} + \frac{100}{111} = 1.$

Thus, all conditions are satisfied, and the solution is correct.