Unit 4

VECTOR CALCULUS

Definitions

Gradient of a scalar

Let f(x, y, z) be a function which is differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f is noted by ∇f and written by grad f and is defined as

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

Divergence of a vector function

Let $\vec{f}$ be a vector function that is differentiable then the divergence of $\vec{f}$ is noted by $\nabla . \vec{v}$ and written by div. $\vec{f}$ and is defined as,

$$\nabla . \vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . \vec{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

Curl of a vector function

Let $\vec{f}$ be a vector function that is differentiable. Then the curl of $\vec{f}$ is noted by $\nabla \times \vec{f}$ and written by curl $\vec{f}$ and is defined as

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad \text{for } \vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}.$$

Directional derivative of a function

Let f be given function. Then the directional derivative of f at a point f in the direction $\vec{a}$ is denoted by $\quad D\frac{f}{\vec{a}} = \text{grad}(f).\hat{a}.$

Here, $\hat{a}$ be unit vector of $\vec{a}$, is defined as $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Exact Integral

Let $\iint_C (f_1dx + f_2dy + f_3dz)$ be an integral with the functions f_1, f_2 and f_3 are continuous and have continuous first order partial derivatives. Then the value under the integral sign, is exact if the conditions

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}$$

is satisfied.

Surface integral

Any integral which is to be evaluated over a surface, is called a surface integral.

Volume integral

Any integral which is to be evaluated over a volume, is called a volume integral.

Note:

(i) If curl $\vec{f} = 0$ i.e. $\nabla \times \vec{f} = 0$ then $\vec{f}$ is irretational.

(ii) If div $\vec{f} = 0$ i.e. $\nabla . \vec{f} = 0$ then $\vec{f}$ is solenoidal.

(iii) Let $\vec{f}$ be a function. Let P be a point on a surfaces and $\vec{n}$ be unit vector at P having direction of outward drawn normal to S at P. Then $\vec{f}.\vec{n}$ called normal component of $\vec{f}$ at P.

(iv) The integral of a normal component $\vec{f}.\vec{n}$ over s is called a flux of $\vec{f}$ over s. That is, $\iint_S \vec{f}.\vec{n}\, ds$ is a flux.

heorem: The necessary and sufficient condition for the vector function $\vec{a}$ of the scalar variable t to have constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt} = 0$.

[2003 Fall Q.No. 3(a)]

roof: (Necessary condition): Let $\vec{a}$ has a constant magnitude. Then we have to show $\vec{a}.\frac{d\vec{a}}{dt} = 0$.

We know,

$$\vec{a}.\vec{a} = (|\vec{a}|)^2$$

Differentiating with respect to t,

$$\Rightarrow \frac{d}{dt}(\vec{a}.\vec{a}) = \frac{d}{dt}[(|\vec{a}|)^2]$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0 \qquad \text{since } |\vec{a}| \text{ is constant.}$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0.$$

Sufficient condition: Let $\vec{a}.\frac{d\vec{a}}{dt} = 0$, then we have to show that $\vec{a}$ has a constant magnitude.

We know, $\quad \vec{a}.\vec{a} = (|\vec{a}|)^2$.

Differentiating with respect to t

$$\frac{d}{dt}(\vec{a}.\vec{a}) = (|\vec{a}|)^2$$

$$\Rightarrow \quad 2\left(\vec{a}\cdot\frac{d\vec{a}}{dt}\right) = 2|\vec{a}|\frac{d}{dt}(|\vec{a}|)$$

$$\Rightarrow \quad 0 = |\vec{a}|\frac{d}{dt}(|\vec{a}|).$$

Thus, we get $|\vec{a}|\frac{d}{dt}|\vec{a}| = 0 \quad \Rightarrow \quad \frac{d}{dt}|\vec{a}| = 0$, since $|\vec{a}| \neq 0$.

Thus we get $|\vec{a}|$ as a constant. That is $\vec{a}$ has a constant magnitude.

Theorem: The necessary and sufficient condition for the vector valued function $\vec{a}$ of the scalar variable t to have a constant direction is $\vec{a}\times\frac{d\vec{a}}{dt} = \vec{0}$.

[2013 Spring Q.No. 2(a)] [2009 Spring Q.No. 3(b)] [2002 Q.No. 3(a)]

Proof: Necessary condition: Let $\vec{a}$ has a constant direction, then we have to show that $\vec{a}\times\frac{d\vec{a}}{dt} = \vec{0}$.

We know, $\vec{a} = a\,\hat{a}$ and $\hat{a}$ is a constant vector in this case where a is a magnitude of $\vec{a}$ and $\hat{a}$ be as unit vector along $\vec{a}$. Then

$$\vec{a}\times\frac{d\vec{a}}{dt} = (a\,\hat{a})\times\left(a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a}\right)$$

$$= a\,\hat{a}\times\left(\frac{da}{dt}\hat{a}\right) \qquad \text{Since } \frac{d\hat{a}}{dt} = \vec{0}$$

$$= \left(a\frac{da}{dt}\right)(\hat{a}\times\hat{a}) = \left(a\frac{da}{dt}\right)\vec{0} = \vec{0}.$$

Thus, $\vec{a}\times\frac{d\vec{a}}{dt} = \vec{0}$.

Sufficient Condition: Let $\vec{a}\times\frac{d\vec{a}}{dt} = \vec{0}$, then we have to show that $\vec{a}$ has a constant direction.

Here,

$$\vec{a}\times\frac{d\vec{a}}{dt} = \vec{0} \quad \Rightarrow \quad (a\,\hat{a})\times\frac{d}{dt}(a\,\hat{a}) = \vec{0}$$

$$\Rightarrow \quad a^2\left(\hat{a}\times\frac{d\hat{a}}{dt}\right) = \vec{0}$$

$$\Rightarrow \quad \hat{a}\times\frac{d\hat{a}}{dt} = \vec{0} \qquad \text{...... (1)} \qquad [\text{Since } a^2 \neq \vec{0}]$$

Also, we have

$$\hat{a}\cdot\frac{d\hat{a}}{dt} = 0 \qquad \text{......... (2)} \qquad [\text{Since } \hat{a} \text{ has constant magnitude}]$$

From (1) and (2), we get

either, $\hat{a} = \vec{0}$. or $\frac{d\hat{a}}{dt} = \vec{0}$.

Here, $\hat{a} \neq \vec{0}$. so $\frac{d\hat{a}}{dt} = \vec{0}$.

Thus by definition $\hat{a}$ is constant. So $\vec{a}$ has a constant direction.

Theorem: Let f be a continuous and differentiable scalar valued function, then curl (gradf) = $\vec{0}$. **[2001 Q.No. 3(b)]**

Proof: Let f be a scalar valued function. Then

$$\text{grad } f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)f$$

$$\Rightarrow \quad \text{grad } f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

So, $\text{curl (gradf)} = \nabla\times\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{i} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)$$

$$= \vec{i}\,0 + \vec{j}\,0 + \vec{k}\,0$$

$$= \vec{0}$$

Thus, we get curl grad(f) = $\vec{0}$.

Theorem: Let $\vec{v}$ be a vector valued function, which is continuous and differentiable, then div.(curl $\vec{v}$) = 0.

Proof: Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector valued function, which is continuous and differentiable. Then

$$\text{Curl } \vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{i} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

Again,

$$\text{div (curl } \vec{v}) = \nabla.\left[\vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x\partial y} - \frac{\partial^2 v_2}{\partial x\partial z} - \frac{\partial^2 v_3}{\partial y\partial x} + \frac{\partial^2 v_1}{\partial y\partial z} + \frac{\partial^2 v_2}{\partial z\partial x} - \frac{\partial^2 v_1}{\partial x\partial y}$$

$$= 0.$$

Thus, div. (curl $\vec{v}$) = 0.

Green's Theorem in a Plane:

Theorem: Let R be a closed bound region in xy plane whose boundary C consists finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that continuous and have continuous partial derivatives everywhere in some domain containing R. Then

$$\oint_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

where integration along the entire boundary C of R is in anti clockwise direction

Proof: Let us define a region R by

$$a \le x \le b, \quad u(x) \le y \le v(x) \quad \text{(fig. 1)}$$

$$\text{and } c \le y \le d, \quad p(y) \le x \le q(y) \quad \text{(fig. 2)}$$

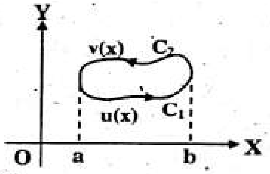


Fig. 1

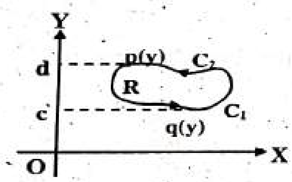


Fig. 2

Here,

$$\iint_R \frac{\partial F_1}{\partial y} dA = \int_a^b \int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy\, dx \quad \text{.......... (1)}$$

For $\displaystyle\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy = [F_1(x, y)]_{u(x)}^{v(x)} = F_1[x, v(x)] - F_1[x, u(x)].$

Thus from equation (1), we get

$$\iint_R \frac{\partial F_1}{\partial y} dA = \int_a^b \{F_1[x, v(x)] - F_1[x, u(x)]\}\, dx$$

$$= \int_a^b F_1[x, v(x)]\, dx - \int_a^b F_1[x, u(x)]\, dx$$

$$= -\int_b^a F_1[x, v(x)]\, dx - \int_a^b F_1[x, u(x)]\, dx$$

Here $y = v(x)$ represents the curve C_2 and $y = u(x)$ represents C_1. Thus

$$\iint_R \frac{\partial F_1}{\partial y} dA = -\int_{C_2} F_1(x, y)dx - \int_{C_1} F_1(x, y)dx = -\oint_C F_1(x, y)\, dx$$

Similarly we get, $\displaystyle\iint_R \frac{\partial F_1}{\partial x} dA = \iint_R \frac{\partial F_1}{\partial x} dx\, dy$

$$= \int_c^d \int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x} dx\, dy$$

$$= \int_c^d F_2[q(y), y]dy + \int_d^c F_2(p(y), y)dy$$

$$= \oint_C F_2(x, y)\, dy$$

Thus we get, $\displaystyle\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C (F_1dx + F_2dy).$

Note:If $\vec{F} = F_1\vec{i} + F_2\vec{j}$. *Then* $\quad$ *curl* $\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$

and $\quad \vec{F}.d\vec{r} = F_1dx + F_2dy$

Therefore Green Theorem can be written as,

$$\iint_R (curl\,\vec{F}.\vec{k})dA = \oint_C \vec{F}.d\vec{r}, \text{ where } \vec{k} \text{ be unit vector along z–axis.}$$

Gauss Divergence Theorem [2009 Spring Q.No. 4(b)]

Statement: Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector valued function that is continuous and has continuous first order partial derivatives in some domain containing T. Then

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint_T \text{div}\,\vec{F}.dV$$

where $\vec{n}$ is the outer unit normal vector on S.

That is the flux of $\vec{F}$ over S equals to the triple integral of the divergence of $\vec{F}$ over T.

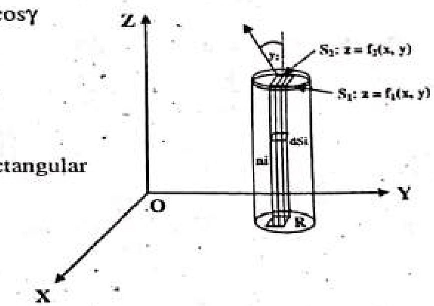
Proof: Let $\vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k}$, where α, β, θ are angles which $\vec{n}$ makes the positive direction of x, y and z axes respectively. Then

$$\vec{F}.\vec{n} = (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).(\cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k})$$

$$\vec{F}.\vec{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

Also,

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



Thus divergence theorem in the rectangular form can be expressed as

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\theta)\, ds$$

To prove the theorem, consider a closed surface S which is such that any line parallel to the coordinate axes cuts S in two points only. Let the equations of the lower and upper portions S_1 and S_2 of S be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Let projection of the surface S on the xy plane be denoted by R. Then,

$$\iiint_T \frac{\partial F_3}{dz}\, dx\, dy\, dz = \iint \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} dz \right\} dxdy$$

$$= \iint [F_3(x, y, z)]_{f_1}^{f_2}\, dxdy$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]dxdy \qquad \text{........ (1)}$$

For upper portion S_2, we have

$$\delta x\delta y = \cos\theta_2\, \delta s_2 = \vec{k}.\vec{n}_2\, \delta s_2$$

and for the lower portion s_1, we have

$$\delta x\delta y = -\cos\theta_1\delta s_1 = -\vec{k}.\vec{n}_1\, \delta s_1$$

Since θ_1 is the obtuse angle between the vector $\vec{n}_1$ and $\vec{k}$.

Thus, $\iint_R F_3(x, y, f_2)\, dx\,dy - \iint_R F_3(x, y, f_1)\, dx\, dy$ can be reduced to

$$\iint_{s2} F_3\vec{k}.\vec{n}_2 ds_2 + \iint_{s1} F_3\vec{k}\,\vec{n}_1 ds_1 = \iint_s F_3\vec{k}.\vec{n}\, ds$$

Therefore we get,

$$\iiint_T \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_s F_3\vec{k}.\vec{n}\, ds$$

Similarly by considering the projection of the surface S on other two coordinate planes, we have

$$\iiint_T \frac{\partial F_2}{\partial y} dx\, dy\, dz = \iint_s F_3\vec{j}.\vec{n}\, ds \quad \text{and} \quad \iiint_T \frac{\partial F_1}{\partial x} dx\, dy\, dz = \iint_s F_3\vec{i}.\vec{n}\, ds$$

Therefore, by adding them we get

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z}\right) dx\, dy\, dz = \iint_s (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).\, \vec{n}\, ds$$

$$= \iint_s \vec{F}.\vec{n}\, ds$$

Thus we get, $\iiint_T \nabla \vec{F} dv = \iint_s \vec{F}.\vec{n}\, ds.$

Stoke's Theorem

Theorem: Let S be a piecewise smooth oriented surface in space and the boundary of S be a piecewise smooth simple closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector valued function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\text{curl}\, \vec{F}.\vec{n})ds = \oint_C \vec{F} d\vec{r},$$

where $\vec{n}$ is a unit normal vector of S and the integration around C is taken in anti–clockwise direction with respect to $\vec{n}$.

Proof: Let

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k} \quad \text{and} \quad \vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\theta\,\vec{k}$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\oint_C (F_1dx + F_2dy + F_3dz)$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\theta\right] ds$$

Let the equation of the surface S be $z = f(x, y)$ and let the projection of S on xy plane be the region R.

Also let the projection of the curve C on the xy plane be the curve denoted by C_1 bounding the region R.

Then,

$$\oint_C F_1(x, y, z)\, dx = \iint_{c1} F_1\{x, y, f(x, y)\}\, dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, f)\, dxdy, \text{ by Greens theorem}$$

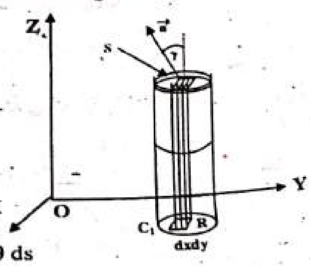
$$= \iint \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}.\frac{\partial f}{\partial y}\right) dxdy \qquad \text{.........(1)}$$

We have the direction cosines of the normal to the surface are given by

$$\frac{\cos\alpha}{\partial f/\partial x} = \frac{\cos\beta}{\partial f/\partial y} = \frac{\cos\theta}{-1}$$

This gives

$$\frac{\partial f}{\partial y} = -\frac{\cos\beta}{\cos\theta}.$$



Also we know, $dxdy = \cos\theta\, ds$ and then equation (1) reduces to

$$\oint_C F_1(x, y, z)dx = -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}.\frac{\cos\beta}{\cos\theta}\right)\cos\theta\, ds$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial y}\cos\theta - \frac{\partial F_1}{\partial z}\cdot\cos\beta\right) ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\theta - \frac{\partial F_1}{\partial y}\cdot\cos\theta\right) ds.$$

Similarly we can get

$$\oint_C F_2(x, y, z)dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\theta - \frac{\partial F_2}{\partial z}\cdot\cos\alpha\right) ds$$

and $$\oint_C F_3(x, y, z)dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cdot\cos\beta\right) ds$$

Adding these we get,

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\vec{n}\,ds.$$

This is the required form.

EXERCISE 4.1

1. If $\vec{r}_1 = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$, $\vec{r}_2 = (2t-3)\vec{i} + \vec{j} - t\vec{k}$.

Find (i) $\frac{d}{dt}(\vec{r}_1.\vec{r}_2)$ [2002 – Short] (ii) $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2)$ at $t = 1$.

Solution: Let $\vec{r}_1 = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{r}_2 = (2t-3)\vec{i} + \vec{j} - t\vec{k}$

Then,

$$\begin{aligned}\vec{r}_1.\vec{r}_2 &= (t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}).((2t-3)\vec{i} + \vec{j} - t\vec{k})\\ &= (t^2, -t, 2t+1).(2t-3, 1, -t)\\ &= 2t^3 - 3t^2 - t - 2t^2 - t\\ &= 2t^3 - 5t^2 - 2t\end{aligned}$$

and

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= (t^2 - 2t - 1)\vec{i} + ((2t-3)(2t+1) + t^3)\vec{j} + (t^2 + t(2t-3))\vec{k}$$

$$= (t^2 - 2t - 1)\vec{i} + (4t^2 - 4t - 3 + t^3)\vec{j} + (3t^2 - 3t)\vec{k}$$

Now,

(i) $\frac{d}{dt}(\vec{r}_1.\vec{r}_2) = \frac{d}{dt}(2t^3 - 5t^2 - 2t) = 6t - 10t - 2$

At $t = 1$,

$$\frac{d}{dt}(\vec{r}_1.\vec{r}_2) = 6 - 10 - 2 = -6$$

(ii) $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \frac{d}{dt}(t^2 - 2t - 1, t^3 + 4t^2 - 4t - 3, 3t^2 - 3t)$

$$= (2t - 2, 3t^2 + 8t - 4, 6t - 3)$$

At $t = 1$,

$$\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = (2 - 2, 3 + 8 - 4, 6 - 3) = (0, 7, 3)$$

If $\vec{A} = t^2\vec{i} + (3t^2 - 2t)\vec{j} + \left(2t - \frac{1}{t}\right)\vec{k}$. Find $\left|\frac{d\vec{A}}{dt}\right|$ at $t = 1$.

ution: Let

$$\vec{A} = t^2\vec{i} + (3t^2 - 2t)\vec{j} + \left(2t - \frac{1}{t}\right)\vec{k}$$

Then,

$$\frac{d\vec{A}}{dt} = 2t\,\vec{i} + (6t - 2)\vec{j} + (2 + t^{-2})\vec{k}$$

So,

$$\left|\frac{d\vec{A}}{dt}\right| = \sqrt{(2t)^2 + (6t-2)^2 + (2+t^{-2})^2}$$

at $t = 1$,

$$\left|\frac{d\vec{A}}{dt}\right| = \sqrt{2^2 + (6-2)^2 + (2+1)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$$

Thus,

$$\left|\frac{d\vec{A}}{dt}\right|_{\text{at } t = 1} = \sqrt{29}.$$

If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$.

[2013 Fall Q. No. 6(a)] [2004 Spring Q.No. 3(a)]

lution: Let $\vec{r} = \vec{a}\,e^{nt} + \vec{b}\,e^{-nt}$ for $\vec{a}$ and $\vec{b}$ are constant vectors.

Then,

$$\left|\frac{d\vec{r}}{dt}\right| = \vec{a}\,ne^{nt} + \vec{b}\,(-n)e^{-nt} = n[\vec{a}\,e^{nt} - \vec{b}\,e^{-nt}]$$

And $$\frac{d^2\vec{r}}{dt^2} = n[\vec{a}\,ne^{nt} - (-n)\,\vec{b}\,e^{-nt}] = n^2[\vec{a}\,e^{nt} + \vec{b}\,e^{-nt}] = n^2\vec{r}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0.$$

. If $\vec{r} = \cos nt\,\vec{i} + \sin nt\,\vec{j}$. Show that $\vec{r} \times \frac{d\vec{r}}{dt} = n\vec{k}$

Solution: Let $\vec{r} = \cos nt\,\vec{i} + \sin nt\,\vec{j} + 0\vec{k}$. Then,

$$\frac{d\vec{r}}{dt} = -\sin nt.\,n\,\vec{i} + \cos nt.\,n\,\vec{j} + 0\vec{k}$$

Now,

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix}$$

$$= 0\vec{i} + 0\vec{j} + (n\cos^2 nt + n\sin^2 nt)\vec{k} = n(\cos^2 nt + \sin^2 nt)\vec{k}$$

$$= n.1\,\vec{k} = n\vec{k}.$$

Thus, $\vec{r} \times \frac{d\vec{r}}{dt} = n\vec{k}$.

5. If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, differentiate with respect to t

(i) $\vec{r}\,.\,\vec{a}$ (ii) $\vec{r} \times \vec{a}$ (iii) $\vec{r} \times \frac{d\vec{r}}{dt}$ (iv) $\vec{r}\,.\,\frac{d\vec{r}}{dt}$

Solution: Let $\vec{r}$ be a vector function of scalar t and $\vec{a}$ be a constant vector. Then,

(i) Derivative of $\vec{r}\,.\,\vec{a}$ w. r. t. 't' be,

$$\frac{d}{dt}(\vec{r}\,.\,\vec{a}) = \frac{d\vec{r}}{dt}\,.\,\vec{a} + \vec{r}\,.\,\frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt}\,.\,\vec{a} + \vec{r}\,.\,0 \qquad [\because \vec{a} \text{ is a constant. So, } \frac{d\vec{a}}{dt} = 0.]$$

$$= \frac{d\vec{r}}{dt}\,.\,\vec{a}$$

(ii) Derivative of $\vec{r} \times \vec{a}$ be,

$$\frac{d}{dt}(\vec{r} \times \vec{a}) = \frac{d\vec{r}}{dt} \times \vec{a} + \vec{r} \times \frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt} \times \vec{a} + \vec{r} \times 0 \qquad [\because \vec{a} \text{ is a constant. So, } \frac{d\vec{a}}{dt} = 0]$$

$$= \frac{d\vec{r}}{dt} \times a$$

(iii) Derivative of $\vec{r} \times \frac{d\vec{r}}{dt}$ be,

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

Since cross product of same vector is zero. So, $\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0$. Then,

$$= 0 + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

(iv) Derivative of $\vec{r}\,.\,\frac{d\vec{r}}{dt}$ be,

$$\frac{d}{dt}\left(\vec{r}\,.\,\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\,.\,\frac{d\vec{r}}{dt} + \vec{r}\,.\,\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}\,.\,\frac{d^2\vec{r}}{dt^2}$$

6. Verify $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$ and $\frac{d}{dt}(\vec{a}\,.\,\vec{b}) = \vec{a}\,.\,\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b}$ where $\vec{a} = 2t^3\vec{i} + t\vec{j} + 3t^2\vec{k}$ and $\vec{b} = 2t\vec{i} + 3\vec{j} + t^3\vec{k}$.

Solution: Let, $\vec{a} = (2t^3\vec{i} + t\vec{j} + 3t^2\vec{k}) = (2t^3, t, 3t^2)$

and $\vec{b} = 2t\vec{i} + 3\vec{j} + t^3\vec{k} = (2t, 3, t^3)$

Then,

$$\vec{a}\,.\,\vec{b} = 4t^4 + 3t + 3t^5$$

and
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & t & 3t^2 \\ 2t & 3 & t^3 \end{vmatrix} = (t^4 - 9t^2)\vec{i} + (6t^3 - 2t^6)\vec{j} + (6t^3 - 2t^2)\vec{k}$$

So that,

$$\frac{d\vec{a}}{dt} = (6t^2, 1, 6t); \qquad \frac{d\vec{b}}{dt} = (2, 0, 3t^2); \qquad \frac{d}{dt}(\vec{a}\,.\,\vec{b}) = 16t^3 + 3 + 15t^4$$

and $\frac{d}{dt}(\vec{a} \times \vec{b}) = (4t^3 - 18t)\vec{i} + (18t^2 - 12t^5)\vec{j} + (18t^2 - 4t)\vec{k}$.

Now,

$$\vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$$

$$= (2t^3, t, 3t^2) \times (2, 0, 3t^2) + (6t^2, 1, 6t) \times (2t, 3, t^3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & t & 3t^2 \\ 2 & 0 & 3t^2 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6t^2 & 1 & 6t \\ 2t & 3 & t^3 \end{vmatrix}$$

$$= [3t^3\vec{i} + (6t^2 - 6t^5)\vec{j} + (-2t)\vec{k}] + [(t^3 - 18t)\vec{i} + (12t^2 - 6t^5)\vec{j} + (18t^2 - 2t)\vec{k}]$$

$$= (4t^3 - 18t)\vec{i} + (18t^2 - 12t^5)\vec{j} + (18t^2 - 4t)\vec{k}$$

$$= \frac{d}{dt}(\vec{a} \times \vec{b})$$

Next,

$$\vec{a}\,.\,\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b} = (2t^3, t, 3t^2)\,.\,(2, 0, 3t^2) + (6t^2, 1, 6t)\,(2t, 3, t^3)$$

$$= (4t^3 + 0 + 9t^4) + (12t^3 + 3 + 6t^4)$$

$$= 16t^3 + 3 + 15t^4 = \frac{d}{dt}(\vec{a}\,.\,\vec{b})$$

Thus, $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b}$.

7. Find the unit tangent vector at any point on the curve x = 3cost, y = 3sint, z = 4t.

Solution: Let, $x = 3\cos t$, $y = 3\sin t$ and $z = 4t$.

Then,

$$\vec{r} = x_i + y\vec{j} + z\vec{k} + (x, y, z) = (3\cos t, 3\sin t, 4t)$$

So, $\frac{d\vec{r}}{dt} = (-3\ \text{sint}, 3\text{cost}, 4)$

Therefore,

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(-3\ \text{sint})^2 + (3\text{cost})^2 + 4^2}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t) + 16} = \sqrt{9+16} = 5.$$

Now, the unit tangent vector be,

$$\left(\frac{d\hat{r}}{dt}\right) = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{1}{5}(-3\ \text{sint}, 3\ \text{cost}, 4).$$

8. Find the angle between the tangents to the curve $x = t, y = t^2, z = t^3$ at $t = \pm 1$.

Solution: Let, $x = t, y = t^2, z = t^3$

Then, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) = (t, t^2, t^3)$.

So, $T = \frac{d\vec{r}}{dt} = (1, 2t, 3t^2)$.

At $t = 1$, $\frac{d\vec{r}}{dt} = T_1 = (1, 2, 3)$ and at $t = -1$, $\frac{d\vec{r}}{dt} = T_2 = (1, -2, 3)$.

Let, θ be the angle between T_1 and T_2 then,

$$\cos\theta = \frac{T_1.T_2}{|T_1||T_2|} = \frac{(1, 2, 3).(1, -2, 3)}{|(1, 2, 3)||(1, -2, 3)|} = \frac{1-4+9}{\sqrt{1+4+9}\sqrt{1+4+9}} = \frac{6}{14} = \frac{3}{7}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{7}\right)$$

Thus, the required angle be $\theta = \cos^{-1}\left(\frac{3}{7}\right)$.

9. A particle moves along the curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitude of velocity and acceleration at $t = 0$. **[2010 Fall Q.No. 3(a)]**

Solution: Given curve is

$x = e^{-t}, y = 2\cos 3t$ and $z = 2\sin 3t$

Then, $\vec{r} = (x, y, z) = (e^{-t}, 2\cos 3t, 2\sin 3t)$.

So that,

$$\frac{d\vec{r}}{dt} = (-e^{-t}, -6\sin 3t, 6\cos 3t) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (e^{-t}, -18\cos 3t, -18\sin 3t).$$

We know that, velocity of a curve is, $\vec{v} = \frac{d\vec{r}}{dt}$ and acceleration is $\vec{a} = \frac{d^2\vec{r}}{dt^2}$.

Therefore, velocity vector be

$$\vec{v} = \frac{d\vec{r}}{dt} = (-e^{-t}, -6\sin 3t, 6\cos 3t)$$

and acceleration vector be

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (e^{-t}, -18\cos 3t, -18\sin 3t)$$

Also, the velocity vector at $t = 0$ is

$$\vec{v}_{\text{at } t=0} = (-e^{-0}, -6\sin 0, 6\cos 0) = (-1, 0, 6)$$

and acceleration vector at $t = 0$ is

$$\vec{a}_{\text{at } t=0} = (1, -18, 0)$$

Therefore, magnitude of velocity at $t = 0$ is

$$|\vec{v}_{\text{at } t=0}| = \sqrt{(-1)^2 + 6^2} = \sqrt{1+36} = \sqrt{37}$$

and magnitude of acceleration at $t = 0$ is

$$|\vec{a}_{\text{at } t=0}| = \sqrt{1^2 + (-18)^2} = \sqrt{1+324} = \sqrt{325} = 5\sqrt{13}$$

0. A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 5$. Find the component of its velocity and acceleration at $t = 1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$. **[2012 Fall Q.No. 3(a)] [2009 Fall Q.No. 3(a)]**

olution: Given curve be

$$\vec{x} = t^3 + 1, \quad \vec{y} = t^2 \quad \text{and} \quad \vec{z} = 2t + 5.$$

Then the position vector of any point of the curve be,

$$\vec{r} = (x, y, z) = (t^3 + 1, t^2, 2t + 5)$$

So that $\frac{d\vec{r}}{dt} = (3t^2, 2t, 2)$ and $\frac{d^2\vec{r}}{dt^2} = (6t, 2, 0)$

at $t = 1$, $\frac{d\vec{r}}{dt}(3, 2, 2)$ and $\frac{d^2\vec{r}}{dt^2} = (6, 2, 0)$

We know that the velocity vector of $\vec{r}$ at $t = 1$ is

$$\vec{v} = \frac{d\vec{r}}{dt} \text{ at } t = 1 \quad \text{i.e.} \quad \vec{v} = (3, 2, 2)$$

and the acceleration vector of $\vec{r}$ at $t = 1$ is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \text{ at } t = 1 \quad \text{i.e.} \quad \vec{a} = (6, 2, 0)$$

Also, given that a vector $\vec{i} + \vec{j} + 3\vec{k} = (1, 1, 3) = \vec{n}$ (say)

So, the unit vector along $(1, 1, 3)$ is

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{(1, 1, 3)}{\sqrt{1+1+9}} = \frac{(1, 1, 3)}{\sqrt{11}}$$

Thus, the velocity component of $\vec{r}$ along $\vec{n}$ is

$$= \vec{v}.\hat{n}$$

$$= (3, 2, 2)\frac{(1, 1, 3)}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

and the acceleration component of $\vec{r}$ along $\vec{n}$ is

$$= \vec{a}.\hat{n}$$

$$= (6, 2, 0)\frac{(1, 1, 3)}{\sqrt{11}} = \frac{6+2+0}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

11. **A particle moves so that its position vector is given by $\vec{r} = \cos wt\,\vec{i} + \sin wt\,\vec{j}$ show that the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$ and show that $\vec{r} \times \vec{v}$ is a constant vector.**

Solution: Given position vector is

$$\vec{r} = \cos wt\,\vec{i} + \sin wt\,\vec{j} = (\cos wt, \sin wt, 0)$$

Then, $\dfrac{d\vec{r}}{dt} = (-w\sin wt, w\cos wt, 0)$

We know that the velocity vector to $\vec{r}$ is $\vec{v} = \dfrac{d\vec{r}}{dt}$

Now,

$$\begin{aligned}\vec{r}.\vec{v} = \vec{r}.\frac{d\vec{r}}{dt} &= (\cos wt, \sin wt, 0).(-w\sin wt, w\cos wt, 0)\\ &= -w\sin wt\cos wt + w\sin wt\cos wt + 0\\ &= 0\end{aligned}$$

This shows that $\vec{r}$ and $\vec{v}$ are perpendicular to each other.

Next,

$$\vec{r}\times\vec{v} = \vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos wt & \sin wt & 0\\ -w\sin wt & w\cos wt & 0\end{vmatrix}$$

$$= 0\vec{i} + 0\vec{j} + (w\cos^2 wt + w\sin^2 wt)\vec{k}$$

$$= w\vec{k} \qquad [\cos^2 wt + \sin^2 wt = 1.]$$

This shows that $\vec{r}\times\vec{v}$ is constant vector.

12. **A particle moves along the curves $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \pi/2$.** **[2011 Spring Q. No. 6(c)]**

Solution: Given curve is

$$x = 4\cos t,\ y = 4\sin t \text{ and } z = 6t$$

Then the position vector of any point of the curve is,

$$\vec{r} = (x, y, z) = (4\cos t, 4\sin t, 6t)$$

Then, $\dfrac{d\vec{r}}{dt} = (-4\sin t, 4\cos t, 6)$ and $\dfrac{d^2\vec{r}}{dt^2} = (-4\cos t, -4\sin t, 0)$

At $t = 0$,

$$\frac{d\vec{r}}{dt} = (0, 4, 6) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (-4, 0, 0)$$

and at $t = \frac{\pi}{2}$,

$$\frac{d\vec{r}}{dt} = (-4, 0, 6) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (0, -4, 0)$$

We know that, velocity along a curve is $\vec{v} = \dfrac{d\vec{r}}{dt}$ and acceleration is, $\vec{a} = \dfrac{d^2\vec{r}}{dt^2}$.

Therefore, velocity at $t = 0$ is, $\vec{v} = (0, 4, 6) = 4\vec{j} + 6\vec{k}$

and velocity at $t = \frac{\pi}{2}$ is, $\vec{v} = (-4, 0, 6) = -4\vec{i} + 6\vec{k}$

Also, acceleration at $t = 0$ is, $\vec{a} = (-4, 0, 0) = -4\vec{i}$

and acceleration at $t = \frac{\pi}{2}$ is, $\vec{a} = (0, -4, 0) = -4\vec{j}$.

...01 Q.No. 3(a)]
A particle moves along the curve, $x = a\cos t$, $y = a\sin t$ and z bt. Find the velocity and acceleration at $t = 0$ and $t = \pi/2$.

...te: See the above solution with replacing 4 by a.

A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$. Find the velocity and acceleration at $t = 1$.

...ution: Part of solution of Q. 10.

If $\vec{a} = x^2\vec{i} - y\vec{j} + xz\vec{k}$ and $\vec{b} = y\vec{i} + x\vec{j} - xyz\vec{k}$ verify that $\dfrac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b}) = \dfrac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b})$

...ution: Let,

$$\vec{a} = x^2\vec{i} - y\vec{j} + xz\vec{k} = (x^2, -y, xz)$$

and $\vec{b} = y\vec{i} + x\vec{j} - xyz\vec{k} = (y, x, -xyz)$

Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x^2 & -y & xz\\ y & x & -xyz\end{vmatrix}$$

$$= (xy^2z - x^2z)\vec{i} + (xyz + x^3yz)\vec{j} + (x^3 + y^2)\vec{k}$$

So that,

$$\frac{\partial(\vec{a}\times\vec{b})}{\partial x} = (y^2z - 2xz)\vec{i} + (yz + 3x^2yz)\vec{j} + 3x^2\vec{k}$$

and $$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = 2yz\vec{i} + (z + 3x^2z)\vec{j} + 0 \quad \text{.........(1)}$$

Also,

$$\frac{\partial(\vec{a}\times\vec{b})}{\partial y} = 2xyz\vec{i} + (xz + x^3z)\vec{j} + 2y\vec{k}$$

and $$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = 2yz\vec{i} + (z + 3x^2z)\vec{j} + 0 \quad \text{.........(2)}$$

From (1) and (2), we have

$$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = \frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y}$$

15. If $\vec{r} = x^2y\vec{i} - 2y^2z\vec{j} + xy^2z^2\vec{k}$. Show that $\left|\dfrac{\partial^2\vec{r}}{\partial x^2} \times \dfrac{\partial^2\vec{r}}{\partial y^2}\right|$ at the (2, 1, −1) is $8\sqrt{2}$

Solution: Let, $\vec{r} = x^2y\vec{i} - 2y^2z\vec{j}, xy^2z^2\vec{k} = (x^2y, -2y^2z, xy^2z^2)$

Then,

$$\frac{\partial\vec{r}}{\partial x} = (2x, 0, y^2z^2), \qquad \frac{\partial\vec{r}}{\partial y} = (x^2, -4yz, 2xyz)$$

and,
$$\frac{\partial^2\vec{r}}{\partial x^2} = (2, 0, 0) \qquad \frac{\partial^2\vec{r}}{\partial y^2} = (0, -4z, 2xz^2)$$

So that,

$$\frac{\partial^2\vec{r}}{\partial x^2} \times \frac{\partial^2\vec{r}}{\partial y^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & -4z & 2z^2 \end{vmatrix} = 0.\vec{i} - 4xz^2\vec{j} - 8z\vec{k}$$

Therefore,

$$\left|\frac{\partial^2\vec{r}}{\partial x^2} \times \frac{\partial^2\vec{r}}{\partial y^2}\right| = \sqrt{0 + (-4xz^2)^2 + (-8z)^2}$$

At the point (2, 1, −1)

$$\left|\frac{\partial^2\vec{r}}{\partial x^2} \times \frac{\partial^2\vec{r}}{\partial y^2}\right| = \sqrt{0 + (4(2)(-1)^2)^2 + (8(-1)^2)}$$

$$= \sqrt{0 + 64 + 864} = \sqrt{2 \times 64} = 8\sqrt{2}.$$

16. Show that the unit tangent to the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ at t = 1 $\dfrac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k})$

Solution: Given curve is

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k} = (t, t^2, t^3)$$

So, the tangent vector of $\vec{r}$ is, $\dfrac{d\vec{r}}{dt} = (1, 2t, 3t^2)$

Therefore, the unit tangent vector of $\vec{r}$ is

$$\widehat{\left(\frac{d\vec{r}}{dt}\right)} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|} = \frac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}}$$

At t = 1, the unit tangent vector of $\vec{r}$ is

$$\widehat{\left(\frac{d\vec{r}}{dt}\right)} = \frac{(1, 2, 3)}{\sqrt{1 + 4 + 9}}$$

$$= \frac{(1, 2, 3)}{\sqrt{14}} = \frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k}).$$

17. **A particle P is moving on a circle of radius a with constant angular velocity $w = \dfrac{d\theta}{dt}$, then show that the acceleration of the particle is $-w^2\vec{r}$.**

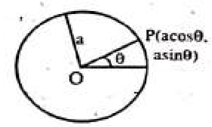
Solution: Since the particle is moving on a circle of radius a and with constant angular velocity $w = \dfrac{d\theta}{dt}$.

So, $\vec{r} = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}$. Then, $\dfrac{d\vec{r}}{dt} = -a\sin\theta\dfrac{d\theta}{dt}\vec{i} + a\cos\theta\dfrac{d\theta}{dt}\vec{j}$

And,
$$\frac{d^2\vec{r}}{dt^2} = -a\cos\theta\left(\frac{d\theta}{dt}\right)^2\vec{i} - a\sin\theta\left(\frac{d\theta}{dt}\right)^2\vec{j}$$

$$= -[a\cos\theta\, w^2\,\vec{i} + a\sin\theta.\, w^2\vec{j}]$$

$$= -w^2(a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}) = -w^2\,\vec{r}$$



Since the acceleration of the particle is $\dfrac{d^2\vec{r}}{dt^2}$ i.e. $-w^2\,\vec{r}$.

EXERCISE 4.2

1. **Find grad f, where**

(i) $f = x^2 + yz$ **(ii) $f = x^3 + y^3 + 3xyz$** **(iii) $f = \log(x^2 + y^2 + z^2)$**

Solution: (i) Given that, $f = x^2 + yz$

Then,

$$\text{Grad (f)} = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + yz)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 + yz) + \vec{j}\frac{\partial}{\partial y}(x^2 + yz) + \vec{k}\frac{\partial}{\partial z}(x^2 + yz)$$

$$= \vec{i}.2x + \vec{j}.z + \vec{k}.y$$

$$= 2x\vec{i} + z\vec{j} + y\vec{k}$$

(ii) Given that, $f = x^3 + y^3 + 3xyz$

Then,

$$\text{Grad (f)} = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz)$$

$$= (3x^2 + 3yz)\vec{i} + (3y^2 + 3zx)\vec{j} + 3xy\,\vec{k}.$$

(iii) Given that, $f = \log(x^2 + y^2 + z^2)$

Then,

$$\text{Grad (f)} = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log(x^2 + y^2 + z^2))$$

$$= \vec{i}\left(\frac{2x}{x^2 + y^2 + z^2}\right) + \vec{j}\left(\frac{2y}{x^2 + y^2 + z^2}\right) + \vec{k}\left(\frac{2z}{x^2 + y^2 + z^2}\right)$$

$$= \frac{2}{x^2 + y^2 + z^2}(x\vec{i} + y\vec{j} + z\vec{k}).$$

2. **Find a unit normal to the surface**

(i) $xy^3z^2 = 4$ at $(-1, -1, 2)$ (ii) $x^2y + 2xz = 4$ at $(2, -2, 3)$

Solution: (i) Let given surface is,

$$f = xy^3z^2 - 4$$

We have, grad (f) is the normal to the given surface.

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy^3z^2 - 4)$$

$$= y^3z^2\,\vec{i} + 3xy^2z^2\,\vec{j} + 2xy^3z\,\vec{k}$$

At point $(-1, -1, 2)$, $\text{grad}(f) = -4\vec{i} - 12\vec{j} + 4\vec{k}$

Thus, $(-4, -12, 4)$ is normal to f at $(-1, -1, 2)$.

And, the unit vector of grad (f) is

$$\hat{n} = \frac{\text{grad}(f)}{|\text{grad}(f)|} = \frac{(-4, -12, 4)}{\sqrt{16 + 144 + 16}} = \frac{4(-1, -3, 1)}{4\sqrt{1 + 9 + 1}} = \frac{(-1, -3, 1)}{\sqrt{11}}$$

(ii) Let the given surface be,

$$f = x^2y + 2xz - 4.$$

We know, the normal vector to the surface f is ∇f.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y + 2xz - 4)$$

$$= (2xy + 2z)\vec{i} + x^2\,\vec{j} + 2x\,\vec{k}$$

At point $(2, -2, 3)$, $\nabla f = (-8 + 6)\vec{i} + 4\vec{j} + 4\vec{k}$

$$= 2(-\vec{i} + 2\vec{j} + 2\vec{k})$$

Now, unit vector of ∇f is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2(\vec{i} + 2\vec{j} + 2\vec{k})}{2\sqrt{1 + 4 + 4}} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

Thus, the unit vector normal to f at $(2, -2, 3)$ is $\frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$.

3. Find the directional derivatives of f at P in the direction $\vec{a}$, where

(i) $f = x^2 + y^2$, $P(1, 1)$, $\vec{a} = 2\vec{i} - 4\vec{j}$

Solution: Given surface be, $f = x^2 + y^2$

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2)$$

$$= 2(x\vec{i} + y\vec{j})$$

At point $P(1, 1)$, $\text{grad}(f) = 2(\vec{i} + \vec{j})$

Also, given that $\vec{a} = 2\vec{i} - 4\vec{j}$. So, unit vector of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - 4\vec{j}}{\sqrt{4 + 16}} = \frac{\vec{i} - 2\vec{j}}{\sqrt{5}}$$

Now, the direction derivative of f at P alone $\vec{a}$ is

$$\nabla f.\hat{a} = \frac{2}{\sqrt{5}}(\vec{i} + \vec{j}).(\vec{i} - 2\vec{j}) = \frac{2}{\sqrt{5}}(1 - 2) = -\frac{2}{\sqrt{5}}$$

) $f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, $P(3, 0, 4)$; $\vec{a} = \vec{i} + \vec{j} + \vec{k}$. **[2006 Spring Q.No. 3(a)]**

lution: Given surface is,

$$f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

We know the directional derivative of f at P in the direction of $\vec{a}$ is $\nabla f\,.\,\hat{a}$ at P.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$= -2(x^2 + y^2 + z^2)^{-3/2}\,(x\vec{i} + y\vec{j} + z\vec{k})$$

At point $P(3, 0, 4)$, $\nabla f = -2(9 + 0 + 16)^{-3/2}(3\vec{i} + 4\vec{k})$

$$= -\frac{2}{125}(3\vec{i} + 4\vec{k})$$

Also, given that, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$. So, unit vector of $\vec{a}$ is

$$\hat{a} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1 + 1 + 1}} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

Now,

$$\nabla f.\hat{a} = -\frac{2}{125}(3\vec{i} + 4\vec{k}).\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$$

$$= -\frac{2}{125\sqrt{3}}(3 + 0 + 4) = -\frac{14}{125\sqrt{3}}$$

Thus, the directional derivative of f at P in the direction of $\vec{a}$ is, $-\frac{14}{125\sqrt{3}}$

iii) $f = xyz$, $P(-1, 1, 3)$, $\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}$

iv) $f = e^x\cos y$, $P(2, \pi, 0)$, $\vec{a} = 2\vec{i} + 3\vec{k}$

v) $f = xy^2 + yz^3$, $P(2, -1, 3)$, $\vec{a} = \vec{i} + 2\vec{i} + 2\vec{k}$ **[2009 Fall – Short]**

vi) $f = 2xy + z^2$, $P(1, -1, 3)$, $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

Solution: (iii) – (vi) – process as (ii).

(vii) $f = 4xz^3 - 3x^2yz^2$ at $(2, -1, 2)$ along z–axis.

Solution: Given that, $f = 4xz^3 - 3x^2yz^2$

Then the directional derivative of f at $(2, -1, 2)$ along z–axis is,

$\nabla f \cdot \hat{a}$ at (2, –1, 2) and where $\vec{a} = \vec{k}$.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(4xz^3 - 3x^2yz^2)$$

$$= (4z^3 - 6xyz^2)\vec{i} + (-3x^2z^2)\vec{j} + (12xz^2 - 6x^2yz)\vec{k}$$

So that,

$$\nabla f.\hat{a} = ((4z^3 - 6xyz^2)\vec{i} - 3x^2z^2\vec{j} + (12xz^2 - 6x^2yz)\vec{k}) \cdot \vec{k}$$

$$[\because |\vec{a}| = 1 \text{ and } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{k}}{1} = \vec{k}]$$

$$= 12xz^2 - 6x^2yz$$

at point (2, –1, 2), $\nabla f \cdot \hat{a} = 12(2)(2)^2 - 6(2)^2(-1)(2) = 96 + 48 = 144$

Thus, the directional derivative of f at (2, –1, 2) along z–axis is 144.

(viii) $f = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the surface x log z – $y^2 + 4 = 0$ at (–1, 2, 1).

Solution: Given that, $f = xy^2 + yz^3$

And the surface is, $\phi = x\log(z) - y^2 + 4$

Then,

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{j}\frac{\partial}{\partial z}\right)(x\log(z) - y^2 + 4) = \log(z)\,\vec{i} - 2y\,\vec{j} + \frac{x}{z}\vec{k}$$

at (–1, 2, 1),

$$\nabla\phi = \log(1)\,\vec{i} - 4\vec{j} - \vec{k} = 0\vec{i} - 4\vec{j} - \vec{k}$$

Also,

$$\text{grad}(f) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{j}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

at (2, –1, 1),

$$\text{grad}(f) = \vec{i} + (-4+1)\vec{j} - 3\vec{k} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Now, directional derivative of f is

$$= (\text{grad}(f))_{at\,(2,-1,1)} \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right)_{at\,(-1,2,1)}$$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \left(\frac{0\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{0 + 16 + 1}}\right) = \frac{1}{\sqrt{17}}(0 + 12 + 3) = \frac{15}{\sqrt{17}}$$

4. **Find the angle between the tangent planes to the surface x log z = $y^2 - 1$ and $x^2y = 2 - z$ at (1, 1, 1).**

Solution: Let the given surface is, $f = x\log(z) - y^2 + 1$

Then,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) = (\log(z)\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}) \quad \text{.........(i)}$$

And the given surface is, $F = x^2y - 2 + z$

Then,

$$\nabla F = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) = 2xy\,\vec{i} + x^2\vec{j} + \vec{k}$$

Let θ be the angle between the tangent planes of surface $x\log(z) = y^2 - 1$ and $x^2y = 2 - z$. Then θ be the angle between ∇f and ∇F.

Therefore,

$$\cos\theta = \frac{\nabla f \cdot \nabla F}{|\nabla f|\,|\nabla F|} = \frac{2xy\log(z) - 2x^2y + x/z}{\sqrt{(\log(z))^2 + 4y^2 + x^2/z^2}\sqrt{4x^2y^2 + x^4 + 1}}$$

at point (1, 1, 1),

$$\cos\theta = \frac{2(0) - 2 + 1}{\sqrt{0 + 4 + 1}\sqrt{4 + 1 + 1}} \qquad [\because \log(1) = 0]$$

$$= \frac{-1}{\sqrt{5}\sqrt{6}} = -\frac{1}{\sqrt{30}}.$$

Thus, angle between the tangent planes to the given surfaces $x\log(z) = y^2 - 1$ and $x^2y = 2 - z$ at (1, 1, 1) is $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$.

Find the angle between the tangent planes to the surface $xy = z^2$ at the point (4, 1, 2) and (3, 3, –3).

Solution: Given surface is, $f = xy - z^2$

Then the normal to the surface f is ∇f.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy - z^2) = y\vec{i} + x\vec{j} - 2z\vec{k}.$$

At point, (4, 1, 2), $\nabla f = \vec{i} + 4\vec{j} - 4\vec{k}$.

and at point (3, 3, –3), $\nabla f = 3\vec{i} + 3\vec{j} + 6\vec{k}$.

Let θ be the angle between ∇f at (4, 1, 2) and at (3, 3, –3)

Then,

$$\cos\theta = \frac{(\nabla f \text{ at } (4,1,2)) \cdot (\nabla f \text{ at } (3,3,-3))}{|\nabla f \text{ at } (4,1,2)|\,|\nabla f \text{ at } (3,3,-3)|}$$

$$= \frac{(\vec{i} + 4\vec{j} - 4\vec{k}) \cdot (3\vec{i} + 3\vec{j} + 6\vec{k})}{|\vec{i} + 4\vec{j} - 4\vec{k}|\,|3\vec{i} + 3\vec{j} + 6\vec{k}|}$$

$$= \frac{3 + 12 - 24}{\sqrt{1 + 16 + 16}\sqrt{9 + 9 + 36}} = \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{9\sqrt{11}\sqrt{2}} = \frac{-1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{22}}\right).$$

Thus, the angle between the normal to $xy = z^2$ at (4, 1, 2) and (3, 3, –3) is, $\cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)$.

5. **If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, show that**

(i) grad r = grad (r) = $\frac{\vec{r}}{r}$ (ii) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

(iii) grad $(r^n) = nr^{n-2}\vec{r}$ (iv) grad $(\vec{a}.\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

Solution: Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$

Then, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$ ....(i)

So that,

$$\frac{\vec{r}}{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} \quad ......(ii)$$

and, $$-\frac{\vec{r}}{r^3} = -\frac{\vec{r}}{|\vec{r}|^3} = \frac{-(x, y, z)}{(x^2+y^2+z^2)^{3/2}} \quad ....(iii)$$

Also, $r^{(n-2)}\vec{r} = |\vec{r}|^{(n-2)}\vec{r} = (x^2+y^2+z^2)^{(n-2)/2}(x, y, z)$ ....(iv)

Now,

(i) $$\text{grad}(r) = \nabla r = \nabla|\vec{r}| = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r}$$

Thus, grad (r) = $\frac{\vec{r}}{r}$

(ii) $$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \nabla\frac{1}{|\vec{r}|} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{1}{2(x^2+y^2+z^2)^{3/2}}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= -\frac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}.$$

Thus, grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$.

(iii) $$\text{grad}(r^n) = \nabla(r^n) = \nabla|\vec{r}|^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= n(x, y, z)\cdot(x^2+y^2+z^2)^{(n-2)/2}$$

$$= nr^{n-2}\vec{r}$$

Thus, grad $(r^n) = nr^{n-2}\vec{r}$

(iv) $$\text{grad}(\vec{a}.\vec{r}) = \nabla((a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(x\vec{i} + y\vec{j} + z\vec{k}))$$

$$= \nabla(a_1x + a_2y + a_3z)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}.$$

Thus, grad $(\vec{a}.\vec{r}) = \vec{a}$.

In what direction from (3, 1, –2) is the directional derivative of $f = x^2y^2z^4$ maximum and what is its magnitude?

ution: Given that, $f = x^2y^2z^4$

Since we have the directional derivative of f is maximum in the direction of grad(f).

Here,

$$\text{grad } f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= 2xy^2z^4\,\vec{i} + 2x^2yz^4\,\vec{j} + 4x^2y^2z^3\,\vec{k}$$

at (3, 1, –2), $\text{grad } f = 2(3)(1)^2(-2)^4\,\vec{i} + 2(3)^2(1)(-2)^4\,\vec{j} + 4(3)^2(1)^2(-2)^3\,\vec{k}$

$$= 96\vec{i} + 288\vec{j} - 288\vec{k}.$$

And its magnitude is

$$|\text{grad } f| = \sqrt{(96)^2 + (288)^2 + (-288)^2}$$

$$= \sqrt{9216 + 82944 + 82944} = \sqrt{175104} = 96\sqrt{19}.$$

Thus in the direction of $96\vec{i} + 288\vec{j} - 288\vec{k}$ from (3, 1, –2) is the maximum directional derivative of $f = x^2y^2z^4$ and its magnitude is $96\sqrt{19}$.

What is the greatest rate of increase of $u = x^2 + yz^2$ at the point (1, –1, 3)?

ution: Given that, $u = x^2 + yz^2$.

Since we have the greatest rate of increase of u at the point (α, β, γ) is the maximum value of the directional derivative at (α, β, γ). So,

greatest rate of increase of u at (α, β, γ)

$= |\nabla u|$ at (α, β, γ)

Here,

$$\nabla u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + yz^2) = 2x\vec{i} + z^2\vec{j} + 2yz\vec{k}$$

At point (1, –1, 3), $\nabla u = 2\vec{i} + 9\vec{j} - 6\vec{k}$

Then, value of $|\nabla u|$ at (1, –1, 3) is,

$|\nabla u| = |(2, 9, -6)|$, at (1, –1, 3)

$= \sqrt{4 + 81 + 26} = \sqrt{121} = 11.$

Thus, the greatest rate of increase of $u = x^2 + yz^2$ at (1, –1, 3) is 11.

The temperature at a point (x, y, z) in space is given by $T = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desire to fly in such a direction that it wing get warm as soon as possible. In what direction should it fly?

lution: Given that, $T = x^2 + y^2 - z$

Then,

$$\nabla T = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z) = 2x\,\vec{i} + 2y\,\vec{j} - \vec{k}$$

at point (1, 1, 2), $\quad \nabla T = 2\vec{i} + 2\vec{j} - \vec{k}$.

Given that a mosquito desires to fly in a direction so that its wing gets warm as soo as possible.

That means, the mosquito wants to fly in the direction where it get maximur tempereture.

Thus, the mosquito should fly in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$.

10. If q is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ a the point (1, –2, 1), show that $\cos\theta = \frac{3}{7\sqrt{6}}$.

Solution: Given surfaces are

$$f = xy^2z - 3x - z^2 \quad \text{and} \quad F = 3x^2 - y^2 + 2z - 1$$

Then,

$$\text{grad } f = (y^2z - 3)\vec{i} + 2xyz\,\vec{j} + (xy^2 - 2z)\vec{k}$$

and $\quad \text{grad } F = 6x\,\vec{i} - 2y\,\vec{j} + 2\vec{k}$

At point, (1, –2, 1), $\quad \text{grad } f = \vec{i} - 4\vec{j} + 2\vec{k}$

and $\quad \text{grad } F = 6\vec{i} + 4\vec{j} + 2\vec{k}$

Let θ be the angle between f and F at (1, –2, 1). Then,

$$\cos\theta = \frac{\text{grad } f \,.\, \text{grad } F}{|\text{grad } f|\,|\text{grad } F|} \quad \text{at point } (1, -2, 1)$$

$$= \frac{(\vec{i} - 4\vec{j} + 2\vec{k})\,.\,(6\vec{i} + 4\vec{j} + 2\vec{k})}{|(\vec{i} - 4\vec{j} + 2\vec{k})|\,|(6\vec{i} + 4\vec{j} + 2\vec{k})|}$$

$$= \frac{6 - 16 + 4}{\sqrt{1 + 16 + 4}\sqrt{36 + 16 + 4}} = \frac{-6}{\sqrt{21}\sqrt{56}} = \frac{-6}{14\sqrt{6}} = \frac{-3}{7\sqrt{6}}.$$

11. Find the value of constants l and u so that the surface $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, –1, 2).

Solution: Given surfaces are

$$\lambda x^2 - \mu yz = (\lambda + 2)x \quad \text{.........(i)}$$

$$4x^2y + z^3 = 4 \quad \text{.......(ii)}$$

Given that the surfaces intersect orthogonally at (1, –1, 2). So, the point lies on both surfaces.

Then at (1, –1, 2), (i) becomes

$$\lambda + 2\mu = \lambda + 2 \Rightarrow \mu = 1.$$

Set, $\quad \phi_1 = \lambda x^2 - (\lambda + 2)x - yz \quad [\because \mu = 1]$

$\quad \phi_2 = 4x^2y + z^3 - 4$

So, $\quad \vec{r}_1 = \text{grad}(\phi_1) = (2\lambda x - \lambda - 2)\vec{i} - z\vec{j} - y\vec{k}$

$\quad \vec{r}_2 = \text{grad}(\phi_2) = 8xy\,\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

at (1, –1, 2)

$$\vec{r}_1 = (\lambda - 2)\vec{i} - 2\vec{j} + \vec{k} \quad \text{and} \quad \vec{r}_2 = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Given that ϕ_1 and ϕ_2 are orthogonal to each other at (1, –1, 2). So, at (1, –1, 2), we should have,

$$\vec{r}_1 . \vec{r}_2 = 0 \quad \Rightarrow \quad -8(\lambda - 2) - 8 + 12 = 0$$

$$\Rightarrow \quad -8(\lambda - 2) + 4 = 0 \Rightarrow \lambda - 2 = -1/2 \Rightarrow \lambda = \frac{3}{2} = 2.5$$

Thus, λ = 2.5 and μ = 1.

EXERCISE 4.3

Find divergence of

(i) $x\vec{i} + y\vec{j} + z\vec{k}$ **(ii) $e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$**

(iii) $\left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$ **(iv) $e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}$**

tion: (i) Let, $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$

Then,

$$\text{Div}(\vec{v}) = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 1 + 1 + 1 = 3.$$

Thus, divergence of $x\vec{i} + y\vec{j} + z\vec{k}$ is 3.

Let $\quad \vec{v} = e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$.

Then the divergence of $\vec{v}$ is,

$$\text{div}.\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(e^x\cos y\,\vec{i} + e^x\sin y\,\vec{j})$$

$$= e^x\cos y + e^x\cos y = 2e^x\cos y.$$

Thus, the divergence of $e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$ is $2e^x\cos y$.

Let $\vec{v} = \left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$. Then the divergence of $\vec{v}$ is,

$$\text{div}.\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)\left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$$

$$= \frac{1}{x^2 + y^2}\left[-\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y}\right] = \frac{1}{x^2 + y^2}(0 + 0) = 0.$$

Thus, the divergence of $\vec{v}$ is 0.

(iv) Let $\vec{v} = e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}$. Then the divergence of $\vec{v}$ is,

$$\text{div}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(e^x\vec{i} + ye^{-x}\vec{j} + 2z\sinh x\,\vec{k}\right)$$
$$= e^x + e^{-x} + 2\sinh x$$
$$= e^x + e^{-x} + 2\sinh x$$
$$= e^x + e^{-x} + 2\left(\frac{e^x - e^{-x}}{2}\right) = e^x + e^{-x} + e^x - e^{-x} = 2e^x.$$

Thus, divergence of $e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}$ is $2e^x$.

2. Find curl of

(i) $\frac{1}{2}(x^2 + y^2 + z^2)(\vec{i} + \vec{j} + \vec{k})$ **(ii)** $(x^2 + y^2 + z^2)^{-3/2}(x\vec{i} + y\vec{j} + z\vec{k})$

(iii) $xyz(x\vec{i} + y\vec{j} + z\vec{k})$

Solution: (i) Let $\vec{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\vec{i} + \vec{j} + \vec{k})$.

Then the curl of $\vec{v}$ is,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1/2\,(x^2+y^2+z^2) & 1/2(x^2+y^2+z^2) & 1/2\,(x^2+y^2+z^2) \end{vmatrix}$$
$$= (y - z)\vec{i} + (z - x)\vec{j} + (x - y)\vec{k}$$

Thus, curl of $\vec{v}$ is $(y - z)\vec{i} + (z - x)\vec{j} + (x - y)\vec{k}$

(ii) – (iii) Similar to (i).

3. Evaluate: (i) div. $(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$ at $(1, 2, 3)$.

(ii) div $(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$ at $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution: (i) Here,

$$\text{div.}\,(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$$
$$= 6x + 10xy + 3xyz^2$$

At point $(1, 2, 3)$, div. $(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k}) = 6 + 20 + 54 = 80$.

(ii) Here,

$$\text{div}\,(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$$
$$= y\sin z + 2y\sin x + 2z\sin xy$$

At point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$, div $(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$

$$= \frac{\pi}{2}\sin\frac{\pi}{2} + 2\frac{\pi}{2}\sin 0 + 2\frac{\pi}{2}\sin 0$$
$$= \frac{\pi}{2}.1 + \pi.0 + \pi.0 = \frac{\pi}{2}.$$

Find the divergence and curl of vectors

(i) $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$

(ii) $\vec{v} = (x^2 + yz)\,\vec{i} + (y^2 + zx)\,\vec{j} + (z^2 + xy)\,\vec{k}$

Solution: (i) Let $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$

Then divergence of $\vec{v}$ is,

$$\text{div.}\,\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k})$$
$$= yz + 3x^2 + 2xz - y^2$$

And, curl of $\vec{v}$ is,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xy^2 - y^2z \end{vmatrix}$$
$$= -2yz\,\vec{i} - (z^2 - xy)\,\vec{j} + (6xy - xz)\,\vec{k}$$

Thus, divergence of $\vec{v}$ is $yz + 3x^2 + 2xz - y^2$ and curl of $\vec{v}$ is $-2yz\,\vec{i} + (xy - z^2)\,\vec{j} + x(6y - z)\,\vec{k}$.

(ii) Similar to (i)

. If $\vec{v} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$. **Show that:** $\nabla.\vec{v} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ and $\nabla\times\vec{v} = \vec{0}$.

[2011 Fall Q.No. 6(a)]

Solution: Let,

$$\vec{v} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Then,

$$\nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$= (x^2 + y^2 + z^2)^{-1/2} - \frac{x}{2}(x^2 + y^2 + z^2)^{-1/2}.2x + (x^2 + y^2 + z^2)^{-1/2} -$$
$$\frac{y}{2}(x^2 + y^2 + z^2)^{-3/2}.2y + (x^2 + y^2 + z^2)^{-1/2} - \frac{z}{2}(x^2 + y^2 + z^2)^{-3/2}.2z$$
$$= \frac{3}{\sqrt{x^2 + y^2 + z^2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x^2 + y^2 + z^2)$$
$$= \frac{3}{\sqrt{x^2 + y^2 + z^2}} - \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$$
$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

And,

$$\nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x.(x^2+y^2+z^2)^{-1/2} & y(x^2+y^2+z^2)^{-1/2} & z(x^2+y^2+z^2) \end{vmatrix}$$

$$= \left[-\frac{z}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2y + \frac{y}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2z\right]\vec{i} -$$

$$\left[-\frac{z}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2x + \frac{x}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2z\right]\vec{j} +$$

$$\left[-\frac{y}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2x + \frac{x}{2}(x^2+y^2+z^2)^{-3/2}\,.\,2y\right]\vec{k}$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}[(-yz+yz)\vec{i}\ (-xz+xz)\vec{j} + (-xy+xy)\vec{k}]$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}(0\vec{i} - 0\vec{j} + 0\vec{k})$$

$$= \vec{0}$$

Thus, $\nabla.\vec{v} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{v} = \vec{0}$.

6. If $\vec{A} = 3xz^2\vec{i} - yz\vec{j} + (x+2z)\vec{k}$. Find curl (curl $\vec{A}$)

Solution: Let $\vec{A} = 3xz^2\vec{i} - yz\vec{j} + (x+2z)\vec{k}$

Then,

$$\text{curl}(\vec{A}) = \nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 3xz^2 & -yz & x+2z \end{vmatrix} = y\vec{i} + (6xz-1)\vec{j} + 0\vec{k}$$

So,

$$\text{curl}(\text{curl}(\vec{A})) = \nabla\times\text{curl}\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & 6xz-1 & 0 \end{vmatrix}$$

$$= -6x\vec{i} + 0\vec{j} + (6z-1)\vec{k}.$$

Thus, curl (curl $\vec{A}$) $= -6x\vec{i} + (6z-1)\vec{k}$.

7. Show that the vector $\vec{v} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$ is solenoidal.

Solution:

Note: If divergence of $\vec{v}$ is zero i.e. div $\vec{v} = 0$ then $\vec{v}$ is called solenoidal.

Let, $\vec{v} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$

Then,

$$\text{div.}\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)((x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k})$$

$$= 1 + 1 - 2 = 0.$$

This shows that $\vec{v}$ is solenoidal.

. If $u = x^2+y^2+z^2$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that div. $(u\vec{v}) = 5u$

Solution: Let, $u = x^2+y^2+z^2$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$

Then,

$$\text{div}(u\vec{v}) = \nabla.(u\vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).((x^2+y^2+z^2)(x\vec{i}+y\vec{j}+z\vec{k}))$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)[(x^3+xy^2+xz^2)\vec{i} + (x^2y+y^3+yz^2)\vec{j} + (x^2z+y^2z+z^3)\vec{k}]$$

$$= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2$$

$$= 5(x^2+y^2+z^2) = 5u$$

Thus, div. $(u\vec{v}) = 5u$.

. Show that the vector $\vec{v} = (\sin y+z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$ is irrotational.

Solution:

Note: If curl of $\vec{v}$ is zero i.e. curl $\vec{v} = 0$ then $\vec{v}$ is called irrotational.

Let, $\vec{v} = (\sin y+z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$

Then $\vec{v}$ is irretational if curl $\vec{v} = \vec{0}$.

Here,

$$\text{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \sin y+z & x\cos y - z & x-y \end{vmatrix}$$

$$= (-1+1)\vec{i} + (1-1)\vec{j} + (\cos y - \cos y)\vec{k} = \vec{0}.$$

This shows that $\vec{v}$ is irrotational.

10. Show that $\vec{v} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ is irrotational.

Solution: Let, $\vec{v} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$

Then $\vec{v}$ is irratational if curl $\vec{v} = \vec{0}$.

Here,

$$\text{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= (3x^2z^2 - 3x^2z^2)\vec{i} - (6xyz^2 - 6xyz^2)\vec{j} + (2xz^3 - 2xz^3)\vec{k}$$

$$= \vec{0}.$$

This shows that $\vec{v}$ is irratational.

11. If $\phi = \log(x^2+y^2+z^2)$, find div (grad ϕ) and curl (grad ϕ).

{2010 Fall: If $\mu = \log(x^2+y^2+z^2)$, find div (grad μ).}

Solution: Let, $\phi = \log(x^2+y^2+z^2)$

Then,

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log(x^2+y^2+z^2))$$

$$= \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

Now,

$$\text{div}(\text{grad}\,\phi) = \nabla\cdot(\text{grad}\,\phi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{x^2+y^2+z^2}\right)$$

$$= 2\left[\frac{(x^2+y^2+z^2) - 2x^2}{(x^2+y^2+z^2)}\right] + 2\left[\frac{(x^2+y^2+z^2) - 2y^2}{(x^2+y^2+z^2)}\right] + 2\left[\frac{(x^2+y^2+z^2) - 2z^2}{(x^2+y^2+z^2)}\right],$$

$$= \frac{2}{(x^2+y^2+z^2)^2}[y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2]$$

$$= \frac{2}{(x^2+y^2+z^2)^2}(x^2+y^2+z^2)$$

$$= \frac{2}{(x^2+y^2+z^2)}$$

And,

$$\text{curl}(\text{grad}\,\phi) = \nabla\times(\text{grad}\,\phi)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= \left(\frac{-4yz}{(x^2+y^2+z^2)^2} - \frac{-4yz}{(x^2+y^2+z^2)^2}\right)\vec{i} - \left(\frac{-4xz}{(x^2+y^2+z^2)^2} - \frac{-4xz}{(x^2+y^2+z^2)^2}\right)\vec{j} + \left(\frac{-2xy}{(x^2+y^2+z^2)^2} - \frac{-2xy}{(x^2+y^2+z^2)^2}\right)\vec{k}$$

$$= \vec{0}.$$

Thus, div (grad ϕ) = $\frac{2}{(x^2+y^2+z^2)^2}$ and curl (grad ϕ) = $\vec{0}$.

2013 Spring Q. No. 3(b)

What is the physical interpretation of curl of averter field. If $\phi = \log(x^2+y^2+z^2)$, find curl (gradϕ).

2014 Fall Q. No. 4(a) OR; 2013 Fall Q. No. 5(b)

If $\phi = \log(x^2+y^2+z^2)$ find curl (grad ϕ).

Solution: See second part of above question Q. 11.

EXERCISE 4.4

Evaluate: (i) $\int_0^1 \{t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k}\}\,dt$ (ii) $\int_0^1 \{t\vec{i} + e^t\vec{j} + e^{-2t}\vec{k}\}\,dt$

Solution: (i) Here,

$$\int_0^1 \{t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k}\}\,dt = \left[\frac{t^2}{2}\vec{i} + \left(\frac{t^3}{3} - t^2\right)\vec{j} + t^3\vec{k}\right]_0^1$$

$$= \frac{\vec{i}}{2} - \frac{2\vec{j}}{3} + \vec{k}$$

(ii) Here,

$$\int_0^1 \{t\vec{i} + e^t\vec{j} + e^{-2t}\vec{k}\}\,dt = \left[\frac{t^2}{2}\vec{i} + e^t\vec{j} + \frac{e^{-2t}}{-2}\vec{k}\right]_0^1$$

$$= \left(\frac{1}{2}\vec{i} + e\vec{j} - \frac{e^{-2}}{2}\vec{k}\right) - \left(\vec{j} - \frac{\vec{k}}{2}\right)$$

$$= \frac{\vec{i}}{2} + (e-1)\vec{j} + \left(\frac{1-e^{-2}}{2}\right)\vec{k}.$$

Evaluate: (i) $\int_0^2 (\vec{r}\cdot\vec{s})\,dt$ (ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt$

where, $\vec{r} = t\vec{i} - t^2\vec{j} + (t-1)\vec{k}$, $\vec{s} = 2t^2\vec{i} + 6t\vec{k}$

Solution: Let $\vec{r} = t\vec{i} - t^2\vec{j} + (t-1)\vec{k}$ and $\vec{s} = 2t^2\vec{i} + 6t\vec{k}$

Then,

$$\vec{r}\cdot\vec{s} = (t\vec{i} - t^2\vec{j} + (t-1)\vec{k})\cdot(2t^2\vec{i} + 0\vec{j} + 6t\vec{k})$$

$$= 2t^3 + 0 + 6t^2 - 6t$$

$$= 2t^3 + 6t^2 - 6t$$

And,

$$\vec{r}\times\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix} = -6t^3\vec{i} - (2t^3 - 2t^2 - 6t^2)\vec{j} + 2t^4\vec{k}$$

$$= -t^3\vec{i} + (8t^2 - 2t^3)\vec{j} + 2t^4\vec{k}$$

Now,

(i) $$\int_0^2 \vec{r}\cdot\vec{s}\,dt = \int_0^2 (2t^3 + 6t^2 - 6t)\,dt = \left[\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}\right]_0^2$$

$$= \frac{2(2)^4}{4} + \frac{6(2)^3}{3} - \frac{6(2)^2}{2}$$

$$= 2(2)^2 + 2(2)^3 - 3(2)^2 = 8 + 16 - 12 = 12.$$

(ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt = \int_0^2 \{-6t^3\,\vec{i} + (8t^2 - 2t^3)\,\vec{j} + 2t^4\,\vec{k}\}\,dt$

$$= \left[-\frac{6t^4}{4}\vec{i} + \left(\frac{8t^3}{3} - \frac{2t^4}{4}\right)\vec{j} + \frac{2t^5}{5}\vec{k}\right]_0^2$$

$$= -\frac{6(2)^4}{4}\vec{i} + \left(\frac{8(2)^3}{3} - \frac{2(2)^4}{4}\right)\vec{j} + \frac{2(2)^5}{5}\vec{k}$$

$$= -24\vec{i} + \left(\frac{64}{3} - 8\right)\vec{j} + \frac{64}{5}\vec{k} = -24\vec{i} + \frac{40}{3}\vec{j} + \frac{64}{5}\vec{k}$$

Thus, (i) $\int_0^2 \vec{r}.\vec{s}\,dt = 12$ (ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt = -24\vec{i} + \frac{40}{3}\vec{j} + \frac{64}{5}\vec{k}$

3. Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} - 24t^2\vec{j} + 4\sin t\,\vec{k}$ given that $\vec{r} = 2\vec{i} + \vec{j}$ and $\frac{d\vec{r}}{dt} = -\vec{i} - 3\vec{k}$ at t = 0.

Solution: Here, $\frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} - 24t^2\vec{j} + 4\sin t\,\vec{k}$ ....(i)

Integrating (i) with respect to t then

$$\frac{d\vec{r}}{dt} = 3t^2\vec{i} - 8t^3\vec{j} - 4\cos t\,\vec{k} + c \quad ....(ii)$$

At t = 0, above equation (ii) gives,

$$\left.\frac{d\vec{r}}{dt^2}\right|_{at\ t=0} = -4\,\vec{k} + c \quad ....(iii)$$

Given that, at t = 0, $\frac{d\vec{r}}{dt} = -\vec{i} - 3\vec{k}$. Then (iii) gives,

$$-\vec{i} - 3\vec{k} = -4\vec{k} + c$$

$$\Rightarrow \quad c = -\vec{i} + \vec{k}$$

Therefore (ii) becomes,

$$\frac{d\vec{r}}{dt} = 3t^2\,\vec{i} - 8t^3\,\vec{j} - 4\cos t\,\vec{k} - \vec{i} + \vec{k} \quad ......(iv)$$

Integrating (iv) w. r. t. t then

$$\vec{r} = t^3\vec{i} - 2t^4\vec{j} - 4\sin t\,\vec{k} - t\,\vec{i} + t\,\vec{k} + c \quad ....(v)$$

At, t = 0, above equation (v) gives,

$$\vec{r}\,|_{at\ t=0} = 0 \quad ....(vi)$$

Given that $\vec{r} = 2\vec{i} + \vec{j}$ at t = 0. Then (vi) gives,

$$2\vec{i} + \vec{j} = c$$

Therefore, (v) becomes,

$$\vec{r} = t^3\vec{i} - 2t^4\vec{j} - 4\sin t\,\vec{k} - t\,\vec{i} + t\,\vec{k} + 2\vec{i} + \vec{j}$$

$$\Rightarrow \quad \vec{r} = (t^3 - t + 2)\,\vec{i} + (1 - 2t^4)\,\vec{j} + (t - 4\sin t)\,\vec{k}.$$

Let $\vec{a} = 12\cos 2t\;\vec{i} - 8\sin 2t\;\vec{j} + 16\,t\,\vec{k}$ be the acceleration of a particle at any time t. find the velocity $\vec{v}$ and displacement $\vec{r}$ at any time t and given $\vec{r} = \vec{0} = \vec{v}$ at t = 0.

Solution: Let $\vec{a} = 12\cos 2t\;\vec{i} - 8\sin 2t\;\vec{j} + 16t\;\vec{k}$ be acceleration at any time t. So,

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = 12\cos 2t\;\vec{i} - 8\sin 2t\;\vec{j} + 16t\;\vec{k} \quad ....(i)$$

Also, given that, $\vec{r} = 0 = \vec{v}$ at t = 0.

Since $\vec{v}$ be the velocity at any time t. Therefore,

$$\vec{r} = 0 = \frac{d\vec{r}}{dt} \quad \text{at } t = 0 \quad ....(ii)$$

Integrating (i) w.r.t. t then

$$\frac{d\vec{r}}{dt} = 6\sin 2t\;\vec{i} + 4\cos 2t\;\vec{j} + 8t^2\;\vec{k} + c \quad ....(iii)$$

at t = 0, (iii) gives,

$$0 = 0 + 4j + c \Rightarrow c = 4j \quad [\because \text{using (ii)}]$$

Then (iii) becomes,

$$\frac{d\vec{r}}{dt} = 6\sin 2t\;\vec{i} + 4\cos 2t\;\vec{j} + 8t^2\;\vec{k} - 4j \quad ....(iv)$$

Again integrating (iv) w.r.t. t then,

$$\vec{r} = -3\cos 2t\;\vec{i} - 2\sin 2t\;\vec{j} + \frac{8t^3}{3}\vec{k} - 4t\;\vec{j} + c \quad ....(v)$$

At t = 0, (v) gives,

$$0 = -3\vec{i} - 0 + 0 - 0 + c \Rightarrow c = 3\vec{i} \quad [\because \text{using (ii)}]$$

Then (v) becomes,

$$\vec{r} = -3\cos 2t\;\vec{i} - 2\sin 2t\;\vec{j} + \frac{8t^3}{3}\vec{k} - 4t\;\vec{j} + 3\vec{i}$$

$$\Rightarrow \quad \vec{r} = (3 - 3\cos 2t)\,\vec{i} - (4t + 2\sin 2t)\,\vec{j} + \frac{8t^3}{3}\vec{k} \quad ....(vi)$$

Thus, (iv) be the velocity and (vi) be the displacement of the particle at time t.

If $\vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$, show that $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$, where $\vec{a}$ is a constant vector.

Solution: Let, $\vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$ ....(i)

Then we wish to show $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$, for $\vec{a}$ is a constant vector.

Let $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$ exists. Differentiating w. r. t. t then,

$$\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\vec{a}) \Rightarrow \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$$

$$\Rightarrow \vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 \qquad \left[\because \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0\right]$$

This holds by (i). Therefore $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{a}$ exists.

6. Solve $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r} = 0$ and $\frac{d\vec{r}}{dt} = \vec{u}$

Solution: Let, $$\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b} \quad \text{.........(i)}$$

for $\vec{a}$ and $\vec{b}$ are constant vectors.

Also given that, $\vec{r} = 0$ and $\frac{d\vec{r}}{dt} = \vec{u}$; at $t = 0$ ....(ii)

Here integrating (i) w.r.t. t then

$$\frac{d\vec{r}}{dt} = \vec{a}\frac{t^2}{2} + \vec{b}t + \vec{c} \quad \text{..........(iii)}$$

At $t = 0$, using (ii) the equation (iii) gives,

$$\vec{u} = \vec{a}.0 + \vec{b}.0 + \vec{c} \Rightarrow \vec{c} = \vec{u}$$

Therefore (ii) becomes,

$$\frac{d\vec{r}}{dt} = \frac{\vec{a}}{2}t^2 + \vec{b}t + \vec{u} \quad \text{..........(iv)}$$

Again integrating (iv) w.r.t. t then,

$$\vec{r} = \frac{\vec{a}}{2}.\frac{t^3}{3} + \vec{b}\frac{t^2}{2} + \vec{u}t + \vec{c} \quad \text{.........(v)}$$

At $t = 0$, using (ii), the equation (v) gives

$$0 = \frac{\vec{a}}{2}.0 + \vec{b}.0 + \vec{u}.0 + \vec{c} \Rightarrow \vec{c} = 0$$

Therefore (v) becomes,

$$\vec{r} = \frac{\vec{a}}{6}t^3 + \frac{\vec{b}}{2}t^2 + \vec{u}t)$$

This is the solution of given equation.

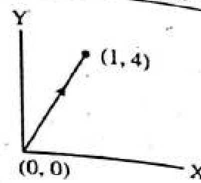
EXERCISE 4.5

A. Calculate $\int_c \vec{F}.d\vec{r}$ for the following data. (If $\vec{F}$ is a force, this gives the work of the displacement along C).

1. $\vec{F} = (y^2, -x^2)$, C is the straight line from (0, 0) to (1, 4)

Solution: Given that, $\vec{F} = (y^2, -x^2)$ .... (i)

and the path is a straight line from (0, 0) to (1, 4).



Here, the equation of path C is

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

$$y - 0 = \frac{4 - 0}{1 - 0}(x - 0) \Rightarrow y = 4x$$

Put $x = t$ then $y = 4t$. Also, t moves from $t = 0$ to $t = 1$.

Then, $\vec{r} = x\vec{i} + y\vec{j} \Rightarrow \vec{r} = t\vec{i} + 4t\vec{j}$

So, differentiating we get, $d\vec{r} = dt\,\vec{i} + 4dt\,\vec{j}$

So that,

$$\vec{F}.d\vec{r} = ((4t)^2\vec{i} - t^2\vec{j}).(dt\,\vec{i} + 4dt\,\vec{j}) = 16t^2dt - 4t^2dt = 12t^2dt.$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^1 12t^2dt = [4t^3]_0^1 = 4.$$

2. $\vec{F} = (xy, x^2y^2)$, C is the quarter circle from (2, 0) to (0, 2) with center (0, 0).

Solution: Given that, $\vec{F} = (xy, x^2y^2) = xy\,\vec{i} + x^2y^2\,\vec{j}$

and the path of force $\vec{F}$ is the quarter circle from (2, 0) to (0, 2) with centre at (0, 0). Clearly, the length of (0, 0) to (2, 0) is 2. So, radius of the circle is 2. Therefore, the equation of path of curve is, $x^2 + y^2 = 4$.

Put $x = t$ then $y = \sqrt{4 - t^2}$. Also, t moves from $t = 2$ to $t = 0$.

Then,

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow \vec{r} = t\vec{i} + \vec{j}\sqrt{4 - t^2}$$

So, $$d\vec{r} = dt\,\vec{i} - \frac{tdt}{\sqrt{4 - t^2}}\vec{j}$$

So that,

$$\vec{F}.d\vec{r} = t\sqrt{4 - t^2}\,\vec{i} + t^2(4 - t^2)\vec{j} = t\sqrt{4 - t^2}\,\vec{i} + (4t^2 - t^4)\vec{j}$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_2^0 \left\{\left(t\sqrt{4 - t^2}\,\vec{i} + (4t^2 - t^4)\vec{j}\right).\left(dt\,\vec{i} - \frac{tdt}{\sqrt{4 - t^2}}\vec{j}\right)\right\}$$

$$= \int_2^0 \left(t\sqrt{4 - t^2} - \frac{t(4t^2 - t^4)}{\sqrt{4 - t^2}}\right)dt = \int_2^0 \{t\sqrt{4 - t^2} - t^3\sqrt{4 - t^2}\}\,dt$$

Put $4 - t^2 = u^2$ then $-2tdt = 2u\,du \Rightarrow tdt = -u\,du$. Also, $t = 2 \Rightarrow u = 0$ and $t = 0 \Rightarrow u = 2$. So,

$$\int_c \vec{F}.d\vec{r} = \int_2^0 \{-u^2 + u^2(4 - u^2)\}\,du$$

$$= \int_2^0 (3u^2 - u^4)du = \left[u^3 - \frac{u^5}{5}\right]_0^2 = 8 - \frac{32}{5} = \frac{40-32}{5} = \frac{8}{5}$$

3. $\vec{F} = [(x-y)^2, (y-x)^2]$, **C;** $xy = 1, 1 \le x \le 4$.

Solution: Let, $\vec{F} = ((x-y)^2, (y-x)^2)$ and C: $xy = 1$ for $1 \le x \le 4$.

Then, $\vec{r} = x\vec{i} + 4\vec{j} = x\vec{i} + \frac{1}{x}\vec{j}$. So, $d\vec{r} = dx\,\vec{i} - \frac{1}{x^2}dx\,\vec{j}$

So that,

$$\vec{F}.d\vec{r} = \{(x-y)^2\vec{i} + (y-x)^2\vec{j}\}.\left\{dx\,\vec{i} - \frac{1}{x^2}dx\,\vec{j}\right\}$$

$$= \left\{\left(x-\frac{1}{x}\right)^2\vec{i} + \left(\frac{1}{x}-x\right)^2\vec{j}\right\}.\left\{i - \frac{1}{x^2}\vec{j}\right\}dx$$

$$= \left\{\left(x-\frac{1}{x}\right)^2 - \frac{1}{x^2}\left(\frac{1}{x}-x\right)^2\right\}dx = \left\{x^2 - 2 + \frac{1}{x^2} - \frac{1}{x^4} + \frac{2}{x^2} - 1\right\}dx$$

$$= (x^2 - 3 + 3x^{-2} - x^{-4})\,dx.$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_1^4 (x^2 - 3 + 3x^{-2} - x^{-4})\,dx$$

$$= \left[\frac{x^3}{3} - 3x + \frac{3x^{-1}}{-1} - \frac{x^{-3}}{-3}\right]_1^4 = \left(\frac{64}{3} - 12 - \frac{3}{4} + \frac{1}{192}\right) - \left(\frac{1}{3} - 3 - 3 + \frac{1}{3}\right)$$

$$= \frac{63}{3} - 12 - \frac{3}{4} + \frac{1}{192} + 6 - \frac{1}{3}$$

$$= 21 - 6 - \frac{144 - 1 + 64}{192}$$

$$= 15 - \frac{207}{192} = \frac{2673}{192} = \frac{891}{64}.$$

4. $\vec{F} = (2z, x, -y)$, **C;** $\vec{r} = (\cos t, \sin t, 2t)$ **from (0, 0, 0) to (1, 0, 4π).**

Solution: Let, $\vec{F} = (2z, x, -y) = 2z\,\vec{i} + x\,\vec{j} - y\,\vec{k}$

and c: $\vec{r} = (\cos t, \sin t, 2t)$ from (0, 0, 0) to (1, 0, 4π).

Here,

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$$

So, $d\vec{r} = (-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})\,dt$

and,

$$\vec{F} = 2(2t)\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}$$

$$= 4t\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}$$

Then

$$\vec{F}.d\vec{r} = (4t\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}).(-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})\,dt$$

$$= (-4t\sin t + \cos^2 t - 2\sin t)\,dt$$

$$= \left(-4\sin t + \frac{1+\cos 2t}{2} - 2\sin t\right)dt$$

Since the particle moves from (0, 0, 0) to (1, 0, 4π) along the curve.

So, $z = 0$ and $z = 4\pi$. i.e. $2t = 0$ and $2t = 4\pi$ $\Rightarrow t = 0$ ad $t = 2\pi$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_c \left(-4t\sin t + \frac{1+\cos 2t}{2} - 2\sin t\right)dt$$

$$= \left[4t\cos t - 4\sin t + \frac{t}{2} + \frac{\sin 2t}{4} + 2\cos t\right]_0^{2\pi}$$

$$= 8\pi + \pi + 2 - 2 \qquad [\because \cos 2\pi = 1, \sin 2\pi = 0]$$

$$= 9\pi.$$

$\vec{F} = (e^x, e^{-y}, e^z)$, **C;** $\vec{r} = (t, t^2, t)$ **from (0, 0, 0) to (1, 1, 1).**

[2003 Spring Q.No. 4(a)]

lution: Similar to Q. 4.

Calculate $\int_c f\,ds$,

$f = x^2 + y^2$, **C:** $y = 3x$ **from (0, 0) to (2, 6).**

Solution: Let, $f = x^2 + y^2$

and given that the path of integration is C: $y = 3x$ from (0, 0) to (2, 6).

Put $x = t$ then $y = 3t$. Also $x = 0 \Rightarrow t = 0$ and $x = 2 \Rightarrow t = 2$

And, $f = t2 + (3t)2 = 10t2$

Since, $\vec{r} = x\vec{i} + y\vec{j} = t\vec{i} + 3t\vec{j} = (\vec{i} + 3\vec{j})t$. So, $d\vec{r} = (\vec{i} + 3\vec{j})\,dt$

Since we have,

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}} = \sqrt{(\vec{i} + 3\vec{j}).(\vec{i} + 3\vec{j})} = \sqrt{1+9} = \sqrt{10}$$

Now,

$$\int_c f\,ds = \int_0^2 f(t)\frac{ds}{dt}\,dt = \int_0^2 10t^2\sqrt{10}\,dt = \left[\frac{10t^3\sqrt{10}}{3}\right]_0^2 = \frac{80\sqrt{10}}{3}$$

. $f = x^2 + y^2 + z^2$, **C: (cost, sint, 2t),** $0 \le t \le 4\pi$.

olution: Given that, $f = x^2 + y^2 + z^2$

and the path of integration is C: (cost, sint, 2t) for $0 \le t \le 4\pi$.

This shows that, $z = 2t$, $x = \cos t$ and $y = \sin t$.

So, $f = \cos^2 t + \sin^2 t + 4t^2 = 1 + 4t^2$

Since, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

So, $\frac{d\vec{r}}{dt} = (-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})$

Since, $\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}}$

$= \sqrt{(-\sin t\ \vec{i} + \cos t\ \vec{j} + 2\vec{k}) . (-\sin t\ \vec{i} + \cos t\ \vec{j} + 2\vec{k})}$

$= \sqrt{\sin^2 t + \cos^2 t + 4} \quad = \sqrt{1+4} = \sqrt{5}$.

Now,

$$\int_c f.\,ds = \int_c f.\frac{ds}{dt}.\,dt = \int_0^{4\pi} (1+4t^2).\sqrt{5}\,dt$$

$$= \sqrt{5}\left[t + \frac{4t^3}{3}\right]_0^{4\pi} = \sqrt{5}\left(4\pi + \frac{256\pi^3}{3}\right).$$

8. $f = 1 + y^2 + z^2$, C: $\vec{r} = (t, \cos t, \sin t)$, $0 \le t \le \pi$.

Solution: Similar to 7.

9. $f = x^2 + (xy)^{1/3}$, C is the hypocycloid $\vec{r} = (\cos^3 t, \sin^3 t)$, $0 \le t \le \pi$.

Solution: Let, $f = x^2 + (xy)^{1/3}$

and the path of integrand is c : $\vec{r} = (\cos^3 t, \sin^3 t)$ for $0 \le t \le \pi$.

Here,

$$\vec{r} + x\vec{i} + y\vec{j} = \cos^3 t\ \vec{i} + \sin^3 t\ \vec{j}$$

So, $\dfrac{d\vec{r}}{dt} = -3\cos^2 t \sin t\ \vec{i} + 3\sin^2 t \cos t\ \vec{j}$

And, $f = x^2 + (xy)^{1/3} = \cos^6 t + \sin t \cos t$.

Since, we have

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = 3\cos t \sin t$$

So that,

$$f.\frac{ds}{dt} = (\cos^6 t + \sin t \cos t)\, 3\cos t \sin t$$

$$= 3\cos^7 t \sin t + 3\sin^2 t \cos^2 t$$

$$= 3\cos^7 t \sin t + \frac{3}{4}\sin^2 2t \quad [\because \sin 2A = 2\sin A \cos A]$$

$$= 3\cos^7 t \sin t + \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right) = 3\cos^7 t \sin t + \frac{3}{8}(1-\cos 4t).$$

Now,

$$\int_c f\,ds = \int_c f\frac{ds}{dt}.\,dt = 3\int_0^{\pi} \cos^7 t \sin t\,dt + \frac{3}{8}\int_0^{\pi}(1-\cos 4t)\,dt$$

Put $\cos t = u$ then $-\sin t\,dt = du$. Also, $t = 0 \Rightarrow u = 1$ and $t = \pi \Rightarrow u = -1$. Therefore,

$$\int_c f\,ds = -3\int_1^{-1} u^7\,du + \frac{3}{8}\left[t - \frac{\sin 4t}{4}\right]_0^{\pi} = -3\left[\frac{u^8}{8}\right]_1^{-1} + \frac{3}{8}\pi \quad [\because \sin 4\pi = 0 = \sin 0]$$

$$= -\frac{3}{8}(1-1) + \frac{3\pi}{8} = \frac{3\pi}{8}$$

Show that $\int_c \vec{F}\,d\vec{r} = 3\pi$, given that $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ and C being the arc of the curve $\vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$ from $t = 0$ to $t = 2\pi$.

Solution: Given that, $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

and c be the arc of $\vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$ for $t = 0$ to $t = 2\pi$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \Rightarrow \quad \vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$$

So, $x = \cos t$, $y = \sin t$ and $z = t$.

Therefore, $\vec{F} = t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}$.

Then,

$$\vec{F}.d\vec{r} = (t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}) . \{d(\cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k})\}$$

$$= (t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}) . (-\sin t\ \vec{i} + \cos t\ \vec{j} + \vec{k})$$

$$= -t\sin t + \cos^2 t + \sin t$$

$$= -t\sin t + \frac{1+\cos 2t}{2} + \sin t$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^{2\pi}\left(-t\sin t + \frac{1+\cos 2t}{2} + \sin t\right)dt$$

$$= \left[(-t)(-\cos t) - (-1)(-\sin t) + \frac{t}{2} + \frac{\sin 2t}{4} - \cos t\right]_0^{2\pi}$$

$$= (2\pi + 0 + \pi + 0 - 1) - (0 - 0 + 0 + 0 - 1) \quad [\because \cos 2\pi = 1, \sin 2\pi = 0]$$

$$= 3\pi - 1 + 1 = 3\pi$$

This shows that $\int_c \vec{F}.d\vec{r} = 3\pi$.

Find the work done by the force $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$. **[2008 Spring Q.No. 3(b)]**

Solution: Given that, $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$ ....(i)

that moves from (0, 0, 0) to (2, 1, 1) along $x = 2t^2$, $y = t$, $z = t^3$.

Then,

$$\vec{F} = (2t+3)\vec{i} + 2t^5\vec{j} + (t^4 - 2t^2)\vec{k}$$

Since, $y = t$ and $y = 0$, $y = 1$. So, $t = 0$ and $t = 1$.

We know that,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \Rightarrow \quad \vec{r} = 2t^2\vec{i} + t\vec{j} + t^3\vec{k}$$

Then,

$$d\vec{r} = (4t\vec{i} + \vec{j} + 3t^2\vec{k})\,dt$$

So that,

$$\vec{F}.d\vec{r} = ((2t+3)\vec{i} + 2t^3\vec{j} + (t^4 - 2t^2)\vec{k}) . (2t\vec{i} + \vec{j} + 3t^2\vec{k})\,dt$$
$$= (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)\,dt$$

We have, the work done by the force $\vec{F}$ along the curve C: $\vec{r}$ is $\int_c \vec{F}.d\vec{r}$.

Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt$$
$$= \left[\frac{3t^7}{7} + \frac{2t^6}{6} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{12t^2}{2}\right]_0^1$$
$$= \frac{3}{7} + \frac{2}{6} - \frac{6}{5} + \frac{8}{3} + \frac{12}{2}$$
$$= \frac{3}{7} + \frac{1}{3} - \frac{6}{5} + \frac{8}{3} + 6$$
$$= \frac{3}{7} - \frac{6}{5} + 3 + 6 = \frac{15-42}{35} + 9 = \frac{-27+315}{35} = \frac{288}{35}.$$

Thus, the work done by F is $\frac{288}{35}$.

E. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$. **[2010 Spring Q.No. 6(a) OR]**

Solution: Given that,

$$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k} \quad \text{.........(i)}$$

This force moves along the curve

$$x^2 = 4y,\ 3x^3 = 8z \quad \text{for } 0 \le x \le 2 \quad \text{..........(ii)}$$
$$\Rightarrow\ y = \frac{x^2}{4},\ z = \frac{3x^3}{8}.$$

Then (i) becomes,

$$\vec{F} = 3x^2\vec{i} + \left(2x.\frac{3x^3}{8} - \frac{x^2}{4}\right)\vec{j} + \frac{3x^3}{8}\vec{k}$$
$$= 3x^2\vec{i} + \frac{3x^4 - x^2}{4}\vec{j} + \frac{3x^2}{8}\vec{k}$$

Since we know that,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + \frac{x^2}{4}\vec{j} + \frac{3x^3}{8}\vec{k}$$

Then,

$$d\vec{r} = \left(\vec{i} + \frac{x}{2}\vec{j} + \frac{9x^2}{8}\vec{k}\right)dx$$

So that,

$$\vec{F}.d\vec{r} = \left(3x^2\vec{i} + \left(\frac{3x^4 - x^2}{4}\right)\vec{j} + \left(\frac{3x^2}{8}\right)\vec{k}\right).\left(\vec{i} + \frac{x}{2}\vec{j} + \frac{9x^2}{8}\vec{k}\right)dx$$
$$= \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\right)dx$$

Now, work done by force $\vec{F}$ along the curve (ii) is.

$$\int_c \vec{F}.d\vec{r} = \int_0^2 \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\right)dx$$
$$= \left[x^3 + \frac{3x^6}{48} - \frac{x^4}{32} + \frac{27x^6}{64\times 6}\right]_0^2$$
$$= \left[x^3 + \frac{x^6}{16} - \frac{x^4}{32} + \frac{9x^6}{128}\right]_0^2$$
$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{9\times 64}{128} = 8 + 4 - \frac{1}{2} + \frac{9}{2} = 16.$$

Thus, the work–done by $\vec{F}$ is 16.

F. Evaluate $\int_c \vec{F}\,d\vec{r}$

i. where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ in the xy plane from (0, 0) to (4, 4).

Solution: Given that the force is,

$$\vec{F} = x^2y^2\vec{i} + y\vec{j} \quad \text{.............. (i)}$$

And the curve c is, $y^2 = 4x$ .... (ii)

Then, $\vec{F} = \left(\frac{y^2}{4}\right)y^2\vec{i} + y\vec{j} = \frac{y^4}{4}\vec{i} + y\vec{j}$

Since, $\vec{r} = x\vec{i} + y\vec{j} = \frac{y^2}{4}\vec{i} + y\vec{j}$

Then, $d\vec{r} = \left(\frac{y}{2}\vec{i} + \vec{j}\right)dy$

Therefore,

$$\vec{F}.d\vec{r} = \left(\frac{y^2}{4}\vec{i} + y\vec{j}\right).\left(\frac{y}{2}\vec{i} + \vec{j}\right)dy$$
$$= \left(\frac{y^3}{8} + y\right)dy = \left(\frac{y^3 + 8y}{8}\right)dy.$$

Given that the force $\vec{F}$ moves along the curve c from (0, 0) to (4, 4). Then,

$$\int_c \vec{F}.d\vec{r} = \int_0^4 \left(\frac{y^3 + 8y}{8}\right)dy$$
$$= \frac{1}{8}\left[\frac{y^4}{4} + 4y^2\right]_0^4$$
$$= \frac{1}{8}\left[\frac{(4)^4}{4} + 4.(4)^2\right] = \frac{1}{8}[64 + 64] = \frac{128}{8} = 16$$

Thus, $\int_c \vec{F}.d\vec{r} = 16.$

(ii) $\vec{F} = (x^2+y^2)\vec{i} + (x^2-y^2)\vec{j}$ **and c is the curve $y^2 = x$ joining (0, 0) and (1, 1)**

Solution: Similar to (i).

(iii) $\vec{F} = \cos y\,\vec{i} - x\sin y\,\vec{j}$ **and c is the curve $y = \sqrt{1-x^2}$ in the xy plane from (1, 0) to (0, 1).**

Solution: Let, $\vec{F} = \cos y\,\vec{i} - x \sin y\,\vec{j}$.

And it moves along $y = \sqrt{1-x^2}$ which is a half range circle having radius r = 1.

Since, $\vec{r} = x\vec{i} + y\vec{j}$. So, $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$.

Then,

$$\int_c \vec{F}.d\vec{r} = \int_c (\cos y\,\vec{i} - x\sin\vec{j}).(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_c (\cos y\,dx - x\sin y\,dy)$$

$$= \int_c d(x\cos y)$$

$$= \int_1^0 d(x\cos\sqrt{1-x^2}) = \left[x\cos\sqrt{1-x^2}\right]_1^0 = -1$$

(iv) $\vec{F} = \sin y\,\vec{i} + x(1+\cos y)\,\vec{j}$ **and c is the curve $x^2+y^2 = a^2$, z = 0.**

Solution: Given that $\vec{F} = \sin y\,\vec{i} + x(1+\cos y)\,\vec{j}$ .... (i)

That moves along c: $x^2+y^2 = a^2$, z = 0

Since we have, $\vec{r} = \vec{i} + \vec{j}$

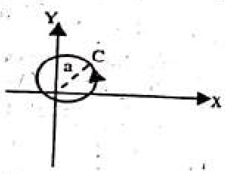
So, $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

Then $\vec{F}.d\vec{r} = (\sin y\,\vec{i} + x(1+\cos y\,\vec{j}).(dx\,\vec{i} + dy\,\vec{j})$

$= \sin y\,dx + x(1+\cos y)\,dy$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_c [d(x\sin y) + x\,dy]$$



Since the path is circular curve. So, its parametric form is,

x = a cost, y = a sint

And t varies from t = 0 to t = 2π.

Therefore,

$$\int_c \vec{F}.d\vec{r} = \int_0^{2\pi} [d(a\cos t\sin(a\sin t) + a\cos t\, a\,d(\sin t)$$

$$= a[\cos t\sin(a\sin t)]_0^{2\pi} + a^2\int_0^{2\pi}\cos t\cos t\,dt$$

$$= a[\cos 2\pi\sin(a\sin 2\pi) - \cos 0\sin(a\sin 0) + a^2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\times 2\pi \qquad [\because \sin 2\pi = 0 = \sin 0]$$

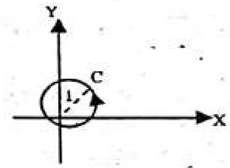
$$= a^2\pi.$$

(v) $\vec{F} = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$, **where c is the circle $x^2+y^2 = 1$ in the z–plane described in the anticlockwise direction.**

Solution: Given that,

$$\vec{F} = \int_c \left(\frac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}\right)(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_c \left(\frac{1}{x^2+y^2}\right)(-y\,dx + x\,dy)$$



Since the path is a circular path with radius r = 1. So, its parametric form is,

x = cost, y = sint

And the variable t varies from t = 0 to t = 2π. Therefore,

$$\int_c F.d\vec{r} = \int_0^{2\pi}\left(\frac{1}{\cos^2 t + \sin^2 t}\right)[-\sin t\,d(\cos t) + \cos t\,d(\sin t)]$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} dt = [t]_0^{2\pi} = 2\pi.$$

(vi) $\vec{F} = (2x-y+z)\vec{i} + (x+y-z^2)\vec{j} + (3x-2y+4z)\vec{k}$, **around the circle $x^2+y^2 = a^2$, z = 0.**

Solution: Similar to (iv).

(vii) $\vec{F} = yz\,\vec{i} + (xz+1)\vec{j} + xy\,\vec{k}$, **and c is the any path from (1, 0, 0) to (2, 1, 4)**

Solution: Given that,

$$\vec{F} = yz\,\vec{i} + (xz+1)\,\vec{j} + xy\,\vec{k}$$

And the path is from (1, 0, 0) to (2, 1, 4).

Here, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

Now,

$$\int_0^{2\pi} \vec{F}\cdot d\vec{r} = \int_{(1,0,0)}^{(2,1,4)} (yz\,\vec{i} + (xz+1)\,\vec{j} + xy\,\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \int_{(1,0,0)}^{(2,1,4)} [yz\,dx + (xz+1)\,dy + xy\,dz]$$

$$= \int_{(1,0,0)}^{(2,1,4)} d(xyz) + \int_{(1,0,0)}^{(2,1,4)} dy = [xyz]_{(1,0,0)}^{(2,1,4)} + [y]_{(1,0,0)}^{(2,1,4)}$$

$$= [2.1.4 - 1.0.0] + (1 - 0)$$

$$= 8 + 1$$

$$= 9.$$

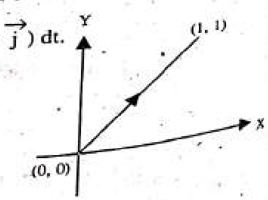
G. Find $\int_c \vec{F}\cdot d\vec{r}$ where, $\vec{F} = y^2\vec{i} + 2xy\,\vec{j}$ from O(0, 0) to P(1, 1) in each of t following cases:
a. along the straight line OP.
b. along the parabola $y^2 = x$.
c. along the x–axis from x = 0 to x = 1 and then along the line x = 1, fro y = 0 to y = 1.

Solution: Here, $\vec{F} = y^2\vec{i} + 2xy\,\vec{j}$ and applied from O(0, 0) to P(1, 1).

(a) The path is a straight line OP. Here, O(0, 0) and P(1, 1). So, the equation of straigh line is x = y.

Set, x = t then y = t. Also, t moves from t = 0 to t = 1.

Since, $\vec{r} = x\,\vec{i} + y\,\vec{j} = t\,\vec{i} + t\,\vec{j}$. So, $d\vec{r} = (\vec{i} + \vec{j})\,dt$.



Now,

$$\int_c F.d\vec{r} = \int_c (y^2\vec{i} + 2xy\,\vec{j})\,d\vec{r}$$

$$= \int_0^1 (t^2\vec{i} + 2t^2\vec{j})\,(\vec{i} + \vec{j})dt$$

$$= \int_0^1 (t^2 + 2t^2)\,dt = \int_0^1 (3t^2)\,dt = [t^3]_0^1 = 1.$$

(b) The path is a parabola $y^2 = x$ from O(0, 0) to P(1, 1).

Set, y = t then $x = t^2$. So that t moves from t = 0 to t = 1.

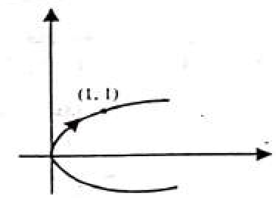
Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^1 (y^2\vec{i} + 2xy\,\vec{j})\,(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_0^1 (y^2\vec{i} + 2xy\,\vec{j}).(2t\,\vec{i} + \vec{j})dt$$

$$= \int_0^1 (2t^3 + 2t^3)\,dt$$

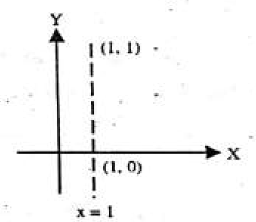
$$= \int_0^1 (4t^3)\,dt = [t^4]_0^1 = 1.$$



(c) Given that the path of the force $\vec{F}$ is along x–axis from x = 0 to x = 1 and then along the line x = 1 from y = 0 to y = 1.

When the force moves along x–axis, y = 0. So, dy = 0

When the force moves along x = 1, x = 1. So, dx = 0



Now,

$$\int_c \vec{F}.d\vec{r} = \int_{x=0}^{1} \vec{F}.d\vec{r} + \int_{y=0}^{1} \vec{F}.d\vec{r}$$

$$= \int_{x=0}^{1} (y^2\vec{i} + 2xy\,\vec{j}).(dx\,\vec{i} + dy\,\vec{j}) + \int_{y=0}^{1} (y^2\vec{i} + 2xy\,\vec{j}).(dx\,\vec{i} + dy\,\vec{j})$$

$$= 0 + \int_{y=0}^{1} (y^2\vec{i} + 2xy\,\vec{j}).(0\,\vec{i} + dy\,\vec{j})$$

$$= \int_{y=0}^{1} 2y\,dy = [y^2]_0^1 = 1.$$

H. If $\vec{F} = (2xy - z)\,\vec{i} + yz\,\vec{j} + x\,\vec{k}$ evaluate $\int_c \vec{F}.d\vec{r}$ along the curve c, where

a. c is the curve x = t, y = 2t, $z = t^2 - 1$, with t increasing from 0 to 1.

b. c consists of two straight line from the origin to the point (1, 0, –1) and from (1, 0, –1) to the point (2, 3, –3).

Solution: Given that, $\vec{F} = (2xy - z)\,\vec{i} + yz\,\vec{j} + x\,\vec{k}$

(a) And the path of $\vec{F}$ is, x = t, y = 2t, $z = t^2 - 1$. When t moves from t = 0 to t = 1.

Since, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = t\,\vec{i} + 2t\,\vec{j} + (t^2 - 1)\,\vec{k}$

So, $d\vec{r} = dt\,\vec{i} + 2dt\,\vec{j} + 2t\,dt\,\vec{k} = (\vec{i} + 2\,\vec{j} + 2t\,\vec{k})\,dt.$

Then,

$$\vec{F}.d\vec{r} = [(2(t)(2t) - (t^2-1))\vec{i} + (2t(t^2-1))\vec{j} + t\vec{k}).(\vec{i} + 2\vec{j} + 2t\vec{k})\,dt$$

$$= [(4t^2 - t^2 + 1)\vec{i} + (2t^3 - 2t)\vec{j} + t\vec{k}].(\vec{i} + 2\vec{j} + 2t\vec{k})\,dt$$

$$= (4t^2 - t^2 + 1 + 4t^3 - 4t + 2t^2)\,dt$$

$$= (4t^3 + 5t^2 - 4t + 1)\,dt$$

$$= \int_0^1 (4t^3 + 5t^2 - 4t + 1)\,dt = \left[t^4 + \frac{5t^3}{3} - 2t^2 + t\right]_0^1$$

$$= 1 + \frac{5}{3} - 2 + 1 = \frac{5}{3}.$$

(b) Let C_1 be the line segment from (0, 0, 0) to (1, 0, –1). So, equation of C_1 is

$$\frac{x-1}{1-0} = \frac{y-0}{0-0} = \frac{z+1}{-1-0} = t \text{ (say)}$$

i.e. $x = t + 1,\ y = 0,\ z = -t - 1.$

So, $dx = dt,\ dy = 0,\ dz = -dt.$

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$.

So that,

$$\vec{F}.d\vec{r} = ((2xy - z)\vec{i} + yz\vec{j} + x\vec{k}).(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= (2xy - z)dx + yzdy + xdz$$

Here the integral along C_1 is,

$$\int_{C_1} \vec{F}.d\vec{r} = \int_{C_1} ((2xy - z)dx + yzdy + xdz)$$

$$= \int_{C_1}[(t+1)dt + (t+1)(-dt)] = \int_{C_1}(t+1-t-1)\,dt = \int_{C_1} 0\,dt = 0$$

Also, C_2 be the line segment from (1, 0, –1) to (2, 3, –3).

The equation of line C_2 is,

$$\frac{x-1}{2-1} = \frac{y-0}{3-0} = \frac{z+1}{-3+1} = 4$$

i.e. $x = 4 + 1,\ y = 3u,\ z = -2u - 1$

Then $dx = du,\ dy = 3du$ and $dz = -2du$.

Here,

$$\vec{F}.d\vec{r} = (2xy - z,\ yz,\ x).(dx, dy, dz)$$

$$= (2(u+1)(3u) + 2u + 1,\ 3u(-2u-1),\ u+1)(du, 3du, -2du)$$

$$= \{(6u^2 + 6u + 2u + 1,\ -6u^2 - 3u,\ u + 1).(1, 3, -2)\}\,du$$

$$= (6u^2 + 8u + 1 - 18u^2 - 9u - 2u - 2)\,du$$

$$= (-12u^2 - 3u - 1)\,du$$

Also, y moves from 0 to 3. So, y = 3u gives u moves from u = 0 to u = 1 along C_2.

So,

$$\int_{C_1} \vec{F}.d\vec{r} = \int_0^1 (-12u^2 - 3u - 1)\,du$$

$$= \left[-4u^3 - \frac{3u^2}{2} - u\right]_0^1 = -4 - \frac{3}{2} - 1 = -\frac{13}{2}$$

Now, given that C consists two lines C_1 and C_2. So,

$$\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} = 0 - \frac{13}{2} = -\frac{13}{2}$$

Evaluate the line integral along c

$\int_c (6x^2y\,dx + xy\,dy)$, where C is the graph of $y = x^3 + 1$ from (–1, 0) to (1, 2).

lution: Given integral is,

$$I = \int_c (6x^2 y dx + xy\,dy)$$

And the path of integration is c that moves along $y = x^3 + 1$ from (–1, 0) to (1, 2).

Set x = t then $y = t^3 + 1$. Then t varies from t = –1 to t = 1.

Then,

$$I = \int_{-1}^{1} [6(t^2)(t^3+1)\,dt + t(t^3+1)(3t^2\,dt)]$$

$$= \left[t^6 + 2t^3 + \frac{3t^7}{7} + \frac{3t^4}{4}\right]_{-1}^{1}$$

$$= \left(1 + 2 + \frac{3}{7} + \frac{3}{4}\right) - \left(1 - 2 - \frac{3}{7} + \frac{3}{4}\right)$$

$$= 3 + \frac{3}{7} + \frac{3}{4} + 1 + \frac{3}{7} - \frac{3}{4} = 4 + \frac{6}{7} = \frac{28+6}{7} = \frac{34}{7}.$$

$\int_c [(x - y)\,dx + xdy]$, where C is the graph of $y^2 = x$ from (4, –2) to (4, 2).

lution: Similar to (a).

Evaluate $\int_c [(xz\ dx + (y - z)\ dy + x\ dz\ ]$, if c is the graph of $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $0 \le t \le 1$.

lution: Given that,

$$I = \int_c [xz\,dx + (y + z)\,dy + x\,dz]$$

And the path of integration c is along the graph of $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $0 \le t \le 1$.

Then, $dx = e^t\,dt$, $dy = -e^{-t}\,dt$, $dz = 2e^{2t}dt$.

Now,

$$I = \int_0^1 [e^t e^{2t} e^t\,dt + (e^{-t} + e^{2t})(-e^{-t}\,dt) + e^t\,2e^{2t}\,dt]$$

$$= \int_0^1 (e^{4t} - e^{-2t} - e^t + 2e^{3t})\,dt$$

$$= \left[\frac{e^{4t}}{4} - \frac{e^{-2t}}{-2} - \frac{e^t}{1} + \frac{2e^{3t}}{3}\right]_0^1$$

$$= \left(\frac{e^4}{4} + \frac{e^{-2}}{2} - e + \frac{2e^3}{3}\right) - \left(\frac{1}{4} + \frac{1}{2} - 1 + \frac{2}{3}\right)$$

$$= \frac{1}{12}(3e^4 + 8e^3 - 12e + 6e^{-2}) - \left(\frac{3 + 6 - 12 + 8}{12}\right)$$

$$= \frac{1}{12}(3e^4 + 8e^3 - 12e + 6e^{-2} - 5).$$

K. Evaluate $\int_c [(x + y + z)\,dx + (x - 2y + 3z)\,dy + (2x + y - z)\,dz]$, where c is th[e] curve from (0, 0, 0) to (2, 3, 4) if

a. C consists of three line segments the first parallel to the x–axis the secon[d] parallel to the y–axis and the third parallel to z–axis.

b. C consists of three line segments the first parallel to the z–axis the secon[d] parallel to the x–axis and the third is parallel to the y–axis.

c. C is the line segments.

Solution: Given that,

$$I = \int_c [(x + y + z)dx + (x - 2y + 3z)\,dy + (2x + y - z)\,dz]$$

Where, the curve varies from (0, 0, 0) to (2, 3, 4).

(a) Given that the movement of the curve is along the line parallel to x– axis i.e. from (0, 0, 0) to (2, 0, 0), then along the line parallel to y–axis i.e. from (2, 0, 0) to (2, 3, 0) and then along the line parallel to z–axis i.e. from (2, 3, 0) to (2, 3, 4).

Therefore,

$$I = \left[\int_{(0,0,0)}^{(2,0,0)} + \int_{(2,0,0)}^{(2,3,0)} + \int_{(2,3,0)}^{(2,3,4)}\right] [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz]$$

$$= \int_0^2 x\,dx + \int_0^3 (2 - 2y)\,dy + \int_0^4 (4 + 3 - z)\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^2 + [2y - y^2]_0^3 + \left[7z - \frac{z^2}{2}\right]_0^4$$

$$= 2 + (6 - 9) + (28 - 8)$$

$$= 2 - 3 + 20$$

$$= 19.$$

Given that the movement of the curve is along the line parallel to z–axis i.e. from (0, 0, 0) to (0, 0, 4), then along the line parallel to y–axis i.e. from (2, 0, 4) to (2, 3, 4).

Therefore,

$$I = \int_{(0,0,0)}^{(0,0,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz] + \int_{(0,0,0)}^{(2,0,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz] + \int_{(2,0,4)}^{(2,3,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz]$$

$$= \int_0^4 (-z)dz + \int_0^2 (x + 4)dx + \int_0^3 (2 - 2y + 12)dy$$

$$= \left[\frac{-z^2}{2}\right]_0^4 + \left[\frac{x^2}{2} + 4x\right] + [14y - y^2]_0^3$$

$$= -8 + (2 + 8) + (42 - 9)$$

$$= 2 + 33$$

$$= 35.$$

Given that the movement of the path curve is along the line segments.

Since c moves fro, (0, 0, 0) to (2, 3, 4). So, $x = 2t$, $y = 3t$, $z = 4t$ for $0 \le t \le 1$.

Now,

$$I = \int_0^1 [2t + 3t + 4t)\,2dt + (2t - 6t + 12t)\,3dt + (4t + 3t - 4t)\,4dt]$$

$$= \int_0^1 [18t + 24\,t + 12t]\,dt = \int_0^1 (54t)\,dt = [27t^2]_0^1 = 27.$$

Evaluate $\int_c (xyz)$ ds, if c is the line segments from (0, 0, 0) to (1, 2, 3).

ution: Given that $I = \int_c (xyz)\,dx$.

And the curve is the line segments from (0. 0, 0) to (1, 2, 3).

Set $x = t$ then $y = 2t$, $z = 3t$. Then t varies from $t = 0$ to $t = 1$.

Since the position vector of the path is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + 2t\vec{j} + 3t\vec{k}$$

Then, $d\vec{r} = \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) dt.$

Since we know that,

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{(\vec{i}+2\vec{j}+3\vec{k}).(\vec{i}+2\vec{j}+3\vec{k})}$$
$$= \sqrt{1+4+9} = \sqrt{14}$$

Now,

$$I = \int_0^1 t.2t.3t\sqrt{14}\, dt = 6\sqrt{14}\int_0^1 t^3\, dt$$
$$= 6\sqrt{14}\left[\frac{t^4}{4}\right]_0^1 = \frac{6\sqrt{14}}{4} = \frac{3}{2}\sqrt{14}.$$

M. If the force at (x, y) is $\vec{F} = xy^2\,\vec{i} + x^2y\,\vec{j}$ find the work done by $\vec{F}$ along the curves in (J) (c).

Solution: Since the work done by force $\vec{F}$ is $\int_c \vec{F}.d\vec{r}$.

Solution is similar to the solution J.

N. The force at a point (x, y, z) in three dimensional is given by $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$. Find the work done by $\vec{F}$ along the twisted cubic $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (2, 4, 8).

Solution: Given that, $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$.

And the force $\vec{F}$ works along $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (2, 4, 8). Thus, t varies from $t = 0$ to $t = 2$. Also, we have the position vector of the curve is,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (t\vec{i} + t^2\vec{j} + t^3\vec{k})\, dt$$

Now, the work done by $\vec{F}$ is,

$$\int_c \vec{F}.d\vec{r} = \int_0^2 (t^2\vec{i} + t^3\vec{j} + t\vec{k})(t^2\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$
$$= \int_0^2 (t^2 + 2t^4 + 3t^3)\, dt = \left[\frac{t^3}{3} + \frac{2t^5}{5} + \frac{3t^4}{4}\right]_0^2.$$
$$= \frac{8}{3} + \frac{64}{5} + \frac{48}{4}$$
$$= \frac{8}{3} + \frac{64}{5} + 12 = \frac{40+192+180}{15} = \frac{412}{15}$$

Thus, the work done by $\vec{F}$ along the given curve is $\frac{412}{15}$.

. Show that following vectors are conservative field

(i) $\vec{F} = Cosy\,\vec{i} - x\,Siny\,\vec{j} - Cosz\,\vec{k}$

(ii) $\vec{F} = (y + sinz)\,\vec{i} + x\,\vec{j} + x\,cosz\,\vec{k}$ **(iii) $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$**

(iv) $\vec{F} = (2xy^2 + yz)\,\vec{i} + (2x^2y + xz + 2yz^2)\,\vec{j} + (2y^2z + xy)\,\vec{k}$

Solution: Given that,

$$\vec{F} = Cosy\,\vec{i} - x\,Siny\,\vec{j} - Cosz\,\vec{k}$$

Then, $\vec{F}$ is conservative is curl $\vec{F} = 0$

Here,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ cosy & -x\,Siny & -cosz \end{vmatrix}$$
$$= 0\,\vec{i} - o\,\vec{j} + (-Siny + Siny)\,\vec{k} = 0.$$

This shows that $\vec{F}$ is Conservative.

olution: (ii) – (iv) – Similar to (i).

. Show that following vectors are conservative and find ϕ such that $\vec{F} = \nabla\phi$.

i) $\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ **ii) $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$**

iii) $\vec{F} = (x^2 - yz)\,\vec{i} + (y^2 - zx)\,\vec{j} + (z^2 - xy)\,\vec{k}$

olution: Given that, $\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

Then, $\vec{F}$ is conservative if curl $\vec{F} = 0$.

Here,

$$\text{Curl } \vec{F} = \nabla.\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = 0$$

This shows that $\vec{F}$ is conservative.

Then, $\vec{F}$ can be written as $\vec{F} = \nabla\varnothing$. So,

$$\nabla\varnothing.d\vec{r} = \left(\frac{\delta\varnothing}{\delta x}\vec{i} + \frac{\delta\varnothing}{\delta y}\vec{j} + \frac{\delta\varnothing}{\delta y}\vec{j}\right).(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= \frac{\delta\varnothing}{\delta x}dx + \frac{\delta\varnothing}{\delta y}dy + \frac{\delta\varnothing}{\delta y}dz = d\varnothing.$$

$$\Rightarrow d\varnothing = \vec{F}.d\vec{r} = (x\,\vec{i} + y\,\vec{j} + z\,\vec{k})(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= xdx + ydy + zdz$$

Integrating we get,

$$\varnothing = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c \quad\Rightarrow\quad \varnothing = \frac{1}{2}(x^2 + y^2 + z^2) + c$$

Solution: (ii) – (iii) – Similar to (i).

Q. Show that the vector $\vec{F} = (y \sin z - \sin x)\vec{i} + (x \sin z + 2yz)\vec{j} + (xy \cos z + y^2)\vec{k}$ is irrotational and find a function and such that $\vec{F} = \nabla\phi$.

Solution: Given that,

$$\vec{F} = (y \text{ Sin}z - \text{Sin}x)\vec{i} + (x \text{ Sin}z + 2yz)\vec{j} + (xy \text{ Cos}z + y^2)\vec{k}$$

Then, $\vec{F}$ is Irrotational if curl $\vec{F} = 0$.

Here,

$$\text{Curl } \vec{F} = \nabla\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y \text{ Sin}z - \text{Sin}x & x \text{ Sin}z + 2yz & xy \text{ Cos}z + y^2 \end{vmatrix}$$

$$= (x \text{ Cos}z + 2y - x \text{ Cos}z - 2y)\vec{i} - (y \text{ Cos}z - y \text{ Cos}z)\vec{j} + (\text{Sin}z - \text{Sin}z)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

This shows that $\vec{F}$ is irroratational.

Then we can write as $\vec{F} = \nabla\varnothing$. So,

$$\vec{F}.d\vec{r} = \nabla\varnothing.d\vec{r}$$

$$\Rightarrow \vec{F}.(dx\vec{i} + dy\vec{j} + dz\vec{k}) = \left(\frac{\delta\varnothing}{\delta x}\vec{i} + \frac{\delta\varnothing}{\delta y}\vec{j} + \frac{\delta\varnothing}{\delta y}\vec{j}\right)(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\Rightarrow (y \text{ Sin}z - \text{Sin}x)dx + (x\text{Sin}z + 2yz)\,dy + (xy \text{ Cos}z + y^2)\,dz = \frac{\delta\varnothing}{\delta x}dx + \frac{\delta\varnothing}{\delta y}dy + \frac{\delta\varnothing}{\delta y}dz = d\varnothing.$$

$$\Rightarrow d\varnothing = (y \text{ Sin}z\,dx + x \text{ Sin}z\,dy + xy \text{ Cos}z\,dz) - \text{Sin}x\,dx + (2yz + y^2dz)$$

$$= d(xy \text{ Sin}z) + d(\text{Cos}x) + d(y^2z)$$

$$= d(xy \text{ Sin}z + \text{Cos}x + y^2z).$$

Integrating we get,

$$\varnothing = xy \text{ Sin}z + \text{Cos}x + y^2z + C.$$

For Exercise 4.6

Process to make the value under integral sign as under differentiation. If the integral is of type, $I = \int_a^b (F_1dx + F_2dy + F_3dz)$

And if the integral is exact. Then,

$$I = \int_a^b d[\int F_1dx + \int(\text{terms free from } x \text{ in } F_2)\,dy + \int(\text{terms free from } x \text{ and } y \text{ in } F_3)\,dz]$$

EXERCISE 4.6

...w that the value under integral sign is exact in the plane and evaluate integral.

$$\int_{(-1,5)}^{(4,3)} (3z^2\,dx + 6xz\,dz)$$ **[2009 Spring – Short]**

...tion: Given Integral is,

$$I = \int_{(-1,5)}^{(4,3)} (3z^2\,dx + 6xz\,dz) \quad \text{... (i)}$$

Here the integrand value of (i) is,

$$3z^2\,dx + 6xz\,dz \quad \text{... (ii)}$$

Comparing (ii) with $F_1dx + F_2dz$ then,

$F_1 = 3z^2$ and $F_2 = 6xz$.

Here,

$\frac{\delta F_1}{\delta z} = 6z$ and $\frac{\delta F_2}{\delta x} = 6z$

This shows that $\frac{\delta F_1}{\delta z} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1dx + \int(\text{terms free from x in } F_2)\,dz]$$

$$= \int_{(-1,5)}^{(4,3)} d(\int 3z^2\,dx) = [3xz^2]_{(-1,5)}^{(4,3)} = 108 + 75 = 183$$

Thus, $I = 183$.

$$\int_{(4,3/2)}^{(4,1/2)} (2x \text{ Sin}\pi y\,dx + \pi x^2 \text{ Cos}\pi y\,dy).$$ **[2010 Fall; 2005 Fall – Short]**

...lution: Given integral is,

$$I = \int_{(4,3/2)}^{(4,1/2)} (2x \text{ Sin}\pi y\,dx + \pi x^2 \text{ Cos}\pi y\,dy) \quad \text{....... (i)}$$

Here the integrand value of (i) is,

$$2x \text{ Sin}\pi y\,dx + \pi x^2 \text{ Cos}\pi y\,dy \quad \text{........ (ii)}$$

Comparing (ii) with $F_1dx + F_2\,dy$ then,

$F_1 = 2x \text{ Sin}\pi y$ and $F_2 = \pi x^2 - \text{Cos}\pi y$.

Here,

$\frac{\delta F_1}{\delta y} = 2\pi x \text{ Cos}\pi y$ and $\frac{\delta F_2}{\delta x} = 2\pi x \text{ Cos}\pi y$

This shows that $\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy]$$

i.e. $$I = \int_{(4,3/2)}^{(4,1/2)} d(\int 2x \sin\pi y\, dx)$$

$$= \int_{(3,3/2)}^{(4,1/2)} d(x^2\, 3\sin\pi y) = [x^2 \sin\pi y]_{(3,3/2)}^{(4,1/2)}$$

$$= 16 \sin\frac{\pi}{2} - 9 \sin\frac{3\pi}{2} = 16 + 9 = 25$$

Thus, $I = 25$.

3. $$\int_{(0,0,0)}^{(4,1,2)} (3y dx + 3x dy + 2z\, dz)$$ [2009 Fall – Short]

[2011 Fall Q.No. 6(b) OR] [2010 Spring Q.No. 6(a)] [2003 Fall Q.No. 4(b) OR]

Solution: Given integral is,

$$I = \int_{(0,0,0)}^{(4,1,2)} (3y dx + 3x dy + 2z\, dz) \quad \text{.......... (i)}$$

Here, the integrand value of (i) is,

$$3y\, dx + 3x\, dx + 2z\, dz \quad \text{.......... (ii)}$$

Comparing (ii) with $F_1 = 3y$, $F_2 = 3x$, and $F_3 = 2z$. Then,

$$\frac{\delta F_1}{\delta y} = 3,\quad \frac{\delta F_2}{\delta x} = 3,\quad \frac{\delta F_1}{\delta z} = 0,\quad \frac{\delta F_3}{\delta x} = 0,\quad \frac{\delta F_2}{\delta z} = 0,\quad \frac{\delta F_3}{\delta y} = 0.$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\ \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x},\ \text{and } \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,0,0)}^{(4,1,2)} d(\int 3y\, dx + \int 0\, dy + \int 2z\, dz)$$

$$= \int_{(0,0,0)}^{(4,1,2)} d(3xy + z^2) = [3xy + z^2]_{(0,0,0)}^{(4,1,2)} = (12 + 4) - 0 = 16.$$

Thus, $I = 16$.

$$\int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx)$$

...ution: Given integrals,

$$I = \int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx) \quad \text{.......... (i)}$$

Here the integrand value of (i) is,

$$e^{x-y+z^2}(dx - dy + 2z\, dz) \text{........... (ii)}$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = e^{x-y+z^2},\quad F_2 = -e^{x-y+z^2} \text{ and } F_3 = 2ze^{x-y+z^2}$$

Then,

$$\frac{\delta F_1}{\delta y} = -e^{x-y+z^2},\quad \frac{\delta F_2}{\delta x} = -e^{x-y+z^2},\quad \frac{\delta F_3}{\delta x} = -2ze^{x-y+z^2},$$

$$\frac{\delta F_1}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_2}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_3}{\delta y} = -2ze^{x-y+z^2}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\quad \frac{\delta F_3}{\delta x} = \frac{\delta F_1}{\delta z},\quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,0,0)}^{(4,1,2)} d(\int e^{x-y+z^2} dx + \int 0\, dy + \int 0\, dz)$$

$$I = \int_{(0,0,0)}^{(4,1,2)} d(e^{x-y+z^2}) = [e^{x-y+z^2}]_{(0,0,0)}^{(2,4,0)} = e^{2-4+0} - e^{0-0+0}$$

$$= e^{-2} - e^0 = e^{-2} - 1$$

Thus, $I = e^{-2} - 1$.

$$\int_{(0,2,3)}^{(1,1,1)} [yz \sinh(xz)\, dx + \cosh(xz)\, dy + xy \sinh(xz)\, dz]$$

...lution: Given integral is,

$$I = \int_{(0,2,3)}^{(1,1,1)} [yz \sinh(xz)\, dx + \cosh(xz)\, dy + xy \sinh(xz)\, dz] \text{ .... (i)}$$

Here, the integrand value is,

$$yz \sinh(xz)\, dx + \cosh(xz)\, dy + xy \sinh(xz)\, dz \text{ .... (ii)}$$

Then,

$$\frac{\delta F_1}{\delta y} = z \text{ Sinh }(xz), \quad \frac{\delta F_2}{\delta x} = z \text{ Sinh }(xz), \quad \frac{\delta F_1}{\delta z} = y \text{ Sinh }(xz) + xyz \text{ Cosh }(xz),$$

$$\frac{\delta F_2}{\delta z} = x \text{ Sinh }(xz), \quad \frac{\delta F_3}{\delta y} = x \text{ Sinh }(xz)$$

$$\frac{\delta F_3}{\delta x} = y \text{ Sinh }(xz) + xyz \text{ Cosh }(xz),$$

This shows that

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}, \quad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}.$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,2,3)}^{(1,1,1)} d[\int yz \text{ Sinh }(xz)\, dx + \int 0\, dy + \int 0\, dz]$$

$$= \int_{(0,2,3)}^{(1,1,1)} d(y \text{ Cosh}(xz)) = [y \text{ Cosh}(xz)]_{(0,2,3)}^{(1,1,1)}$$

$$= \text{Cosh } 1 - 2 \text{ Cosh } 0. = \text{Cosh } 1 - 2.$$

Thus, $I = \text{Cosh } 1 - 2$.

6. $$\int_{(0,0,1)}^{(1,\pi/4,2)} [2xyz^2 dx + (x^2z^2 + z \text{ Cosyz})\, dy + (2x^2yz + y \text{ Cosyz})\, dz]$$

Solution: Given integral is,

$$I = \int_{(0,0,1)}^{(1,\pi/4,2)} [2xyz^2 dx + (x^2z^2 + z \text{ Cosyz})\, dy + (2x^2yz + y \text{ Cosyz})\, dz] \quad \text{..... (i)}$$

Here, the integrand value of (i) is,

$$2xyz^2 dx + 6x^2z^2 + z \text{ Cosyz})\, dy + (2x^2yz + y \text{ Cosyz})\, dz \quad \text{...... (ii)}$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = 2xyz^2, \quad F_2 = x^2z^2 + z \text{ Cosyz}, \quad F_3 = 2x^2yz + y \text{ Cosyz}$$

Then,

$$\frac{\delta F_1}{\delta y} = 2xz^2, \quad \frac{\delta F_2}{\delta x} = 2xz^2, \quad \frac{\delta F_1}{\delta z} = 4xyz, \quad \frac{\delta F_3}{\delta x} = 4xyz,$$

$$\frac{\delta F_2}{\delta z} = 2x^2z + \text{Cosyz} + yz \text{ Cosyz}, \quad \frac{\delta F_3}{\delta y} = 2x^2z + \text{Cosyz} + yz \text{ Cosyz}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}, \quad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So the integrand value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,0,0)}^{(1,\pi/4,2)} d[\int 2xyz^2\, dx + \int z \text{ cosyz}\, dy + \int 0\, dz]$$

$$I = \int_{(0,0,0)}^{(1,\pi/4,2)} ((2xyz^2\, dx + x^2 z^2 dy + 2x^2yz\, dz + z \text{ Cosyz}\, dy + y \text{ Cosyz}\, dz)$$

$$= \int_{(0,0,0)}^{(1,\pi/4,2)} d(x^2yz^2 + \text{Sinyz}) = [x^2yz + \text{Sinyz}]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$= \frac{4\pi}{4} + 3 \text{ Sin}\frac{\pi}{2} - 0 - \text{Sin}0 = \pi + 1.$$

Thus, $I = \pi + 1$.

$$\int_{(0,1)}^{(2,3)} [(2x + y^3)\, dx + (3xy^2 + 4)\, dy]$$ [2009 Spring – Short]

Solution: Here,

$$I = \int_{(0,1)}^{(2,3)} [(2x + y^3)\, dx + (3xy^2 + 4)\, dy] \quad \text{............ (i)}$$

The integrand value of (i) is,

$$(2x + y^3)\, dx + (3xy^2 + 4)\, dy \quad \text{............ (ii)}$$

Comparing (ii) with $F_1\, dx + F_2\, dy$ then we get,

$$\frac{\delta F_1}{\delta y} = 3y^2 \quad \text{and} \quad \frac{\delta F_2}{\delta x} = 3y^2$$

This shows that $\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy]$$

i.e. $$I = \int_{(0,1)}^{(2,3)} d[\int(2x + y^3)\, dx + \int 4\, dy]$$

$$I = \int_{(0,1)}^{(2,3)} (2x\, dx + (y^3 dx + 3xy^2 dy) + 4dy)$$

$$= \int_{(0,1)}^{(2,3)} d(xy^3 + x^2 + 4y) = [xy^3 + x^3 + 4y]_{(0,1)}^{(2,3)}$$

$$= (54 + 4 + 12) - (0 + 0 + 4) = 70 - 4 = 66.$$

Thus, I = 66.

8. $$\int_{(-1,2)}^{(3,1)} [(y^2 + 2xy)\, dx + (x+2+2xy)\, dy]$$

Solution: Similar to 7.

9. $$\int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)\, dx + 9x^2y^2\, dy + (4xz + 1)\, dz]$$

Solution: Similar to 6.

10. $$\int_{(0,1,1/2)}^{(\pi/2,3,2)} [y^2 \cos x\, dx + (2y \sin x + e^{2z})\, dy + 2ye^{2z}\, dz]$$ **[2012 Fall Q.No. 4(a) OR]**

Solution: Here,

$$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} [y^2 \text{Cos}x\, dx + (2y\, \text{Sin}x + e^{2z})\, dy + 2y\, e^{2z}\, dz] \quad \text{........ (i)}$$

The integrand value of (i) is,

$$y^2\, \text{Cos}x\, dx + (2y\, \text{Sin}x + e^{2z})\, dy + 2y\, e^{2z}\, dz$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = y^2\, \text{Cos}x, \qquad F_2 = 2y\, \text{Sin}x + e^{2z}, \qquad F_3 = 2ye^{2z}$$

Then,

$$\frac{\delta F_1}{\delta y} = 2y\text{Cos}x,\ \frac{\delta F_2}{\delta x} = 2y\, \text{Cos}x,\ \frac{\delta F_1}{\delta z} = 0,\ \frac{\delta F_3}{\delta x} = 0,\ \frac{\delta F_2}{\delta z} = 2e^{2z},\ \frac{\delta F_3}{\delta y} = 2e^{2z}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}, \qquad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \qquad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} d[\int y^2\, \text{Cos}x\, dx + \int e^{2z}\, dy + \int 0\, dz]$$

$$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} (y^2\, \text{Cos}x\, dx + 2y\, \text{Sin}x\, dy + e^{2z}\, dz)$$

$$= \int_{(0,1,1/2)}^{(\pi/2,3,2)} d\,(y^2\, \text{Sin}x + ye^{2z}) = [y^2\text{Sin}x + ye^{2z}]_{(0,1,1/2)}^{(\pi/2,3,2)}$$

$$= (9\, \text{Sin}\tfrac{\pi}{2} + 3\, e^4) - (\text{Sin}0 + e^1) = 3e^4 + 9 - e$$

Thus, $I = 3e^4 - e + 9$.

EXERCISE 4.7

Using Greens theorem, evaluate the following integrals:

$\oint_c$ (ydx + 2xdy), C: the boundary of the square $0 < x < 1$, $0 < y < 1$ (counterclockwise).

ution: Given that, the integral is,

$$I = \oint_c (y\, dx + 2x\, dy) \quad \text{......... (i)}$$

where, c is the path $0 \le x \le 1$, $0 \le y \le 1$ (in counter clockwise).

Comparing the given integral I with the integral $\oint_c [F_1\, dx + F_2\, dy]$ then we get,

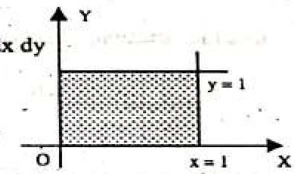
$$F_1 = y \text{ and } F_2 = 2x$$

By Green's theorem we have,

$$\oint_c [F_1\, dx + F_2\, dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \int_0^1\int_0^1 (2-1)\, dx\, dy$$

$$= \int_0^1\int_0^1 1\, dx\, dy = \int_0^1 dy = 1.$$



Thus, $\oint_c$ ydx+ 2x dy = 1 for $0 \le x \le 1$, $0 \le y \le 1$.

$\oint_c [2xy\, dx + (e^x + x^2)\, dy]$, C: the boundary of the triangle with vertices (0, 0), (1, 0), (1, 1) (clockwise).

Solution: Given that,

$$I = \oint_c [2xy\,dx + (e^x + x^2)\,dy] \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

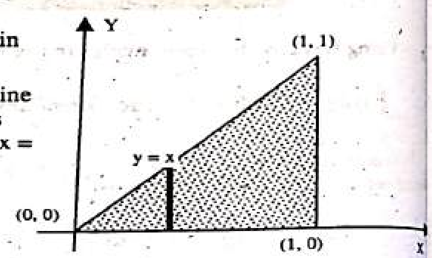
And the region is bounded by a triangle having vertices (0, 0), (1, 0) and (1, 1) in clockwise direction.

Comparing the given integral I with the integral $\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 2xy \quad \text{and} \quad F_2 = e^x + x^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$



Since the region of I is shown in figure in which has counterclockwise direction.

In the figure y varies from y = 0 to the line joining (0, 0) and (1, 1). That is y varies from y = 0, to y = x. and x moves from x = 0 to x = 1.

Then, (i) becomes,

$$\oint_c [2xy\,dx + (e^x + x^2)\,dy]$$

$$= \int_0^1 \int_0^x e^x dy dx = \int_0^1 e^x [y]_0^x\,dx = [xe^x - e^x]_0^1 = (e - e) - (0 - 1) = 1$$

Thus, $\oint_c [2xy\,dx + (e^x + x^2)\,dy] = 1.$

Since the direction of the force is in clockwise. So,

$$\oint_c [2xy\,dx + (e^x + x^2)\,dy] = -1.$$

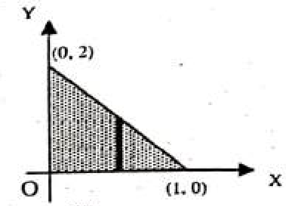
3. $\oint_c [(3x^2 + y)\,dx + 4y^2\,dy]$, **C: the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2): counterclockwise.**

[2009 Spring Q.No. 4(a); 2006 Spring Q.No. 4(a) OR]

Solution: Given that,

$$I = \oint_c [(3x^2 + y)\,dx + 4y^2\,dy] \quad \ldots\ldots\ldots\ldots \text{(i)}$$

And the region is the triangle having vertices (0, 0), (1, 0) and (0, 2) in counter wise direction.



Comparing the given integral I with the integral $\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 3x^2 + y \quad \text{and} \quad F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \ldots\ldots \text{(ii)}$$

In the figure, y varies from y = 0 to the line joining points (1, 0) and (0, 2). That is, y varies from y = 0 to y = −2x + 2. And x moves from x = 0 to x = 1.

Then (i) becomes,

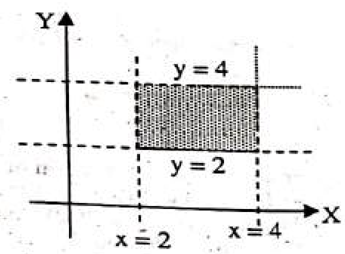
$$I = \iint_R [0 - 1]\,dA = \int_0^1 \int_0^{2-2x} (-1)\,dy\,dx = -\int_0^1 (2 - 2x)\,dx$$

$$= -[2 - 2x]_0^1$$

$$= -(2 - 1) = -1$$

Thus, $\oint_c [(3x^2 + y)dx + 4y^2\,dy] = -1.$

4. $\oint_c (x^2 + y^2)\,dy$, **C: the boundary of the of the square $2 \le x \le 4, 2 \le y \le 4$.**



Solution: Given that,

$$I = \oint_c (x^2 + y^2)dy \quad \ldots\ldots \text{(i)}$$

And the boundary of c are $2 \le x \le 4, 2 \le y \le 4$.

Comparing the given integral I with the integral

$\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 3x^2 + y \quad \text{and} \quad F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \ldots\ldots \text{(ii)}$$

Now (i) becomes,

$$I = \oint_c (x^2 + y^2)dy = \iint_R (2x)\,dA \quad [\because F_1 = 0]$$

$$= \int_2^4\int_2^4 2x\,dxdy \quad [\because \text{ using the boundaries}]$$

$$= \int_2^4 [x^2]_2^4 = 12\int_2^4 dy = 12 \times (4-2) = 24$$

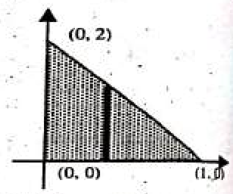
Thus, $\oint_c (x^2 + y^2)\,dy = 24.$

5. $\oint_c [(x^3-3y)\,dx + (x+\text{Siny})\,dy]$, **C: the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2).**

Solution: Given that,

$$I = \oint_c [(x^3 - 3y)\,dx + (x + \text{Siny})\,dy] \quad \ldots \text{(i)}$$

And the boundaries of has vertices (0, 0), (1, 0) and (0, 2).
Comparing the given integral I with the integral



$\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 3x^2 + y \text{ and } F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx \quad \ldots\ldots\text{(ii)}$$

From the figure, the region of integration (path) of $\vec{F}$ has boundaries with vertices at (0, 0), (1, 0) and (0, 2). On the region y varies from y = 0 to y = 2 – 2x (line joining the points (1, 0) and (0, 2). And x moves from x = 0 to x = 1.
Therefore, (iii) becomes,

$$I = 4\int_0^1\int_0^{2-2x} dydx = 4\int_0^1 (2-2x)\,dx = 4\,[2x - x^2]_0^1 = 4(2-1) = 4.$$

Thus, $\oint_c [(x^3 - 3y)\,dx + (x + \text{siny})\,dy] = 4$

B. Using Green's theorem, evaluate the live integral $\oint_c \vec{F}(r).d\vec{r}$ counterclockwis around the boundary C of the region R, where

1. $\vec{F} = (x^2e^y, y^2e^x)$, **C the rectangle with vertices (0, 0), (2, 0), (2, 3), (0, 3).**
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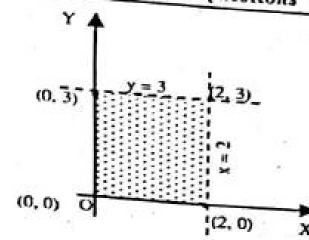
Solution: Given that.

$$\vec{F} = (x^2e^y, y^2e^x)$$



Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2e^y & y^2e^x & 0 \end{vmatrix}$$

$$= (y^2e^x - x^2e^y)\vec{k}$$

So,

$$\text{Curl } \vec{F}.\vec{k} = (y^2e^x - x^2e^y)\vec{k}.\vec{k} = y^2e^x - x^2e^y$$

Now, by Green's theorem, we have,

$$\oint_c \vec{F}\,d\vec{r} = \iint_R (\text{curl }\vec{F}.\vec{k})\,dA = \iint_R (y^2e^x - x^2e^y)\,dA \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Given that the path of $\vec{F}$ is C: the rectangle having vertices (0, 0), (2, 0), (2, 3) and (0, 3).
From the figure, y varies from y = 0 to y = 3 and x moves from x = 0 to x = 2
Therefore (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = \int_0^2\int_0^3 (y^2e^x - x^2e^y)\,dydx$$

$$= \int_0^2 \left[\frac{y^3e^x}{3} - x^2e^y\right]_0^3 dx = \int_0^2 (9e^x - x^2e^3 + x^2)\,dx$$

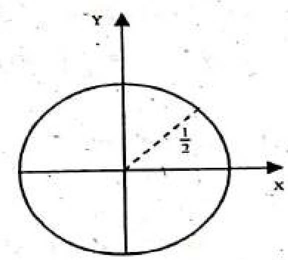
$$= \left[9e^x - \frac{x^3}{3}e^3 + \frac{x^3}{3}\right]_0^2$$

$$= 9e^2 - \frac{8}{3}e^3 + \frac{8}{3} - 9$$

$$= 9(e^2 - 1) + \frac{8}{3}(1 - e^3)$$

Thus, $\oint_c \vec{F}.d\vec{r} = 9\,(e^2 - 1) + \frac{8}{3}(1 - e^3).$

$\vec{F} = (y, -x)$, C the circle $x^2 + y^2 = \frac{1}{4}$



olution: Given that,

$$\vec{F} = (y, -x)$$

Then,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & -x & 0 \end{vmatrix} = (-1-1)\,\vec{k} = -2\vec{k}$$

And,

$$\text{Curl } \vec{F}.\vec{k} = -2\vec{k}.\vec{k} = -2$$

By Green's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = -2\iint_R dA \quad \text{.......... (i)}$$

Given that the of $\vec{F}$ is $x^2 + y^2 = \frac{1}{4}$. That is the path is a circle having radius $\frac{1}{2}$. So, changing the Cartesian from to polar with $x = r\cos\theta$ and $y = r\sin\theta$. Then $dxdy = rdrd\theta$.

Also, radius of region is $r = \frac{1}{2}$. And the angle θ varies from $\theta = 0$ to $\theta = 2\pi$.

Therefore, (i) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\int_0^{1/2}\int_0^{2\pi} r\,drd\theta = -2\int_0^{1/2} r.\,2\pi dr = -4p\left[\frac{r^2}{2}\right]_0^{1/2} = -4\pi.\frac{1}{8} = \frac{-\pi}{2}$$

Thus, $\oint_C \vec{F}.d\vec{r} = \frac{-\pi}{2}$

3. $\vec{F}$ = grad (sinx.cosy), C is the ellipse $25x^2 + 9y^2 = 225$.

Solution: Given that,

$$\vec{F} = \text{grad}\,(\sin x \cos y)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{i}\frac{\partial}{\partial y}\right)(\sin x \cos y) = \cos x \cos y\,\vec{i} - \sin x \sin y\,\vec{j}$$

So,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \cos x \cos y & -\sin x \sin y & 0 \end{vmatrix}$$

$$= (-\cos x \sin y + \cos x \sin y)\,\vec{k} = 0\,\vec{k}.$$

Therefore, $\quad \text{Curl}\,\vec{F}.\vec{k} = 0$

By Green's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = \iint_R 0\,dA = 0.$$

Thus, $\oint_C \vec{F}.d\vec{r} = 0.$

4. $\vec{F} = (\tan 0.2x, x^5 y)$, R: $x^2 + y^2 \le 25$, $y \ge 0$.

Solution: Given that,

$$\vec{F} = (\tan 0.2x, x^5 y)$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \tan 0.2x & x^5 y & 0 \end{vmatrix} = 5x^4 y\,\vec{k}$$

So,

$$\text{Curl}\,\vec{F}.\vec{k} = 5x^4 y\,\vec{k}.\vec{k} = 5x^4 y$$

By Green's theorem, we have,

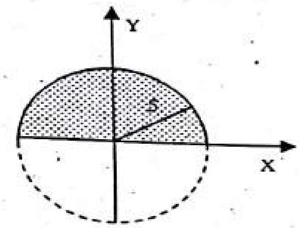
$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = 5\iint_R (x^4 y)\,dA \quad \text{........ (i)}$$

Given that the path of $\vec{F}$ is in the region $x^2 + y^2 \le 25$, $y \ge 0$. Clearly the region is a half circle having radius $r = 5$.

Thus, $r = 5$ and θ varies from $\theta = 0$ to $\theta = \pi$.

Transforming the coordinate in to polar from then,

$x = r\cos\theta$, $y = r\sin\theta$ and $dydx = rdrd\theta$.



Then, (i) becomes,

$$\oint_C \vec{F}.d\vec{r} = \int_0^5\int_0^\pi 5r^4\cos^4\theta\; r\sin\theta.\; r\,d\theta\,dr$$

$$= \int_0^5\int_0^\pi 5r^6\cos^4\theta\,\sin\theta\,d\theta\,dr$$

Put $\cos\theta = u$ then $-\sin\theta\,d\theta = du$. Also, $\theta = 0 \Rightarrow u = 1$, $\theta = \pi \Rightarrow u = -1$

Then,

$$\oint_C \vec{F}.d\vec{r} = -\int_0^5 5r^6 \int_1^{-1} u^4\,dudr$$

$$= -5\int_0^5 r^6\left[\frac{u^5}{5}\right]_1^{-1} dr = -5\int_0^5 r^6\left(\frac{-1-1}{5}\right) dr$$

$$= 5\times\frac{2}{5}\left[\frac{r^7}{7}\right]_0^5 = \frac{2\times 5^7}{7}.$$

Thus, $\oint_C \vec{F}.d\vec{r} = \frac{2\times 5^7}{7}.$

$\vec{F} = [\frac{e^y}{x}\; e^y \log x + 2x)$, R: $1 + x^4 \le y \le 2$.

lution: Given that,

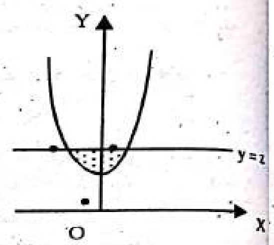
$$\vec{F} = \left(\frac{e^y}{x}, e^y \log x + 2x\right)$$

Then,

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{e^y}{x} & e^y \log x + 2x & 0 \end{vmatrix} = \left(\frac{e^y}{x} + 2 - \frac{e^y}{x}\right)\vec{k} = 2\vec{k}$$

So, $\text{Curl } \vec{F} . \vec{k} = 2\vec{k} . \vec{k} = 2$

Now, by Green's theorem we have,



$$\oint_c \vec{F} d\vec{r} = \iint_R (\text{curl } \vec{F} . \vec{k}) dA = 2 \iint_R dA \quad \text{... (i)}$$

Also, given that the path of region of $\vec{F}$ is $1 + x^4 \le y \le 2$

For the curve $1 + x^4 = y$

x	0	± 1	± 2
y	1	2	17

And the curve $y = 2$ is a straight line.

From the figure, the region is bounded by $1 + x^2 \le y \le 2$ and solving the curves $y = 2$ and $y = x^2 + 1$ then we get $x = \pm 1$.

Now, (i) becomes,

$$\oint_c \vec{F} . d\vec{r} = 2 \int_{-1}^{1} \int_{1+x^4}^{2} dy\, dx$$

$$= 2 \int_{-1}^{1} [y]_{1+x^4}^{2} dx = 2 \int_{-1}^{1} (2 - 1 - x^4)\, dx$$

$$= 2 \int_{-1}^{1} (1 - x^4)\, dx$$

$$= 2\left[x - \frac{x^5}{5}\right]_{-1}^{1} = \left[\left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right)\right]$$

$$= 2\left(2 - \frac{2}{5}\right)$$

$$= 4\left(5 - \frac{1}{5}\right) = \frac{16}{5}.$$

Thus, $\oint_c \vec{F} . d\vec{r} = \frac{16}{5}$.

C. Use Green's theorem to evaluate the line integrals:

1. $\oint_c [x^2 + y^2) dx + xy^2 dy]$; **where C is the closed curve determined by $y^2 = x$ and $y = -x$ with $0 \le x \le 1$.**

Solution: Given that,

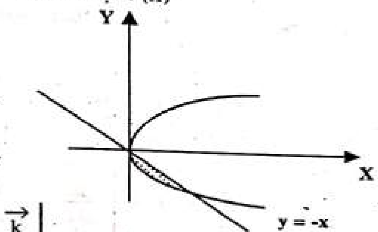
$$I = \oint_c [(x^2 + y^2)\, dx + xy^2 dy] \quad \text{............ (i)}$$

Where, the path c is determined by $y^2 = x$ and $y = -x$ for $0 < x < 1$.

Clearly, $y^2 = x$ is a parabola having vertex at (0, 0) and line of symmetry is $y = 0$. And, the line $y = -x$ passes through (0, 0) and (1, −1). From the figure, y varies from $y = -\sqrt{x}$ to $y = -x$. And x moves $x = 0$ to $x = 1$.

By Green's theorem, we have,

$$\oint_c \vec{F} d\vec{r} = \iint_R (\text{curl } \vec{F} . \vec{k}) dA \quad \text{.............. (ii)}$$



Comparing (i) with $\oint_c \vec{F} . d\vec{r}$ then, we get,

$$\vec{F} = (x^2 + y^2)\vec{i} + xy^2\vec{j}$$

Then,

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 + y^2 & xy^2 & 0 \end{vmatrix} = (y^2 - 2y)\vec{k}$$

So,

$$\text{Curl } \vec{F} . \vec{k} = (y^2 - 2y)\vec{k} . \vec{k} = y^2 - 2y$$

Then (ii) becomes,

$$\oint_c [(x^2 + y^2) dx + xy^2 dy] = \int_0^1 \int_{-\sqrt{x}}^{-x} (y^2 - 2y)\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^3}{3} - y^2\right]_{\sqrt{x}}^{-x} dx$$

$$= \int_0^1 \left(\frac{-x^3}{3} - x^2 + \frac{x\sqrt{x}}{3} + x\right) dx$$

$$= \left[\frac{-x^4}{12} - \frac{x^3}{3} + \frac{x^{5/2}}{15/2} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{-1}{12} - \frac{1}{3} + \frac{2}{15} + \frac{1}{2}$$

$$= \frac{-5 - 20 + 8 + 30}{60} = \frac{13}{60}.$$

Thus, $I = \frac{13}{60}$.

2. $\oint_C [x^2y^2\,dx + (x^2 - y^2)\,dy]$; where C is the square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

Solution: Given that,

$$I = \oint_C [x^2y^2 + (x^2 - y^2)\,dy] \quad \ldots\ldots\ldots\ldots (i)$$

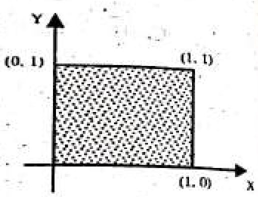
With C is a square having vertices (0, 0), (1, 0), (1, 1) and (0, 1)

Comparing (i) with $\oint_C \vec{F}.d\vec{r}$ then, we get,

$$\vec{F} = x^2y^2\,\vec{i} + (x^2 - y^2)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2y^2 & x^2 - y^2 & 0 \end{vmatrix} = (2x - 2x^2y)\,\vec{k}$$



So,

$$\text{Curl}\,\vec{F}.\vec{k} = (2x - 2x^2y)\,\vec{k}.\vec{k} = (2x - 2x^2y)$$

Now, by Green's theorem,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA$$

So,

$$\oint_C [x^2y^2dx + (x^2 - y^2)\,dy] = \iint_R (2x - 2x^2y)\,dA \quad \ldots\ldots (ii)$$

Given that the region of the force is the square shown in figure. In which, y varies from y = 0 to the line joining the points (0, 1) and (1, 1). That is, from y = 0 to y = 1. And x moves from x = 0 to x = 1.

Therefore (ii) becomes,

$$\oint_C [x^2y^2dx + (x^2 - y^2)\,dy] = \int_0^1\int_0^1 (2x - 2x^2y)\,dy\,dx$$

$$= \int_0^1 [2x - 2x^2y]_0^1\,dx = \int_0^1 (2x - x^2)\,dx$$

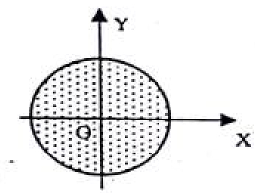
$$= \left[x^2 - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus, $I = \frac{2}{3}$.

$\oint_C xy\,dx + (y + x)\,dy)]$, where C is the circle $x^2 + y^2 = 1$.

Solution: Given that,

$$I = \oint_C [xy\,dx + (y + x)\,dy] \quad \ldots\ldots\ldots (i)$$



where c is a circle $x^2 + y^2 = 1$

Comparing (i) with $\oint_C \vec{F}.d\vec{r}$, then we get,

$$\vec{F} = xy\,\vec{i} + (y + x)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy & y + x & 0 \end{vmatrix} = (1 - x)$$

Then,

$$\text{Curl}\,\vec{F}.\vec{k} = (1 - x)\,\vec{k}.\vec{k} = 1 - x$$

By Green's theorem,

$$\oint_C \vec{F}\,d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA$$

So,

$$\oint_C [xy\,dx + (y + x)\,dy] = \iint_R (1 - x)\,dA$$

$$= \int_0^1\int_0^{2\pi} (1 - r\cos\theta)\,r\,d\theta\,dr \text{ [Changing in polar form]}$$

$$= \int_0^1 [\theta - r\sin\theta]_0^{2\pi}\,r\,dr = \int_0^1 2\pi r\,dr \;[\because \sin 2\pi = 0 = \sin 0]$$

$$= [\pi r^2]_0^1 = \pi.$$

Thus, $I = \pi$.

4. $\oint_C [xy\,dx + \sin y\,dy]$, where C is the triangle with vertices (1, 1), (2, 2), (3, 0).

Solution: Given that,

$$I = \oint_C (xy\,dx + \sin y\,dy) \quad \ldots\ldots\ldots\ldots (i)$$

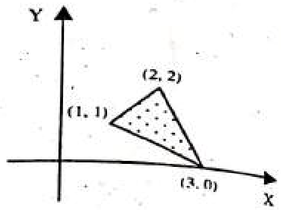
with c is a triangle having vertices at (1, 1), (2, 2) and (3, 0).

Comparing (i) with $\oint_c \vec{F}\,d\vec{r}$ then we get,

$$\vec{F} = xy\,\vec{i} + \sin y\,\vec{j}$$

So,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy & \sin y & 0 \end{vmatrix} = -x\,\vec{F}$$



Then $\quad \text{Curl}\,\vec{F}\,.\,\vec{k} = -x\,\vec{k}\,.\,\vec{k} = -x$

Now, by Green's theorem,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}\,.\,\vec{k})dA$$

So, $\quad I = \iint_R (-x)\, dA \qquad \text{.........} \text{ (ii)}$

Since the region C is shown in figure.

Here, the equation of line joining (1, 1) and (2, 20 is, y = x.

The equation of line joining (1, 1) and (3, 0) is, $y = \frac{-1}{2}(x - 3)$.

The equation of line joining (2, 2) and (3, 0) is, $y = 6 - 2x$.

From the figure, C is bounded from $y = \frac{3-x}{2}$ to $y = x$ in which x moves from $x = 1$ to $x = 2$. And the region is moves from $x = 2$ to $x = 3$ in which it is bounded by the lines $y = \frac{3-x}{2}$ to $y = 6 - 2x$.

Therefore, (ii) becomes,

$$I = -\int_1^2 \int_{[(3-x)/2]}^{x} dx - \int_2^3 \int_{[(3-x)/2]}^{6-x} x\, dy\, dx$$

$$= -\int_1^2 x\,[y]_{[(3-x)/2]}^{6-x} dx - \int_2^3 [y]_{[(3-x)/2]}^{6-x} dx$$

$$= -\int_1^2 x\left(x - \frac{3-x}{2}\right) dx - \int_2^3 x\left(6 - 2x - \frac{3-x}{2}\right) dx$$

$$= -\int_1^2 \left(\frac{2x^2 - 3x + x^2}{2}\right) dx - \int_2^3 \left(\frac{12x - 4x^2 - 3x + x^2}{2}\right) dx$$

$$I = \oint_c [(x + y)\, dx + (y + x^2)\, dy] \qquad \text{.......... (i)}$$

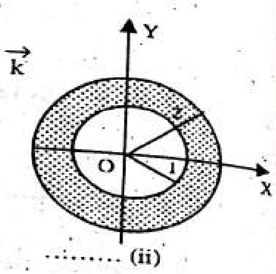
Where C is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Comparing (i) with $\oint_c \vec{F}\, d\vec{r}$ then we get,

$$\vec{F} = (x + y)\,\vec{i} + (y + x^2)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x+y & y+x^2 & 0 \end{vmatrix} = (2x - 1)\,\vec{k}$$



So,

$$\text{Curl}\,\vec{F}.\vec{k} = (2x - 1)\,\vec{k}.\vec{k} = 2x - 1$$

Since, by Green's theorem we have,

$$\oint_c \vec{F}\, d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\, dA = \iint_R (2x - 1)\, dA \qquad \text{......... (ii)}$$

Given that the force $\vec{F}$ works on the region in between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Clearly the first circle has radius 1 and second has radius 2.
Therefore, the feasible region is in between r = 1 to r = 2.
Also, the region moves from $\theta = 0$ to $\theta = 2\pi$.
Therefore changing the integrand in (ii) in to polar from as $x = r\cos\theta$ and $dxdy = r dr d\theta$.
So that,

$$\oint_c \vec{F}\, d\vec{r} = \int_0^{2\pi}\int_1^2 (2r\cos\theta - 1)\, r dr d\theta$$

$$= \int_1^2\int_0^{2\pi} (2r^2 \text{Cos}\theta - r)\, d\theta\, dr$$

$$= \int_1^2 [2r^2\sin\theta - r\theta]_0^{2\pi}\, dr = -2\pi\int_1^2 r\, dr \quad [\because \sin 2\pi = \sin 0]$$

$$= -\pi[r^2]_1^2 = -\pi\,(4 - 1) = -3\pi.$$

Thus, $\oint_c [(x + y)\, dx + (y + x^2)\, dy] = -3\pi$

$\int$ [15xy dx + x^3 dy], where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points (0, 0) and (2, 4).

ion: Given that,

$$I = \oint_c (5xy\, dx + x^3\, dy) \qquad \text{.............. (i)}$$

Where c is the closed curve obtained by the graph of the curve $y = x^2$ and $y = 2x$ in between (0, 0) to (2, 4).

Comparing (i) with $\oint_c \vec{F}\, d\vec{r}$ then we get,

$$\vec{F} = 5xy\,\vec{i} + x^3\,\vec{j}$$

So,

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 5xy & x^3 & 0 \end{vmatrix} = (3x^2 - 5x)\,\vec{k}$$

Then, $\text{Curl}\,\vec{F}.\vec{k} = (3x^2 - 5x)\,\vec{k}\,\vec{k} = 3x^2 - 5x$

Since by Green's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\, dA = \iint_R (3x^2 - 5x)\, dA \qquad \text{........... (ii)}$$

Given that $\vec{F}$ work on the region of common part of $y = x^2$ and $y = 2x$ in between (0, 0) to (2, 4).

Therefore, (ii) becomes,

$$I = \iint_R (3x^2 - 5x)\, dA$$

$$= \int_0^2\int_{2x}^{x^2} (3x^2 - 5x)\, dy\, dx = \int_0^2 [3x^2y - 5x^3]_{2x}^{x^2}\, dx$$

$$= \int_0^2 [(3x^4 - 5x^3) - (6x^3 - 10x^2)]\, dx$$

$$= \int_0^2 (3x^4 - 11x^3 + 10x^2)\, dx$$

$$= \left[\frac{3x^5}{5} - \frac{11x^4}{4} + \frac{10x^3}{3}\right]_0^2$$

$$= \frac{96}{5} - \frac{11\times 16}{4} + \frac{80}{3} = \frac{288 - 660 + 400}{15} = \frac{28}{15}$$

Thus, $I = \frac{28}{15}$.

8. $\oint_C [2xy\,dx + (x^2+y^2)\,dy]$, where C is the ellipse $4x^2 + 9y^2 = 36$.

Solution: Given that,

$$I = \oint_C [2xy\,dx + (x^2+y^2)\,dy] \quad \text{......... (i)}$$

where C is $4x^2 + 9y^2 = 36$.

Comparing (i) with $\oint_C \vec{F}.d\vec{r}$ then we get,

$$\vec{F} = 2xy\,\vec{i} + (x^2+y^2)\,\vec{j}$$

Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xy & x^2+y^2 & 0 \end{vmatrix} = (2x-2x)\,\vec{k} = 0\,\vec{k}$$

So, Curl $\vec{F}.\vec{k} = 0\,\vec{k}.\vec{k} = 0$

By Green's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA$$

So, $\oint_C [2xy\,dx + (x^2+y^2)\,dy] = \iint_R 0\,dA = 0.$

EXERCISE 4.8

Evaluate $\iint_S \vec{F}.\vec{n}\,dA$, where

1. $\vec{F} = (3x^2, y^2, 0)$, S: $\vec{r} = (u, v, 2u + 3v)$, $0 \le u \le 2, -1 \le v \le 1$.

Solution: Given that,

$$\vec{F} = (3x^2, y^2, 0) = 3x^2\vec{i} + y^2\vec{j} + 0\,\vec{k}$$

And $\vec{r} = (u, v, 2u+3v) = u\,\vec{i} + v\,\vec{j} + (2u+3v)\,\vec{k}$.

Then, $\vec{r_u} = (\vec{i} + 2\vec{k})$ and $\vec{r_v} = \vec{j} + 3\,\vec{k}$.

So that,

$$\vec{N} = \vec{r_u}\,\vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2\vec{i} - 3\vec{j} + \vec{k}.$$

Since we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iint_R \vec{F}(\vec{r}).\vec{N}\,du\,dv \quad \text{......... (i)}$$

Since, $\vec{r} = x\,\vec{i} - y\,\vec{j} + z\,\vec{k}$. And given that, $\vec{r} = u\,\vec{i} + v\,\vec{j} + (2u+3v)\,\vec{k}$.

So that, $\vec{F} = 3x^2\,\vec{i} + y^2\,\vec{j}$

This implies that,

$$\vec{F}(\vec{r}).\vec{N} = (3u^2\,\vec{i} + v^2\,\vec{j}).(-2\,\vec{i} - 3\,\vec{j} + \vec{k}) = -6u^2 - 3v^2$$

Also, given that the region is $0 \le u \le 2$, $-1 \le v \le 1$.

Thus, (i) become,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_{-1}^{1}\int_0^2 (-6u^2 - 3v^2)\,du\,dv$$

$$= \int_{-1}^{1} [-2u^3 - 3v^2u]_0^2\,dv = \int_{-1}^{1} (-16 - 6v^2)\,dv$$

$$= [-16v - 2v^3]_{-1}^{1}$$

$$= (-16-2) - (16+2) = -18 - 18 = -36$$

Thus, $\iint_S \vec{F}.\vec{n}\,dA = -36.$

$\vec{F} = (e^{2y}, e^{-2z}, e^{2x})$, S : $\vec{r} = (3\cos u, 3\sin u, v)$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$.

lution: Given that, $\vec{F} = (e^{2y}, e^{-2z}, e^{2x})$ and $\vec{r} = (3\,\text{Cos}u, 3\,\text{Sin}u, v)$.

So, $\vec{r_u} = (-3\,\text{Sin}u, 3\,\text{Cos}u, 0)$ and $\vec{r_v} = (0, 0, 1)$

Then,

$$\vec{N} = \vec{r_u}\times\vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3\,\text{Cos}u\,\vec{i} + 3\,\text{Sin}u\,\vec{j} = (3\,\text{Cos}v, \text{Sin}u, 0).$$

Since we know that $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = (x, y, z)$ and given that $\vec{r} = (3\,\text{Cos}u, 3\,\text{Sin}u, v)$ then we get

$$x = 3\,\text{Cos}u, \quad y = 3\,\text{Sin}u, \quad z = v$$

Then, $\vec{F}(\vec{r}) = (e^{6\,\text{Sin}u}, e^{-2v}, e^{6\,\text{Cos}u})$

So that,

$$\vec{F}(\vec{r}).\vec{N} = (e^{6\,\text{Sin}u}, e^{-2v}, e^{6\,\text{Cos}u}).(3\,\text{Cos}u, 3\,\text{Sin}u, 0)$$

$$= 3\,\text{Cos}u\,e^{6\,\text{Sin}u} + 3\,\text{Sin}u\,e^{-2v}$$

Since we have, $\iint_S \vec{F}.\vec{n}\,dA = \iint_R \vec{F}(\vec{r}).\vec{N}\,du\,dv$ .... (i)

Also given that the region is $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$.

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^2\int_0^{\pi/2}(3\cos u\, e^{6\sin u} + 3\sin u\, e^{-2v})\,du\,dv$$

$$= \int_0^2 [\frac{e^{6\sin u}}{2} + (-3)\cos u\, e^{-2v}]_0^{\pi/2} dv$$

$$= \int_0^2 \left(\frac{1}{2}e^6 + 3e^{-2v} - \frac{1}{2}\right) dv$$

$$[\because \sin\frac{\pi}{2} = 1 = \cos 0,\ \sin 0 = 0 = \cos\frac{\pi}{2}]$$

$$= \left[\frac{1}{2}e^6 v + \frac{3e^{-2v}}{-2} - \frac{v}{2}\right]_0^2$$

$$= e^6 - \frac{3}{2}(e^{-4} - 1) - 1 \quad = e^6 - \frac{3}{2}e^{-4} + \frac{1}{2}.$$

Thus, $\iint_S \vec{F}.\vec{n}\,dA = e^6 - \frac{3}{2}e^{-4} + \frac{1}{2}$.

3. $\vec{F} = (x - z, y - x, z - y)$, S: $\vec{r} = (u\cos v, u\sin v, u)$, $0 \le u \le 3$, $0 \le v \le 2\pi$.

[2004 Spring Q.No. 4(a)]

Solution: Similar to Q. No. 1 and Q. No. 2.

4. $\vec{F} = (0, x, 0)$, S: $x^2 + y^2 + z^2 = 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$.

Solution: Given that

$\vec{F} = (0, x, 0)$ and $x^2 + y^2 + z^2 = 1$ for $x \ge 0, y \ge 0, z \ge 0$.

Set, $x = u$, $y = v$ then $z = \sqrt{1 - u^2 - v^2}$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + \sqrt{1 - u^2 - v^2}\,\vec{k}.$$

Then, $\vec{r_u} = \vec{i} - \frac{u\vec{k}}{\sqrt{1-u^2-v^2}}$ and $\vec{r_v} = \vec{j} - \frac{v\vec{k}}{\sqrt{1-u^2-v^2}}$.

Since the sphere $x^2 + y^2 + z^2 = 1$ has radius $r = 1$. And given that the region is only the part $x \ge 0, y \ge 0, z \ge 0$ that implies the angle $\theta = \frac{\pi}{2}$.

So, set the Cartesian form u, v is polar form as,

$u = r\cos\theta$, $v = r\sin\theta$

So, $\vec{r_u} = \vec{i} - \frac{r\cos\theta\,\vec{k}}{\sqrt{1-r^2}}$, $\vec{r_v} = \vec{j} - \frac{r\sin\theta\,\vec{k}}{\sqrt{1-r^2}}$, $du\,dv = r\,dr\,d\theta$.

Since we have, $\iint_S \vec{F}.\vec{n}\,dA = \iint_R \vec{F}(\vec{r}).\vec{N}\,du\,dv$ .... (i)

where $\vec{N} = \vec{r_u} \times \vec{r_v}$

Since, $\vec{F} = (0, x, 0)$ and $\vec{r} = (x, y, z) = (u, v, \sqrt{1-u^2-v^2})$.

Then, $\vec{F}(\vec{r}) = u\vec{j}$

And,

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -u/\sqrt{1-u^2-v} \\ 0 & 1 & -v/\sqrt{1-u^2-v^2} \end{vmatrix} = \frac{u\vec{i}}{\sqrt{1-u^2-v^2}} + \frac{v\vec{j}}{\sqrt{1-u^2-v^2}} + \vec{k}$$

Then,

$$\vec{F}(\vec{r}).\vec{N} = (u\vec{j}).\left(\frac{u\vec{i}}{\sqrt{1-u^2-v^2}} + \frac{v\vec{j}}{\sqrt{1-u^2-v^2}} + \vec{k}\right)$$

$$= \frac{uv}{\sqrt{1-u^2-v^2}} = \frac{r^2\sin\theta\cos\theta}{\sqrt{1-r^2}} = \frac{r^2\sin 2\theta}{2\sqrt{1-r^2}}.$$

Now (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^{\pi/2}\int_0^1 \frac{r^2\sin 2\theta}{2\sqrt{1-r^2}}\,r\,dr\,d\theta$$

$$= \int_0^1 \frac{r^3\,dr}{2\sqrt{1-r^2}} \int_0^{\pi/2}\sin 2\theta\,d\theta.$$

$$= \int_0^1 \frac{r^3\,dr}{2\sqrt{1-r^2}}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{-1}{4}\int_0^1 \frac{r^3\,dr}{\sqrt{1-r^2}}(\cos\pi - \cos 0) = \frac{2}{4}\int_0^1 \frac{r^3\,dr}{\sqrt{1-r^2}}$$

Put $r = \sin\theta$ then $dr = \cos\theta\,d\theta$. Also $r = 0 \Rightarrow \theta = 0$, $r = 1 \Rightarrow 1 = \frac{\pi}{2}$. Then,

$$\iint_S \vec{F}.\vec{n}\,dA = \frac{1}{2}\int_0^{\pi/2}\frac{\sin^3\theta.\cos\theta\,d\theta}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\cos^0\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{3+0+2}{2}\right)} \quad [\because \text{ Using beta and gamma function}]$$

$$= \frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{1!\sqrt{\pi}}{2\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}$$

$$[\because \Gamma(m) = m!; \Gamma(m+1) = m\Gamma(m); \Gamma(1/2) = \sqrt{\pi}]$$

$$= \frac{1}{3}$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = \frac{1}{3}$.

5. $\vec{F} = (x, y, z)$, S: $\vec{r} = (u \cos v, u \sin v, u^2)$, $0 \le u \le 4$, $-\pi \le v \le \pi$.

Solution: Similar to Q. No. 2.

6. $\vec{F} = (18z, -12, 3y)$ **and S is the surface of the plane $2x + 3y + 6z = 12$ in the firs octant.**

Solution: Given that, $\vec{F} = (18z, -12, 3y)$

and the surface is, $2x + 3y + 6z = 12$

in the first octant set, $x = u$, $y = v$ then $z = \frac{12 - 2u - 3v}{6}$.

Since we have,

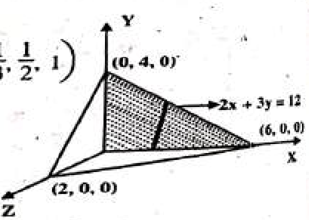
$$\vec{r} = (x, y, z) = \left(u, v, \frac{12 - 2u - 3v}{6}\right)$$

So, $\vec{r_u} = \left(1, 0, \frac{-2}{6}\right) = \left(1, 0, \frac{-1}{3}\right)$ and $\vec{r_v} = \left(0, 1, \frac{-3}{6}\right) = \left(0, 1, \frac{-1}{2}\right)$

Then,

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \end{vmatrix} = \frac{\vec{i}}{3} + \frac{\vec{j}}{3} + \vec{k} = \left(\frac{1}{3}, \frac{1}{2}, 1\right)$$



So that,

$$\vec{F}(\vec{r}).\vec{N} = (36 - 64 - 9v, -12, 3v).\left(\frac{1}{3}, \frac{1}{2}, 1\right)$$

$$= 12 - 2u - 3v - 6 + 3v = 6 - 2u$$

The projection of the plane $2x + 3y + 6z = 12$ is xy–plane is,

$$2x + 3y = 12, z = 0.$$

In which y varies from $y = 0$ to $y = \frac{12 - 2x}{3}$ and on the region, x moves fro. $x = 0$ to $x = 6$.

Since $x = u$, $y = v$. therefore (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^6 \int_0^{(12-2u)/3} (6 - 2u)\, dv du$$

$$= \int_0^6 [6v - 2uv]_0^{(12-2u)/3} du$$

$$= \int_0^6 \left(24 - 4u - 8u + \frac{4u^2}{3}\right) du$$

$$= \int_0^6 \left(24 - 12u + \frac{4u^2}{3}\right) du$$

$$= \left[24u - 6u^2 + \frac{4u^3}{3}\right]_0^6$$

$$= 144 - 216 + 96 = 24$$

Thus

$$\iint_S \vec{F}.\vec{n}\, dA = 24.$$

$\vec{F} = (12x^2y, -3yz, 2z)$ **and S is the portion of the plane $x + y + z = 1$ included in the first octant.** **[2010 Fall Q.No. 4(b)]**

ution: Given that $\vec{F} = (12x^2y, -3yz, 2z)$.

And surface is $x + y + z = 1$ in first octant.

Set $x = u$ and $y = v$ then $z = 1 - u - v$

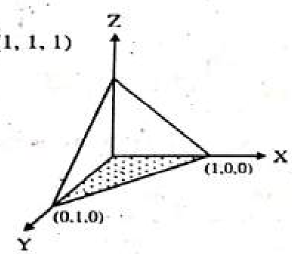
Here,

$$\vec{r} = (x, y, z) = (u, v, 1 - u - v)$$

So, $\vec{r}_u = (1, 0, -1)$ and $\vec{r}_v = (0, 1, -1)$

Then,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1)$$



By surface integral we have,

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R \vec{F}.\vec{N}\, dx\, dy \qquad \text{........(i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{r}_u \times \vec{r}_v = (1, 1, 1)$

Here,

$$\vec{F}.\vec{N} = (12x^2y, -3yz, 2z).(1, 1, 1)$$
$$= (12u^2v, -3v(1-u-v), 2(1-u-v)).(1, 1, 1)$$
$$= 12u^2v - 3v(1-u-v) + 2(1-u-v)$$
$$= 12u^2v - 3v + 3uv + 3v^2 + 2 - 2u - 2v$$
$$= 12u^2v + 3v^2 + 3uv - 5v - 2u + 2$$

Also, the projection of the surface in xy–plane is $x + y = 1$. So, the surface in uv-plane is $u + v = 1$ in which u varies from $u = 0$ to $u = 1 - v$ and v moves from $v = 0$ to $v = 1$. Also, $dx\,dy = du\,dv$.

Now, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^1\int_0^{1-v} (12u^2v + 3v^2 + 3uv - 5v - 2u + 2)\,du\,dv$$
$$= \int_0^1 \left[4u^3v + (3v^2-5v)u + (3v-2)\frac{u^2}{2} + 2u\right]_0^{1-v} dv$$
$$= \int_0^1 \left[4(1-v)^3v + (3v^2-5v)(1-v) + \frac{(3v-2)}{2}(1-v)^2 + 2(1-v)\right]dv$$
$$= \int_0^1 [4(1-v^3-3v+3v^2)v + 3v^2 - 5v - 3v^3 + 5v^2) + \frac{1}{2}(3v-2)(1-2v+v^2) + 2 - 2v]\,dv$$
$$= \frac{1}{2}\int_0^1 [8v - 8v^4 - 24v^2 + 24v^3 + 16v^2 - 10v - 6v^3 + 3v - 6v^2 + 3v^3 - 2 + 4v - 2v^2 + 4 - 4v]\,dv$$
$$= \frac{1}{2}\int_0^1 (-8v^4 + 21v^3 - 16v^2 + v + 2)\,dv$$
$$= \frac{1}{2}\left[-\frac{8v^5}{5} + \frac{21v^4}{4} - \frac{16v^3}{3} + \frac{v^2}{2} + 2v\right]_0^1$$
$$= \frac{1}{2}\left[-\frac{8}{5} + \frac{21}{4} - \frac{16}{3} + \frac{1}{2} + 2\right] = \frac{-96 + 315 - 320 + 30 + 120}{120} = \frac{49}{120}$$

Thus, $\iint_S \vec{F}.\vec{r}\,dA = \frac{49}{120}$.

8. $\vec{F} = (yz, zx, xy)$ **and S is the part of the surface** $x^2 + y^2 + z^2 = 1$**, which lies in the first octant.** **[2014 Fall Q. No. 5(b)]**

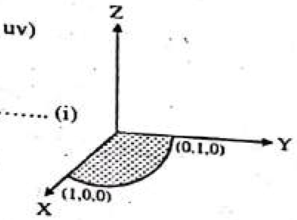
Solution: The projection of the surface on xy–plane is the region R bounded by x–axis, y–axis and the arc of circle $x^2 + y^2 = 1$.

Given that, $\vec{F} = (yz, zx, xy)$ and the sphere $x^2 + y^2 + z^2 = 1$.

Set $x = u$, $y = v$ then $z = \sqrt{1-u^2-v^2}$

Since, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + \sqrt{1-u^2-v^2}\,\vec{k}$.

Then,
$$\vec{F}(\vec{r}) = (v\sqrt{1-u^2-v^2}, u\sqrt{1-u^2-v^2}, uv)$$



Since we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iint_R \vec{F}(\vec{r}).\vec{N}\,du\,dv \quad \text{........ (i)}$$

So that
$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -u/\sqrt{1-u^2-v^2} \\ 0 & 1 & -v/\sqrt{1-u^2-v^2} \end{vmatrix}$$
$$= \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right)$$

Therefore,

Also, given that, $\vec{F} = (yz, zx, xy) = (v\sqrt{1-u^2-v^2}, u\sqrt{1-u^2-v^2}, uv)$.

Then, $\vec{F}.\vec{N} = uv + uv + uv = 3uv$

And from figure, u varies on the region from $u = 0$ to the curve $u = \sqrt{1-v^2}$ and on the region, v moves from $v = 0$ to $v = 1$.

Then, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,ds = \int_0^1\int_0^{\sqrt{1-v^2}} 3uv\,du\,dv$$
$$= \frac{3}{2}\int_0^1 [u^2]_0^{\sqrt{1-v^2}}\,v\,dv$$
$$= \frac{3}{2}\int_0^1 (1-v^2)\,v\,dv$$
$$= \frac{3}{2}\int_0^1 (v - v^3)\,dv$$
$$= \frac{3}{2}\left[\frac{v^2}{2} - \frac{v^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

Thus, $\iint_S \vec{F}.\vec{n}\,dA = \frac{3}{8}$.

Find $\iint_S (\vec{F}, \vec{n})\,ds$**, where** $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ **and S is the upper half of the sphere** $x^2 + y^2 + z^2 = a^2$**.**

Solution: Given that $\vec{F} = (x, y, z)$.

And the surface is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

The projection of the surface in xy–plane is the circle $x^2 + y^2 = a^2$ in which y varies $= -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$ and x moves from $x = -a$ to $x = a$.

Set $x = u$ and $y = v$ then $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - u^2 - v^2}$

Here,

$$\vec{F} = (x, y, z) = (u, v, \sqrt{a^2 - u^2 - v^2})$$

Now, by surface integral we have,

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R \vec{F}.\vec{N}\, dx\text{-}dy \qquad \text{......(i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{r}_u \times \vec{r}_v$ and $dx\, dy = du\, dv$

Since we have, $\vec{r} = (x, y, z) = (u, v, \sqrt{a^2 - u^2 - v^2}))$

Then,

$$\vec{r}_u = \left(1, 0, \frac{-u}{\sqrt{a^2 - u^2 - v^2}}\right) \text{ and } \vec{r}_v = \left(0, 1, \frac{-v}{\sqrt{a^2 - u^2 - v^2}}\right)$$

So,

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -u/\sqrt{a^2 - u^2 - v^2} \\ 0 & 1 & -v/\sqrt{a^2 - u^2 - v^2} \end{vmatrix}$$

$$= \frac{u}{\sqrt{a^2 - u^2 - v^2}}\vec{i} + \frac{v}{\sqrt{a^2 - u^2 - v^2}}\vec{j} + \vec{k}$$

Then,

$$\vec{F}.\vec{N} = \frac{u^2}{\sqrt{a^2 - u^2 - v^2}} + \frac{v^2}{\sqrt{a^2 - u^2 - v^2}} + \sqrt{a^2 - u^2 - v^2}$$

$$= \frac{u^2 + v^2 + a^2 - u^2 - v^2}{\sqrt{a^2 - u^2 - v^2}} = \frac{a^2}{\sqrt{a^2 - u^2 - v^2}}$$

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_{-a}^{a} \int_{-\sqrt{a^2 - u^2}}^{\sqrt{a^2 - u^2}} \frac{a^2}{\sqrt{a^2 - u^2 - v^2}}\, dv\, du \qquad \text{......(ii)}$$

Put $u = r\cos\theta$, $v = r\sin\theta$ then $r^2 = u^2 + v^2$. Also, $dv\, du = r\, dr d\theta$.

Moreover, the radius of the circle $u^2 + v^2 = a^2$ is $r = a$ and θ varies from $\theta = 0$ to $\theta = 2\pi$. Then (ii) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^{2\pi} \int_0^{a} \frac{a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

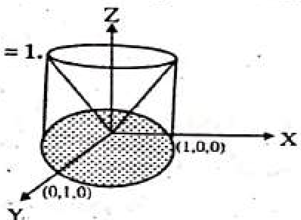
Put $a^2 - r^2 = p$ then $-2r\, dr = dp$. Also, $r = 0 \Rightarrow p = a^2$, $r = a \Rightarrow p = 0$. Then

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^{2\pi} \int_{a^2}^{0} a^2 p^{-1/2}\left(-\frac{dp}{2}\right) d\theta$$

$$= -\frac{a^2}{2}\left[\frac{p^{1/2}}{1/2}\right]_{a^2}^{0} [\theta]_0^{2\pi} = -a^2 (0 - a)(2\pi - 0) = 2\pi a^3.$$

Thus, $$\iint_S \vec{F}.\vec{n}\, dA = 2\pi a^3.$$

Find $\int\int_S (\vec{F}, \vec{n})\, ds$, where $\vec{F} = 2\vec{i} + 5\vec{j} + 3\vec{k}$ and S is the portion of the cone $Z = \sqrt{x^2 + y^2}$ that is inside the cylinder $x^2 + y^2 = 1$.



Solution: Given that, $\vec{F} = 2\vec{i} + 5\vec{j} + 3\vec{k}$

And S is the portion of the cone $z = \sqrt{x^2 + y^2}$ inside in the cylinder $x^2 + y^2 = 1$.

That means, the projection of the portion in xy–plane is $x^2 + y^2 = 1$.

On the projection y varies from $y = -\sqrt{1 - x^2}$ to $y = \sqrt{1 - x^2}$. And x moves from $x = -1$ to $x = 1$.

Now, by surface integral

$$\iint_S \vec{F}.\vec{n}\, ds = \iint_R (\vec{F}.\vec{N})\, dy\, dx \qquad \text{........(i)}$$

where, $\vec{N} = \vec{r}_u \times \vec{r}_v$

Here, $\vec{r} = (x, y, z)$.

Put $x = u$ and $y = v$ then $z = \sqrt{u^2 + v^2}$

So, $\vec{r} = (u, v, \sqrt{u^2 + v^2})$

Then, $\vec{r}_u = \left(1, 0, \frac{u}{\sqrt{u^2 + v^2}}\right)$ and $\vec{r}_v = \left(0, 1, \frac{v}{\sqrt{u^2 + v^2}}\right)$

Therefore,

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & u/\sqrt{u^2 + v^2} \\ 0 & 1 & v/\sqrt{u^2 + v^2} \end{vmatrix} = \left(\frac{-u}{\sqrt{u^2 + v^2}}, \frac{-v}{\sqrt{u^2 + v^2}}, 1\right)$$

Then,

$$\vec{F}.\vec{N} = \frac{-2u - 5v + 3\sqrt{u^2 + v^2}}{\sqrt{u^2 + v^2}}$$

Now, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, ds = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(\frac{-2u-5v+3\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\right) dv\, du$$

Set $u = r\cos\theta$, $v = r\sin\theta$. Then on the circle, $r = 0$ to $r = 1$ and θ varies from $\theta = 0$ $\theta = 2\pi$.

Therefore,

$$\iint_S \vec{F}.\vec{n}\, ds = \int_0^1\int_0^{2\pi} \frac{-2r\cos\theta - 5r\sin\theta + 3r}{r}\, r\, d\theta\, dr$$

$$= \int_0^1 r\,[-2\sin\theta + 5\cos\theta + 3\theta]_0^{2\pi}\, dr = \int_0^1 r\, dr.\, 6\pi = 6\pi.\frac{1}{2} = 3\pi.$$

11. Find the flux of $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ through the surface S is the first octa portion of the plane $2x + 3y + z = 6$.

Similar to Q. 10.

12. Let S be the part of the graph of $z = 9 - x^2 - y^2$ with $z \ge 0$. If $\vec{F} = 3x\vec{i} + 3y\vec{j} + z\vec{k}$. Find the flux of $\vec{F}$ through S. **[2009 Fall Q.No. 4(a)]**

Solution: Given that $\vec{F} = (3x, 3y, z)$.

And S is part of $z = 9 - x^2 - y^2$ with $z \ge 0$.

Clearly, the projection of the parabolid in xy–plane is a circle $x^2 + y^2 = 9$.

By surface integral,

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R (\vec{F}.\vec{N})\, dx\, dy \quad \text{......(i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y$

Since $\vec{r} = (x, y, z) \Rightarrow \vec{r} = (x, y, 9 - x^2 - y^2)$

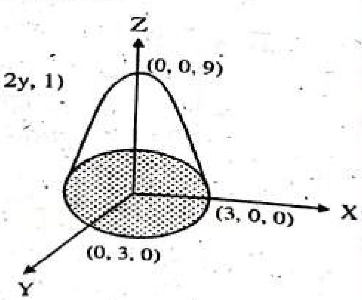
Then $\vec{r}_x = (1, 0, -2x)$, $\vec{r}_y = (0, 1, -2y)$.

For the circle, set $x = r\cos\theta$, $y = r\sin\theta$ then $z = 9 - r^2$.

On the circle, radius $r = 3$ and angular variation $\theta = 2\pi$.

Also,

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1)$$



Then

$$\begin{aligned} \vec{F}.\vec{N} &= 6x^2 + 6y^2 + z \\ &= 6(x^2 + y^2) + 9 - (x^2 + y^2) \\ &= 6r^2 + 9 - r^2 \\ &= 9 + 5r^2 \end{aligned}$$

Now, (i) becomes,

$$\begin{aligned} \iint_S \vec{F}.\vec{n}\, dS &= \int_0^{2\pi}\int_0^3 (9 + 5r^2)\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\left[\frac{9r^2}{2} + \frac{5}{4}r^4\right]_0^3 d\theta \\ &= \int_0^{2\pi}\left(\frac{81}{2} + \frac{405}{4}\right) d\theta \\ &= \left(81 + \frac{405}{2}\right).\frac{1}{2}\times 2\pi = \frac{162 + 405}{2}\pi = \frac{567\pi}{2} \end{aligned}$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = \frac{567\pi}{2}$.

. **$\vec{F} = (x, z, y)$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$.**

lution: Given that $\vec{F} = (x, z, y)$.

And the surface S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$.

Here,

$$\nabla.\vec{F} = 1 + 0 + 0 = 1.$$

Now,

$$\begin{aligned} \iint_S \vec{F}.\vec{n}\, dA &= \text{volume of the hemisphere.} \\ &= \frac{1}{2}\times\frac{4}{3}\times\pi\times(2)^3 \\ &= \frac{16\pi}{3} \end{aligned}$$

. **$\vec{F} = 3x\vec{i} + xz\vec{j} + z^2\vec{k}$, S is the surface of the region bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy–plane.**

lution: Similar to Q. 12.

EXERCISE – 4.9

valuate $\int\int_S \vec{F}.\vec{n}\, dA$, by using Gauss divergence theorem of the following data:

. **$\vec{F} = (x^2, 0, z^2)$, S is the box $|x| \le 1$, $|y| \le 3$, $|z| \le 2$.**

Solution: Given that $\vec{F} = (x^2, 0, \ge 2)$ and the surface is the box $|x| \le 1$, $|y| \le 3$, $|z| \le 2$.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \text{div}\,\vec{F}\,dv \quad \text{......... (i)}$$

Here, $\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(x^2, 0, z^2) = 2x + 2z$

Now (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_{-1}^{1}\int_{-3}^{3}\int_{-2}^{2}(2x + 2z)dzdydx$$

$$= \int_{-1}^{1}\int_{-3}^{3}[2xz + z^2]_{2}^{-2}\,dydx$$

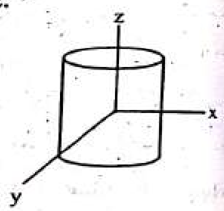
$$= \int_{-1}^{1}\int_{-3}^{3}(4x + 4 + 4x - 4)\,dydx$$

$$= \int_{-1}^{1}\int_{-3}^{3} 8x\,dydx$$

$$= \int_{-1}^{1}[8xy]_{3}^{-3}\,dx = \int_{-1}^{1}[24x + 24x]dx$$

$$= \int_{-1}^{1}48x\,dx = [24x^2]_{1}^{-1} = 24 - 24 = 0.$$

2. $\vec{F}$ **= (cosy, sinx, cosz), S is the surface** $x^2 + y^2 \leq 4$, $|z| \leq 2$.



Solution: Given that, $\vec{F} = (\text{Cosy, Sinx, Cosz})$

And the surface is $x^2 + y^2 \leq 4$, $|z| \leq 2$.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \text{div}\,\vec{F}\,dv \quad \text{... (i)}$$

Here, $\text{Div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\text{Cosy, Sinx, Cosz}) = -\text{Sinz}.$

Since $x^2 + y^2 = 4$ is a circle in xy–plane in which y varies from $y = 0$ to $y = \pm\sqrt{4 - x^2}$ and on the surface x moves from $x = -2$ to 2.

Therefore (i) become

$$\iint_S \vec{F}.\vec{n}\,dA = \int_{-2}^{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(-\text{Sinz})\,dy\,dx\,dz$$

$$= \int_{-2}^{2}\text{Sinz}\,dz\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy\,dx$$

4. $\overrightarrow{F} = (x^3, y^3, z^3)$, **S is the sphere** $x^2 + y^2 + z^2 = 9$.

Solution: Given that, $\overrightarrow{F} = (x^3, y^3, z^3)$.

And the surface is a sphere $x^2 + y^2 + z^2 = 9$, that has radius 3.

By Gauss divergence theorem, we have,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \iiint_T \text{div}\overrightarrow{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\text{div}\overrightarrow{F} = \nabla.\overrightarrow{F} = \left(\overrightarrow{i}\frac{\delta}{\delta x} + \overrightarrow{j}\frac{\delta}{\delta y} + \overrightarrow{k}\frac{\delta}{\delta z}\right).(x^3, y^3, z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3 \times 9 = 27.$$

Clearly, the sphere has limits $x = \pm 3$, $y = \pm\sqrt{9 - x^2}$ and $z = \pm\sqrt{1 - x^2 - y^2}$.

Then (i) becomes,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 27\,dz\,dy\,dx$$

$$= 27\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

Since $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$, is a sphere of radius 3. So the volume of the sphere is $\frac{4}{3}\pi r^2$. That is,

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi\,9 = 12\pi.$$

Thus,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = 27\,(12\pi) = 372\pi.$$

5. $\overrightarrow{F} = (4xz, -y^2, yz)$, **S is the surface of the cube bounded by the planes** $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: Given that, $\overrightarrow{F} = (4xz, -y^2, yz)$.

And the surface is bounced by $x = 0$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

By Gauss divergence we have,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \iiint_T \text{div}\overrightarrow{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\text{div}\overrightarrow{F} = \nabla.\overrightarrow{F} = \left(\overrightarrow{i}\frac{\delta}{\delta x} + \overrightarrow{j}\frac{\delta}{\delta y} + \overrightarrow{k}\frac{\delta}{\delta z}\right).(4xz, -y^2, yz)$$

$$= 4z - 2y + y = 4z - y.$$

Then (i) becomes,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \int_0^1\int_0^1\int_0^1 (4z - y)\,dzdydx$$

$$= \int_0^1\int_0^1 [2z^2 - yz]_0^1\,dydx$$

$$= \int_0^1\int_0^1 (2 - y)dydx$$

$$= \int_0^1\left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_0^1\left(2 - \frac{1}{2}\right)dx = \int_0^1 \frac{3}{2}dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}$$

$\overrightarrow{F} = (4x, -2y^2, z^2)$ **and s is the surface bounding the region** $x^2 + y^2 = 4$, $z = 0, z = 3$.

ion: Given that, $\overrightarrow{F} = (4x, -2y^2, z^2)$.

And the surface is bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Clearly the circle $x^2 + y^2 = 4$ is bounded by $y = \pm\sqrt{4 - x^2}$ in which x moves from $x = -2$ to $x = 2$.

By Gauss divergence theorem we have,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \iiint_T \text{div}\overrightarrow{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\text{div}\overrightarrow{F} = \nabla.\overrightarrow{F} = \left(\overrightarrow{i}\frac{\delta}{\delta x} + \overrightarrow{j}\frac{\delta}{\delta y} + \overrightarrow{k}\frac{\delta}{\delta z}\right).(4x, -2y^2, z^2)$$

$$= 4 - 4y + 2z.$$

Then (i) becomes,

$$\iint_S \overrightarrow{F}.\overrightarrow{n}\,dA = \int_0^3\int_{-2}^{2}\int_{-\sqrt{4-x}}^{\sqrt{4-x}} (4 - 4y + 2z)\,dy\,dx\,dz$$

$$= \int_0^3\int_{-2}^{2} [4y - 2y^2 + 2yz]_{\sqrt{4-x^2}}^{-\sqrt{4-x^2}}$$

$$= \int_0^3\int_{-2}^{2} [8\sqrt{4 - x^2} - 0 + 4z\sqrt{4 - x^2}]dxdz$$

$$= \int_{-2}^{2}\int_{0}^{3}[8\sqrt{4-x^2} + 4z\sqrt{4-x^2}\,]\, dzdx$$

$$= \int_{-2}^{2}[8\sqrt{4-x^2}\; z + 2z^2\sqrt{4-x^2}\,]_0^3 dz$$

$$= \int_{-2}^{2}(24\sqrt{4-x^2} + 18\sqrt{4-x^2})dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\, dx$$

$$= 42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$= 42\,[(0 + 2\mathrm{Sin}^{-1}(1)) - (0 + 2\,\mathrm{Sin}^{-1}(-2))]$$

$$= 42\,[2\mathrm{Sin}^{-1}(1) + 2\,\mathrm{Sin}^{-1}(1)] \qquad [\because \mathrm{Sin}(-\theta) = -\mathrm{Sin}\theta]$$

$$= 168\,\mathrm{Sin}^{-1}(1) = 168\,\frac{\pi}{2} = 84\pi.$$

7. $\vec{F} = (9x, y\cosh^2 x, -z\sinh^2 x)$, **S: the ellipsoid** $4x^2 + y^2 + 9z^2 = 36$.

Solution: Similar to Q.4.

[Hints: Use $\cosh^2 x - \sinh^2 x = 1$. And obtain limits as in Q.4.]

8. $\vec{F} = (\sin x, y, z)$, **S is the surface of** $0 \le x \le \pi/2,\ x \le y \le z,\ 0 \le z \le 1$.

Solution: Given that, $\vec{F} = (\mathrm{Sin}x, y, z)$ and surface is, $0 \le x \le \frac{\pi}{2},\ x \le y \le z,\ 0 \le z \le 1$.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \iiint_T \mathrm{div}\vec{F}\, dv \qquad \text{......... (i)}$$

Here,

$$\mathrm{div}\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\mathrm{Sin}x, y, z) = \mathrm{Cos}x + 1 + 1 = 2 + \mathrm{Cos}x$$

Then (i) becomes,

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \int_0^1\int_0^{\frac{\pi}{2}}\int_x^z (2 + \mathrm{Cos}x)\, dy\, dx\, dz$$

$$= \int_0^1\int_0^{\frac{\pi}{2}}[2y + y\,\mathrm{Cos}x]_x^z\, dx\, dz$$

$$= \int_0^1 [2xz + z\,\mathrm{Sin}x - x^2 - x\,\mathrm{Sin}x - \mathrm{Cos}x]_0^{\pi/2}\, dx$$

$$= \int_0^1\left[xz + z - \frac{\pi^2}{4} - \frac{\pi}{2} + 1\right]dx \quad [\because \mathrm{Sin}\frac{\pi}{2} = 1 = \mathrm{Cos}0]$$

$$[\because \mathrm{Sin}0 = 0 = \mathrm{Cos}\frac{\pi}{2}]$$

$$= \left[\frac{xz^2}{2} + \frac{z^2}{2} - \frac{\pi^2 z}{4} - \frac{\pi z}{2} + z\right]_0^1$$

$$= \frac{\pi}{2} + \frac{1}{2} - \frac{\pi^2}{4} - \frac{\pi}{2} + 1$$

$$= \frac{3}{2} - \frac{\pi^2}{4}$$

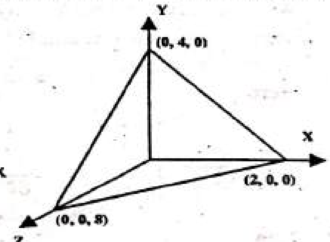
$\iint_S \phi\, dv$, **where** $\phi = 45x^2y$ **and v is the closed region bounded by the planes** $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.

tion: Given that, $\varnothing = 45x^2y$

And the surface is bounded by $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.

Clearly the region is bounded $x = 0$ and $x = 2$, $y = 0$ and $y = 4 - 2x$, $z = 0$ and $z = 8 - 4x - 2y$.

Now,



$$\iiint \varnothing\, dv = \int_0^2\int_0^{4-2x}\int_0^{(8-4x-2y)} 45x^2y\, dzdydx$$

$$= 45\int_0^2\int_0^{4-2x} x^2y\,[z]_0^{8-4x-2y}\, dydx$$

$$= 45\int_0^2\int_0^{4-2x} x^2y\,(8 - 4x - 2y)\, dydx$$

$$= 45\int_0^2\int_0^{4-2x}(8x^2y - 4x^3y - 2x^2y^2)\, dydx$$

$$= 45\int_0^2\left[4x^2y^2 - 2x^3y^2 - \frac{2x^2y^3}{3}\right]_0^{4-2x} dx$$

$$= 45\int_0^2\left[(4x^2 - 2x^3)(4 - 2x)^2 - \frac{2x^2}{3}(4 - 2x)^3\right]dx$$

$$= 45\int_0^2[(4x^2 - 2x^3)(16 - 16x + 4x^2) - \frac{2x^2}{3}(64 - 8x^3 - 96x + 48x^2)]\, dx$$

$$= \frac{45}{3}\int_0^2 [192x^2 - 192x^3 + 48x^4 - 96x^3 + 96x^4 - 24x^5 - 123x^2 + 16x^5 + 192x^3 - 96x^4]\,dx$$

$$= 15\int_0^2 (64x^2 - 96x^3 + 48x^4 - 8x^5)\,dx$$

$$= 120\int_0^2 (8x^2 - 12x^3 + 6x^4 - x^5)\,dx$$

$$= 120\left[\frac{8x^3}{3} - 3x^4 + \frac{6x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= 120\left[\frac{64}{3} - 48 + \frac{192}{5} - \frac{64}{6}\right]$$

$$= 120\left(\frac{32}{3} - 48 + \frac{192}{5}\right) = 120\left(\frac{160 - 720 + 576}{15}\right) = 8\times 16 = 128.$$

C. If $\vec{F} = (2x^2 - 3z, -2xy, -4x)$, then evaluate $\iiint (\nabla\times\vec{F})\,dv$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution: Given that, $\vec{F} = (2x^2 - 3z, -2xy, -4x)$.

And the region is bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Then the region is bounded by $x = 0$ and $x = 2$, $y = 0$ and $y = 2 - x$, $z = 0$ and $z = 4 - 2x - 2y$.

Here,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta y} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = 0\,\vec{i} + (4-3)\,\vec{j} + (-2y)\,\vec{k} = (0, 1, -2y)$$

Now,

$$\iiint (\nabla\times\vec{F})\,dv = \int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} (\vec{j} - 2y\,\vec{k})\,dzdydx$$

$$= \int_0^2\int_0^{(2-x)} [z\,\vec{j} - 2yz\,\vec{k}]_0^{4-2x-2y}\,dydx$$

$$= \int_0^2\int_0^{(2-x)} [(4 - 2x - 2y)\,\vec{j} - 2y\,(4 - 2x - 2y)\,\vec{k}]\,dydx$$

$$= \int_0^2\int_0^{(2-x)} [(4 - 2x - 2y)\,\vec{j} - (8y - 4xy - 4y^2)\,\vec{k}]\,dydx$$

$$= \int_0^2 [(4 - 2xy - y^2)\,\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}]_0^{2-x}\,dx$$

$$= \int_0^2 [[4(2-x) - 2x(2-x) - (2-x)^2]\,\vec{j} - [4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\,\vec{k}]]_0^{2-x}\,dx$$

$$= \int_0^2 [(8 - 4x - 4x + 2x^2 - 4 - x^2 + 4x)\,\vec{j} - (16 + 4x^2 - 16x - 8x - 2x^3 + 8x^2 - \frac{4}{3}(8 - x3 - 12x + 6x^2)\,\vec{k}]dx$$

$$= \int_0^2 [(4 - 4x + x^2)\,\vec{j} - \frac{1}{3}(16 - 24x + 12x^2 - 2x^3)\,\vec{k}]\,dx$$

$$= [\left(4x - 2x^2 + \frac{x^3}{3}\right)\vec{j} - \frac{1}{3}\left(16x - 12x^2 + 4x^3 - \frac{2x^4}{4}\right)\vec{k}]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{1}{3}(32 - 48 + 32 - 8)\,\vec{k}$$

$$= \frac{8}{3}\,\vec{j} - \frac{8}{3}\,\vec{k} = \frac{8}{3}(\vec{j} - \vec{k})$$

Using the Gauss divergence theorem, find $\iint_S (\vec{F}.\vec{n})\,ds$, where $\vec{F} = y\sin x\,\vec{i} + y^2z\,\vec{j} + (x + 3z)\,\vec{k}$ and S is the surface of the region bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

ution: Given that, $\vec{F} = y\,\text{Sin}x\,\vec{i} + y^2z\,\vec{j} + (x + 3z)\,\vec{k}$.

And the surface of region is bounded by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

By Gauss divergence theorem, we have,

$$\iint_S (\vec{F}.\vec{n})\,ds = \iiint_T \text{div}\,\vec{F}\,dv \quad \text{......... (10)}$$

Here,

$$\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right)(y\,\text{Sin}x\,\vec{i} + y^2z\,\vec{j} + (x + 3z)\,\vec{k})$$

$$= y\,\text{Cos}x + 2yz + 3$$

Now (i) becomes,

$$\iint_S (\vec{F}.\vec{n})\,ds = \int_{-1}^1\int_{-1}^1\int_{-1}^1 (y\,\text{Cos}x + 2yz + 3)\,dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[y\ \text{Sinx} + 2xyz + 3x]_{-1}^{1}\ dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[y\ \text{Sin1} - y\ \text{Sin}(-1) + 2yz - 2(-1)\ yz + 3 - 3(-1)]dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2y\ \text{Sin1} + 4yz + 6)\ dydz$$

$$= \int_{-1}^{1}[y^2\ \text{so1} + 2y^2z + 6y)]_{-1}^{1}\ dz$$

$$= \int_{-1}^{1}(\text{Sin1} - \text{Sin1} + 2z - 2z + 6 + 6)\ dz$$

$$= \int_{-1}^{1}12dz = [12z]_{-1}^{1} = 12(1+1) = 24.$$

2. $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$**; S is the graph of** $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

Solution: Given that, $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = (yz, xz, xy)$.

And the surface of region is the graph bounced by $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

By Gauss divergence theorem, we have

$$\iint_S \vec{F}.\vec{n}\ ds = \iiint_T \text{div}\,\vec{F}\,dv \qquad \text{......... (i)}$$

Hare,

$$\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k})$$

$$= 0 + 0 + 0 = 0.$$

Therefore (i) becomes,

$$\iint_S (\vec{F}.\vec{n})\ ds = \iiint_T 0\ dv = 0.$$

EXERCISE 4.10

Evaluate $\oint_c \vec{F}.d\vec{r}$ **by using stocks theorem:**

1. $\vec{F} = (z^2, 5x, 0)$**, S is the square** $0 \le x \le 1, 0 \le y \le 1, z = 1$.

[2004 Spring Q.No. 4(b)]

Solution: Given that, $\vec{F} = (z^2, 5x, 0)$ and the surface is $0 \le x \le 1, 0 \le y \le 1, z = 1$.

$$\oint_c \vec{F}.d\vec{r} = \iint_S (\nabla.\vec{F}).\vec{N}dxdy \qquad \text{......... (i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y$.

Here,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ z^2 & 5x & 0 \end{vmatrix} = 2z\,\vec{j} + 5\,\vec{k} = (0, 2z, 5).$$

Since we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

$$\Rightarrow \vec{r} = x\,\vec{i} + y\,\vec{j} + \vec{k}.$$

So, $\vec{r}_x = \vec{i}$ and $\vec{r}_y = \vec{j}$.

Then, $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{i} \times \vec{j} = \vec{k} = (0, 0, 1)$.

So that, $(\nabla\times\vec{F}).\vec{N} = (0, 2z, 5).(0, 0, 1) = 5$.

Now (i) becomes

$$\oint_c \vec{F}.d\vec{r} = \int_0^1\int_0^1 5dxdy = 5\int_0^1[x]_0^1\,dy = 5\int_0^1 dy = 5[y]_0^1 = 5$$

Thus $\oint_c \vec{F}.d\vec{r} = 5$.

$\vec{F} = (e^z, e^z\,\text{siny}, e^z\,\text{cosy})$**, S:** $z = y^2, 0 \le x \le 4, 0 \le y \le 2$.

Solution: Given that $\vec{F} = (e^z, e^z\,\text{Siny}, e^z\,\text{Cosy})$

And the surface is, $0 \le x \le 4, 0 \le y \le 2, z = y^2$.

By Stoke's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_S (\nabla\times\vec{F}).\vec{N}\ dxdy \qquad \text{........ (i)}$$

where,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ e^z & e^z\,\text{siny} & e^z\,\text{cosy} \end{vmatrix}$$

$$= (-e^z\,\text{Siny} - e^z\,\text{Siny})\,\vec{i} + e^z\,\vec{j} + 0\,\vec{k}$$

We have,

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + y^2\,\vec{k} \qquad [\because z = y^2]$$

Then, $\vec{r}_x = \vec{i}$ and $\vec{r}_y = \vec{j} + 2y\,\vec{k}$.

So that,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2y \end{vmatrix} = -2y\vec{j} + \vec{k}$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-2e^z \text{Siny}\,\vec{i} + e^z\vec{j} + 0\vec{k}).(0\vec{i} - 2y\vec{j} + \vec{k}) = -2ye^z$$

Now, (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^4\int_0^2 2ye^z\,dydx = -\int_0^4\int_0^2 2ye^{y^2}dydx$$

Set $y^2 = t$ then $2ydy = dt$. Also $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow t = 4$. Then,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^4\int_0^4 e^t\,dtdx = -\int_0^4 [e^t]_0^4\,dx = -\int_0^4 (e^4 - e^0)\,dx = -[xe^4 - x]_0^4$$

$$\Rightarrow \oint_c \vec{F}.d\vec{r} = -(4e^4 - 4) = 4(1 - e^4).$$

3. $\vec{F} = (y^2, z^2, x^2)$, S the portion of the parabolid $x^2 + y^2 = z$, $y \geq 0$, $z \leq 1$.

Solution: Given that, $\vec{F} = (y^2, z^2, x^2)$ and the surface is, $x^2 + y^2 = z$, $y \geq 0$, $z \leq 1$.

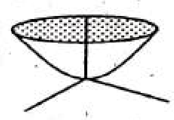
By Stokes theorem, we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\vec{i} - 2x\vec{j} - 2y\vec{k}.$$

Since S is a paraboloid with z is linear and z may have maximum value 1. Therefore $z = 1$.



We have, $\vec{r} = x\vec{i} + y\vec{j} + \vec{k}$. So,

$$\vec{r_x} = \vec{i} \quad \text{and} \quad \vec{r_y} = \vec{j}.$$

Then,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \vec{i} \times \vec{j} = \vec{k}.$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-2z\vec{i} - 2x\vec{j} - 2y\vec{k}).\vec{k} = -2y$$

Given that the surface is $x^2 + y^2 = z$, $z \leq 1$, $y \geq 0$.

This gives that y varies from $y = 0$ to $y = \sqrt{1 - x^2}$ and x moves from $x = 1$ to $x = 1$.

Therefore,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (-2y)\,dy\,dx$$

$$= -\int_{-1}^{1} [y^2]_0^{\sqrt{1-x^2}}\,dx$$

$$= -\int_{-1}^{1} (1 - x^2)\,dx$$

$$= -\left[x - \frac{x^3}{3}\right]_{-1}^{1} = -\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$$

$$= -\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= -\left(2 - \frac{2}{3}\right) = -\frac{4}{3}$$

Thus, $$\oint_c \vec{F}.d\vec{r} = -\frac{4}{3}.$$

$\vec{F} = (-5y, 4x, z)$, C is the circle $x^2 + y^2 = 4$, $z = 1$.

ution: Given that, $\vec{F} = (-5y, 4x, z)$ and the surface is $x^2 + y^2 = 4$, $z = 1$.

By Stoke's theorem, we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -5y & 4x & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + (4 + 5)\vec{k} = (0, 0, 9).$$

We have, $\vec{r} = x\vec{i} + y\vec{j} + \vec{k}$. So,

$$\vec{r_x} = \vec{i} \quad \text{and} \quad \vec{r_y} = \vec{j}.$$

So that, $\vec{N} = \vec{r_x} \times \vec{r_y} = \vec{i} \times \vec{j} = \vec{k} = (0, 0, 1)$.

Therefore, $(\nabla\times\vec{F}).\vec{N} = (0, 0, 9).(0, 0, 1) = 9$.

Given that the surface is $x^2 + y^2 = 4$, $z = 1$.

Clearly, the surface is a circle in which y varies from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$ and on the region x moves from $x = -2$ to $x = 2$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = 9\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$

$$= 9\int_{-2}^{2}(2\sqrt{4-x^2})\,dx$$

$$= 18\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

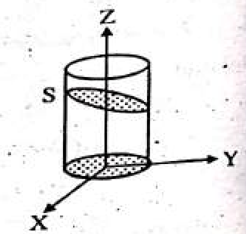
$$= 18\,[0 + 2\,\mathrm{Sin}^{-1}(1) - 0 - 2\,\mathrm{Sin}^{-1}(-1)]$$

$$= 18\times 4\times\frac{\pi}{2} \quad [\because \mathrm{Sin}^{-1}(-\theta) = -\mathrm{Sin}^{-1}(\theta)]$$

$$= 36\pi$$

Thus, by (i), $\oint_C \vec{F}.d\vec{r} = = 36\pi$.

5. $\vec{F} = (4z, -2x, 2x)$, C is the circle $x^2 + y^2 = 1$, $z = y + 1$.



Solution: Given that, $\vec{F} = (4z, -2x, 2x)$ and the surface is,
$x^2 + y^2 = 1, z = y + 1.$

By Stoke's theorem we have

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{.... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$.

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 4z & -2x & 2x \end{vmatrix} = 0\,\vec{i} + (4-2)\,\vec{j} + (-2)\,\vec{k} = (0, 2, -2)$

Since we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + (y+1)\,\vec{k}$. Then,

$\vec{r_x} = \vec{i} = (1, 0, 0)$ and $\vec{r_y} = \vec{j} + \vec{k} = (0, 1, 1)$.

So that,

$$\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0\,\vec{i} - \vec{j} + \vec{k} = (0, -1, 1).$$

Then; $(\nabla.\vec{F}).\vec{N} = (0, 2, -2).(0, -1, 1) = 0 - 2 - 2 = -4$

Given surface on xy plane is $x^2 + y^2 = 1$ which is a circle in which y varies from $y = -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$ and x moves on the region from $x = -1$ to $x = 1$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-4)\,dydx$$

$$= -4\int_{-1}^{1}[y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= -4\int_{-1}^{1}2\sqrt{1-x^2}\,dx$$

$$= -8\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{-1}^{1}$$

$$= -8\left[0+\frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(-1)\right]$$

$$= -8.\mathrm{Sin}^{-1}(1) \qquad [\because \mathrm{Sin}^{-1}(-\theta) = -\mathrm{Sin}^{-1}\theta]$$

$$= -8.\frac{\pi}{2} \qquad [\because \mathrm{Sin}^{-1}(1) = \frac{\pi}{2}]$$

$$= -4\pi$$

s, by (i), $\oint_C \vec{F}.d\vec{r} = -4\pi$.

= (0, xyz, 0), C is the boundary of the triangle with vertices (1, 0, 0), 1, 0), (0, 0, 1).

Sol: Given that, $\vec{F} = (0, xyz, 0)$.

d the surface is a triangle having vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Stoke's theorem we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{......... (i)}$$

ere, $\vec{N} = \vec{r_x}\times\vec{r_y}$

re, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 0 & xyz & 0 \end{vmatrix} = -xy\,\vec{i} + yz\,\vec{k} = (-xy, 0, yz).$

nce the equation of plane that passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1) be

$$x + y + z = 1 \quad \text{........ (ii)}$$

ince we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + (1 - x - y)\,\vec{k}$ [∵ uSing (ii)]

hen,

$\vec{r_x} = \vec{i} - \vec{k} = (1, 0, -1)$ and $\vec{r_y} = \vec{j} - \vec{k} = (0, 1, -1)$.

o that,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1).$$

Therefore, $(\nabla\times\vec{F}).\vec{N} = (-xy, 0, yz)\,(1, 1, 1)$

$$= -xy + 0 + yz$$
$$= y(-x + z)$$
$$= y(-x + 1 - x - y) = y(1 - 2x - y) = y - 2xy - y^2$$

Since the surface is the plane $x + y + z = 1$. On the xy–plane, the projection of the plane is $x + y = 1$ in which x varies from $x = 0$ to $x = 1 - y$ and y moves from $y = 0$ to $y = 1$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dydx = \int_0^1\int_0^{1-y} (y - 2xy - y^2)\,dxdy$$
$$= \int_0^1 [xy - x^2y - xy^2]_0^{1-y}\,dy$$
$$= \int_0^1 [(1 - y)\,y - (1 - y)^2\,y - (1 - y)\,y^2]dy$$
$$= \int_0^1 (y - y^2 - y + 2y^2 - y^3 - y^2 + y^3)\,dy$$
$$= \int_0^1 0\,dy = 0\int_0^1 dy = 0.$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 0.$

7. $\vec{F} = (y^3, 0, x^3)$, **C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1).**

Solution: Similar to Q. No. 6.

8. $\vec{F} = (x^2 + y^2, -2xy, 0)$, **C is the rectangle hounded by the lines $x = \pm a$, $y = 0$, $y = b$.**

Solution: Given that $\vec{F} = (x^2 + y^2, -2xy, 0)$.

And the surface is a rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

By Stoke's theorem we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + (-2y - 2y)\,\vec{k} = (0, 0, -4y)$

Since z is independent to x and y. Therefore.

$$\vec{r_x} = \vec{i} = (1, 0, 0) \quad \text{and} \quad \vec{r_y} = \vec{j} = (0, 1, 0).$$

Then, $\vec{N} = \vec{r_x}\times\vec{r_y} = \vec{i}\times\vec{j} = \vec{k} = (0, 0, 1).$

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \int_0^b\int_{-a}^a (-4y)\,dxdy$$
$$= \int_0^b [-4xy]_{-a}^a\,dy = \int_0^b -4y(a + a)\,dy$$
$$= -4a\,[y^2]_0^b = -4a\,(b^2 - 0) = -4ab^2$$

Thus (i) gives, $\oint_c \vec{F}.d\vec{r} = -4ab^2$.

$\vec{F} = (2x - y, -yz^2 - y^2z)$, **S is the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on xy plane.**

R Verify Stoke's theorem for the vector function, $\vec{F} = (2x - y)\,\vec{i} - yz^2\,\vec{j} - y^2z\,\vec{k}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane and C its boundary. **[2001 Q.No. 4(b) OR]**

Solution: Given that $\vec{F} = (2x - y, -yz^2, -y^2z)$.

And the region is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ that is bounded by its projection on xy–plane.

By Stoke's theorem, we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$

$$= (-2yz + 2yz)\,\vec{i} + (0 - 0)\,\vec{j} + (0 + 1))\,\vec{k} = (0, 0, 1).$$

Since we have,

Then, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + \sqrt{1 - x^2 - y^2}\,\vec{k}$

$$\vec{r_x} = \vec{i} - \frac{x}{\sqrt{1-x^2-y^2}}\vec{k} \quad \text{and} \quad \vec{r_y} = \vec{j} - \frac{y}{\sqrt{1-x^2-y^2}}\vec{k}$$

So that, $\vec{N} = \vec{r_x} \times \vec{r_y}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -x/\sqrt{1-x^2-y^2} \\ 0 & 1 & -y/\sqrt{1-x^2-y^2} \end{vmatrix} = \frac{x}{\sqrt{1-x^2-y^2}}\vec{i} + \frac{y}{\sqrt{1-x^2-y^2}}\vec{j} + \vec{k}.$$

Then,

$$(\nabla\times\vec{F}).\vec{N} = 1.$$

Since the region is the projection of $x^2 + y^2 + z^2 = 1$ on xy–plane. So the region of integration is $x^2 + y^2 = 1$, $z = 0$.

This is a circle with radius $r = 1$.

Setting, $x = \cos\theta$ and $y = \sin\theta$ then $dx\,dy = r\,dr\,d\theta$. Also, θ varies from $\theta = 0$ to $\theta = 2\pi$. Then,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}[\theta]_0^{2\pi} = \frac{1}{2}2\pi = \pi.$$

Then (i) gives, $\oint_C \vec{F}.d\vec{r} = \pi$.

10. $\vec{F} = (y^2, x^2, (x + z))$, C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

Solution: Similar to Q. No. 7.

11. $\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$, S is the first octant portion of the plane $x + y + z = 1$.

Solution: Given that $\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$.

And the surface is the portion of the plane $x + y + z = 1$ in the first octant.

By Stoke's theorem we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{.........(i)}$$

where $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\,\vec{i} - 2x\,\vec{j} - 2y\,\vec{k}.$$

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (1 - x - y)\vec{k} \quad [\because x + y + z = 1]$$

Then,

$$\vec{r_x} = \vec{i} - \vec{k} \quad \text{and} \quad \vec{r_j} = \vec{j} - \vec{k}.$$

So that,

$$\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}.$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-2z\,\vec{i} - 2x\,\vec{j} - 2y\,\vec{k}).(\vec{i} + \vec{j} + \vec{k})$$
$$= -2z - 2x - 2y$$
$$= -2(x + y + z) = -2\,(1) = -2.$$

The projection of the surface plane $x + y + z = 1$ on xy–plane is $x + y = 1$, $z = 0$. In which y varies from $y = 0$ to $y = 1 - x$ and x moves from $x = 0$ t $x = 1$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \int_0^1\int_0^{1-x}(-2)\,dydx$$
$$= -2\int_0^1 [y]_{-1}^{1-x}\,dx$$
$$= -2\int_0^1 (1 - x)dx = -2\left[x - \frac{x^2}{2}\right]_0^1 = -2\left(1 - \frac{1}{2}\right) = -1.$$

Thus, by (i), $\oint_C \vec{F}.d\vec{r} = -1$.

$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, S is the hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

ution: Given that $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$.

And the surface is a hemisphere, $z = (a^2 - x^2 - y^2)^{1/2}$.

By Stoke's theorem we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_C \vec{F}.d\vec{r} \quad \text{.........(i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

Since we have,

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + \sqrt{a^2 - x^2 - y^2}\,\vec{k}$$

Then,

$$\vec{r_x} = \vec{i} - \frac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad \vec{r_y} = \vec{j} - \frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

So that,

$$\vec{N} = \vec{r_x}\times\vec{r_y}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -x/\sqrt{a^2-x^2-y^2} \\ 0 & 1 & -y/\sqrt{a^2-x^2-y^2} \end{vmatrix} = \frac{x}{\sqrt{a^2-x^2-y^2}}\vec{i} + \frac{y}{\sqrt{a^2-x^2-y^2}}\vec{j} + \vec{k}$$

Then, $(\nabla\times\vec{F}).\vec{N} = \frac{x}{\sqrt{a^2-x^2-y^2}} + \frac{y}{\sqrt{a^2-x^2-y^2}} + 1 = \frac{x+y}{\sqrt{a^2-x^2-y^2}} + 1.$

Given surface is a hemisphere $z = \sqrt{a^2-x^2-y^2}$ that has radius $r = a$.

Set $x = r\cos\theta$, $y = r\sin\theta$ then $dx\,dy = r\,dr\,d\theta$. And the angular region moves from $\theta = 0$ to $\theta = 2\pi$.

Then,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \int_0^{2\pi}\int_0^{a}\left[\frac{r(\cos\theta+\sin\theta)}{\sqrt{a^2-r^2(\cos^2\theta+\sin^2\theta)}} + 1\right] r\,dr\,d\theta$$

$$= \int_0^{a}\int_0^{2\pi}\left(\frac{r^2(\cos\theta+\sin\theta)}{\sqrt{a^2-r^2}} + r\right) d\theta\,dr$$

$$= \int_0^{a}\left[\frac{r^2(\sin\theta-\cos\theta)}{\sqrt{a^2-r^2}} + r\theta\right]_0^{2\pi}$$

$$= \int_0^{a}\left[\frac{r^2\times 0}{\sqrt{a^2-r^2}} + 2r\pi\right] dr = \int_0^{a}(2r\pi)\,dr = \pi[r^2]_0^a = \pi a^2$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = \pi a^2$.

13. $\vec{F} = 2y\vec{i} + e^z\vec{j} - \tan^{-1}x\vec{k}$ and S is the portion of the paraboloid $z = 4 - x^2 - y^2$ cut off by the xy–plane.

Solution: Given that $\vec{F} = 2y\vec{i} + e^z\vec{j} - \tan^{-1}x\vec{k}$

and the surface is $z = 4 - x^2 - y^2$ that cut off by xy–plane.

By stake's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl}\,\vec{F}.\vec{N})\,ds \quad \ldots\ldots\ldots (i)$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{k} = (0, 0, 1)$.

Here,

$$\text{curl}.\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & e^z & -\tan^{-1}x \end{vmatrix} = \left(e^z, \frac{1}{1+x^2}, -2\right).$$

Then, curl $\vec{F}.\vec{N} = -2$.

Given that the surface $z = 4 - x^2 - y^2$ is cut off by xy–plane. So, on the projection of the surface in xy–plane is $x^2 + y^2 = 4$. This is a circle with radius 2 and angular variation is 2π. Therefore, (i) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\int_0^{2}\int_0^{2\pi} r\,d\theta\,dr \quad \text{being the parabolid is downward}$$

$$= -2\left[\frac{r^2}{2}\right]_0^2 [\theta]_0^{2\pi} = -2\left(\frac{4-0}{2}\right)(2\pi - 0) = -8\pi.$$

$\vec{F} = y^2\vec{i} + 2x\vec{j} + 5y\vec{k}$, S is the hemisphere $z = (4 - x^2 - y^2)^{1/2}$.

Solution: Similar to Q. No. 12

EXERCISE 4.11

Find the line integral $\int_c \vec{F}.d\vec{r}$

$\vec{F} = (y\cos xy, x\cos xy, e^z)$, C is the straight line segment from $(\pi, 1, 0)$ to $\left(\frac{1}{2}, \pi, 1\right)$

Solution: Given that $\vec{F} = (y\text{Cos}xy, x\text{Cos}xy, e^z)$.

And the line is from $(\pi, 1, 0)$ to $\left(\frac{1}{2}, \pi, 1\right)$

Since we have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

So that,

$$\vec{F}.d\vec{r} = (y\text{Cos}xy, x\text{Cos}xy, e^z).(dx, dy, dz)$$
$$= y\text{Cos}xy\,dx + x\text{Cos}xy\,dy + e^z dz = d(\text{Sin}xy) + d(e^z).$$
$$= d(\text{Sin}xy + e^z).$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_{(\pi,1,0)}^{(1/2,\pi,1)} d(\text{Sin}xy + e^z) = [\text{Sin}xy + e^z]_{(\pi,1,0)}^{(1/2,\pi,1)}$$

$$= \left(\sin\frac{\pi}{2} + e^1\right) - (\text{Sin}\pi + e^0)$$
$$= 1 + e - 0 - 1 = e.$$

Thus, $\int_c \vec{F}.d\vec{r} = e.$

$\vec{F} = (y^2, 2xy + \sin x, 0)$, C the boundary of $0 \le x \le \pi/2$, $0 \le y \le 2$, $z = 0$.

Solution: Given that, $\vec{F} = (y^2, 2xy + \text{Sin}x, 0)$

And the surface is bounded by the boundaries $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 2$, $z = 0$.

Since the surface is a closed surface, by Stoke's theorem we have,

$$\int_c \vec{F}.d\vec{r} = \iint_s (\nabla\times\vec{F}).\vec{N}\, dx\, dy \;\ldots\ldots (i)$$

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$ [Being z = 0]

Then, $\vec{N} = \vec{r_x}\times\vec{r_y} = \vec{k} = (0, 0, 1)$.

Here,

$$(\nabla\times\vec{F}).\vec{N} = (0, 0, \text{Cos}x).(0, 0, 1) = \text{Cos}x.$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\, dx\, dy = \int_0^2\int_0^{\pi/2} \text{Cos}x\, dx\, dy$$

$$= \int_0^2 [\text{Sin}x]_0^{\pi/2}\, dy = \int_0^2 dy \quad [\because \text{Sin}\frac{\pi}{2} = 1, \text{Sin}0 = 0]$$

$$= [y]_0^2 = 2$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 2$.

3. $\vec{F} = (\cos\pi y, \sin\pi x, \cos\pi x)$, **C the boundary of** $0 \le x \le \frac{1}{2}, 0 \le y \le 4, z = x$.

Solution: Given that, $\vec{F} = (\text{Cos}\pi x, \text{Sin}\pi x, \text{Cos}\pi x)$.

And the region is bounded by $0 \le x \le \frac{1}{2}$, $0 \le y \le 4$, $z = x$

Since the region is a closed surface, by Stoke's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_s (\nabla\times\vec{F}).\vec{N}\, dxdy \;\ldots\ldots (i)$$

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + x\vec{k}$ [Being z = x]

Then, $\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} = (-1, 0, 1)$.

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (0, \pi\,\text{Sin}\pi x, \pi\,(\text{Cos}\pi x + \text{Sin}\pi y).(-1, 0, 1).$$
$$= \pi(\text{Cos}\pi x + \text{Sin}\pi y).$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\, dxdy = \pi\int_0^4\int_0^{1/2} (\text{Cos}\pi x + \text{Sin}\pi y)\, dxdy$$

$$= \pi\int_0^4 \left[\frac{\sin\pi x}{\pi} + x\sin\pi y\right]_0^{1/2} dy$$

$$= \frac{\pi}{\pi}\int_0^4 \left[\sin\frac{\pi}{2} + \pi.\frac{1}{2}\sin\pi y\right] dy \quad [\because \sin 0 = 0]$$

$$= \int_0^4 (1 + \frac{\pi}{2}\,\text{Sin}\pi y)\, dy \quad [\because \text{Sin}\frac{\pi}{2} = 1]$$

$$= \left[y - \frac{\pi}{2}\cdot\frac{\cos\pi y}{\pi}\right]_0^4$$

$$= \left(4 - \frac{1}{2}\cos 4\pi\right) - \left(0 - \frac{1}{2}\cos 0\right)$$

$$= 4 - \frac{1}{2} - 0 + \frac{1}{2} \quad [\because \text{Cos}0 = \text{Cos}4\pi = 1]$$

Thus, by (i) $\oint_c \vec{F}.d\vec{r} = 4$.

$\vec{F} = (8xy, 4x^2, 2\cos 2z)$, **C the helix** $\vec{r} = (\cos t, \sin t, t)$, $0 \le t \le \pi/4$.

...tion: Given that, $\vec{F} = (8xy, 4x^2, 2\,\text{Cos}2z)$.

And the surface is a helix, $\vec{r} = (\text{Cos}t, \text{Sin}t, t)$ for $0 < t < \frac{\pi}{4}$.

Since we have $\vec{r} = (x, y, z)$. So, comparing with given term then, we see that,

$x = \text{Cos}t, \quad y = \text{Sin}t, \quad z = t.$

Therefore, $\vec{F} = (8\,\text{Cos}t\,\text{Sin}t, 4\,\text{Cos}^2t, 2\,\text{Cos}2t)$.

Since $\vec{r} = (\text{Cos}t, \text{Sin}t, t)$. So, $d\vec{r} = (-\text{Sin}t, \text{Cos}t, 1)\, dt$. So that,

$$\begin{aligned}
\vec{F}.d\vec{r} &= [-8\,\text{Cos}t\,\text{Sin}^2t + \text{Cos}^3t + 2\,\text{Cos}2t]\, dt \\
&= [-8\,\text{Cos}t\,(1 - \text{Cos}^2t) + 4\,\text{Cos}^3t + 2\,\text{Cos}2t]\, dt \\
&= [-8\,\text{Cos}t + 8\,\text{Cos}^2t + 4\,\text{Cos}^3t + 2\,\text{Cos}2t]\, dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cos}t + 12\,\text{Cos}^3t]\, dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cos}t + 12\,\text{Cos}t\,(1 - \sin^2t)]\, dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cos}t + 12\,\text{Cos}t - 12\,\text{Sin}^2t\,\text{Cos}t]\, dt \\
&= [2\,\text{Cos}2t + 4\,\text{Cos}t - 12\,\text{Sin}^2t\,\text{Cos}t]\, dt
\end{aligned}$$

Now,

$$\oint_c \vec{F}.d\vec{r} = \int_0^{\pi/4} (2\,\text{Cos}2t + 4\,\text{Cos}t - 12\,\text{Sin}^2t\,\text{Cos}t)\, dt$$

$$= [\text{Sin}2t + 4\,\text{Sin}t\,]_0^{\pi/4} - 12\int_0^{\pi/4} \text{Sin}^2t\cos t\, dt$$

$$= \text{Sin}\frac{\pi}{2} + 4\,\text{Sin}\frac{\pi}{4} - 12\int_0^{\pi/4} \text{Sin}^2t\cos t\, dt$$

$$= 1 + 4\left(\frac{1}{\sqrt{2}}\right) - 12\int_0^{\pi/4} \text{Sin}^2 t \cos t \, dt$$

Set Sint = u then Cost dt = du. Also, $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$. Then,

$$\oint_c \vec{F}.d\vec{r} = 1 + 2\sqrt{2} - 12\int_0^{1/\sqrt{2}} u^2 \, du$$

$$= 1 + 2\sqrt{2} - 12\left[\frac{u^3}{3}\right]_0^{1/\sqrt{2}}$$

$$= 1 + 2\sqrt{2} - 12\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= 1 + 2\sqrt{2} - 4\frac{1}{2\sqrt{2}}$$

$$= 1 + 2\sqrt{2} - \sqrt{2} = 1 + \sqrt{2}(2-1) = 1 + \sqrt{2}.$$

5. $\vec{F} = (e^x, e^y, e^z)$, **C:** $x = \log y, z = \log y, 1 \le y \le 2$.

Solution: Given that, $\vec{F} = (e^x, e^y, e^z)$ and region is $x = \log y = z$; $1 \le y \le 2$.

Then, $\vec{F} = (e^{\log y}, e^y, e^{\log y}) = (y, e^y, y)$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \log y\,\vec{i} + y\vec{j} + \log y\,\vec{k}$$

So, $d\vec{r} = \left(\frac{1}{y}\vec{i} + \vec{j} + \frac{1}{y}\vec{k}\right) dy$

Then,

$$\vec{F}.d\vec{r} = \left[(y, e^y, y).\left(\frac{1}{y}, 1, \frac{1}{y}\right)\right] dy = (1 + e^y + 1)dy$$
$$= (2 + e^y)\, dy.$$

Now,

$$\oint_c \vec{F}.d\vec{r} = \int_1^2 (2 + e^y)\, dy = \int_1^2 (1 + e^y + 1)\, dy$$

$$= \int_1^2 (2 + e^y)\, dy$$

$$= [2y + e^y]_1^2 = 4 + e^2 - 2 - e^1 = 2 + e^2 - e^1$$

Thus, $\oint_c \vec{F}.d\vec{r} = 2 + e^2 - e^1$.

6. $\vec{F} = (x^3, e^{3y}, e^{-3z})$, **C :** $x^2 + 9y^2 = 9, z = x^2$.

Solution: Given that, $\vec{F} = (x^3, e^{3y}, e^{-3z})$ and the region is $x^2 + 9y^2 = 9$, $z = x^2$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + x^2\vec{k}$$

So, $\vec{r_x} = \vec{i} + 2x\vec{k} = (1, 0, 2x)$, $\vec{r_y} = \vec{j} = (0, 1, 0)$

By Stoke's theorem we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\, dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$

Here, $\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 0 \end{vmatrix} = (-2x, 0, 1).$

And,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^3 & e^{3y} & e^{-3x^2} \end{vmatrix} = (0, 6x\, e^{-3x^2}, 0)$$

Then,

$$(\nabla\times\vec{F}).\vec{N} = (0, 6x\, e^{-3x^2}, 0).(-2x, 0, 1) = 0 + 0 + 0 = 0.$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\, dxdy = \iint_s 0\, dxdy = 0$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 0$.

$\vec{F}$ = **(sinπx, z, 0), C the boundary of the triangle with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0).**

Solution: Given that, $\vec{F} = (\text{Sin}\pi x, z, 0)$.

And the surface is a triangle that has vertices (0, 0, 0), (1, 0, 0) and (1, 1, 0). Therefore, the surface is the plane z = 0 that passes all through points.

We have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j}$ $\quad [\because z = 0]$

So $\vec{r_x} = \vec{i} = (1, 0, 0)$ and $\vec{r_y} = \vec{j} = (0, 1, 0)$

By Stoke's theorem we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\, dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$

Here, $\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \vec{k} = (0, 0, 1).$

And,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \sin\pi x & z & 0 \end{vmatrix} = -\vec{i}+0\vec{j}+\vec{k} = (-1, 0, 0).$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-1, 0, 0).(0, 0, 1) = 0+0+0 = 0.$$

Now, $\iint_S (\nabla\times\vec{F}).\vec{N}\, dxdy = \iint_S 0\, dx\, dy = 0.$

Then (i) gives, $\oint_C \vec{F}.d\vec{r} = 0.$

B. Find $\iint_S \vec{F}.\vec{n}\, dA$, where

1. $\vec{F} = (x, y)$, S: $z = 2x + 5y$, $0 \le x \le 2$, $-1 \le y \le 1$.

Solution: Given that, $\vec{F} = (x, y) = (x, y, 0)$

and S is $z = 2x + 5y$ for $0 \le x \le 2, -1 \le y \le 1$.

Since we have, $\vec{r} = (x, y, z) = (x, y, 2x + 5y)$

So, $\vec{r}_x = (1, 0, 2)$ and $\vec{r}_y = (0, 1, 5)$.

Now, $I = \iint_S \vec{F}.\vec{n}\, dA$ ....(i)

with $\vec{n} = \vec{r}_x\times\vec{r}_y$

Here,

$$\vec{n} = \vec{r}_x\times\vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{vmatrix} = -2\vec{i} - 5\vec{j} + \vec{k}$$

Then, $\vec{F}.\vec{n} = -2x - 5y$.

Therefore (i) becomes,

$$I = \int_0^2\int_{-1}^1 (-2x-5y)\, dy\, dx$$

$$= -\int_0^2 \left[2xy + \frac{5y^2}{2}\right]_{-1}^1 dx$$

$$= -\int_0^2 \left(2x+\frac{5}{2}+2x-\frac{5}{2}\right) dx \quad = -4\int_0^2 x\, dx$$

$$= -4\left[\frac{x^2}{2}\right]_0^2 = -4\left(\frac{4}{2}-0\right) = -8.$$

$\vec{F} = (0, 20y, 2z^3)$, S: the surface of $0 \le x \le 6$, $0 \le y \le 1$, $0 \le z \le y$.

lution: Given that, $\vec{F} = (0, 20y, 2z^3)$.

And the surface is bounded by $0 \le x \le 6$, $0 \le y \le 1$, $0 \le z \le y$.

By Gauss divergence theorem we have,

$$\iiint_T \operatorname{div}\vec{F}\, dv = \iint_S \vec{F}.\vec{n}\, dA \quad \text{....... (i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x}+\vec{j}\frac{\delta}{\delta y}+\vec{k}\frac{\delta}{\delta z}\right).(0, 20y, 2z^3)$$
$$= 0 + 20 + 6z^2 = 20 + 6z^2$$

Then,

$$\iiint_T \operatorname{div}\vec{F}\, dv = \int_0^6\int_0^1\int_0^{6y} (20+6z^2)\, dz\, dy\, dx$$

$$= \int_0^6\int_0^1 [20z + 2z^3]_0^y\, dydx$$

$$= \int_0^6\int_0^1 (20y + 2y^3)\, dy\, dx$$

$$= \int_0^6 \left[10y^2 + \frac{2y^4}{4}\right]_0^1 dx$$

$$= \int_0^6 \left[10 + \frac{2}{4} - 0\right] dx$$

$$= \frac{42}{4}\int_0^6 dx = \frac{21}{2}[x]_0^6 = \frac{21}{2}\times 6 = 63.$$

Thus by (i), $\iint_S \vec{F}.\vec{n}\, dA = 63.$

$\vec{F} = (0, x^2, -xz)$, S: $\vec{r} = (u, u^2, v)$, $0 \le u \le 1, -2 \le v \le 2$.

lution: Given that, $\vec{F} = (0, x^2, -xz)$. And $\vec{r} = (u, u^2, v)$.

So $\vec{r_u} = (1, 24, 0)$ and $\vec{r_v} = (0, 0, 1)$

Then,

$$\vec{N} = \vec{r_u}\times\vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 24 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (24, -1, 0).$$

Comparing $\vec{r} = (u, u^2, v)$ with $\vec{r} = (x, y, z)$ then we get,

$x = u, \quad y = u^2, \quad z = v.$

Then, $\vec{F}(\vec{r}) = (0, u^2, -uv)$.

So,

$$\vec{F}(\vec{r}).\vec{N} = (0, u^2, -uv).(2u, -1, 0) = 0 - u^2 + 0 = -u^2.$$

Now,

$$\iint_R \vec{F}(\vec{r}).\vec{N}\, du\, dv = \int_0^1 \int_{-2}^{2} (-u^2)\, dv\, du$$

$$= \int_0^1 [-u^2 v]_{-2}^{2}\, du = \int_0^1 [-u^2 (2+2)\, du$$

$$= -4\int_0^1 u^2\, du = -4\left[\frac{u^3}{3}\right]_0^1 = \frac{-4}{3}.$$

Then by (i), $\iint_S \vec{F}.\vec{N}\, dA = \frac{-4}{3}$.

4. $\vec{F} = (1, 1, 1)$, **S:** $x^2 + y^2 + 4z^2 = 4, z \geq 0$.

Solution: Given that $\vec{F} = (1, 1, 1) = \vec{i} + \vec{j} + \vec{k}$.

And the surface is $x^2 + y^2 + 4z^2 = 4,\ z \geq 0$.

By Gauss divergence theorem, we have,

$$\iiint_T \text{div}\vec{F}\, dv = \iint_S \vec{F}.\vec{n}\, dA \quad \text{.........} (i)$$

Here,

$$\text{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\vec{i} + \vec{j} + \vec{k})$$

$$= 0 + 0 + 0 = 0$$

Then,

$$\iiint_T \text{div}\vec{F}\, dv = \iiint_T 0\, dv = 0$$

Thus by (i), $\iint_S \vec{F}.\vec{n}\, dA = 0$.

5. $\vec{F} = (x + z, y + z, x + y)$, **S is the sphere** $x^2 + y^2 + z^2 = 9$.

Solution: Given that, $\vec{F} = (x + z, y + z, x + y)$.

And the surface is a sphere, $x^2 + y^2 + z^2 = 9$.

Then, div. $\vec{F} = \nabla.\vec{F} = 1 + 1 + 0 = 2$

Clearly on the sphere, $z = \pm\sqrt{9 - x^2 - y^2}$ and on the projection in xy–plane, $y = \pm\sqrt{9 - x^2}$. And, x moves from $x = -3$ to 3.

Clearly, the sphere has symmetrical time hemisphere.

So, $z = 0$ to $\sqrt{9 - x^2 - y^2}$, $y = 0$ to $\sqrt{1 - x^2}$ and $x = 0$ to 3.

Now, by Gauss divergence theorem,

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint_V \text{div}.\vec{F}\, dv$$

$$= 2^3 \int_0^3 \int_0^{\sqrt{3 - x^2}} \int_0^{\sqrt{9 - x^2 - y^2}} 2\, dz\, dy\, dx$$

$$= 16 \int_0^3 \int_0^{\sqrt{9 - x^2}} \sqrt{9 - x^2 - y^2}\, dy\, dx$$

$$= 16 \int_0^3 \left[\frac{y}{2}\sqrt{9 - x^2 - y^2} + \left(\frac{3 - x^2}{2}\right)\sin^{-1}\left(\frac{y}{\sqrt{9 - x^2}}\right)\right]_0^{\sqrt{9 - x^2}} dx$$

$$= 16 \int_0^3 \left(\frac{9 - x^2}{2}\right)\sin^{-1}(1)\, dx$$

$$= \frac{16\pi}{4}\int_0^3 (9 - x^2)\, dx \qquad [\because \sin^{-1}(1) = \pi/2]$$

$$= 4\pi\left[9x - \frac{x^3}{3}\right]_0^3 = 4\pi(27 - 9) = 72\pi.$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = 72\pi$.

OTHER IMPORTANT QUESTION FROM FINAL EXAM

VELOCITY, ACCELARATION, GRADIENT, DIVERGENCE, CURL, DIRECTIONA; DERIVATIVES, SOLENOIDAL, IRROTATIONAL, CONSERVATIVE

14 Fall Q.No. 4(b)

For curve $x = 3t, y = 3t^2, z = 2t^3$ show that $[\vec{r}, \vec{r}', \vec{r}''] = 180$ at $t = 1$.

lution: Given that $x = 3t, y = 3t^2, z = 2t^3$. Then $\vec{r} = (3t, 3t^2, 2t^3)$.

Then,

$\vec{r}' = (3, 6t, 6t^2)$ and $\vec{r}'' = (0, 6, 12t)$.

Now,

$$[\vec{r}, \vec{r}', \vec{r}''] = \begin{vmatrix} 3t & 3t^2 & 2t^3 \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix}$$

$$= 3t(72t^2 - 36t^2) - 3(36t^3 - 12t^3) = 108t^3 + 72t^3. = 180t^3.$$

Thus, at t = 1,

$$[\vec{r}, \vec{r}', \vec{r}''] = 180.$$

2014 Spring Q. No. 2(a)

Define directional derivative of f in the direction of $\vec{a}$, find the directional derivative of $f = 4xz^3 - 3x^2yz^2$ in the direction of z–axis at P(2, –1, 2).

Solution: First Part: See the definition of directional derivative.

Second Part: See Exercise 4.2 Q. No. 3(vii).

2014 Fall Q. No. 4(a)

Define directional derivative of a function f in the direction of $\vec{a}$. Find the directional derivative of a function $f = x^2 - y^2 + 2z^2$ at the point A(1, 2, 3) in the direction of $\vec{a} = \vec{i} + \vec{j} + \vec{k}$.

Solution: First Part: See the definition of directional derivative.

Second Part: Given surface is, $f = x^2 - y^2 + 2z^2$

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = 2x\vec{i} - 2y\vec{j} + 4z\vec{k}$$

At point A(1, 2, 3), grad (f) $= 2\vec{i} - 4\vec{j} + 12\vec{k}$.

Also given that $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

Then the unit vector of $\vec{a}$ is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1+1+1}} = \frac{1}{3}(\vec{i} + \vec{j} + \vec{k}).$$

Now the directional derivative of f along $\vec{a}$ at p is,

$$\nabla f . \hat{a} = \left(2\vec{i} - 4\vec{j} + 12\vec{k}\right) . \left(\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})\right)$$

$$= \frac{1}{\sqrt{3}}(2 - 4 + 12) = \frac{1}{\sqrt{3}}(10).$$

2012 Fall Q.No. 3(b)

Show that the vector $F = (x^2 - yz)i + (x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is conservative and find $\varnothing$ such that $F = \nabla\varnothing$.

Solution: Given that,

$$\vec{F} = (x^2 - yz)\vec{i} + (x^2y + xz + 2yz^2)\vec{j} + (2y^2z + xy)\vec{k}$$

function $\vec{F}$ is conservative only if curl $\vec{F} = 0$.

e,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= (4yz + x - x - 4yz)\vec{i} + (-y - y)\vec{j} + (2xy + z - z)\vec{k}$$

$$= -2y\vec{j} + 2xy\vec{k}.$$

This shows that the function is not a conservative.

e: This shows question should be corrected as

$$\vec{F} = (2xy^2 + yz)\vec{i} + (2x^2y + xz + 2yz^2)\vec{i} + (2y^2z + xy)\vec{k}$$

see Ex. 4.5, Q. O(iv).

l Q.No. 4(a); 2010 Spring Q.No. 6(b)

$\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$, find (i) div (curl $\vec{v}$) and (ii) curl (curl $\vec{v}$).

: Given that $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$.

n,

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= (xz^2 - xy^2)\vec{i} + (x^2y - yz^2)\vec{j} + (y^2z - x^2z)\vec{k}$$

w,

$$\text{Div. (curl } \vec{v}) = \nabla . (\text{curl } \vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\text{curl } \vec{v})$$

$$= (z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)$$

$$= 0$$

$$\text{curl (curl } \vec{v}) = \nabla \times (\text{curl } \vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^2 - xy^2 & x^2y - yz^2 & y^2z - x^2z \end{vmatrix}$$

$$= 4(yz\vec{i} + zx\vec{j} + xy\vec{k}).$$

pring Q.No. 3(a)

efine gradient of a scalar function. If $\varnothing = x^3 + y^3 + z^3 - 3xyz$. Find ∇(grad $\varnothing$) ; and curl (grad $\varnothing$).

n: Given that $\phi = x^3 + y^3 + z^3 - 3xyz$

hen,

$$\text{grad } \phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial y}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\vec{i} + (3y^2 - 3zx)\vec{j} + (3z^2 - 3xy)\vec{k}$$

Now,

$$\text{div(grad } \phi) = \nabla.(\text{grad } \phi)$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial y}\right).(\text{grad } \phi)$$
$$= 6x + 6y + 6z = 6(x + y + z)$$

And,

$$\text{curl(grad } \phi) = \nabla\times(\text{grad } \phi)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$
$$= (3x - 3x)\vec{i} - (3y + 3y)\vec{j} + (-3z + 3z)\vec{k} = 0.$$

Thus, div(grad ϕ) = $6(x + y + z)$ and curl (grad ϕ) = 0.

2010 Fall Q.No. 3(b)

Define directional derivative of the function f in the direction $\vec{a}$. Derive the expression of directional derivative of f in the direction $\vec{a}$. Find directional derivate of $f = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the surface S: $x\log z - y^2 + 4 = 0$ at (–1, 2, 1).

Solution: First Part: See the definition of directional derivative.

Second Part: See the derivation.

Third Part: See the solution of Exercise 4.2, Q. 3(viii).

2010 Spring Q. No. 4(c)

Find the directional derivative of the function $f = x^2 + 3y^2 + 4z^2$ at (1, 0, 1) in the direction of $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$.

Solution: Similar to 2010 Fall.

2009 Fall Q.No. 3(b)

Define Divergence and Curl of a vector. If $\varnothing = \log(x^2 + y^2 + z^2)$ find div (grad$\varnothing$) and curl (grad$\varnothing$).

Solution: First Part: See the definition of divergence and curl of a vector.

Second Part: See the solution of Exercise 4.3, Q. 11.

2005 Fall Q.No. 3(b)

If $\vec{v}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$ find div $\vec{v}$ and curl $\vec{v}$.

Solution: See the problem part of 2011 Spring.

2006 Spring Q.No. 3(b)

If $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$, and $\vec{v} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ find curl $(\vec{u}\times\vec{v})$ and grad$(\vec{u}.\vec{v})$.

Solution: Given that,

$$\vec{u} = y\vec{i} + z\vec{j} + x\vec{k} \text{ and } \vec{v} = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

hen,

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ y & z & x \\ yz & zx & xy \end{vmatrix}$$
$$= (xyz - zx^2)\vec{i} - (xy^2 - xyz)\vec{j} + (xyz - yz^2)\vec{k}$$

ow,

$$\text{curl}(\vec{u}\times\vec{v}) = \nabla\times(u\times\vec{v})$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz - zx^2 & xyz - xy^2 & xyz - yz^2 \end{vmatrix}$$
$$= (xz - z^2 - xy)\vec{i} + (xy - x^2 - yz)\vec{j} + (yz - y^2 - zx)\vec{k}.$$

d,

$$\vec{u}.\vec{v} = (y, z, x).(yz, zx, xy) = y^2z + z^2x + x^2y$$

,

$$\text{grad }(\vec{u}.\vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{u}.\vec{v})$$
$$= (z^2 + 2xy)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 2zx)\vec{k}.$$

pring Q. No. 2(a) OR; 2008 Spring Q.No. 3(a)

fine directional derivative of function f(x) in the direction $\vec{a}$. Find directional rivative of $f = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the rface S: $x\log z - y^2 + 4 = 0$ at (–1, 2, 1).

n: See the first and third part of 2010 Fall.

pring Q.No. 3(a) OR

fine Divergence and Curl of a vector function. If f be a continuous and fferential scalar values function then prove that curl (grad f) = 0.

n: First Part: See the definitions.

Second Part: See the relative theorem.

all Q.No. 3(a)

efine divergence and curl of a vector. Define directional derivative of f in the rection of $\vec{a}$. Find the directional derivative of $f = x^2 + 3y^2 + 4z^2$ in the rection $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$ at P(1, 0, 0).

on: First Part: See the definition of divergence, curl of a vector directional rivative.

nd Part: Given surface is, $f = x^2 + 3y^2 + 4z^2$

hen,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = 2x\vec{i} + 6y\vec{j} + 8z\vec{k}$$

At point P(1, 0, 0), grad (f) $= 2\vec{i}$.

Also given that $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$

Then the unit vector of $\vec{a}$ is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-\vec{i} - \vec{j} + \vec{k}}{\sqrt{1+1+1}} = \frac{1}{3}(-\vec{i} - \vec{j} + \vec{k})$$

Now the directional derivative of f along $\vec{a}$ at p is,

$$\nabla f . \hat{a} = 2\vec{i} . \left(\frac{1}{\sqrt{3}}(-\vec{i} - \vec{j} + \vec{k})\right) = -\frac{1}{\sqrt{3}}$$

2004 Spring Q.No. 3(a) OR

If $\vec{v} = x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}$, find: i. div $\vec{v}$ ii. curl $\vec{v}$.

Solution: Given that $\vec{v} = x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}$

Then,

$$\text{div}.(\vec{v}) = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{v} = 2xy + 0 + 2y = 2y(x+1).$$

And,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= (2z - x)\vec{i} + (0 - 0)\vec{j} + (z - x^2)\vec{k}$$

$$= (2z - x)\vec{i} + (z - x^2)\vec{k}.$$

2004 Fall Q.No. 3(a)

If $\vec{r_1} = x^2yz\vec{i} - 2xz^2\vec{f} + xz^2\vec{k}$ and $\vec{r_2} = 2z\vec{i} - y\vec{j} + x^2\vec{k}$ find the value $\frac{\delta^2}{\delta y\delta x}(\vec{r_1}\times\vec{r_2})$.

Solution: Given that, $\vec{r}_1 = x^2yz\vec{i} - 2xz^2\vec{j} + xz^2\vec{k}$

Then,

$$\vec{r}_1\times\vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2yz & -2xz^2 & xz^2 \\ 2z & -y & x^2 \end{vmatrix}$$

$$= (-2x^3z^2 + xyz^2)\vec{i} - (x^4yz - 2xz^3)\vec{j} + (-x^2y^2z + 4xz^3)\vec{k}$$

Now,

$$\frac{\partial^2}{\partial y\partial x}(\vec{r}_1\times\vec{r}_2) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(\vec{r}_1\times\vec{r}_2)\right)$$

$$= \frac{\partial}{\partial y}\{(-6x^2z^2 + xz^2)\vec{i} + (2z^3 - 4x^3yz)\vec{j} + (4z^3 - 2xy^2z)\vec{k}\}$$

$$= 0.\vec{i} - 4x^3z\vec{j} - 4xyz\vec{k}$$

$$= -4xz(x^2\vec{j} + y\vec{k}).$$

all Q.No. 3(b)

$\vec{u} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$ find curl (curl $\vec{u}$).

n: Similar to 2011 Fall Q. No. 4(a–ii).

ll Q.No. 3(a) OR

d the directional derivative of f(xyz) $= 2x^2 + 3y^2 + z^2$ at the point (2, 1, 3) in direction of vector $\vec{a} = \vec{i} - 2\vec{k}$.

: Similar to problem part of 2007 Fall Q. No. 3(a).

ll Q.No. 3(b)

ine divergence and curl of a vector $\vec{v}$. If $\vec{v}$ is the vector function, then prove div (curl $\vec{v}$) = 0.

: **First Part:** See the definition of divergence and curl of a vector.

ond Part: Similar to 2007 Fall 3(a).

ing Q.No. 3(a)

$\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and $\vec{r_2} = t^2\vec{i} - t\vec{j} + (t-1)\vec{k}$. Find $\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$.

: Given that

$\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^2\vec{k}$ and $\vec{r_2} = t^2\vec{i} - t\vec{j} + (t-1)\vec{k}$

n,

$$\vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ t^2 & -t & t-1 \end{vmatrix}$$

$$= (t^2 - t^3 + 3t^4) - (2t^2 - 2t + t - 1 - 3t^5)\vec{j} + (t^4 - 2t^2 - t)\vec{k}$$

v,

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = (2t - 3t^2 + 12t^3) - (4t - 1 - 15t^4)j + (4t3 - 4t - 1)k.$$

ing Q.No. 3(b)

Solution: Given that $\phi = \log(x^2 + y^2 + z^2)$

Then,

$$\text{grad } \phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\frac{1}{x^2 + y^2 + z^2}\right)(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

Now,

$$\text{div. (grad } \phi) = \nabla.\text{ (grad } \phi)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left\{\left(\frac{1}{x^2 + y^2 + z^2}\right)(x\vec{i} + y\vec{j} + z\vec{k})\right\}$$

$$= \left(\frac{2}{x^2 + y^2 + z^2}\right)(1 + 1 + 1)$$

$$= \frac{6}{x^2 + y^2 + z^2}$$

2002 Q.No. 3(a) OR

Find the derivative of $\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right]$.

Solution: Let $R = [\vec{r}, \vec{r}', \vec{r}'']$

Then

$$\frac{dR}{dt} = \left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^3\vec{r}}{dt^3}\right]$$

Since in a scalar triple product if two component has same value then the product value is zero. So,

$$\left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right] = 0 = \left[\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

Therefore, the derivative of $\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right]$ is $\left[\vec{r}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^3\vec{r}}{dt^3}\right]$.

2002 Q.No. 3(b)

If $\vec{f} = x^2y\vec{i} - xz\vec{j} + 4yz\vec{k}$ find div (curl $\vec{f}$).

Solution: Given that $\vec{f} = x^2y\vec{i} - xz\vec{j} + 4yz\vec{k}$

Then,

$$\text{curl } \vec{f} = \nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -xz & 4yz \end{vmatrix}$$

$$= (4z + x)\vec{i} + (0 - 0)\vec{j} + (-z - x^2)\vec{k}$$

$$= (4z + x)\vec{i} - (x^2 + z)\vec{k}$$

Now,

$$\text{div.(curl } \vec{f}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\text{curl } \vec{f}) = 1 - 1 = 0.$$

002 Q.No. 3(a)

Find the directional derivatives of the function $f = xy + yz + zx$ in the direction of the vector $\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$ at the point (3, 1, 2).

lution: Similar to 2007 Fall 3(a).

002 Q.No. 3(a) OR

Define divergence and curl of vector $\vec{v}$. If $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$, find (i) div $\vec{v}$ and (ii) curl $\vec{v}$.

lution: See the definition of Divergence.

For problem part, see exam question solution of 2011 Fall.

SIMPLE INTEGRATION, LINE INTEGRAL, WORK DONE, FLUX, EXACTNESS

14 Spring Q.No. 3(a)

Show that the value under the integral sign is exact and evaluate the integral

$$\int_{(4,0,3)}^{(-1,1,2)} [(yz + 1)dx + (xz + 1)dy + (xy + 1)dz]$$

lution: Given integral is, $I = \displaystyle\int_{(4,0,3)}^{(-1,1,2)} [(yz + 1)dx + (xz + 1)dy + (xy + 1)dz]$ ... (i)

Comparing the value under the integral sign on (i) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = yz + 1,\quad F_2 = xz + 1,\quad F_3 = xy + 1$$

Then,

$$\frac{\partial F_1}{\partial y} = z,\ \frac{\partial F_1}{\partial z} = y,\quad \frac{\partial F_2}{\partial z} = x,\ \frac{\partial F_2}{\partial x} = z,\quad \frac{\partial F_3}{\partial x} = y,\ \frac{\partial F_3}{\partial y} = x$$

Here,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},\quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

So, the value in (i) is exact.

Now,

$$I = \int_{(4,0,3)}^{(-1,1,2)} [(yz + 1)dx + (xz + 1)dy + (xy + 1)dz]$$

$$= \int_{(4,0,3)}^{(-1,1,2)} d\left[\int (yz+1)dx + \int dy + \int dz\right]$$

$$= \int_{(4,0,3)}^{(-1,1,2)} d\,[xyz + x + y + z]$$

$$= [xyz + x + y + z]_{(4,\,0,3)}^{(-1,1,2)}$$

$$= (-2-1+1+2) - (0+4+0+3) = -7.$$

2006 Spring Q.No. 4(a)

What do you mean by exact integral? Show that the expression within the integral sign is exact and evaluate it,

$$\int_{(0,2,3)}^{(0,2,3)} (yz \sinh zx\, dx + \cosh zx\, dy + xy \sinh zx\, dz).$$

Solution: First Part – See the definition of exact definition.

Second Part – See solution of Exercise 4.6 Q. No. 5.

2005 Fall Q.No. 3(a)

Let f be a continuous and differentiable scalar valued function, then show that curl (grad f) = 0. And find unit normal on the surface $\vec{r} = e^x\vec{i} + e^z\vec{j} + e^y\vec{k}$ at (2, 3, 4). If $\vec{r} = 5t^2\vec{i} + t\,\vec{j} - t^3\vec{k}$, find $\int_1^2 \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) dt$.

Solution: Given that, $\vec{r} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$

Then, $\frac{d\vec{r}}{dt} = 10t\,\vec{i} + \vec{j} - 3t^2\vec{k}$

So that,

$$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= (-3t^3 + t^3)\vec{i} + (-10t^4 + 10t^4)\vec{j} + (5t^2 - 10t^2)\vec{k}$$

$$= -2t^3\vec{i} + 5t^4\vec{j} - 5t^2\vec{k}$$

Now,

$$\int_1^2 \left(\vec{r}\times\frac{d\vec{r}}{dt}\right) dt = \int_1^2 (-2t^3\vec{i} + 5t^4\vec{j} - 5t^2\vec{k})\, dt$$

$$= \left[-\frac{2t^4}{4}\vec{i} + t^5\vec{j} - \frac{5t^3}{3}\vec{k}\right]_1^2$$

$$= \frac{-2(16-1)}{4}\vec{i} + (32-1)\vec{j} - \frac{5(8-1)\vec{k}}{3}$$

$$= -\frac{15}{2}\vec{i} + 31\vec{j} - \frac{35}{3}\vec{k}$$

Spring Q.No. 4(b) OR

Show that the form under the integral sign is exact and evaluate

$$\int_{(0,\pi)}^{(3,\pi/2)} [e^x\cos y\, dx - e^x \sin y\, dy].$$

ion: Given integral is

$$I = \int_{(0,\pi)}^{(3,\pi/2)} [e^x \cos y dx - e^x \sin y\, dy] \quad \text{........(i)}$$

Comparing the value under the integral sign in (i) with $F_1dx + F_2dy$ then we get,

$F_1 = e^x \cos y$ and $F_2 = -e^x \sin y$

Then $\frac{\partial F_1}{\partial y} = -e^x \sin y$ and $\frac{\partial F_2}{\partial x} = -e^x \sin y$

Thus, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. So, the value is exact.

Now, (i) becomes,

$$I = \int_{(0,\pi)}^{(3,\pi/2)} d(e^x \cos y) = [e^x \cos y]_{(0,\pi)}^{(3,\pi/2)} = e^3 \cos\frac{\pi}{2} - e^0 \cos\pi = 0 - 1(-1) = 1$$

Thus,

$$I = \int_{(0,\pi)}^{(3,\pi/2)} (e^x \cos y\, dx - e^x \sin y\, dy) = 1.$$

Fall Q.No. 4(b) OR

Show that the form under the integral sign is exact and then evaluate

$$\int_{(0,0,0)}^{(a,b,c)} (2xy^2\, dx + 2x^2y\, dy + dz).$$

tion: Given integral is, $I = \int_{(0,0,0)}^{(a,b,c)} (2xy^2dx + 2x^2y\, dy + dz)$ ————(i)

Comparing the value under the integral sign on (i) with $F_1dx + F_2dy + F_3dz$ then we get,

$F_1 = 2xy^2$, $F_2 = 2x^2y$, $F_3 = 1$

Then,

$\frac{\partial F_1}{\partial y} = 4xy$, $\frac{\partial F_1}{\partial z} = 0$, $\frac{\partial F_2}{\partial z} = 4xy$, $\frac{\partial F_3}{\partial x} = 0 = \frac{\partial F_3}{\partial y}$

Here,

$$\frac{\partial F_1}{\partial y}=\frac{\partial F_2}{\partial x},\quad \frac{\partial F_1}{\partial z}=\frac{\partial F_3}{\partial x},\quad \frac{\partial F_2}{\partial z}=\frac{\partial F_3}{\partial y}$$

So, the value in (i) is exact.

Now,

$$I=\int_{(0,0,0)}^{(a,b,c)} d(x^2y^2+z)=[x^2y^2+z]_{(0,0,0)}^{(a,b,c)}=a^2b^2+c.$$

2011 Fall Q.No. 6(b); 2010 Spring Q.No. 5(b)

Calculate $\int_c \vec{f}.d\vec{r}$, where $\vec{f}$ = [cosh x, sinh y, e^z], C : $\vec{r}$ = [t, t^2, t^3] from (0, 0, 0) to (2, 4, 8).

Solution: Given that $\vec{f}$ = (cohx, sinhy, e^z) and $\vec{r}$ = (t, t^2, t^3).

Then $d\vec{r}$ = (1, 2t, $3t^2$) dt.

Since we know $\vec{r}$ = (x, y, z). So, comparing it with $\vec{r}$ = (t, t^2, t^3) we get,

$x = t,\ y = t^2,\ z = t^3$.

Then $\vec{f}$ = (cosht, sinh t^2, e^{t^3}) and t moves from 0 to 2.

So, $\vec{f}\,d\vec{r} = (\cosh t + 2t\sinh t^2 + 3t^2e^{t^3})\,dt$

Now,

$\int_c \vec{f}\,.\,d\vec{r}$ from (0, 0, 0) to (2, 4, 8) is

$$\int_{(0,0,0)}^{(2,4,8)} \vec{r}\,.\,d\vec{r}=\int_0^2(\cosh t+2t\sinh t^2+3t^2e^{t^3})\,dt$$

Put $t^2 = u$ and $t^3 = v$ then,

$$=\int_0^2 \cosh t\,dt+\int_0^4 \sinh u\,du+\int_0^8 e^v\,dv$$

$$=[\sinh t]_0^2+[\cosh u]_0^4+[e^v]_0^8$$

$$=\sinh 2+\cosh 4-1+e^8-1$$

$$=\sinh 2+\cosh 4+e^8-2.$$

2011 Spring Q.No. 4(a)

Prove that $\int_c \vec{F}.\,d\vec{r}=2\pi^2-8\pi$, where $\vec{F}$ = (x − y, y − z, z − x); C: (2cost, t, 2sint) from (2, 0, 0) to (2, 2π, 0).

Solution: Similar to 2011 Fall 6(b).

2011 Spring Q.No. 4(b)

Evaluate $\oint_c(-xy^2\,dx + x^2y\,dy)$, where c is the boundary of the region in the first quadrant bounded by $y = 1 - x^2$ counter clockwise.

Solution: By Greens theorem in plane, we get

$$\oint_c (F_1dx+F_2dy)=\iint_R\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dx\,dy \quad \text{........ (1)}$$

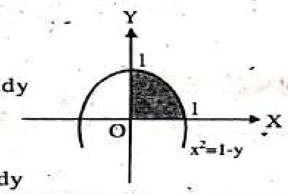
Then,

$$\oint_c(-xy^2dx+x^2ydy)=\iint_R(2xy+2xy)dxdy$$

$$=4\iint_R dy\,dxdy \quad \text{......... (2)}$$

We have C is the boundary of the region in the first quadrant bounded by $y = 1 - x^2$, show in figure below.

From (2),



$$\oint_c(-xy^2dx+x^2\,ydy)=4\int_0^1\int_0^{\sqrt{1-y}} xydxdy$$

$$=4\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y}}dy$$

$$=\frac{4}{2}\int_0^1 y(1-y)\,dy$$

$$=2\int_0^1(y-y^2)\,dy=2\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=2\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{3}$$

Thus, $\oint_c(-xy^2dx+x^2\,ydy)=\frac{1}{3}$.

2011 Fall Q.No. 5(b)

Find the flux integral of $\vec{F}$= [x, y, z] through the surface S, where S is the first octant portion of the plane 2x + 3y + z = 6.

Solution: Similar to the solution of 2011 Spring Q. 3(b).

2005 Fall Q.No. 4(b)

If $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 3z\vec{k}$, find the line integral of $\vec{F}$ along the curve $y = 3x$, $z = 2x$ from (0, 0, 0) to (1, 3, 2).

Solution: Given that $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 3z\vec{k}$

And along the curve $y = 3x$, $z = 2x$.

Since we have, $\vec{r} = (x, y, z) = (x, 3x, 2x)$. So, $d\vec{r} = (1, 3, 2)\,dx$.

Now, line integral of $\vec{F}$ along $y = 3x$, $z = 2x$ from (0, 0, 0) to (1, 3, 2) is

$$\int_c \vec{F}.d\vec{r} = \int_0^1 ((12x^2, 24x, 6x).(1, 3, 2))\,dx$$

$$= \int_0^1 (12x^2 + 72x + 12x)\,dx = [4x^3 + 36x^2 + 6x^2]_0^1$$

$$= 4 + 36 + 6 = 46.$$

Thus, $\int_c \vec{F}.d\vec{r} = 46$.

2004 Fall Q.No. 4(a)

Calculate $\int_c \vec{F}(\vec{r}).d\vec{r}$ if $\vec{F} = [xy, x^2y^2]$, where "c" is the quarter circle from (2, 0) to (0, 2) with centre at (0, 0).

Solution: Similar to 2012 Fall Q.4(a).

2004 Spring Q.No. 3(b)

Find the work done by the force $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 2\vec{k}$ along the curve $y = 2x$, $z = 2x$ from (0, 0, 0) to (3, 6, 6).

Solution: Given that $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 2\vec{k}$

And along the curve $y = 2x$, $z = 2x$

Since we have, $\vec{r} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$

i.e. $\vec{r} = x\vec{i} + 2x\vec{j} + 2x\vec{k}$

Then, $d\vec{r} = (\vec{i} + 2\vec{j} + 2\vec{k})\,dx$.

So that,

$$\vec{F}.d\vec{r} = (4xy\vec{i} + 8y\vec{j} + 2\vec{k}).(\vec{i} + 2\vec{j} + 2\vec{k})\,dx$$

$$= (8x^2\vec{i} + 16x\vec{j} + 2\vec{k}).(\vec{i} + 2\vec{j} + 2\vec{k})\,dx$$

$$= (8x^2 + 32x + 4)\,dx$$

Now, work done by $\vec{F}$ along $y = 2x$, $z = 2x$ from (0, 0, 0) to (3, 6, 6) is,

$$\int_c \vec{F}.d\vec{r} = \int_0^3 (8x^2 + 32x + 4)\,dx = \left[\frac{8x^3}{3} + 16x^2 + 4x\right]_0^3$$

$$= 8 \times 9 + 16 \times 9 + 12$$

$$= 9[8 + 16] + 12 = 216 + 12 = 228.$$

Thus, the work done by the force $\vec{F}$ along the given curve is 228.

2 Q.No. 4(a)

Evaluate the line integral $\oint_c \vec{F}.d\vec{r}$ counter clockwise around the boundary C of the region R where $\vec{F}$ = [Siny cosx], R be the triangle with vertices (0, 0), (π, 0), (π, 1).

tion: See the problem part of 2007 Fall.

Q.No. 4(a) OR

Evaluate the line integral of $\vec{F} = [3y^2, x - y^4]$ over C the square with vertices (1, 1), (−1, 1), (−1, −1), (1, −1) counter clockwise.

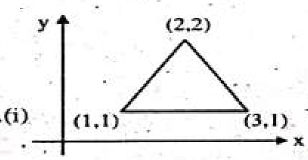
tion: Similar to 2007 Fall.

SURFACE INTEGRAL BY USING GREEN'S THEOREM

Fall Q.No. 4(a)

State Green theorem. Evaluate $\int_c [\sqrt{y}\,dx + \sqrt{x}\,dy]$ where C is the triangle with vertices (1,1), (3,1) and (2,2).

tion: We have to evaluate $\int_c (\sqrt{y}dx + \sqrt{x}dy)$ around the triangle having verticals (1, 1), (3, 1) and (2, 2).



By Green's theorem we have,

$$\int_c \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA \quad \text{..........(i)}$$

Comparing $\int_c (\sqrt{y}\,dx + \sqrt{x}\,dy)$ with $\int_c \vec{F}\,d\vec{r}$ then we get,

$\vec{F} = (\sqrt{y}, \sqrt{x})$ and $\vec{r} = (x, y)$.

Then,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sqrt{y} & \sqrt{x} & 0 \end{vmatrix} = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\right)\vec{k}$$

Then, $\quad \text{curl } \vec{F}.\vec{k} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right)$.

Since the region of I is as shown in figure with shaded portion in which y varies from x = y [equation of line passes through (1, 1) and (2, 2)] to y = 4 – x [equation of line passes through (3, 1) and (2, 2)] and on the region x moves from x = 1 to x = 2.

Then, (i) gives,

$$\oint_C \vec{F}\,d\vec{r} = \oint_C (\sqrt{y}\,dx + \sqrt{x}dy)$$

$$= \int_1^2 \int_x^{4-x} \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right) dy\,dx$$

$$= \frac{1}{2}\int_1^2 \left[\frac{y}{\sqrt{x}} - \frac{y^{1/2}}{1/2}\right]_x^{4-x} dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4-x-x}{\sqrt{x}} - 2\left(\sqrt{4-x} - \sqrt{x}\right)\right) dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4}{\sqrt{x}} - 2\sqrt{x} - 2\sqrt{4-x} + 2\sqrt{x}\right) dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4}{\sqrt{x}} - 2\sqrt{4-x}\right) dx$$

$$= \frac{1}{2}\left[\frac{4x^{1/2}}{1/2} - 2\left(\frac{\sqrt{x}\sqrt{4-x}}{2} + \frac{4}{2}\sin^{-1}\frac{\sqrt{x}}{2}\right)\right]_1^2$$

$$= \frac{1}{2}\left[8(\sqrt{2}-1) - (\sqrt{2}\sqrt{2} - \sqrt{3}) - 4\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[8\sqrt{2} - 2 + \sqrt{3} - 4\frac{\pi}{4} + \frac{4\pi}{6}\right] \quad \left[\because \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4},\ \sin^{-1}\frac{1}{2} = \frac{\pi}{6}\right]$$

$$= 4\sqrt{2} + \sqrt{3} - 5 - \frac{\pi}{6}$$

Thus, $\oint_C (\sqrt{y}\,dx + \sqrt{x}\,dy) = \frac{\pi}{6}$, around the triangle.

2011 Fall Q.No. 4(b)

State Green theorem. Use it to evaluate the integral $\vec{F}.d\vec{r}$, where $\vec{F} = [x^2, y^2]$, c: the square whose vertices are (0, 0), (1, 0), (1, 1), and (0, 1).

Solution: First Part: See the statement of Green's theorem.

Second Part: Similar problem for rectangle to 2012 Fall Q.No. 4(a).

closed curve consisting of the graph of $y = x^2$ and $y = 2x$ between the points (0,0) and (2,4).

lution: First Part: See the statement of Green's theorem.

Second Part: See Exercise 4.7 Q. No. 7.

08 Spring Q.No. 4(a)

State Green's theorem and use it to evaluate the integral $\oint_c (2xy^3dx + 3x^2y^2dy)$, c: $x^2 + y^2 = 1$ counter clockwise.

lution: First Part: See the statement of Green's theorem.

Second Part: See Exercise 4.7 Q. No. 4.

97 Fall Q.No. 3(b); 2003 Fall Q.No. 4(a)

State Green's theorem. Use it to evaluate the line integral $\oint_c \vec{F}(r).d\vec{r}$ counter clock wise where $\vec{F}$ = [siny, cosx] and C is the triangle with vertices (0, 0), (π, 0) and (π, 1).

ution: First Part: See the statement of Green's theorem.

Second Part: By Greens theorem

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA \qquad \dots\dots (1)$$

where, $\vec{F} = \text{siny}\,\vec{i} + \text{cosx}\,\vec{j}$.

Then,

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \text{siny} & \text{cosx} & 0 \end{vmatrix} = \vec{k}\,(-\text{sinx} - \text{cosy})$$

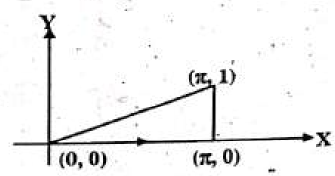
Thus from (1),

$$\oint_c \vec{F}.d\vec{r} = -\iint_R (\text{sinx} + \text{cosy})\,dA \qquad \dots\dots (2)$$

We have, R is the triangle with vertices (0, 0), (π, 0), (π, 1).

So, (2) becomes,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^1 \int_{\pi y}^{\pi} (\text{cosy} + \text{sinx})\,dxdy$$



$$= -\int_0^1 [x\cos y - \cos x]_{\pi y}^{\pi} dy$$

$$= -\int_0^1 (\pi\cos y - \cos\pi - \pi y\cos y + \cos\pi y)\, dy$$

$$= -\int_0^1 (\pi\cos y + 1 - \pi y\cos y + \cos\pi y)\, dy$$

$$= -\left[\pi\sin y + y - \pi(y\sin y + \cos y) + \frac{\sin\pi y}{\pi}\right]_0^1$$

$$= -(\pi\sin 1 + 1 - \pi\sin 1 - \pi\cos 1 + \pi)$$

Thus, $\oint_C \vec{F}.d\vec{r} = (\pi\cos 1 - \pi - 1)$.

2002 Q.No. 3(b)

State Green's theorem and then use it to evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = e^{x+y}\vec{i} + e^{x-y}\vec{j}$ where C is the boundary of the triangle region $x \le y \le 2x$, $0 \le x \le 1$.

Solution: Given that, $\vec{F} = e^{x+y}\vec{i} + e^{x-y}\vec{j} = e^x e^y\vec{i} + e^x.e^{-y}\vec{j}$

And boundaries of the region be $x \le y \le 2x$ $0 \le x \le 1$.

By Green's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\, dA \qquad \text{———(i)}$$

Here,

$$\text{curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x.e^y & e^x.e^{-y} & 0 \end{vmatrix} = (e^x e^{-y} - e^x e^y)\,\vec{k}$$

Then $\text{curl } \vec{F}.\vec{k} = e^x e^{-y} - e^x e^y$

Now, (i) becomes with boundaries $x \le y \le 2x$, $0 \le x \le 1$,

$$\oint_C \vec{F}.d\vec{r} = \int_0^1 \int_x^{2x} e^x [e^{-y} - e^y]\, dy\, dx$$

$$= \int_0^1 e^x \left[\frac{e^{-y}}{-1} - e^y\right]_x^{2x} dx$$

$$= \int_0^1 e^x\, [(-e^{-2x} - e^{2x}) - (-e^{-x} - e^x)]dx$$

$$= \int_0^1 (-e^{-x} - e^{3x} + e^0 + e^{2x})\, dx$$

$$= \int_0^1 (e^{2x} - e^{3x} - e^{-x} + 1)\, dx$$

$$= \left[\frac{e^{2x}}{2} - \frac{e^{3x}}{3} - \frac{e^{-x}}{-1} + x\right]_0^1 = \left(\frac{e^2}{2} - \frac{e^3}{3} + e^{-1} + 1\right) - \left(\frac{1}{2} - \frac{1}{3} + 1 + 0\right)$$

$$= \frac{e^2}{2} - \frac{e^3}{3} - \frac{1}{e} - \frac{1}{6}$$

01 Q.No. 4(a)

Evaluate the following integral by using Green's theorem. $\int[(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region defined by $y^2 = x$, $y = x^2$.

lution: First Part: See the statement of Green's theorem.

Second Part: Similar to 2002 Q.3(b).

ilar Questions

3 Spring Q. No. 2(b)

State Greens theorem in plane. Evaluate $\oint_C [(3x^2 + y)dx + 4y^2\, dy]$, where C is the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2) counterclockwise.

4 Spring Q. No. 2(b)

State Green's theorem in a plane, and find $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (2x - y - z)i + (x + y - z^2)\vec{j} + (3x - 2y + 4z)k$ around the circle $x^2 + y^2 = a^2$, $z = 0$.

3 Spring Q. No. 3(a)

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, where $\vec{F} = (18z, -12, 3y)$, s is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

4 Fall Q. NO. 5(a)

Evaluate: $\iint_S \vec{F}.ndA$ where $\vec{F} = (x^2, e^y, 1)$, S: $x + y + z = 1$ $x \ge 0$, $y \ge 0$, $z \ge 0$.

14 Spring Q. No. 3(b)

Evaluate the surface integral $\iint_S (\vec{F}.\vec{n})dA$, where $\vec{F} = (x^2, 0, 3y^2)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

CLOSED CURVE INTEGRAL BY USING STOKE'S THEOREM

2012 Fall Q.No. 4(b)

State Stoke's theorem. Evaluate $\oint_C \vec{F} dr$ where $F = (y^2 + z^2 + x^2)$ and C the portion of the sphere $x^2 + y^2 + (z-1)^2 = 1, y \geq 0, z \leq 1$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: Similar to 2002, 4(b).

2009 Fall Q.No. 4(b)

State Stokes theorem. Evaluate $\oint_C \vec{F}.d\vec{r}$, where $\vec{F} = (z, x, y)$, S: the hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 12.

2009 Spring Q.No. 4(b) OR

Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = \left(y, \frac{z}{2}, \frac{3y}{2}\right)$, C is the circle of $x^2 + y^2 + z^2 = 6z$, $z = x + 3$.

Solution: We know by Stoke's theorem

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{r})dA \quad \text{........(1)}$$

Here, $\vec{F} = y\vec{i} + \frac{z}{2}\vec{j} + \frac{3y}{2}\vec{k}$.

Then,

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z/2 & 3/2\cdot y \end{vmatrix} = \vec{i}\left(\frac{3}{2}-\frac{1}{2}\right) - \vec{j}(0) + \vec{k}(-1)$$

$$= \vec{i} - \vec{k} = \vec{G}\,(\text{say}).$$

Then equation (1) can be written as, by surface integral,

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\vec{G}.\vec{n})\,dA = \iint_R \vec{G}.\vec{N}\,dx\,dy \quad \text{.......(2)}$$

where $\vec{N} = \vec{r}_x \times \vec{r}_y$

and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + (x+3)\vec{k}$.

Differentiate partially, w. r. t. x and y, we get

$\vec{r}_x = \vec{i} + \vec{k}$ and $\vec{r}_y = \vec{j}$.

Then $\vec{N} = \vec{r}_x \times \vec{r}_y = (\vec{i} + \vec{k}) \times (\vec{j}) = \vec{k} - \vec{i}$.

So that (2) reduces as,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (-2)\,dxdy = -2\iint_R dx\,dy \quad \text{.......... (3)}$$

The given surface $x^2 + y^2 + z^2 = 6z$, $z = x + 3$ can be written as,

$$x^2 + y^2 + (x+3)^2 = 6(x+3)$$
$$\Rightarrow x^2 + y^2 + x^2 = 9$$
$$\Rightarrow 2x^2 + y^2 = 9$$
$$\Rightarrow \frac{x^2}{9/2} + \frac{y^2}{9} = 1$$

Thus (3) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\times[\text{Area of the ellipse } 2x^2 + y^2 = 9]$$

$$= -2\times\pi\frac{3}{\sqrt{2}}.3$$

Hence, $\oint_C \vec{F}.d\vec{r} = -9\sqrt{2}\,\pi$.

008 Spring Q.No. 4(b) OR

State Stokes theorem. Evaluate $\oint_C \vec{F}.d\vec{r}$, where $\vec{F} = (y^3, 0, x^3)$ and C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1).

olution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 7.

006 Spring Q.No. 4(b)

State Stokes theorem and evaluate: $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = y\vec{i} + \frac{z}{2}\vec{j} + \frac{3y}{2}\vec{k}$, C is the boundary of the circle $x^2 + y^2 + z^2 = 6z$, $z = x + 3$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See 2009 Spring Q. No. 4(b) OR.

2003 Fall Q.No. 4(b)

State Stock's theorem. Evaluate $\int_C F.r'(s)\,ds$ where $F = [2y^2, x, -z^3]$, C the circle $x^2 + y^2 = a^2$, $z = b\,(> 0)$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: Similar to 2007 Fall 3(a).

2002 Q.No. 4(b)

State Stoke's theorem and hence evaluate the surface integral is $\iint(\nabla \times \vec{F}).n\,dA$, where $\vec{F} = [y^2, -x^2, 0]$. S the semi–circular disc $x^2 + y^2 \le 4$, $y \ge 0$ and $z = 0$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: We know by stokes theorem

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl}\,\vec{F}.\vec{n})\,dA \quad \ldots\ldots\ldots(1)$$

We have

$$\vec{F} = y^2\vec{i} - x^2\vec{j}$$

Then

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -x^2 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(-2x-2y)$$

$$= -2(x+y)\vec{k} = \vec{G}\ (\text{say}).$$

Then from equation (1) we get

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\vec{G}.\vec{n})\,dA, \quad \text{where } \vec{G} = -2(x+y)\vec{k}$$

$$= \iint_R (\vec{G}.\vec{N})\,dx\,dy \quad \ldots\ldots\ldots(2)$$

(by definition of surface integral.)

where $\vec{N} = \vec{r}_x \times \vec{r}_y$

Since, $\vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$. Then, $\vec{r}_x = \vec{i}$ and $\vec{r}_y = \vec{j}$.

So, $\vec{N} = \vec{i}\times\vec{j} = \vec{k}$ and $\vec{G} = -2(x+y)\vec{k}$.

Now (2) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\iint_R (x+y)\,dxdy = -2\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x+y)\,dxdy$$

Since the surface is $x^2 + y^2 \le 4$ and $y \ge 0$. Therefore,

$$\oint_C \vec{F}.d\vec{r} = -2\int_0^2\left[\frac{x^2}{2} + yx\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= -2\int_0^2 y^2\sqrt{4-y^2}\,dy$$

$$= -4\int_0^2 y\sqrt{4-y^2}\,dy$$

$$= \left[\frac{4}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{4}{3}[0 - 4^{3/2}] = -\frac{4}{3}\times 2^3 = -32/3$$

Hence, $\oint_C \vec{F}.d\vec{r} = -32/3$.

Q.No. 4(b)

Evaluate the line integral using Stoke's theorem $\int_C \vec{F}.\vec{r}(s)\,ds$, where $\vec{F} = [y, xz^3, -zy^3]$; C, the circle $x^2 + y^2 = 4$, $z = -3$.

Solution: We know, by stokes theorem

$$\oint_C (\vec{F}.\vec{r}'(s))\,ds = \iint_R (\text{curl}\,\vec{F}.\vec{n})\,ds \quad \ldots\ldots\ (1)$$

Here, $\vec{F} = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$

Then,

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^3 & -zy^3 \end{vmatrix}$$

$$= \vec{i}(-3zy^2 - 3xz^2) - \vec{j}(0) + \vec{k}(z^3 - 1) = \vec{G}\ (\text{say})$$

Then (1) becomes,

$$\oint_C (\vec{F}.\vec{r})ds = \iint_S (\vec{F}.\vec{n})ds = \iint_R (\vec{F}.\vec{N})\,dx\,dy \quad \ldots\ldots\ (2)$$

Given that C is the circle $x^2 + y^2 = 4$, $z = -3$.

Then, $\vec{N} = \vec{k}$ and $\vec{G}.\vec{N} = (z^3 - 1)$.

Therefore, (2) reduces as,

$$\oint_C (\vec{F}.\vec{r})ds = \iint (z^3 - 1)\,dxdy$$

$$= -28\iint dx\,dy$$

$$= -28 \times (\text{Area of the circle } x^2 + y^2 = 4)$$

$$= -28 \times \pi(2)^2$$

$$= -112\pi$$

Thus we get, $\oint_C (\vec{F}.\vec{r})ds = -112\pi$.

20 Q.No. 4(b) OR

Verify Stoke's theorem for the vector function. Evaluate $\vec{F}$ dr where $F = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane and C its boundary.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 9.

Similar Questions

2014 Spring Q. No. 3(b) OR

State Stokes theorem. Find $\oint_C \vec{F}.d\vec{r}$ if $\vec{F} = (y^2, z^2, x^2)$, S is the first portion of the plane $x + y + z = 1$.

2013 Fall Q. No. 6(b)

Find $\oint_C \vec{F}.d\vec{r}$ if $\vec{F} = [y^2, 2xy + \sin x, 0]$, where c is the boundary of $0 \le x \le \frac{\pi}{2}, 0 \le y \le 2$ by using Stoke's Theorem.

2013 Fall Q. No. 5(a)

State Stoke's theorem. Evaluate $\oint_C \vec{F}.d\vec{r}$ where $F = [y, xz^3, -zy^3]$, C the circle $x^2 + y^2 = 4, z = -3$.

Solution: See Statement of Stoke's theorem and see 2002 Q. No. 4(b).

VOLUME INTEGRAL BY USING GUASS DIVERGENCE THEOREM

2013 Fall Q.No. 4(b); 2012 Fall Q.No. 4(b) OR; 2004 Fall Q.No. 4(b); 2003 Spring Q.No. 4(b)

Evaluate $\iint_S \vec{F}.\vec{n}\,dA$ if $\vec{F} = [x^2, e^y, 1]$; S: $x + y + z = 1, x \ge 0, y \ge 0, z \ge 0$.

Solution: Given that

$$\vec{F} = (x^2, e^y, 1)$$

and the surface is, $x + y + z = 1$, for $x \ge 0, y \ge 0, z \ge 0$

By Gauss divergence theorem we have,

$$\iint_R \vec{F}.\vec{n}\,dA = \iiint_V \text{div}\,\vec{F}\,dV \quad ....(i)$$

Here,

$$\text{div}.\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2\vec{i} + e^y\vec{j} + \vec{k})$$

$$= 2x + e^y$$

The plane $x + y + z = 1$ with $x \ge 0, y \ge 0, z \ge 0$ is as shown in figure.

Clearly z varies from $z = 0$ to the plane $x + y + z = 1 \Rightarrow 1 - x - y$. And the variable y varies in xy-plane from $y = 0$ to $x + y = 1 \Rightarrow y = 1 - x$.

Also, on the region x moves from $x = 0$ to $x = 1$.

Then, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y}(2x + e^y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}[2xz + ze^y]_0^{1-x-y}$$

$$= \int_0^1\int_0^{1-x}[2x(1 - x - y) + (1 - x - y)e^y]\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}[2x - 2x^2 - 2xy + (1 - x)e^y - ye^y]\,dy\,dx$$

$$= \int_0^1[2xy - 2x^2y - xy^2 + (1 - x)e^y - ye^y + e^y]_0^{1-x}\,dx$$

$$= \int_0^1[(2x - x^2)(1 - x) - x(1 - x)^2 + (2 - x)e^{1-x} - (2 - x)e^0 - (1 - x)e^{1-x}]\,dx$$

$$= \int_0^1[2x - 2x^2 - x^2 + x^3 - x - x^3 + 2x^2 + (2 - x - 1 + x)e^{1-x} - 2 + x]\,dx$$

$$= \int_0^1[2x - x^2 - 2 + e\,e^{-x}]\,dx$$

$$= \left[x^2 - \frac{x^3}{3} - 2x + e.\frac{e^{-x}}{-1}\right]_0^1 = \left(1 - \frac{1}{3} - 2 - e.e^{-1}\right) + e$$

$$= -\frac{7}{3} + e \qquad [\because e.e^{-1} = 1]$$

Thus, $\iint_S \vec{F}.\vec{n}\,dA = e - \frac{7}{3}$.

2013 Fall Q.No. 5(a)

...ate

Here,

$$\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(4x\,\vec{i} + 2y^2\,\vec{j} + z^2\vec{k})$$

$$= 4 + 4y + 2z$$

Since the surface is a cube with $|x| \le 1, |y| \le 1, |z| \le 1$.

So, $-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$.

Now, by (i)

$$\iint_S (\vec{F}.\vec{n})\,ds = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (4 + 4y + 2z)\,dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [4x + 4xy + 2xz]_{-1}^{1}\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [4(1 + 1) + 4y(1 + 1) + 2z(1 + 1)]\,dydz$$

$$= 4\int_{-1}^{1}\int_{-1}^{1} (2 + 2y + z)\,dydz$$

$$= 4\int_{-1}^{1} [2y + y^2 + zy]_{-1}^{1}\,dz$$

$$= 4\int_{-1}^{1} [2(1 + 1) + (1 - 1) + z(1 + 1)]\,dz$$

$$= 4\int_{-1}^{1} (4 + 2z)\,dz = 4[4z + z^2]_{-1}^{1}$$

$$= 4[4(1 + 1) + (1 - 1)] = 4 \times 8 = 32.$$

Thus, $\iint_S (\vec{F}.\vec{n})\,ds = 32.$

2011 Spring Q.No. 3(b); 2007 Fall Q.No. 4(a)

Evaluate the surface integral $\iint_S (\vec{F}.\vec{n})ds$, where $\vec{F} = (x^2, 0, 3y^2)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

Solution: Similar to 2012 Fall Q. No. 4(b).

2010 Spring Q.No. 5(a)

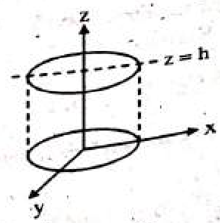
State **Gauss Divergence theorem. Use it to evaluate $\iint_S \vec{F}.\vec{n}\,dA$, where $\vec{F} = (4x, -2y^2, z^2)$, S is the surface bounding the region $x^2 + y^2 = 4, z = 3, z = 0$.**

Solution: First Part: See the statement of Gauss Divergence theorem.

Second Part: See Exercise 4.9 Q. No. A–6.

2008 Spring Q.No. 4(b)

Evaluate $\iint_S \vec{F}.\vec{n}\,dA$ where $\vec{F} = (y^3, x^3, z^3)$, s: $x^2\ 4y^2 = 1$, $x \ge 0$, $y \ge 0$, $0 \le z \le h$.



Solution: Given that $\vec{F} = (y^3, x^3, z^3)$

and surface is $x^2 + 4y^2 = 1$ for $x \ge 0, y \ge 0, 0 \le z \le h$.

By Gauss divergence theorem,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_V \text{div}\,\vec{F}\,dV \quad ....(i)$$

Here,

$$\text{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(y^3\vec{i} + x^3\vec{j} + z^3\vec{k}) = 3z^2$$

Given surface is a ellipsoid having height h on first quadrant. So, x varies on the region from $x = 0$ to $x = \sqrt{1 - 4y^2}$ and y moves from $y = 0$ to $y = \frac{1}{2}$. Also, the region moves from $z = 0$ to $z = h$.

Now, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^{1/2}\int_0^{\sqrt{1-4y^2}}\int_0^{h} 3z^2\,dzdxdy$$

$$= \int_0^{1/2}\int_0^{\sqrt{1-4y^2}} [z^3]_0^h\,dx\,dy$$

$$= h^3\int_0^{1/2} [x]_0^{\sqrt{1-4y^2}}\,dy$$

$$= h^3\int_0^{1/2} \sqrt{1 - 4y^2}\,dy$$

$$= 2h^3\int_0^{1/2} \sqrt{\frac{1}{4} - y^2}\,dy$$

$$= 2h^3\left[\frac{y\sqrt{\frac{1}{4}-y^2}}{2} + \frac{1/4}{2}\sin^{-1}\frac{y}{1/2}\right]_0^{1/2}$$

$$= 2h^3\left[0 + \frac{1}{8}\sin^{-1}(1)\right]$$

$$= \frac{h^3}{4}\cdot\frac{\pi}{2} = \frac{\pi h^3}{8} \qquad [\because \sin^{-1}(1) = \frac{\pi}{2}].$$

2007 Fall Q.No. 4(b)

State Gauss divergence theorem for the surface integral. Evaluate $\iint_S (\vec{F}.\vec{n})\, ds$, where $\vec{F} = (e^x, e^y, e^z)$ and S is the surface of the cube $|x| \le 1, |y| \le 1$, and $|z| \le 1$.

Solution: First Part: See the statement of Gauss Divergence theorem.

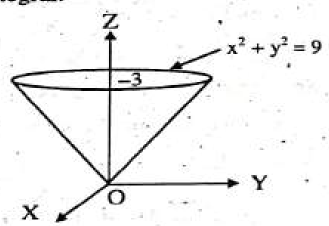
Second Part: Similar to 2011 Fall Q. No. 5(a).

2005 Fall Q.No. 4(a)

Define surface integral of $\vec{F}$ on the surface S. Evaluate $\iint_S \vec{F}.\vec{n}\, dA$ where $\vec{F} = (x, 3y, 6z)$ and S is the surface of the cone $\sqrt{x^2+y^2} \le Z$, $0 \le z \le 3$.

Solution: First Part: See the definition of surface integral.

Second Part: By Gauss divergence theorem



$$\iint_S \vec{F}.\vec{n}\, dA = \iiint \operatorname{div}\vec{F}\, dv$$

We have,

$$\vec{F} = x\vec{i} + 3y\vec{j} + 6z\vec{k}$$

Then, $\operatorname{div}.\vec{F} = 1 + 3 + 6 = 10.$

and we have given surface is $\sqrt{x^2+y^2} \le z$, $0 \le z \le 3$ is shown in figure.

Then from equation (1), we get

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint 10dv = 10\iiint dv$$

$= 10\times$volume of cone represented by the surface S

$$= 10 \times \frac{1}{3}\pi(3)^2.3$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = 90\pi.$

Note: If div $\vec{F}$ is not a constant value. Then we put the limits of x, y and z are as follows.

$$0 \le z \le 3; -\sqrt{z} \le y \le \sqrt{z}; -\sqrt{z^2-y^2} \le x \le \sqrt{z^2-y^2}.$$

2002 Q.No. 4(a)

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, where $\vec{F} = [3x^2, y^2, 0]$; S: $\vec{r} = [u, v, 2u + 3v]$, for $0 \le u \le 2; -1 \le v \le 1$.

Solution: Similar to 2004 Spring 4(a).

2001 Q.No. 4(b)

State Gauss's divergence theorem and use this to evaluate $\iint_S [(x^3 - yz)\vec{i} - 2x^2y\vec{j}].\vec{n}\, dA$, where S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 1, z = 1$.

Solution: First Part: See the statement of Gauss Divergence theorem.

Second Part: Similar to 2011 Fall 5(a).

Similar Questions

2014 Fall Q. No. 5(b) OR

State Gauss Divergence Theorem. Evaluate $\int_C \vec{F}.\vec{n}\, dA$ by using Green's Theroem if $\vec{F} = \left[\frac{e^y}{x}, e^y \ln x + 2x\right]$, R: $1 + x^4 \le y \le 2$.

SHORT QUESTIONS FROM FINAL EXAMINATION

2012 Fall: If $\vec{r} \times \frac{d^2r^4}{dt^2}$, show that $\vec{r} \times \frac{d^2\vec{r}}{dt} = 0$.

Question is incomplete.

2011 Spring: Find the curl of $\vec{F} = 2y\vec{j} + 5x\vec{k}$.

Solution: Given that $\vec{F} = 2y\vec{j} + 5x\vec{k}$.

Then,

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2y & 5x \end{vmatrix} = 0\vec{i} - 5\vec{j} + 0\vec{k}.$$

Thus, $\operatorname{curl}\vec{F} = -5\vec{j}$.

2010 Spring: Find the unit tangent vector to the curve $\vec{r} = [t, t^2, t^3]$.

Solution: See the solution part of Q. 11, Exercise 4.3.

2009 Spring: Find the directional derivative of the scalar valued function $f(x) = x^2 + y^2$, at (1, 2) in the direction $\vec{a} = 2\vec{i} - \vec{j}$.

Solution: Given that $f = x^2 + y^2$ and $\vec{a} = 2\vec{i} - \vec{j}$

Then $\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)f = 2x\vec{i} + 2y\vec{j}$

and $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - \vec{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}(2\vec{i} - \vec{j})$

Now, directional derivative of f at (1, 2) along $\vec{a}$ is,

$$D_{\vec{a}}f = \text{grad}(f) \cdot \hat{a} \text{ at } (1, 2)$$
$$= \frac{1}{\sqrt{5}}(4x - 2y) \quad \text{at } (1, 2)$$
$$= \frac{1}{\sqrt{5}}(4 - 4) = 0.$$

2008 Spring: Find the divergence of the vector $\vec{v} = (x^2 + yz)\vec{i} + (y^2 + zx)\vec{j} + (z^2 + xy)\vec{k}$.

Solution: See the solution part of Q. 4(ii), Exercise 4.3.

2007 Fall: If $\vec{r} = \vec{a}\cos wt + b\sin wt$, show that $\vec{r} \times \frac{d\vec{r}}{dt} = w\,\vec{a} \times \vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution: Let $\vec{r} = \vec{a}\cos wt + \vec{b}\sin wt$. Then, $\frac{d\vec{r}}{dt} = -w\vec{a}\sin wt + w\vec{b}\cos wt$.

Now,

$$\vec{r} \times \frac{d\vec{r}}{dt} = w(\vec{a} \times \vec{b})\cos^2 wt - w(\vec{b} \times \vec{a})\sin^2 wt \qquad [\because \vec{a} \times \vec{a} = 0]$$
$$= w(\vec{a} \times \vec{b})\cos^2 wt + w(\vec{a} \times \vec{b})\sin^2 wt$$
$$= w(\vec{a} \times \vec{b})(\cos^2 wt + \sin^2 wt) = w(\vec{a} \times \vec{b}).$$

Thus, $\vec{r} \times \frac{d\vec{r}}{dt} = w(\vec{a} \times \vec{b})$.

2006 Spring: If $f(x, y, z) = xyz$, show that $\nabla . (\nabla f) = 0$.

Solution: Let $f = xyz$.

Then, $\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz) = yz\vec{i} + zx\vec{j} + xy\vec{k}$

and,

$$\nabla . \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . (yz\vec{i} + zx\vec{j} + xy\vec{k})$$
$$= \frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}zx + \frac{\partial}{\partial z}xy = 0 + 0 + 0 = 0.,$$

Thus, $\nabla . \nabla f = 0$.

Alternative solution

Let $f = xyz$. Since we have div(grad f) = 0.

That is $\nabla . (\nabla f) = 0$

5 Spring: Find $\frac{d}{dt}(\vec{r} . \vec{r})$ where $\vec{r} = t\vec{i} + 3t^2\vec{j} + 4t^3\vec{k}$.

tion: Let $\vec{r} = (t, 3t^2, 4t^3)$. Then $\vec{r} . \vec{r} = (t, 3t^2, 4t^3) . (t, 3t^2, 4t^3)$
$$= t^2 + 9t^4 + 16t^6$$

So, $\frac{d}{dt}(\vec{r} . \vec{r}) = 2t + 36t^3 + 96t^5$.

Fall: If $\vec{v} = 3t^2\vec{i} + 3t\vec{j} - (3t + 2)\vec{k}$, evaluate $\int_2^3 \vec{v}\,dt$.

ion: Let $\vec{v} = (3t^2, 3, -3t - 2)$. Then,

$$\int_2^3 \vec{v}\,dt = 3\vec{i}\int_2^3 t^2 dt + 3\vec{j}\int_2^3 t\,dt - \vec{k}\left[3\int_2^3 t\,dt - 2\int_2^3 dt\right]$$
$$= 3\vec{i}\left[\frac{t^3}{3}\right]_2^3 + 3\vec{j}\left[\frac{t^2}{2}\right]_2^3 - \vec{k}\left[3\frac{t^2}{2} - 2t\right]_2^3$$
$$= \vec{i}(27 - 8) + \frac{3\vec{j}}{2}(9 - 4) - \vec{k}\left(\frac{27}{2} - 6 - 6 + 4\right)$$
$$= 19\vec{i} + \frac{15}{2}\vec{j} - \frac{11}{2}\vec{k}.$$

Fall: Find the directional derivative of $f(x, y, z) = 2x^{-2} + 3y^2 + z^2$ at P (1, 2, 3) in the direction of $\vec{a} = \vec{i} - 2\vec{k}$.

ion: Similar to 2009 Spring.

Fall: If the divergence of $\vec{F} = 2x\vec{i} + y\vec{j} + pz\vec{k}$ is zero find the value of p.

ion: Let $\vec{F} = (2x, y, pz)$.

Then div. $\vec{F} = \nabla . \vec{F} = \left(\vec{i}\frac{\partial}{\partial z} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . \vec{F} = 2 + 1 + P = 3 + P$

Given that div. $\vec{F} = 0$. Then $3 + P = 0 \Rightarrow P = -3$.

Thus, value of P is -3.

2004 Spring: Find the gradient of f = xy + yz + zx.

Solution: Let f = xy + yz + zx. Then,

$$\text{grad } f = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$$

$$= (y + z)\vec{i} + (x + z)\vec{j} + (x + y)\vec{k}.$$

Thus, gradient of f is (y + z, x + z, x + y).

2004 Spring: If $\vec{a} = 3t^2\vec{i} + 4t^3\vec{j}$ and $\vec{b} = 5t^2\vec{i} + 4t\vec{j}$ find $d/dt\,(\vec{a}.\vec{b})$.

Solution: Let $\vec{a} = (3t^2, 4t^3)$, $\vec{b} = (5t^2, 4t)$

Then:

$$\vec{a}.\vec{b} = (3t^2, 4t^3).(5t^2, 4t) = 15t^4 + 16t^4 = 31t^4.$$

Now,

$$\frac{d}{dt}(\vec{a}.\vec{b}) = 124\,t^3$$

2004 Spring: If $\vec{r} = t^2\vec{i} + (2t + 1)\vec{j} + 3t\vec{k}$. Find $|d^2\vec{r}/dt^2|$.

Solution: Let $\vec{r} = (t^2, 2t + 1, 3t)$.

Then,

$$\frac{d\vec{r}}{dt} = (2t, 2, 3) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (2, 0, 0).$$

Thus, $\frac{d^2\vec{r}}{dt^2} = 2\vec{i}$.

2003 Fall: Find the gradient of $f = x^3 + y^3 + z^3 - 3xyz$.

Solution: Similar to 2004 Spring.

2003 Fall: If $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$, $\frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$. Show that $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$.

Solution: Let $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$.

Now,

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

$$= (\vec{c}\times\vec{a})\times\vec{b} + \vec{a}\times(\vec{c}\times\vec{b})$$

$$= (\vec{b}.\vec{c})\vec{a} - (\vec{b}.\vec{a})\vec{c} + (\vec{a}.\vec{b})\vec{c} - (\vec{a}.\vec{c})\vec{b}$$

[∵ Using cross product of three vectors]

$$= (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{b} \quad = \vec{c}\times(\vec{a}\times\vec{b}).$$

$$= \vec{c}\times(\vec{a}\times\vec{b}).$$

Thus, $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$.

Spring: Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at (1, 2, 1) in the direction $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$.

tion: Similar to 2009 Spring.

Spring: If the divergence of $\vec{F} = 2px\vec{i} + y\vec{j} + z\vec{k}$ is zero, find the value of p.

tion: Similar to 2004 Fall.

: If $\vec{V} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ find div $\vec{V}$.

tion: Let $\vec{v} = (x^2y, y^2z, z^2x)$.

Then,

$$\text{div.}\vec{r} = \nabla.\vec{v}. = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).\vec{v} = (2xy + 2yz + 2zx).$$

: Find the gradient of $f = x^2 + y^2 + z^2$.

tion: Similar to 2004 Spring.

: If $\vec{v} = x^2yz\,\vec{i} - xy^2z\,\vec{j} - xyz^2\vec{k}$ find div. $\vec{v}$.

tion: Similar to 2004 Spring.

: Evaluate $\int_c (y^2dx - x^2\,dy)$ counter the clockwise along the circle $x^2 + y^2 + = 1$ from (1, 0) to (0, 1).

tion: Given integral is, $\int_c (y^2dx - x^2dy)$.

Comparing it with $\int_c (F_1dx + F_2dy)$ then we get, $F_1 = y^2$, $F_2 = -x^2$.

Also, given that the integral moves along $x^2 + y^2 = 1$ in counterclockwise direction. In which y varies from $y = -\sqrt{1 - x^2}$ to $y = \sqrt{1 - x^2}$ and moves from x = −1 to x = 1.

Now, by Green's theorem,

$$\oint_c (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx$$

Then,

$$\oint_c (y^2dx - x^2dy) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dy\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-2x - 2y)\, dy\, dx$$

$$= \int_{-1}^{1} [-2xy - y^2]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dx$$

$$= -\int_{-1}^{1} (4x\sqrt{1-x^2} + 1 - x^2 - 1 + x^2)\, dx$$

$$= -\int_{-1}^{1} 4x\sqrt{1-x^2}\, dx$$

Put $1 - x^2 = t^2$ then $-2x\, dx = 2t\, dt \Rightarrow x\, dx = -t\, dt$. Also, $x = 0 \Rightarrow t = 1$, $x = 1 \Rightarrow t = 0$.

Now,

$$\oint_c (y^2dx - x^2dy) = -2\int_{1}^{0} 4t\,(-tdt)$$

$$= -8\int_{1}^{0} t^2\, dt = -8\left[\frac{t^3}{3}\right]_0^1 = -8\left(0 - \frac{1}{3}\right) = \frac{8}{3}.$$

Similar Questions

2013 Fall Q. No. 7(a): Find unit tangent vector to the curve $\vec{r} = t^2\vec{i} + 2t\vec{j} - t^3\vec{k}$ at t = 1.

2013 Spring Q. No. 7(d): Check the exactness condition for value under the integral sign $\int_{(0,\pi)}^{(3,\pi/2)} (e^x \cos y dx - e^x \sin y dy)$ and evaluate the integral if it is exact.

2014 Fall Q. No. 7(b): If $\phi = e^{xy}$, find grad ϕ.

2014 Spring Q. No. 7(b): If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where $\bar{a}$ & $\bar{b}$ are constant vectors, show $\frac{d^2\bar{r}}{dt^2} - n^2\bar{r} = 0$.

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