Welcome

To

MKS TUTORIALS

By

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Topics covered under playlist of Multiple Integral: Double Integral, Triple Integral, Change of Order of Integration, Area, Volume, Beta Function, Gamma Function, Relation between Beta and Gamma Function, Problems on Beta and Gamma Function.

Complete playlist of Multiple Integrals (in Hindi):
https://www.youtube.com/playlist?list=PLhSp9OSVmeyLwtQbXv7VRXgoueSFaJGB4

Complete playlist of Multiple Integrals (in English):
https://www.youtube.com/playlist?list=PL0d0PH4hlFOcnTIn5b1LOWhC9prqS2Esn

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MULTIPLE INTEGRALS

(A) Double Integrals

$$\text{Let } I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dx\, dy$$

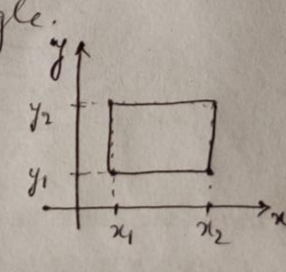
Case I: When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x,y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2.

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx$$

Case II: When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t. x keeping y fixed between limits x_1, x_2 and then the resulting expression is integrated w.r.t. y within the limits y_1, y_2.

$$I = \int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} f(x,y)\, dx\, dy$$

Case III: When both pairs of limits are constants, the region of integration is rectangle.



$$I = \int_a^b \int_c^d f(x,y)\, dx\, dy$$

y x

Double Integrals

Que ① Evaluate: (i) $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$ (ii) $\int_1^2 \int_1^3 xy^2\,dx\,dy$

Sol^n (i) Let $I = \int_0^5 \left[\int_0^{x^2} (x^3 + xy^2)\,dy \right] dx$

$$= \int_0^5 \left| x^3 y + \frac{x}{3} y^3 \right|_0^{x^2} dx = \int_0^5 \left[x^3(x^2) + \frac{x}{3}(x^2)^3 \right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5$$

$$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6 \left(\frac{1}{6} + \frac{25}{24} \right) = 5^6 \left(\frac{4+25}{24} \right) = 5^6 \left(\frac{29}{24} \right) \text{ Ans.}$$

(ii) Given: $I = \int_1^2 \left[\int_1^3 xy^2\,dx \right] dy = \int_1^2 \left| \frac{x^2}{2} y^2 \right|_1^3 dy$

$$= \int_1^2 \left(\frac{3^2 - 1^2}{2} \right) y^2\,dy = \int_1^2 4y^2\,dy = 4 \left| \frac{y^3}{3} \right|_1^2$$

$$= \frac{4}{3}(2^3 - 1^3) = \frac{4}{3}(7) = \frac{28}{3} \text{ Ans.}$$

Que (2) Evaluate : $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol^n Let $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy \right] dx$$

$\left\{ \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \right.$

Here

$a^2 = 1+x^2$

$a = \sqrt{1+x^2}$

$$= \int_0^1 \left| \frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right|_0^{\sqrt{1+x^2}} \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) - \tan^{-1}\left(\frac{0}{\sqrt{1+x^2}}\right) \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \cancel{\tan^{-1}(0)}^{\,0} \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\pi/4)\,dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\pi}{4} \left| \log\left(x + \sqrt{1+x^2}\right) \right|_0^1$$

$$= \frac{\pi}{4} \left[\log\left(1+\sqrt{1+1^2}\right) - \log\left(0+\sqrt{1+0^2}\right) \right]$$

$$= \frac{\pi}{4} \left[\log\left(1+\sqrt{2}\right) - \cancel{\log(1)}^{\,0} \right]$$

$I = \frac{\pi}{4} \log(1+\sqrt{2})$ Ans

Que (3) Evaluate : $\int_0^4 \int_0^{x^2} e^{y/x} \, dy \, dx$

Sol^n Let $I = \int_0^4 \left[\int_0^{x^2} e^{y/x} \, dy \right] dx$ $\left\{ \int e^{ax} dx = \frac{e^{ax}}{a} \right.$

$= \int_0^4 \left| \frac{e^{y/x}}{1/x} \right|_0^{x^2} dx$

$= \int_0^4 x \left(e^{x^2/x} - e^0 \right) dx = \int_0^4 x (e^x - 1) \, dx$

$= \int_0^4 (xe^x - x) \, dx$

$I = \int_0^4 xe^x \, dx - \int_0^4 x \, dx$

ILATE

Solving $\int \underset{\mathrm{I}}{x} \, \underset{\mathrm{II}}{e^x} dx = x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx \right] dx$

$= xe^x - \int e^x dx = xe^x - e^x$

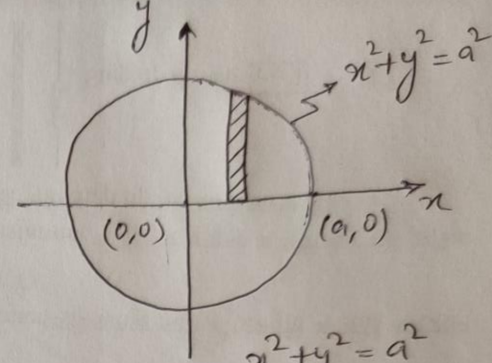
$\therefore \; I = \left| xe^x \right|_0^4 - \left| e^x \right|_0^4 - \left| \frac{x^2}{2} \right|_0^4$

$= 4e^4 - (e^4 - 1) - \left(\frac{4^2 - 0^2}{2} \right)$

$= 4e^4 - e^4 + 1 - 8$

$= 3e^4 - 7$ Ans.

Que (4) Find $\iint xy\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=a^2$.



Sol.ⁿ

Limit varies from

$x=0$ to $x=a$

and $y=0$ to $y=\sqrt{a^2-x^2}$

$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\sqrt{a^2-x^2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} xy\,dy\right] dx$$

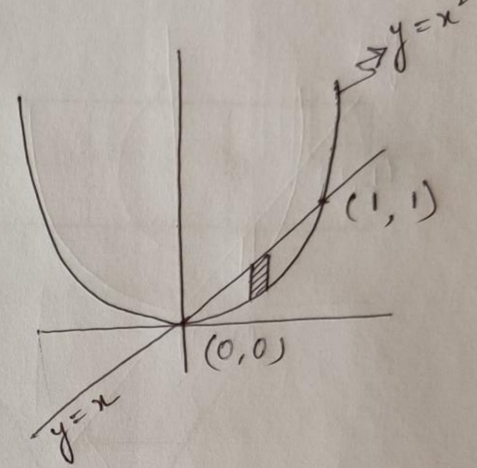
$$= \int_0^a \left| x\frac{y^2}{2} \right|_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}(a^2-x^2)\,dx = \int_0^a \left(\frac{a^2x}{2} - \frac{x^3}{2}\right) dx$$

$$= \left| \frac{a^2x^2}{4} - \frac{x^4}{8} \right|_0^a = \frac{a^4}{4} - \frac{a^4}{8} = \frac{a^4}{4}\left(1-\frac{1}{2}\right)$$

$$= \frac{a^4}{8} \text{ Ans}$$

Que (5) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$
(Eq.^{n} of parabola) (Eq.^{n} of a line)
(Symmetric abt y-axis)



Sol.^{n}

We have

$y = x^2$ —— ①

and $y = x$ —— ⑪

On solving $x^2 = x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ and $x = 1$

When $x=0 \Rightarrow y=0$ } - from eq.^{n} ⑪
and $x=1 \Rightarrow y=1$ }

Hence, the intersection points are $(0,0)$ and $(1,1)$.

$$\int_0^1\int_{x^2}^{x} xy(x+y)\,dx\,dy = \int_0^1\left[\int_{x^2}^{x}(x^2y+xy^2)\,dy\right]dx$$

$$= \int_0^1 \left| \frac{x^2}{2}y^2 + \frac{x}{3}y^3 \right|_{x^2}^{x} dx = \int_0^1 \left[\left(\frac{x^2}{2}\cdot x^2 - \frac{x^2}{2}\cdot x^4\right) + \left(\frac{x}{3}\cdot x^3 - \frac{x}{3}\cdot x^6\right)\right]dx$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3}\right)dx = \left| \frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right|_0^1$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{7-5}{70} + \frac{8-5}{120} = \frac{1}{35} + \frac{1}{40}$$

$$= \frac{40+35}{35\times 40} = \frac{\cancel{75}^{\cancel{15}\,3}}{\cancel{35}_7 \times \cancel{40}_8} = \frac{3}{56} \quad \text{Ans}$$

(B) TRIPLE INTEGRALS

Let $I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z)\, dx\, dy\, dz$

Case I : If x_1, x_2 are constants ; y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then this integral is evaluated as follows :

First $f(x,y,z)$ is integrated w.r.t z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t x from x_1 to x_2 :

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Case II : If all three pairs of limits of x, y and z are constants, then

$$I = \int_a^b \left[\int_c^d \left\{ \int_e^f f(x,y,z)\, dx \right\} dy \right] dz$$

Que① Evaluate: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

(z) (x) (y)

Sol.ⁿ Let $I = \int_{-1}^{1}\left[\int_{0}^{z}\left\{\int_{x-z}^{x+z}(x+y+z)\,dy\right\}dx\right]dz$

$= \int_{-1}^{1}\int_{0}^{z}\left|xy+\frac{y^2}{2}+zy\right|_{x-z}^{x+z}dx\,dz$

$= \int_{-1}^{1}\int_{0}^{z}\Big[x(\cancel{x}+z-\cancel{x}+z)+\frac{1}{2}\{(x+z)^2-(x-z)^2\}$
$+z(\cancel{x}+z-\cancel{x}+z)\Big]dx\,dz$

$= \int_{-1}^{1}\int_{0}^{z}\left\{2xz+\frac{1}{2}(4xz)+2z^2\right\}dx\,dz$

$= \int_{-1}^{1}\left[\int_{0}^{z}(4xz+2z^2)\,dx\right]dz$

$= \int_{-1}^{1}\left|4\frac{x^2}{2}z+2z^2x\right|_{0}^{z}dz$

$= \int_{-1}^{1}(2z\cdot z^2+2z^2z)\,dz = \int_{-1}^{1}(2z^3+2z^3)\,dz$

$= \int_{-1}^{1}4z^3\,dz = \cancel{4}\left|\frac{z^4}{\cancel{4}}\right|_{-1}^{1} = (1)^4-(-1)^4$

$= 0$ Ans

Que (2) Evaluate : $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

(z) (y) (x)

Sol.ⁿ Given:

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\int_{-a}^{a}(x^2+y^2+z^2)\,dx\right]dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left|\frac{x^3}{3}+y^2\cdot x+z^2\cdot x\right|_{-a}^{a}dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left[\left\{\frac{a^3-(-a^3)}{3}\right\}+y^2(a+a)+z^2(a+a)\right]dy\,dz$$

$$= \int_{-c}^{c}\left[\int_{-b}^{b}\left(\frac{2a^3}{3}+2ay^2+2az^2\right)dy\right]dz$$

$$= \int_{-c}^{c}\left|\frac{2a^3}{3}\cdot y+2a\frac{y^3}{3}+2az^2\cdot y\right|_{-b}^{b}dz$$

$$= \int_{-c}^{c}\left[\frac{2a^3}{3}(b+b)+\frac{2a}{3}\left\{b^3-(-b)^3\right\}+2az^2(b+b)\right]dz$$

$$= \int_{-c}^{c}\left(\frac{4}{3}a^3b+\frac{2}{3}a\cdot 2b^3+4abz^2\right)dz$$

$$= \left|\frac{4}{3}a^3bz+\frac{4}{3}ab^3\cdot z+4ab\frac{z^3}{3}\right|_{-c}^{c}$$

$$= \frac{4}{3}a^3b(c+c)+\frac{4}{3}ab^3(c+c)+\frac{4}{3}ab\left\{c^3-(-c)^3\right\}$$

$$= \frac{8}{3}a^3bc+\frac{8}{3}ab^3c+\frac{4}{3}ab\,2c^3$$

$$= \frac{8}{3}a^3bc+\frac{8}{3}ab^3c+\frac{8}{3}abc^3$$

$$= \frac{8}{3}abc(a^2+b^2+c^2) \quad \text{Ans.}$$

Que ③ Evaluate : $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$

Sol^n (x) (y) (z)

Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} xyz \; dz \right] dy\, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left| xy\frac{z^2}{2} \right|_0^{\sqrt{1-x^2-y^2}} dy\, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{2} (1-x^2-y^2) \, dy\, dx$

$= \frac{1}{2} \int_0^1 \left[\int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3) \, dy \right] dx$

$= \frac{1}{2} \int_0^1 \left| \frac{x}{2}y^2 - \frac{x^3}{2}y^2 - \frac{x}{4}y^4 \right|_0^{\sqrt{1-x^2}} dx$

$= \frac{1}{2} \int_0^1 \left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2 \right] dx$

$= \frac{1}{2} \int_0^1 \left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-2x^2+x^4) \right] dx$

$= \frac{1}{2} \int_0^1 \left[\frac{x}{2} - \frac{x^3}{2} - \cancel{\frac{x^3}{2}} + \frac{x^5}{2} - \frac{x}{4} + \cancel{\frac{x^3}{2}} - \frac{x^5}{4} \right] dx$

$= \frac{1}{2} \int_0^1 \left[\frac{x}{2}\left(1 - \frac{1}{2}\right) - \frac{x^3}{2} + \frac{x^5}{2}\left(1 - \frac{1}{2}\right) \right] dx$

$= \frac{1}{2} \int_0^1 \left(\frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4} \right) dx = \frac{1}{2} \left(\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24} \right)_0^1$

$= \frac{1}{2} \left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{8}} + \frac{1}{24} \right) = \frac{1}{48}$ Ans

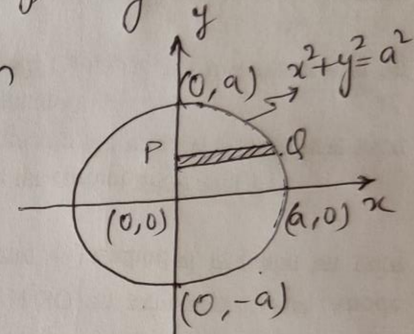
Change the order of Integration

Que ① Change the order of integration of

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy.$$

$y \quad x$

Solⁿ Given: $I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$



Here, the old limit varies from

$y=-a$ to $y=a$

and $x=0$ to $x=\sqrt{a^2-y^2}$

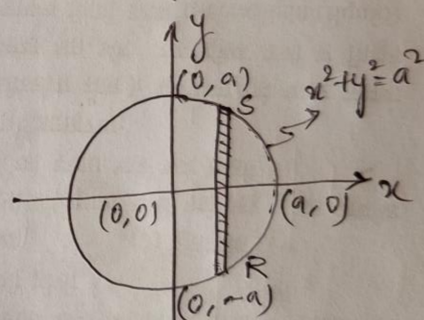
$(x^2+y^2=a^2)$

Let RS be the new strip.

∴ Limit varies from

$x=0$ to $x=a$

and $y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$



On changing the strip,

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$$

$$= \int_{0}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\, dy\, dx$$

Ans

$x^2+y^2=a^2$
$y^2=a^2-x^2$
$y=\pm\sqrt{a^2-x^2}$

Que(2) Change the order of integration

$$I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$$

$\quad\ x \quad\ y$

Sol.ⁿ

let $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

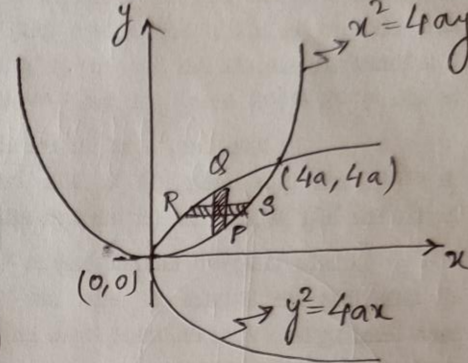
Old strip varies from

$x = 0$ to $x = 4a$

and $y = x^2/4a$ to $y = 2\sqrt{ax}$

$(x^2 = 4ay)$ Parabola $\quad (y^2 = 4ax)$ Parabola

(Sym. abt. y-axis) (Sym. abt. x-axis)



Here, $y^2 = 4ax$ and $x^2 = 4ay$

On equating both,

$$2\sqrt{ax} = \frac{x^2}{4a} \Rightarrow 4a\cancel{x} = \frac{x^{\cancel{4}\,3}}{16a^2} \Rightarrow 64a^3 = x^3$$

$$\Rightarrow x = 4a \Rightarrow y = 4a$$

Let RS be the new strip, the limit varies from

$y = 0$ to $y = 4a$ and $x = y^2/4a$ to $x = 2\sqrt{ay}$

$$\therefore I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_0^{4a} \left| x \right|_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left| 2a^{1/2}\cdot\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right|_0^{4a} = \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{(4a)^3}{12a}$$

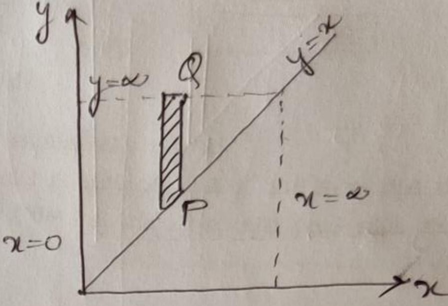
$$= \frac{4}{3}a^{1/2}\cdot 8a^{3/2} - \frac{\overset{16}{\cancel{64}}a^3}{{}_3\cancel{12}a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \quad \text{Ans}$$

Que ③ Change the order of integration of

$$I = \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dx \, dy$$

Solⁿ

Let $I = \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dy \, dx$

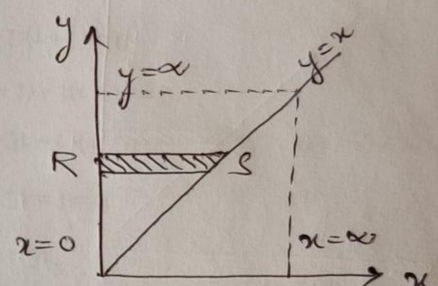


Old limit varies from

$x=0$ to $x=\infty$

and $y=x$ to $y=\infty$

(line)



Let RS be the new strip

and limit varies from

$y=0$ to $y=\infty$

and $x=0$ to $x=y$

On changing the order of integration,

$$I = \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dy \, dx = \int_0^{\infty} \left[\int_0^{y} \frac{e^{-y}}{y} \, dx \right] dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} \left| x \right|_0^{y} dy = \int_0^{\infty} \frac{e^{-y}}{\cancel{y}} (\cancel{y}) \, dy$$

$$= \int_0^{\infty} e^{-y} \, dy = \left| \frac{e^{-y}}{-1} \right|_0^{\infty} = - \left| \frac{1}{e^{y}} \right|_0^{\infty}$$

$$= -\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}} \right) = -(0-1) = 1 \text{ Ans}$$

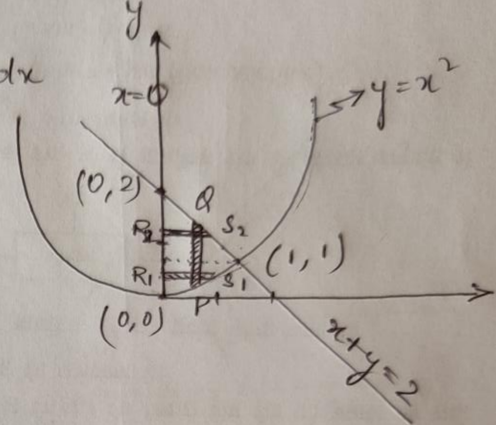
Imp

Que (4) Change the Order of Integration of

$$I = \int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$$

Sol.ⁿ (x) (y)

Let $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$



Old limit varies from

$x=0$ to $x=1$

and $y = x^2$ to $y = 2-x$ → line

$\downarrow$ Parabola (Sym. abt y-axis)

$x+y=2$ $\begin{Bmatrix} x=0, y=2 \\ x=1, y=1 \end{Bmatrix}$

We have $y = x^2$ and $y = 2-x$

$\Rightarrow x^2 = 2-x$

$\Rightarrow x^2 + x - 2 = 0 \Rightarrow x = 1 \Rightarrow y = 1$

Hence the intersection point is $(1,1)$

New limit varies from

$y=0$ to $y=1$ and $x=0$ to $x=\sqrt{y}$ (for R_1S_1 strip)

Also, $y=1$ to $y=2$ and $x=0$ to $x = 2-y$ (for R_2S_2 strip)

$$\therefore I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \int_0^1 \left[\int_0^{\sqrt{y}} xy \, dx\right] dy + \int_1^2 \left[\int_0^{2-y} xy \, dx\right] dy$$

$$= \int_0^1 y \left|\frac{x^2}{2}\right|_0^{\sqrt{y}} dy + \int_1^2 y \cdot \left|\frac{x^2}{2}\right|_0^{2-y} dy = \int_0^1 \frac{y}{2} \cdot y \, dy + \int_1^2 \frac{y}{2}(2-y)^2 dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{y}{2}(4 - 4y + y^2) dy = \left|\frac{y^3}{6}\right|_0^1 + \int_1^2 \left(2y - 2y^2 + \frac{y^3}{2}\right) dy$$

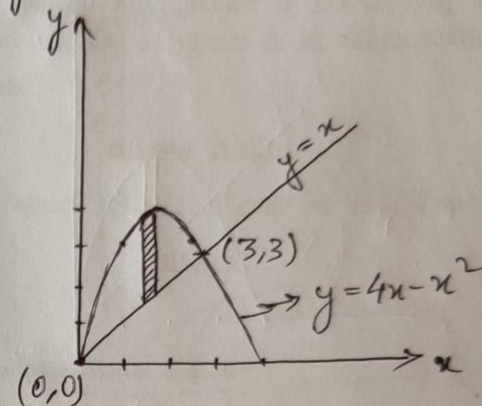
$$= \frac{1}{6} + \left| y^2 - \frac{2}{3}y^3 + \frac{y^4}{8}\right|_1^2 = \frac{1}{6} + (4-1) - \frac{2}{3}(8-1) + \frac{1}{8}(16-1)$$

$$= \frac{1}{6} + 3 - \frac{14}{3} + \frac{15}{8} = \frac{1+18}{6} - \left(\frac{14}{3} - \frac{15}{8}\right) = \frac{19}{6} - \left(\frac{112-45}{24}\right)$$

$$= \frac{19}{6} - \frac{67}{24} = \frac{76-67}{24} = \frac{9}{24} = \frac{3}{8} \quad \text{Ans}$$

Area enclosed by Plane Curves

Que① Find the area included between parabola $y = 4x - x^2$ and the line $y = x$.



Solⁿ For parabola, $y = 4x - x^2$

x:	0	1	2	3	4
y:	0	3	4	3	0

We have, $y = 4x - x^2$ —— ①

and $y = x$ —— ⑪

On equating both,

$x = 4x - x^2 \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

$\Rightarrow x = 0, 3$

from eq ⑪ when $x = 0 \Rightarrow y = 0$

and $x = 3 \Rightarrow y = 3$

$\therefore$ Intersection points are $(0,0)$ and $(3,3)$

Required Area, $A = \iint dx\,dy = \int_0^3 \left[\int_x^{4x-x^2} dy\right] dx$

$= \int_0^3 |y|_x^{4x-x^2} dx$

$= \int_0^3 (4x - x^2 - x)\,dx = \int_0^3 (3x - x^2)\,dx$

$= \left|3 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right|_0^3 = 3 \cdot \frac{3^2}{2} - \frac{3^3}{3} = \frac{27}{2} - \frac{27}{3}$

$= \frac{81 - 54}{6} = \frac{\cancel{27}^{9}}{\cancel{6}_{2}} = 9/2$ sq. units

Ans.

Que ② Find the area enclosed by parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol^n We have

$y^2 = 4ax \rightarrow$ Parabola (Sym. abt. x-axis)

and $x^2 = 4ay \rightarrow$ Parabola (Sym. abt. y-axis)

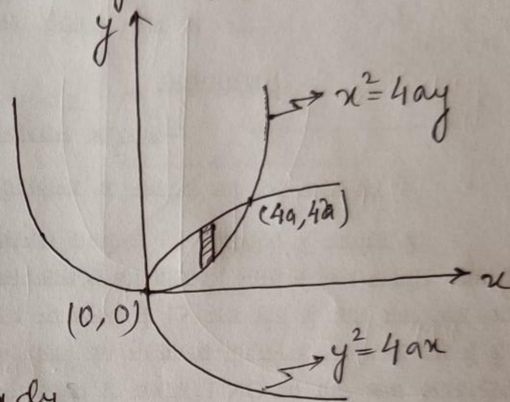
On equating both,

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^{\cancel{4}3}}{16a^2} = 4a\cancel{x} \Rightarrow x^3 = 64a^3$$

$$\Rightarrow x = 4a \Rightarrow y = 4a \quad (\because y^2 = 4ax)$$

Intersection points $(4a, 4a)$



$\therefore$ Required Area, $A = \iint dx\,dy$

$$= \int_0^{4a}\left[\int_{x^2/4a}^{2\sqrt{ax}} dy\right]dx = \int_0^{4a} \left| y \right|_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx = \left| 2a^{1/2}\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right|_0^{4a}$$

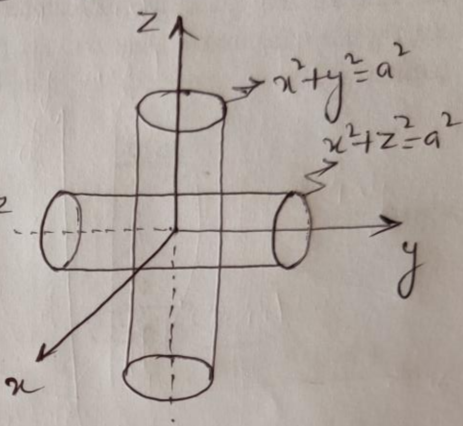
$$= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{4}{3}a^{1/2}(8a^{3/2}) - \frac{\overset{16}{\cancel{64}}a^3}{\underset{3}{\cancel{12}}a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ sq. units}$$

Ans

Volume of Solids

Que (1) Find the volume common to the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.



Sol.n Given : $x^2+y^2=a^2$

and $x^2+z^2=a^2$

Required volume $= \iiint dx\,dy\,dz$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 2^3 \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^{\sqrt{a^2-x^2}} dz\right] dy\,dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \left| z \right|_0^{\sqrt{a^2-x^2}} dy\,dx = 8\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy\right] dx$$

$$= 8 \int_0^a \sqrt{a^2-x^2} \left| y \right|_0^{\sqrt{a^2-x^2}} dx = 8\int_0^a (a^2-x^2)\,dx$$

$$= 8 \left| a^2 \cdot x - \frac{x^3}{3} \right|_0^a = 8\left(a^3 - \frac{a^3}{3}\right) = 8a^3\left(1-\frac{1}{3}\right)$$

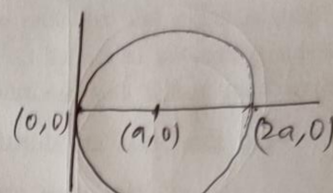
$= \frac{16}{3}a^3$ cu. units. Ans

Que② Prove that the volume enclosed by cylinder $x^2+y^2=2ax$ and $z^2=2ax$ is $\frac{128}{15}a^3$.

Solⁿ Given: $x^2+y^2=2ax$

$\Rightarrow (x-a)^2+y^2=a^2$

and $z^2=2ax$



Required volume $=\iiint dx\,dy\,dz$

$$=\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\int_{-\sqrt{2ax}}^{\sqrt{2ax}} dz\,dy\,dx = 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\sqrt{2ax}} dz\,dy\,dx$$

$$=4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} |z|_0^{\sqrt{2ax}}\,dy\,dx = 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\sqrt{2ax}\,dy\,dx$$

$$V=4\int_0^{2a}\sqrt{2ax}\,|y|_0^{\sqrt{2ax-x^2}}\,dx = 4\int_0^{2a}\sqrt{2ax}\sqrt{2ax-x^2}\,dx$$

Put $x=2a\sin^2\theta \Rightarrow dx = 2a\,2\sin\theta\cos\theta\,d\theta$
$= 4a\sin\theta\cos\theta\,d\theta$.

$$\therefore V = 4\int_0^{\pi/2}\sqrt{4a^2\sin^2\theta}\sqrt{4a^2\sin^2\theta-4a^2\sin^4\theta}\,.\,4a\sin\theta\cos\theta\,d\theta$$

$$=4\int_0^{\pi/2} 2a\sin\theta\,.\,2a\sin\theta\sqrt{1-\sin^2\theta}\,.\,4a\sin\theta\cos\theta\,d\theta.$$

$$V = 64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta.$$

$$\boxed{\text{From Gamma function, } \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}}$$

$$V = 64a^3\cdot\frac{\Gamma 2\,\Gamma 3/2}{2\,\Gamma 7/2}$$

$$=32a^3\cdot\frac{1\cdot \cancel{1/2}\times\cancel{\sqrt{\pi}}}{5/2\times 3/2\times\cancel{1/2}\times\cancel{\sqrt{\pi}}} = \frac{128}{15}a^3 \text{ cu. units.}$$

Ans

Beta Function

The beta function is defined as

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \quad \text{where, } m>0,\ n>0$$

Put $x = 1-y$

$dx = -dy$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(\cancel{1}-\cancel{1}+y)^{n-1}(-dy)$$

$$= -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

$$\Rightarrow \boxed{\beta(m,n) = \beta(n,m)} \longrightarrow \text{Imp. property}$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$.

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot \sin\theta\cos\theta\,d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta.}$$

Also, $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Gamma Function (Γ)

The gamma function is defined as

$$\boxed{\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx} \qquad n > 0$$

Put $n=1$:

$$\Gamma 1 = \int_0^\infty e^{-x} \cdot x^{1-1}\, dx = \int_0^\infty e^{-x}\, dx = \left| \frac{e^{-x}}{-1} \right|_0^\infty$$

$$= -\left| \frac{1}{e^x} \right|_0^\infty = -\left(\cancelto{0}{\frac{1}{e^\infty}} - \frac{1}{e^0} \right) = 1 \qquad \Gamma 1 = 1$$

✱ **Reduction formula for Γn :** ILATE

$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^{n+1-1}\, dx = \int_0^\infty \underset{\text{II}}{e^{-x}} \cdot \underset{\text{I}}{x^n}\, dx$$

$$= \left| x^n \left(\frac{e^{-x}}{-1} \right) - \int n x^{n-1} \left(\frac{e^{-x}}{-1} \right) dx \right|_0^\infty = n \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$= n\,\Gamma n \qquad \boxed{\Gamma(n+1) = n\,\Gamma n}$$

✱ **Values of Γn in terms of factorial :**

$$\Gamma 2 = 1 \times \Gamma 1 = \lfloor 1$$

$$\Gamma 3 = 2 \times \Gamma 2 = \lfloor 2$$

$$\boxed{\Gamma n = \lfloor (n-1)}$$

$n \to$ +ve integer

✱ $\Gamma 1/2 = \sqrt{\pi}$, $\Gamma 3/2 = 1/2 \times \sqrt{\pi}$, $\Gamma 5/2 = 3/\cancel{2} \times 1/2 \times \sqrt{\pi}$

$= 1.772$

✱

$$\boxed{\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta\, d\theta = \frac{\Gamma \frac{m+1}{2}\, \Gamma \frac{n+1}{2}}{2\, \Gamma \frac{m+n+2}{2}}}$$

Que ① Prove that : $B(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

OR

Find the relation between Beta and Gamma function.

Sol^n We know,

$$\Gamma n = \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n \, dx \quad \text{(General form)}$$

Multiplying $e^{-z} \cdot z^{m-1}$ on both sides and integrating w.r.t z from 0 to ∞.

$$\Gamma n \cdot \int_0^\infty e^{-z} \cdot z^{m-1} \, dz = \int_0^\infty \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n \cdot e^{-z} \cdot z^{m-1} \, dz \, dx$$

$$\Rightarrow \Gamma n \, \Gamma m = \int_0^\infty \int_0^\infty e^{-z(x+1)} \cdot x^{n-1} \cdot z^{m+n-1} \, dz \, dx$$

Put $z(x+1) = y \Rightarrow z = \dfrac{y}{1+x} \Rightarrow dz = \dfrac{dy}{1+x}$

$$\Gamma n \, \Gamma m = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(x+1)} \cdot z^{m+n-1} \, dz \right] dx$$

$$= \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-y} \cdot \frac{y^{m+n-1}}{(1+x)^{m+n-1}} \cdot \frac{dy}{(1+x)} \right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left[\int_0^\infty e^{-y} \cdot y^{(m+n)-1} \, dy \right] dx$$

$$\Gamma m \, \Gamma n = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, \Gamma(m+n) \, dx$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx = B(m,n)$$

$$\boxed{B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}}$$

Hence Proved

Que② Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n \lfloor n}{(m+1)^{n+1}}$, where n is a positive integer & $m > -1$.

Proof: L.H.S. $= \int_0^1 x^m (\log x)^n dx$

Let $\log x = -y$

$x = e^{-y}$

$dx = e^{-y}(-1)dy = -e^{-y}dy$

$$\int_0^1 x^m (\log x)^n dx = \int_\infty^0 (e^{-y})^m (-y)^n (-e^{-y}dy)$$

$$= -\int_0^\infty (e^{-y})^m (-1)^n \cdot y^n (-e^{-y}dy)$$

$$= (-1)^n \int_0^\infty e^{-y(m+1)} \cdot y^n dy.$$

Put $y(m+1) = z \Rightarrow y = \frac{z}{m+1} \Rightarrow dy = \frac{dz}{m+1}$

$$\int_0^1 x^m (\log x)^n dx = (-1)^n \int_0^\infty e^{-z} \frac{z^n}{(m+1)^n} \cdot \frac{dz}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-z} \cdot z^{(n+1)-1} dz$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) = \frac{(-1)^n}{(m+1)^{n+1}} \lfloor n = \text{R.H.S.}$$

$$\Rightarrow \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n \lfloor n}{(m+1)^{n+1}}$$ Hence Proved.

Que(3) Given that $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, show that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad \text{where } 0<n<1$$

Sol^n Given: $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$

We know,

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\Rightarrow \Gamma n\,\Gamma m = \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Put $m = 1-n$

$$\Gamma n\,\Gamma(1-n) = \Gamma(1-\cancel{n}+\cancel{n})\int_0^\infty \frac{x^{n-1}}{(1+x)^{1-\cancel{n}+\cancel{n}}}\,dx$$

$$= \Gamma 1 \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \qquad \left(\because \Gamma 1 = 1\right)$$

$$= \frac{\pi}{\sin n\pi} \qquad \left\{ \because \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \right.$$

Hence Proved

Que(4) Evaluate (i) $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$ (ii) $\Gamma -\frac{3}{2}$ (iii) $\beta\left(\frac{9}{2},\frac{7}{2}\right)$

Sol^n (i) $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

$\because$ We know,

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\therefore \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)$$

Here, $n = \frac{1}{4}$

$$\therefore \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{\pi}{\sin n\pi} = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2} \quad \text{Ans}$$

(ii) $\Gamma -\frac{3}{2}$

$\because$ We know,

$$\Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\Gamma -\frac{3}{2} = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-3/2} = \frac{\Gamma -\frac{1}{2}}{-3/2} \quad \text{--- (1)}$$

Now, $\Gamma -\frac{1}{2} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2} = \frac{\Gamma\frac{1}{2}}{-1/2} = -2\sqrt{\pi}$

eq^n (1) becomes,

$$\Gamma -\frac{3}{2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4}{3}\sqrt{\pi} \quad \text{Ans}$$

(iii) $\beta\left(\frac{9}{2},\frac{7}{2}\right)$

$\because$ We know,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

$$\Rightarrow \beta\left(\frac{9}{2},\frac{7}{2}\right) = 2\int_0^{\pi/2} \sin^8\theta \cdot \cos^6\theta\, d\theta$$

$$= \cancel{2}\,\frac{\Gamma\frac{8+1}{2}\,\Gamma\frac{6+1}{2}}{\cancel{2}\,\Gamma\frac{8+6+2}{2}} = \frac{\Gamma\frac{9}{2}\,\Gamma\frac{7}{2}}{\Gamma 8}$$

$$= \frac{\cancel{7}/2 \times \cancel{5}/2 \times \cancel{3}/2 \times 1/2 \times \sqrt{\pi} \times 5/2 \times 3/2 \times 1/2 \times \sqrt{\pi}}{\cancel{7} \times 6 \times \cancel{5} \times 4 \times \cancel{3} \times 2}$$

$$= \frac{\overset{5}{\cancel{15}}}{2^{10} \times \cancel{6}2}\pi = \frac{5\pi}{2^{11}} = \frac{5\pi}{2048} \quad \text{Ans}$$

$$\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

$\because \Gamma n = \lfloor n-1$

THANK YOU SO MUCH