1

Multiple Integrals

(Double and Triple Integrals and Change of Order of Integration)

1.1 Double Integrals

The concept of double integral is an extension of the concept of a definite integral to the case of two arguments (*i.e.*, a two dimensional space). Let a function f (x, y) of the independent variables x and y be continuous inside some domain (region) A and on is boundary. Divide the domain A into n subdomains $A_1, A_2, \ldots A_n$ of areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point inside the rth elementary area δA_r. From the sum

$$S_n = f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_r, y_r)\,\delta A_r + \ldots + f(x_n, y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r. \qquad \ldots(1)$$

Now take the limit of the sum (1) as $n \to \infty$ in such a way that the largest of the areas δAr approaches to zero. This limit, if it exists, is called the *double integral* of the function f (x, y) over the domain A. It is denoted by $\iint_A f(x,y)$ dA and is real as "the double integral of f (x, y) over A".

Suppose the domain (region) A is divided into rectangular partitions by a network of lines parallel to the coordinate axes. Let dx be the length of area in Cartesian coordinated. The integral $\iint f(x,y)\,dA$ is written as $\iint_A f(x,y)\,dx\,dy$ and is called the double integral of f (x, y) over the region A.

1.2 Properties of a Double Integral

1. If the region A is partitioned into two parts, say A_1 and A_2, then

$$\iint_A f(x,y)\;dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\;dx\,dy.$$

Similarly, for a sub-division of A into three or more parts.

2. The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term. Thus

$$\iint_A [f_1(x,y)+f_2(x,y)+f_3(x,y)+\ldots]\,dx\,dy$$

$$= \iint_A f_1(x,y)\,dx\,dy + \iint_A f_2(x,y)\,dx\,dy + \iint_A f_3(x,y)\,dx\,dy + \ldots$$

3. A constant factor may be taken outside the integral sign. Thus

$$\iint_A m f(x,y)\,dx\,dy = m\iint_A f(x,y)\,dx\,dy,$$

where m is a constant.

1.3 Evaluation of Double Integrals

(a) If the region A be given by the inequalities $a \le x \le b$, $c \le y \le d$, then the double integral

$$\iint_A f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dx\,dy$$

$$= \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx, \qquad \ldots(1)$$

or

$$\iint_A f(x,y)\,dx\,dy = \int_c^d\int_a^b f(x,y)\,dy\,dx$$

$$= \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy \qquad \ldots(2)$$

i.e., in this case the order of integration is immaterial, provided the limits of integration are changed accordingly.

Note: In formula (1) the definite integral $\int_c^d f(x,y)\,dy$ is calculated first. During this integration x is regarded as a constant. While in the formula (2) the definite integral $\int_a^b f(x,y)\,dx$ is calculated first and during this integration y is regarded as a constant.

(b) If the region A is bounded by the curves $y = f_1(x)$, $y = f_2(x)$, x = a and x = b, then

$$\iint_A f(x,y)\,dx\,dy = \int_a^b\int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

$$= \int_a^b\left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx,$$

where the integration with respect to y is performed first treating x as a constant.

Similarly, if the region A is bounded by the curves $x = f_1(y)$, $x = f_2(y)$, y = c, y = d, we have

$$\iint_A f(x,y)\,dx\,dy = \int_c^d\int_{f_1(y)}^{f_2(y)} f(x,y)\,dy\,dx$$

$$= \int_c^d\left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\right]dy,$$

where the integration with respect to x is performed first treating y as a constant.

Remember :

While evaluating double integrals, first integrate w.r.t. the variable haring variable limits (treating the other variable as constant) and then integrate w.r.t. the variable with constant limits.

Remark :

In the double integral $\int_a^b\int_c^d f(x,y)\,dx\,dy$, it is generally understood that the limits of integration c to d are those of y and the limits of integration a to b are those of x. However this is not a standard convention. Some authors regard these limits in the reverse order *i.e.* they regard the limits c to d as those of x and the limits a to b as those of y. So it is better to write this double integral as $\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dx\,dy$ so that there in no confusion about the limits. However in the double integral $\int_a^b\int_{f_1(x)}^{f_2(x)} F(x,y)\,dx\,dy$, there is no confusion about the limits. Obviously the variable limits are those of y because they are in terms of x and so the constant limits must be those of x. Here the first integration must be performed with respect to y regarding x as constant.

1.4 To Express a Double Integral in Terms of Polar Coordinates

Let a function f (r, θ) of the polar coordinates (r, θ) be continuous inside some region A and on its boundary. Let the region A be bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.

Divide the area A into elements by a series of concentric circular arcs with centre at origin and successive radii differing by equal amounts and a series of straight lines drawn through the origin at equal intervals of angles. Let δr be the distance between two consecutive circles and δθ be the angle between two consecutive lines. There is thus a network of elementary areas

(say n in number) of which a typical one is PQRS. If P is the point (r, θ), the area of the element PQRS situated at the point P is $\frac{1}{2}(r+\delta r)^2\delta\theta - \frac{1}{2}r^2\,\delta\theta$, $= r\delta\,\theta\,\delta r$, by neglecting the term $\frac{1}{2}(\delta r)^2\,\delta\theta$ being an infinitesimal of higher order.

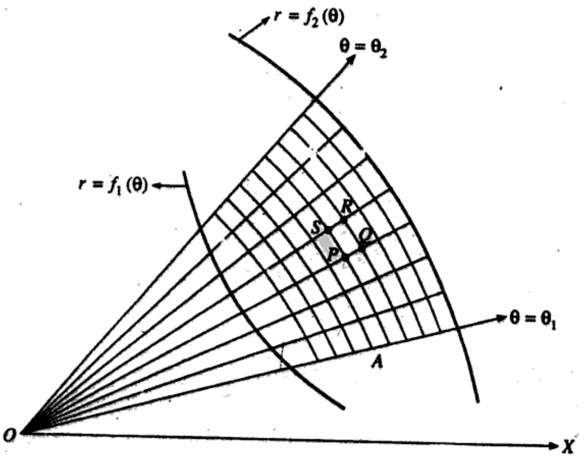

Fig. 1.1

Now by the definition of the double integers of f (r, θ) over the region A, we have

$$\iint_A f(r,\theta)\,dA = \lim_{\substack{n\to\infty\\ \delta r\to 0,\,\delta\theta\to 0}} \sum_{k=1}^{n} f(r_k,\theta_k)\,r_k\,\delta\theta\,\delta r,$$

where rk δq δr is the area of the element started at the point (r_k, θ_k).

Using the area of integration, this double integral is generally written as

$$\int_{\theta 1}^{\theta 2}\int_{f1(\theta)}^{f2(\theta)} f(r,\theta)\,d\theta\,dr, \text{ or } \int_{\theta 1}^{\theta 2} d\theta\int_{f1(\theta)}^{f2(\theta)} f(r,\theta)\,dr.$$

The first integration is performed with respect to r, keeping θ as a constant. After substituting the limits for r, the second integration with respect to θ is performed.

Remark:

The area of the typical element PQRS situated at the point P (r, θ) can also be found as below :

We have OP = r, OQ = r + δr so that PQ = δr. Also PS is the arc of a circle of radius r subtending an angle δθ at the centre of the circle and so arc PS = r δθ. Therefore the area of the element PQRS is δr. r δθ *i.e.*, r δθ δr.

1.5 Triple Integrals

Let the function f (x, y, z) of the point P (x, y, z) be continuous for all points within a finite region V and on its boundary. Divide the region V into n parts; let $\delta V_1, \delta V_2, \dots, \theta V_n$ be their volumes. Take a point in each part and form the sum

$$S_n = f(x_1, y_1, z_1)\,\delta V_1 + f(x_2, y_2, z_2)\,\delta V_2 + \dots + f(x_n, y_n, z_n)\,\delta V_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r . \qquad \dots(1)$$

Then the limit to which the sum (1) tends when n tends to infinity and the dimensions of each sub-division tend to zero, is called the triple integral of the function f (x, y, z) over the region V. This is denoted by

$$\iiint_V f(x,y,z)\,dV \text{ or } \iiint_V f(x,y,z)\,dx\,dy\,dz.$$

1.6 Evaluation of Triple Integrals

(a) If the region V be specified by the inequalities

$a \le x \le b,\ c \le y \le d,\ e \le z \le f,$

then the triple integral

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b\int_c^d\int_e^f f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b dx\int_c^d dy\int_e^f f(x,y,z)\,dz.$$

Here the order of integration is immaterial and the integration with respect to any of x,y and z can be programed first.

(b) If the limits of z are given as functions of x and y, the limits of y as functions of x while x takes the constant values say from x = a to x = b, then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b dx\int_{y_1(x)}^{y_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz.$$

The integration with respect to z is performed first regarding x and y as constants, then the integration w.r.t. y is performed regarding x as a constant and in the last we perform the integration w.r.t. x.

1.7 Change of Order of Integration

If in a double integral the limits of integration of both x and y are constant, we can generally integrate $\iint f(x,y)\,dx\,dy$ in either order. But if the limits of y are functions of x, we must first integrate w.r.t. y regarding x as constant and then integrate w.r.t. x. In this case the order of integration can be chaged only if we find the new limits of x as functions of y and the new constant of y.

1.8 Change of Variables in a Double Integral

Sometimes, the evaluation of a double integral becomes more convenient by a suitable change of variable from one system to another system.

Let the variables in the double integral $\iint_A f(x,y)\,dx\,dy$ be changed from x,y to u, v where x = ϕ (y, v) and y = ψ (u,v).

Then on substituting for x and y, the double integral is transformed to $\iint_{A'} F(u,v)\,J\,du\,dv$, where J (u, v) is the Jacobian of x,y w.r.t. u,v *i.e.*,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix},$$

and A' is the region in the uv-plane corresponding to the region A in the xy-plane. Thus remember that dxdy = J dudv.

Special Case : *Change to polar coordinates from the Cartesian coordinates.*

To change the variables from cartesians to polar coordinates we put x = r cos θ, y = r sin θ. In this case

$$= J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

and therefore dxdy = J dθ dr = r dθ dr.

This change is specially usefully useful when the region of integration is a circle or a part of a circle.

MISCELLANEOUS EXAMPLES

Example 1:

Evaluate $\iint r^2\,d\theta\,dr$ *over the area of the circle r = a cos θ.*

Solution:

The circle r = a cos θ passes through the pole and the diameter through the pole is initial line. The region of integration can be covered by radial strips originating from r = 0 and terminating at r = a cos θ. From the equation of the circle, we have r = 0 when cos θ = 0 *i.e.*, θ = ± π/2. Therefore for the given area θ varies from – π/2 to π/2. Therefore, the required integral

$$= \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{a\cos\theta} r^2\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\left[\int_0^{a\cos\theta} r^2\,dr\right]d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{a^3\cos^3\theta}{3}\,d\theta = \frac{2a^3}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{2a^3}{3}\cdot\frac{2}{3.1} = \frac{4a^3}{9}$$

Example 2:

Evaluate $\iint \frac{r\,d\theta\,dr}{\sqrt{(a^2+r^2)}}$ *over one loop of the lemniscate* $r^2 = a^2$ *cos 2θ.*

Solution:

In the equation of the lemniscate $r^2 = a^2\cos^2\theta$, putting r = 0, we get cos 2θ, putting r = 0, we get cos 2θ = 0 *i.e.*, 2θ = ± π/4. Therefore, one loop of the given lemniscate θ varies from –π/4 and r varies from 0 to a √(cos 2θ).

Therefore the required integral

$$= \int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}\frac{r\,d\theta\,dr}{\sqrt{(a^2+r^2)}}$$

$$= \int_{-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}\frac{1}{2}(a^2+r^2)^{-1/2}(2r)\,d\theta\,dr$$

$$= \int_{-\pi/4}^{\pi/4}\left[(a^2+r^2)^{1/2}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}[a(1+\cos 2\theta)^{1/2} - a]\,d\theta$$

$$= 2a\int_0^{\pi/4}[(2\cos^2\theta)^{1/2} - 1]\,d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta - 1)\,d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4} = 2a\left[\sqrt{2}.\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= 2a\left[1 - \frac{\pi}{4}\right] = \frac{a}{2}(4 - \pi).$$

Example 3:

Find by double integration the area lying inside the cardioid r = a (1 + cos θ) and outside the circle r = a.

Solution:

Eliminating r between the given equations of the cardioid r = a (1 + cos θ) and the circle r = a, we have a = a (1 + cos θ) or cos θ=0 *i.e.*, θ = ± π/2.

Thus, the region of integration A is enclosed by r = a, r = a (1 + cos θ), θ = – π/2, θ = π/2.

$$\therefore \text{ the required area} = \iint_A r\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}[a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/2}^{\pi/2}(1+\cos^2\theta + 2\cos\theta - 1)\,d\theta$$

$$= \frac{a^2}{2}.2\int_0^{\pi/2}[\cos^2\theta + 2\cos\theta]\,d\theta$$

$$= a^2\left[\frac{1}{2}.\frac{1}{2}\pi + 2\{\sin\theta\}_0^{\pi/2}\right] = a^2\left[\frac{1}{4}\pi + 2\right] = \frac{a^4}{4}(\pi + 8).$$

Example 4:

Find the mass of a loop of the lemniscate $r^2 = a^2\sin^2\theta$ *if density* $\rho = kr^2$.

Solution:

In the equation of the lemniscate $r^2 = a^2\sin^2\theta$, putting r = 0, we get sin 2θ = 0 *i.e.*, 2θ = 0, π *i.e.*, θ = 0, $\frac{1}{2}\pi$.Therefore, for one loop of the given lemniscate θ varies from 0 to π/2 and r varies from 0 to a √(sin 2θ).

∴ mass of a loop of the lemniscate

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sqrt{(\sin 2\theta)}}\rho r\,d\theta\,dr = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sqrt{(\sin 2\theta)}} kr^2.r\,d\theta\,dr$$

$$= k\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sqrt{(\sin 2\theta)}} r^3 d\theta\, dr = k.\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{a\sqrt{(\sin 2\theta)}} d\theta$$

$$= \frac{ka^4}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta = \frac{ka^4}{8}\int_0^{\pi/2}(1-\cos 4\theta)d\theta$$

$$= \frac{ka^4}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{ka^4}{8}.\frac{\pi}{2} = \frac{\pi k a^4}{16}$$

Example 5:

Integrate r sin θ over the area of the cardioid r = a (1+ cos θ) lying above the initial line.

Solution:

For the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line θ varies from 0 to π. Also for that required area r varies from $r = 0$ to $r = a(1 + \cos\theta)$. If A denotes the region consisting of the area of the cardioid lying above the initial line, then the required integral

$$= \iint_A r\sin\theta\, dA = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, r\, d\theta\, dr$$

$$= \int_0^{\pi}\sin\theta\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta = \frac{a^3}{3}\int_0^{\pi}\sin\theta(1+\cos\theta)^3 d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2}\sin\phi\cos^7\phi.2d\phi,\ \text{Puting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2d\phi$$

$$= 32.\frac{a^3}{3}\left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2} = \frac{32a^3}{3}\left[0+\frac{1}{8}\right] = \frac{4a^3}{3}.$$

Example 6:

Evaluate $\iint_R r^2\sin\theta\, d\theta\, dr$ *where R is the circle r = 2a cos θ.*

Solution:

The given integral $I = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\sin\theta\, d\theta\, dr$

$$= \int_{-\pi/2}^{\pi/2}\left[\int_{r=0}^{2a\cos\theta} r^2 dr\right]\sin\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}\sin\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{8a^3}{3}\cos^3\theta\sin\theta\, d\theta$$

= 0 because $\cos^3\theta\sin\theta$ is an odd function of θ.

$$\left[\text{Note that } \int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) = -f(x)\right]$$

Example 7:

Find by double integration the area lying inside the circle r = a sin θ and outside the cardioid r = a (1 + cos θ).

Solution:

The given circle is $r = a\sin\theta$ and the cardioid is $r = a(1 - \cos\theta)$. Note that the given circle passes through the pole and the diameter through the pole makes an angle $\pi/2$ with the initial line. Eliminating r between the two equations, we have

$$a\sin\theta = a(1-\cos\theta)$$

or $$1 = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\frac{\theta}{2}$$

or $\frac{1}{2}\theta = \frac{1}{4}\pi$ *i.e.*, $\theta = \pi/2$.

Thus, the two curves meet at the point where $\theta = \pi/2$. Also for both the curves $r = 0$ when $\theta = 0$ and so the two curves also meet at the pole O where θ 0. To cover the required area the limits of integration for r are a $(1 - \cos\theta)$ to a $\sin\theta$ and for θ are 0 to $\pi/2$. Therefore the required area

$$\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\, d\theta\, dr = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}[a^2\sin^2\theta - a^2(1-\cos\theta)^2]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]d\theta$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\pi}{2} + 2.1 - \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{a^2}{4}\left[1 - \frac{\pi}{2}\right] = \frac{a^2}{4}(4-\pi).$$

Example 8:

Transform the integral $\int_0^2\int_0^{\sqrt{(2x-x^2)}}\frac{x\, dx\, dy}{\sqrt{(x^2+y^2)}}$ *by changing to polar coordinates and hence evaluate it.*

Solution:

From the limits of integration it is obvious that the region of integration is bounded by $y = 0$, $y = \sqrt{(2x - x^2)}$ and $x = 0$, $x = 2$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 - 2x = 0$ between the lines $x = 0$, $x = 2$ and lying above the axis of x*i.e.*, the line $y = 0$,

Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is $r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0$, or $r = 2\cos\theta$.

From the figure it is obvious that r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\pi/2$. Note that at the point A of the circle $\theta = 0$ and at the point O, $r = 0$ and so from $r = 2\cos\theta$, we get $\theta = \pi/2$ at O.

The polar equivalent of elementary area dxdy is $r\, d\theta\, dr$.

$\therefore \iint_A f(x,y)dx\, dy = \iint_A f(r\cos\theta, r\sin\theta)r\, d\theta\, dr$, where A is the region of integration.

Hence transforming to polar coordinates, the given double integral

$$= \int_\theta^{\pi/2}\int_{r=0}^{2\cos\theta}\frac{r\cos\theta}{r} r\, d\theta\, dr$$

$$= \int_\theta^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$\int_\theta^{\pi/2}\frac{1}{2}\cos\theta.4\cos^2\theta d\theta = 2\int_0^{\pi/2}\cos^3 d\theta = 2.\frac{2}{3} = \frac{4}{3}.$$

Example 9:

Find by double integration the area lying inside the cardioid r = 1 + cos θ and outside the parabola r (1 + cos θ) = 1.

Solution:

Eliminating r between the given equations of the cardioid and the parabola, we have $(1 + \cos\theta) = 1/(1 + \cos\theta)$ or $(1 + \cos\theta)^2 = 1$

or $\cos^2\theta + 2\cos\theta = 0$ or $\cos\theta(2 + \cos\theta) = 0$

or $\cos\theta = 0$, because $\cos\theta$ cannot be equal to -2

or $\theta = \pm\pi/2$.

Thus, the two curves intersect at the point where $\theta = -\pi/2$ and $\theta = \pi/2$.

Therefore, the required area is enclosed by $r = 1/(1 + \cos\theta)$, $r = (1 + \cos\theta)$, $\theta = -\pi/2$, $\theta = \pi/2$.

Hence the required area

$$= \int_{-\pi/2}^{\pi/2}\int_{1/(1+\cos\theta)}^{(1+\cos\theta)} r\, d\theta\, dr = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}r^2\right]_{1/(1+\cos\theta)}^{(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]d\theta$$

$$= 2\cdot\frac{1}{2}\int_0^{\pi/2}\left[(1+2\cos\theta+\cos^2\theta) - \frac{1}{(2\cos^2\frac{1}{2}\theta)^2}\right]d\theta$$

$$= \int_0^{\pi/2}(1+2\cos\theta)d\theta + \int_0^{\pi/2}\cos^2\theta\, d\theta - \frac{1}{4}\int_0^{\pi/2}\sec^4\frac{2}{2}\theta\, d\theta$$

$$= [\theta + 2\sin\theta]_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{4}\int_0^{\pi/2}(1+\tan^2\frac{1}{2}\theta)\sec^2\frac{1}{2}\theta\, d\theta$$

$$= \frac{\pi}{2} + 2 + \frac{\pi}{4} - \frac{1}{4}\int_0^{\pi/2}\left[\sec^2\frac{1}{2}\theta + 2(\tan^2\frac{1}{2}\theta)(\frac{1}{2}\sec^2\frac{1}{2}\theta)\right]d\theta$$

$$= \frac{3\pi}{4} + 2 - \frac{1}{4}\left[2\tan\frac{1}{2}\theta + \frac{2}{3}\tan^3\frac{1}{2}\theta\right]_0^{\pi/2}$$

$$= \frac{3\pi}{4} + 2 - \frac{1}{4}\left[2+\frac{2}{3}\right] = \frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3} = \frac{(9\pi+16)}{12}.$$

Example 10:

Transform the following double integrals to polar coordinates and hence evaluate them.

(i) $\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}(a^2 - x^2 - y^2)dx\, dy$.

(ii) $\int_0^1\int_x^{\sqrt{(a^2-y^2)}}(x^2+y^2)dx\, dy$

(iii) $\int_0^a\int_0^{\sqrt{(2a-x^2)}} y^2\sqrt{(x^2+y^2)}dx\, dy$

Solution:

(i) The given double integral $I = \int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}[a^2 - (x^2+y^2)]dx\, dy$.

From the limits of integration it is obvious that the region of integration R is bounded by $x = 0$, $x = \sqrt{(a^2-y^2)}$ and $y = 0$, $y = a$.

Thus, the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is $r = a$.

From the figure it is obvious that for the area OAB, r varies from 0 to a and θ varies from 0 to $\pi/2$. Also the polar equivalent of dx dy is r dθ dr.

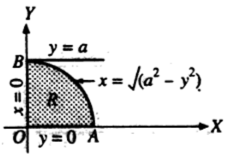

Fig. 1.2

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a}(a^2 - r^2)\, r\, d\theta\, dr,$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a}[a^2 r - r^3]\, d\theta\, dr = \int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{a} d\theta$$

$$= \int_0^{\pi/2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]d\theta = \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}[\theta]_0^{\pi/2} = \frac{\pi a^4}{8}.$$

(ii) The given double integral $I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{(2x-x^2)}}(x^2+y^2)\,dx\,dy$.

Here the region of integration R is bounded by $y = x$, $y = \sqrt{(2x-x^2)}$ and $x = 0$, $x = 1$ *i.e.*, the region of integration is the area OBCO of the circle $x^2 + y^2 - 2x = 0$ bounded by the lines $y = x$, $x = 0$ and $x = 1$.

Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is $r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0$,

or $r = 2\cos\theta$.

The point B is on the line $y = x$ which makes an algle $\pi/4$ with OX and so, at B, $\theta = \pi/4$. At the point O of the circle $r = 2\cos\theta$, we have $r = 0$ and so $\theta = \pi/2$. Thus for the region R, r varies from 0 to $2\cos\theta$ and θ varies from $\pi/4$ to $\pi/2$. Also the polar equivalent of dx dy is r dθ dr.

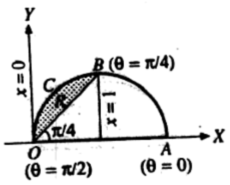

Fig. 1.3

Hence transforming to polar coordinates, we have

$$I = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2 . r\, d\theta\, de = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} r^3 d\theta\, dr$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{2\cos\theta} d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2}\cos^4\theta\, d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}(1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right]d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right]d\theta$$

$$= \left[\frac{2}{3}\theta + 2.\frac{\sin 2\theta}{2} + \frac{1}{2}.\frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{3\pi}{4} - \frac{3\pi}{8} - 1\right] = \frac{3\pi}{8} - 1.$$

(iii) The given double integral $I = \int_{x=0}^{a}\int_{y=0}^{\sqrt{(a^2+x^2)}} y^2\sqrt{(x^2+y^2)}\,dx\,dy$.

Here the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$. Thus the region of integration R is the area of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant. The polar equation of this circle is $r = a$ and for the region R, r varies from 0 to a and θ varies from 0 to $\pi/2$. Putting $x = r\cos\theta$, $y = r\sin\theta$ and replacing dx dy by dθ dr, we have

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\sin^2\theta . r . r\, d\theta\, dr$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^4\sin^2\theta\, d\theta\, dr$$

$$= \int_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a \sin^2\theta\, d\theta = \frac{a^5}{5}\int_0^{\pi/2}\sin^2\theta\, d\theta$$

$$= \frac{a^5}{5}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^5}{20}.$$

Example 11:

Changes the following integrals into polar coordinates.

(i) $\int_0^a\int_0^{\sqrt{(a^2-y^2)}}(x^2+y^2)\,dy\,dx$.

(ii) $\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}\,dx\,dy$.

Solution:

(i) Let $I = \int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}(x^2+y^2)\,dx\,dy$.

Changing to polar coordinates, we have

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 . r\, d\theta\, dr$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\, d\theta\, dr.$$

(ii) Let $I = \int_{y=0}^{4a}\int_{x=y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}\,dx\,dy$.

Here the region of integration R is bound by $x = y^2/4a$, $x = y$, $y = 0$ and $y = 4a$ *i.e.*, the region of integration is the area of the parabola $y^2 = 4ax$ cut off by the straight line $y = x$.

Changing to polar coordinated, the equation $y^2 = 4ax$ becomes

$(r\sin\theta)^2 = 4a\,(r\cos\theta)$

or $r = \dfrac{4a\cos\theta}{\sin^2\theta}$.

At the point B, $\theta = \pi/4$.

At the point O of the parabola $r = \dfrac{4a\cos\theta}{\sin^2\theta}$, we have $r = 0$ a nd so $\theta = \pi/2$.

Thus, for the region R, r varies from 0 to $\dfrac{4a\cos\theta}{\sin^2\theta}$ and θ varies from $\pi/4$ to $\pi/$. Also the polar equivalent of dx dy is r dθ dr.

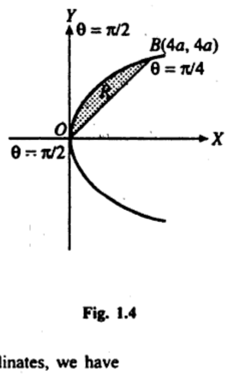

Fig. 1.4

Hence transforming to polar coordinates, we have

$$I = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{(4a\cos\theta)/\sin^2\theta}\frac{r^2(\cos^2\theta - \sin^2)}{r^2}\, r\, d\theta\, dr$$

$$= \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{(4a\cos\theta)/\sin^2\theta} r\cos 2\theta\, d\theta\, dr.$$

Example 12:

Transform to polar coordinates and integrate

$$\iint\sqrt{\left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)}dx\,dy$$

the integral being extended over all positive values of x and y subject to $x^2 + y^2 \le 1$.

Solution:

Here the region of integration R of the given double integral is the area of the circle $x^2 + y^2 = 1$ lying in the positive quadrant. The polar equation of this circle is $r = 1$ and for the region R, r varies from 0 to 1 and θ varies from 0 to $\pi/2$. Also $dx\,dy = r\,d\theta\,dr$.

Hence transforming to polar coordinates, the given double integral

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\sqrt{\left(\frac{1-r^2}{1+r^2}\right)} r\, d\theta\, dr$$

$$= \int_{r=0}^{1} r\sqrt{\left(\frac{1-r^2}{1+r^2}\right)}[\theta]_{\theta=0}^{\pi/2}dr,$$

first integrating w.r.t. θ taking r as constant

$$= \frac{\pi}{2}\int_0^1\frac{r(1-r^2)}{\sqrt{(1-r^4)}}dr$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\frac{(1-\sin t)}{\cos t}\cdot\frac{1}{2}\cos t\, dt,$$

Putting $r^2 = \sin t$ so that $2r\,dr = \cos t\,dt$

$$= \frac{\pi}{4}[t + \cos t]_0^{\pi/2} = \frac{\pi}{4}\left[\frac{\pi}{2} + 0 - 0 - 1\right] = \frac{\pi}{8}(\pi - 2).$$

Example 13:

By changing to polar coordinates, evaluate

$$\iint xy(x^2+y^2)^{n/2}\,dx\,dy,\ n + 3 > 0,$$

over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Deduce the value of

$$\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dx\,dy$$

over the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution:

Let $I = \iint_R xy(x^2+y^2)^{n/2}\,dx\,dy$,

where the region of integration R is the area of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

We have $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = a^2$. The polar equation of the circle $x^2 + y^2 = a^2$ is $r = a$ and for the region R, r varies from 0 to a and θ varies from 0 to $\pi/2$. Also $dx\,dy = r\,d\theta\,dr$.

∴ transforming to polar coordinates, we have

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta.\,r\sin\theta.(r^2)^{n/2}\,r\,d\theta\,dr$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\cos\theta.\,r^{n+3}\sin\theta\,d\theta\,dr$$

$$= \int_0^{\pi/2}\left[\frac{r^{n+4}}{n+4}\right]_{r=0}^{a}\cos\theta\sin\theta\,d\theta$$

$$= \frac{a^{n+4}}{n+4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \frac{a^{n+4}}{n+4}\cdot\frac{1}{2} = \frac{a^{n+4}}{2(n+4)}.$$

Now let $I_1 = \iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dx\,dy$, the integral being extended over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Put $\frac{x}{a} = u, \frac{y}{b} = v$. Then $dx = a\,du$, $dy = b\,dv$.

∴ $I_1 = \iint au.bv.(u^2+v^2)\,ab\,du\,dv$, the integral being extended to all positive values of u and v subject to the condition $u^2 + v^2 \le 1$

$$= a^2b^2\iint uv(u^2+v^2)\,du\,dv.$$

Now putting a = 1 and n = 2 in the value of I, we have

$$\iint uv(u^2+v^2)\,du\,dv = \frac{1}{12}.$$

$$\therefore I_1 = a^2b^2.\frac{1}{12} = \frac{a^2b^2}{12}.$$

Example 14:

Evaluate $\int_{y=0}^{3}\int_{x=0}^{2}\int_{z=0}^{1}(x+y+z)\,dz\,dx\,dy$.

Solution:

The given integral

$$= \int_{y=0}^{3}\int_{x=0}^{2}\left\{\int_0^1(x+y+z)\,dz\right\}dx\,dy$$

$$= \int_{y=0}^{3}\int_{x=0}^{2}\left\{xz+yz+\frac{z^2}{2}\right\}_0^1 dx\,dy = \int_0^3\left\{\int_0^2\left(x+y+\frac{1}{2}\right)dx\right\}dy$$

$$= \int_0^3\left\{\frac{x^2}{2}+xy+\frac{x}{3}\right\}_0^2 dy = \int_0^3(3+2y)\,dy = \left[3y+\frac{2y^2}{2}\right]_0^3 = 18.$$

Example 15:

Show that $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2}x^2yz\,dz\,dy\,dx = 1$.

Solution:

We have $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2}x^2yz\,dz\,dy\,dx$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left\{\int_1^2 x^2yz\,dz\right\}dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left[x^2y.\frac{z^2}{2}\right]_1^2 dy\,dx = \frac{1}{2}\int_0^1\left[\int_0^2(3x^2y)\,dy\right]dx$$

$$= \frac{3}{2}\int_0^1\left[x^2.\frac{y^2}{2}\right]_0^2 dx = \frac{3}{4}\int_0^1 4x^2\,dx = 3\left[\frac{x^3}{3}\right]_0^1 = 3.\frac{1}{3} = 1.$$

Example 16:

Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$.

Solution:

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1\left[\int_0^1 e^{x+y+z}\,dx\right]dy\,dz$$

$$= \int_0^1\int_0^1\left[\int_0^1 e^{x+y+z}\right]_0^1 dy\,dz = \int_0^1\left[\int_0^1\{e^{1+y+z}-e^{y+z}\}\,dy\right]dz$$

$$= \int_0^1\left[e^{1+y+z}-e^{y+z}\right]_0^1 dz$$

$$= \int_0^1\{(e^{2+z}-e^{1+z})-(e^{1+z}-e^z)\}\,dz$$

$$= \int_0^1(e^{2+z}-2e^{1+z}+e^z)\,dz = \int_0^1(e^2-2e+1)e^z\,dz$$

$$= (e^2-2e+1)\int_0^1 e^z\,dz = (e-1)^2\left[e^z\right]_0^1 = (e-1)^2(e-e^0)$$

$$= (e-1)^2(e-1) = (e-1)^3.$$

Example 17:

Evaluate $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$.

Solution:

Here x − z to x + z are the limits of integration of y, 0 to z are those of x and − 1 to 1 are those of z. The given triple integral is

$$= \int_{-1}^{1}\int_0^z\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$$

$$= \int_{-1}^{1}\int_0^z\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\,dz$$

$$= \int_{-1}^{1}\int_0^z\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)-x(x-z)-\frac{(x-z)^2}{2}-z(x-z)\right]dx\,dz$$

$$= \int_{-1}^{1}\left[\int_0^z(4xz+2z^2)\,dx\right]dz = \int_{-1}^{1}\left[2zx^2+2z^2x\right]_0^z dz$$

$$= \int_{-1}^{1}(2z.z^2+2z^2.z)\,dz = 4\int_{-1}^{1}z^3\,dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 1.\,[1-1] = 0.$$

Example 18:

Evaluate the following integrals.

(i) $\int_0^1\int_0^{1-x}\int_0^{1-x-y}xyz\,dx\,dy\,dz$;

(ii) $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$;

(iii) $\int_0^{\log 2}\int_0^x\int_0^{x+\log y}e^{x+y+z}\,dx\,dy\,dz$;

(iv) $\int_0^1\int_{y^2}^1\int_0^{1-x}x\,dy\,dx\,dz$.

(v) $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{dx\,dy\,dz}{(1+x+y+z)^3}$.

(vi) $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}}xyz\,dz\,dy\,dx$

(vii) $\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dx\,dy$.

Solution:

(i) We have $\int_0^1\int_0^{1-x}\int_0^{1-x-y}xyz\,dx\,dy\,dz$

$$= \int_0^1\int_0^{1-x}xy\left[\frac{z^2}{2}\right]_0^{1-x-y}dx\,dy,$$

integrating w.r.t. z regarding x and y as constants

$$= \frac{1}{2}\int_0^1\int_0^{1-x}xy\{(1-x)-y\}^2\,dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x}x[y(1-x)^2-2(1-x)y^2+y^3]\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^2y^2}{2}-\frac{2(1-x)y^3}{3}+\frac{y^4}{4}\right]_0^{1-x}dx$$

integrating w.r.t. y regarding x as constants

$$= \frac{1}{24}\int_0^1 x[6(1-x)^4-8(1-8)^4+3(1-x)^4]\,dx$$

$$= \frac{1}{24}\int_0^1 x(1-x)^4\,dx$$

$$= \frac{1}{24}\int_0^{\pi/2}\sin^2\theta\cos^8\theta.2\sin\theta\cos\theta\,d\theta,$$

putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$

$$= \frac{1}{12}\int_0^{\pi/2} \sin^3\theta\cos^9\theta\, d\theta = \frac{1}{12}\cdot\frac{2.8.6.4.2}{12.10.8.6.4.2} = \frac{1}{720}.$$

(ii) Here the integrand $x^2 + y^2 + z^2$ is a symmetrical expression in x, y and z and therefore the limits of integration can be as signed at pleasure. We have the given integral

$$= \int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=0}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz,$$

because $x^2 + y^2 + z^2$ is an even function of x

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{x^3}{3}+(y^2+z^2)x\right]_0^a dy\,dz.$$

integrating w.r.t. x regarding y and z as constants

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{a^3}{3}+ay^2+az^2\right]dy\,dz$$

$$= 4\int_{z=-c}^{c}\int_{0}^{b}\left[\frac{a^3}{3}+az^2+ay^2\right]dy\,dz$$

because $\frac{a^3}{3}+az^2+ay^2$ is an even function of y

$$= 4\int_{z=-c}^{c}\left[\frac{a^3}{3}y+az^2y+\frac{ay^3}{3}\right]dz,$$

integrating w.r.t. y regarding z as constant

$$= 4\int_{z=-c}^{c}\left[\frac{a^3b}{3}+abz^2+\frac{ab^3}{3}\right]dz = 8\int_0^c\left[\frac{a^3b}{3}+abz^2+\frac{ab^3}{3}\right]dz$$

$$= 8\left[\frac{a^3b}{3}z+ab\frac{z^3}{3}+\frac{ab^3}{3}z\right]_0^c$$

$$= \frac{8}{3}(a^2bc+abc^3+ab^3c) = \frac{8}{3}abc(a^2+b^2+c^2).$$

(iii) We have $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$

$$= \int_0^{\log 2}\int_0^x\left[e^{x+y+z}\right]_0^{x+\log y} dx\,dy,$$

integrating w.r.t. z regarding x and y as constants

$$= \int_0^{\log 2}\int_0^x[e^{x+y+x+\log y}-e^{x+y}]\,dx\,dy$$

$$= \int_0^{\log 2}\int_0^x[e^{2x}e^ye^{\log y}-e^xe^y]\,dx\,dy$$

$$= \int_0^{\log 2}\int_0^x[e^{2x}ye^y-e^xe^y]\,dx\,dy,$$

$$= \int_0^{\log 2}\left[\int_0^x e^{2x}y\,e^y\,dy-\int_0^x e^xe^y\,dy\right]dx$$

$$= \int_0^{\log 2}\left[e^{2x}\{ye^y\}_0^x-e^{2x}\int_0^x e^y\,dy-e^x\{e^y\}_0^x\right]dx$$

integrating w.r.t. y regarding x as a constant; to integrate ye^y we have applied integration by parts

$$= \int_0^{\log 2}\left[e^{2x}.xe^x-e^{2x}\{e^y\}_0^x-e^x(e^x-1)\right]dx$$

$$= \int_0^{\log 2}[xe^{3x}-e^{2x}(e^x-1)-e^{2x}+e^x]\,dx$$

$$= \int_0^{\log 2}[x\,e^{3x}-e^{3x}+e^x]\,dx$$

$$= \int_0^{\log 2}x\,e^{3x}dx-\int_0^{\log 2}e^{3x}dx+\int_0^{\log 2}e^xdx$$

$$= \frac{1}{3}\left[x\,e^{3x}\right]_0^{\log 2}-\frac{1}{3}\int_0^{\log 2}e^{3x}dx-\int_0^{\log 2}e^{3x}dx+\int_0^{\log 2}e^xdx$$

$$= \frac{1}{3}(\log 2)e^{3\log 2}-\frac{4}{3}\left[\frac{e^{3x}}{3}\right]_0^{\log 2}+\left[e^x\right]_0^{\log 2}$$

$$= \frac{1}{3}(\log 2)e^{3\log 8}-\frac{4}{9}(e^{3\log 2}-1)+(e^{\log 2}-1)$$

$$= \frac{8}{3}\log 2-\frac{4}{9}(8-1)+(2-1) = \frac{8}{3}\log 2-\frac{28}{9}+1$$

$$= \frac{8}{3}\log 2-\frac{19}{9}.$$

(iv) We have $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dy\,dx\,dz$

$$= \int_0^1\int_{y^2}^1 x[z]_0^{1-x}\,dy\,dx,$$

integrating w.r.t. z regarding x and y as constants

$$= \int_0^1\int_{y^2}^1 x(1-x)\,dy\,dx$$

$$= \int_0^1\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{y^2}^1 dy,$$

integrating w.r.t. x regarding y as constant

$$= \int_0^1\left[\frac{1}{2}-\frac{1}{3}-\frac{y^4}{2}+\frac{y^6}{3}\right]dy = \int_0^1\left[\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right]dy$$

$$= \left[\frac{1}{6}y-\frac{y^5}{10}+\frac{y^7}{21}\right]_0^1 = \frac{1}{6}-\frac{1}{10}+\frac{1}{21} = \frac{35-21+10}{210} = \frac{24}{210} = \frac{4}{35}.$$

(v) We have $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{dx\,dy\,dz}{(1+x+y+z)^3}$

$$= \int_0^1\int_0^{1-x}\left[-\frac{1}{2(1+x+y+z)^2}\right]_0^{1-x-y} dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x}\left[-\frac{1}{4}+\frac{1}{(1+x+y+z)^2}\right]dx\,dy$$

$$= \frac{1}{2}\int_0^1\left[-\frac{1}{4}y-\frac{1}{(1+x+y)}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1\left[-\frac{1}{4}(1-x)-\frac{1}{2}+\frac{1}{(1+x)}\right]dx$$

$$= \frac{1}{2}\int_0^1\left[-\frac{3}{4}+\frac{1}{4}x+\frac{1}{(1+x)}\right]dx$$

$$= \frac{1}{2}\left[-\frac{3}{4}x+\frac{1}{4}\cdot\frac{x^2}{2}+\log(1+x)\right]_0^1$$

$$= \frac{1}{2}\left[-\frac{3}{4}+\frac{1}{8}+\log 2\right] = \frac{1}{2}\left(\log 2-\frac{5}{8}\right).$$

(vi) The given integral I

$$= \int_{x=0}^1\int_{y=0}^{\sqrt{(1-z^2)}}\int_{z=0}^{\sqrt{(1-x^2-y^2)}} xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^1\int_{y=0}^{\sqrt{(1-z^2)}} xy\left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{(1-x^2-y^2)}} dy\,dx$$

$$= \int_{x=0}^1\int_{y=0}^{\sqrt{(1-z^2)}}\frac{1}{2}xy(1-x^2-y^2)\,dy\,dx$$

$$= \int_{x=0}^1\frac{1}{2}x\left[(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right]_{y=0}^{\sqrt{(1-x^2)}} dx$$

$$= \int_0^1\frac{1}{2}x\left[\frac{1}{2}(1-x^2)^2-\frac{1}{4}(1-x^2)^2\right]dx$$

$$= \int_0^1\frac{1}{2}x\cdot\frac{1}{4}(1-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^{\pi/2}\sin\theta.\cos^4\theta\cos\theta\,d\theta,$$

putting $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$

$$= \frac{1}{8}\int_0^{\pi/2}\sin\theta\cos^5\theta\,d\theta = \frac{1}{8}\cdot\frac{1.4.2}{6.4.2} = \frac{1}{48}.$$

(vii) The given integral I

$$= \int_{y=1}^e\int_{x=1}^{\log y}\int_{z=1}^{e^x}\log z\,dz\,dx\,dy$$

$$= \int_{y=1}^e\int_{x=1}^{\log y}[z\log z-z]_{z=1}^{e^x}\,dx\,dy$$

$$= \int_{y=1}^e\int_{x=1}^{\log y}[xe^x-e^x+1]\,dx\,dy$$

$$= \int_{y=1}^e[xe^x-2e^x+x]_{x=1}^{\log y}\,dy$$

$$= \int_1^e[y\log y-2y+\log y-e+2e-1]\,dy \qquad [\because e^{\log y}=y]$$

$$= \int_1^e[y\log y+\log y-2y+e-1]\,dy$$

$$= \left[\frac{y^2}{2}\log y-\frac{y^2}{4}+y\log y-y-y^2+ey-y\right]_1^e$$

$$= \frac{1}{2}e^2-\frac{1}{4}e^2+e-e-e^2+e^2-e+\frac{1}{4}+1+1-e+1$$

$$= \frac{1}{4}e^2-2e+\frac{13}{4}.$$

Example 19:

Evaluate $\int_1^3\int_{1/x}^1\int_0^{\sqrt{(xy)}} xyz\,dx\,dy\,dz$.

Solution:

The given triple integral is

$$\int_1^3\int_{1/x}^1\left[\int_0^{\sqrt{(xy)}} xyz\,dz\right]dx\,dy = \int_1^3\int_{1/x}^1\left[xy.\frac{z^2}{2}\right]_0^{\sqrt{(xy)}}dx\,dy$$

$$= \frac12\int_1^3\left[\int_{1/x}^1 x^2y^2dy\right]dx = \frac12\int_1^3\left[x^2.\frac{y^3}{3}\right]_{1/x}^1$$

$$=\frac16\int_1^3\left[x^2-\frac1x\right]dx = \frac16\left[\frac{x^3}{3}-\log x\right]_1^3$$

$$=\frac16\left[(9-\log 3)-\left(\frac13-\log 1\right)\right] = \frac16\left[\left(9-\frac13\right)-\log 3\right]$$

$$=\frac16\left[\frac{26}{3}-\log 3\right].$$

Example 20:

Evaluate $\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}dr\int_0^{(a^2-r^2)/a} r\,dz$.

Solution:

The given triple integral is

$$=\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}dr[rz]_0^{(a^2-r^2)/a}$$

$$=\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}\frac{r(a^2-r^2)}{a}dr$$

$$=\frac1a\int_0^{\pi/2}\left[\frac{a^2r^2}{2}-\frac{r^2}{4}\right]_0^{a\sin\theta}d\theta = \frac{a^3}{4}\int_0^{\pi/2}(2\sin^2\theta-\sin^4\theta)d\theta$$

$$=\frac{a^3}{4}\left[2\cdot\frac12\cdot\frac\pi2-\frac34\cdot\frac12\cdot\frac\pi2\right]=\frac{5a^3\pi}{64}.$$

Example 21:

Evaluate $\int_0^a\int_0^x\int_0^{x+y}e^{x+y+z}dx\,dy\,dx$.

Solution:

The given triple integral is

$$=\int_0^a\int_0^x\left[\int_0^{x+y}e^{x+y+z}dz\right]dx\,dy = \int_0^a\int_0^x\left[e^{x+y+z}\right]_{z=0}^{x+y}dx\,dy$$

$$=\int_0^a\int_0^x[e^{2(x+y)}-e^{(e+y)}]dx\,dy = \int_0^a\left[\frac12e^{2(x+y)}-e^{(x+y)}\right]_0^x dx$$

$$=\int_0^a\left[\frac12(e^{4x}-e^{2x})-(e^{2x}-e^x)\right]dx=\int_0^a\left(\frac12e^{4x}-\frac32e^{2x}+e^x\right)dx$$

$$=\left[\frac12\cdot\frac14e^{4x}-\frac34\cdot\frac12e^{2x}+e^x\right]_0^a$$

$$=\left[\left(\frac18e^{4a}-\frac34e^{2a}+e^a\right)-\left(\frac18e^0-\frac34e^0+e^0\right)\right]$$

$$=\frac18e^{4a}-\frac34e^{2a}+e^a-\left(\frac18-\frac34+1\right)=\frac18(e^{4a}-6e^{2a}+8e^a-3)$$

Example 22:

Evaluate $\int_0^4\int_0^{2\sqrt z}\int_0^{\sqrt{(4z-x^2)}}dz\,dx\,dy$.

Solution:

The given triple integral is

$$=\int_0^4\int_0^{2\sqrt z}\left[\int_0^{\sqrt{(4z-x^2)}}dy\right]dz\,dx=\int_0^4\int_0^{2\sqrt z}[y]_0^{\sqrt{(4z-x^2)}}dz\,dx$$

$$=\int_0^4\left[\int_0^{2\sqrt z}\sqrt{(4z-x^2)}dx\right]dz$$

$$=\int_0^4\left[\frac x2\sqrt{(4z-x^2)}+\frac{4z}{2}\sin^{-1}\frac{x}{2\sqrt z}\right]_0^{2\sqrt z}dz$$

$$=\int_0^4\left[0+\frac{4z}{2}\sin^{-1}\frac{2\sqrt z}{2\sqrt z}\right]_0^{2\sqrt z}dz=\int_0^4 2z.\frac\pi2dz=\int_0^4\pi z\,dz$$

$$=\pi\left[\frac{z^2}{2}\right]_0^4=\frac\pi2[16]=8\pi.$$

Example 23:

Evaluate $\int_0^a\int_0^{a-x}\int_0^{a-x-y}x^2dx\,dy\,dz$.

Solution:

The given triple integral

$$=\int_0^a\int_0^{a-x}\int_0^{a-x-y}x^2dx\,dy\,dz=\int_0^a\int_0^{a-x}x^2[z]_0^{a-x-y}dx\,dy$$

integrating w.r.t. z regarding x and y as constants

$$=\int_0^a\int_0^{a-x}x^2[a-x-y]dx\,dy=\int_0^a\int_0^{a-x}x^2[(a-x)-y]dx\,dy$$

$$=\int_0^a\int_0^{a-x}x^2\left[(a-x)y-\frac12y^2\right]_0^{a-x}dx,$$

integrating w.r.t. y regarding x as constant

$$=\int_0^a x^2[(a-x)^2-\frac12(a-x)^2\,dx$$

$$=\int_0^a x^2.\frac12(a-x)^2dx=\frac12\int_0^a x^2(a^2-2ax+x^2)dx$$

$$=\frac12\int_0^a(x^2a^2-2ax^3+x^4)dx=\frac12\left[a^2\frac{x^3}{3}-2a\frac{x^4}{4}+\frac{x^5}{5}\right]_0^a$$

$$=\frac12\left[\frac13a^5-\frac12a^5+\frac15a^5\right]=\frac12\left(\frac13-\frac12+\frac15\right)a^5=\frac1{60}a^5.$$

Example 24:

Evaluate $\iint(x+y+a)dx\,dy$ *over the circular area* $x^2+y^2\le a^2$.

Solution:

Here the region of integration R can be expressed as $-a\le y\le a$, $-\sqrt{(a^2-y^2)}\le x\le\sqrt{(a^2-y^2)}$,

where the first integration is to be performed w.r.t. x regarding y as constant.

$$\therefore\iint_R(x+y+a)dx\,dy$$

$$=\int_{y=-a}^a\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}}(x+y+a)dx\,dy$$

$$=\int_{-a}^a\left[\frac{x^2}{2}+(y+a)x\right]_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}}dy,$$

[integrating w.r.t. x treating y as a constant]

$$=\int_{-a}^a\left[\frac{a^2-y^2}{2}+(y+a)\sqrt{(a^2-y^2)}dy\right.$$

$$\left.-\frac{a^2-y^2}{2}+(y+a)\sqrt{(a^2-y^2)}\right]$$

$$=\int_{-a}^a 2(y+a)\sqrt{(a^2-y^2)}dy$$

$$=\int_{-a}^a 2y.\sqrt{(a^2-y^2)}dy+2a\int_{-a}^a\sqrt{(a^2-y^2)}dy$$

$$=0+2a.\,2\int_0^a\sqrt{(a^2-y^2)}dy,$$

the first integal vanishes because the integrand is an odd function of y

$$=4a\left[\frac{y\sqrt{(a^2-y^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac ya\right]_0^a$$

$$=4a\left[0+\frac12a^2\sin^{-1}1-0\right]$$

$$=4a.\frac12a^2.\frac12\pi=\pi a^3.$$

Example 25:

Evaluate $\iint x^2y^2\,dxdy$ *over the region bounded by x = 0, y = 0 and* $x^2+y^2=1$.

Solution:

The given region for integration is the area of the positive quadrant of the circel $x^2+y^2=1$ in the xy-plane. This region R can be expressed either as

$0\le x\le\sqrt{(1-y^2)},\ 0\le y\le 1$

or as $0\le y\le\sqrt{(1-x^2)},\ 0\le x\le 1.$

the first integration to be performed w.r.t. x regarding y as constant

$$=\int_0^1y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}}dy=\int_0^1\frac12y^2(1-y^2)^{3/2}dy.$$

Put $y=\sin\theta$ so that $dy=\cos\theta\,d\theta$.

When $y=0$, $\theta=0$ and when $y=1$, $\theta=\pi/2$.

$$\therefore\iint_R x^2y^2dx\,dy=\int_0^{\pi/2}\frac13\sin^2\theta(1-\sin^2\theta)^{3/2}.\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta = \frac{1}{3}\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{96}.$$

Example 26:

Evaluate $\iint x^2y^2\,dx\,dy$ *over the region* $x^2 + y^2 \le 1$.

Solution:

Here the given region of integration R is the whole area of the circle $x^2 + y^2 = 1$. This region R can be expressed as $-\sqrt{(1-y^2)} \le x \le \sqrt{(1-y^2)}$, $-1 \le y \le 1$.

$$\therefore \iint_R x^2y^2\,dx\,dy = \int_{y=-1}^{1}\int_{x=-\sqrt{(1-y)}}^{\sqrt{(1-y^2)}} x^2y^2\,dx\,dy$$

$$= \int_{y=-1}^{1}\int_{x=0}^{\sqrt{(1-y^2)}} x^2y^2\,dxdy,$$

by a property of definite integrals because x^2 is an even function of x

$$= 2\int_{-1}^{1}\frac{1}{3}y^2(1-y^2)^{3/2}\,dy, \text{ proceeding as in Ex. 12 (a)}$$

$$= \frac{4}{3}\int_0^1 y^2(1-y^2)^{3/2}\,dy$$

because $y^2(1-y^2)^{3/2}$ is an even function of y

$$= \frac{\pi}{24}.$$

Example 27:

Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$*, by double integration.*

Solution:

From the equation of the ellipse, we have

$$\frac{y}{b} = \pm\sqrt{\left\{1-\frac{x^2}{a^2}\right\}}.$$

So the region of integration R to cover the area of the ellipse can be considered as bounded by $y = -b\sqrt{(1-x^2/a^2)}$, $y = b\sqrt{(1-x^2/a^2)}$, $x = -a$ and $x = a$.

Therefore the required area of the ellipse

$$= \iint_R dx\,dy = \int_{x=-a}^{a}\int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} 1.dx\,dy$$

$$= \int_{-a}^{a}\left[2\int_0^{b\sqrt{(1-x^2/a^2)}} 1.dy\right]dy = 2\int_{-a}^{a}[y]_0^{b\sqrt{(1-x^2/a^2)}}\,dx$$

$$= 2\int_{-a}^{a} b\sqrt{\left(1-\frac{x^2}{a^2}\right)}dx = 2.2\int_0^a b\sqrt{\left(1-\frac{x^2}{a^2}\right)}dx$$

$$= \frac{4b}{a}\int_0^a\sqrt{(a^2-x^2)}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[0+\frac{a^2}{2}\{\sin^{-1}1-\sin^{-1}0\}\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{b} = \pi ab.$$

Example 28:

Compute the value of $\iint_R y\,dx\,dy$*, where R its the region in the first quadrant bounded by the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

Solution:

If the first integration is to be performed w.r.t. y regarding x as a constant, then the given region of integration can be expressed as $0 \le x \le a$, $0 \le y \le b\sqrt{(1-x^2/a^2)}$.

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} y\,dx\,dy$$

$$= \int_0^a\left[\frac{y^2}{2}\right]_0^{b\sqrt{(1-x^2/a^2)}}dx = \frac{1}{2}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{b^2}{2a^2}\int_0^a(a^2-x^2)\,dx = \frac{b^2}{2a^2}\left[a^2x-\frac{x^3}{3}\right] = \frac{b^2}{2a^2}\cdot\frac{2a^3}{3} = \frac{ab^2}{3}.$$

Example 29:

Evaluate $\iint (x+y)^2\,dx\,dy$ *over the area bounded by the ellipse* $x^2/a^2 + y^2/b^2 = 1$*. Hence find the mass of an elliptic plate whose density per unit area is given by* $\rho = k(x+y)^2$.

Solution:

The region of integration can be considered as bounded by $y = -b\sqrt{(1\ x^2/a^2)}$, $y = b\sqrt{(1-x^2/a^2)}$, $y = b\sqrt{(1-x^2/a^2)}$, $x = -a$ and $x = a$.

$$\therefore \iint (x+y)^2\,dx\,dy = \int_{-a}^{a}\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}(x^2+y^2+2xy)\,dx\,dy,$$

the first integration to be performed w.r.t. y regarding x as a constant

$$= \int_{-a}^{a} 2\int_0^{b\sqrt{(1-x^2/a^2)}}(x^2+y^2)\,dx\,dy,$$

[$\therefore$ 2xy being an odd function of y, its integration under the giver limits of y is 0]

$$= 2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{b\sqrt{(1-x^2/a^2)}}dx$$

$$= 2\int_{-a}^{a}\left\{x^2b\sqrt{\left(1-\frac{x^2}{a^2}\right)}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

$$= 4\int_0^a\left\{x^2b\sqrt{\left(1-\frac{x^2}{a^2}\right)}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

$$= 4b\int_0^{\pi/2}\left\{a^2\sin^2\theta\cos\theta+\frac{b^2}{3}\cos^3\theta\right\}a\cos\theta\,d\theta,$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

$$= 4ab\int_0^{\pi/2}\left\{a^2\sin^2\theta\cos^2\theta+\frac{b^2}{3}\cos^4\theta\right\}d\theta$$

$$= 4ab\left[a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta+\frac{b^2}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta\right]$$

$$= 4ab\left[a^2.\frac{1.1}{4.2}.\frac{\pi}{2}+\frac{b^2}{3}.\frac{3.1}{4.2}.\frac{\pi}{2}\right], \text{[by Walli's formula]}$$

$$= 4ab\left[\frac{1}{16}\pi a^2+\frac{1}{16}\pi b^2\right] = \frac{1}{4}\pi ab(a^2+b^2).$$

The mass of an elliptic plate whose density is given by

$$\rho = k(x+y)^2$$

$= \iint_A k(x+y)^2\,dx\,dy$, where the integrations to be performed over the area A of the ellipse

$$= k.\frac{1}{4}.\pi ab(a^2+b^2).$$

Example 30:

Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x + y \le 1$.

solution:

The region of integration is the area bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

To cover this region of integration R , x varies from 0 to 1 and y varies from 0 to $1 - x$.

$$\therefore \iint_R xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dx\,dy = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x}dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^2\,dx = \frac{1}{2}\int_0^1 x(1-2x+x^2)\,dx = \frac{1}{2}\left[\frac{x^2}{2}-2.\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{24}.$$

Example 31:

Evaluate $\iint e^{2x+3y}\,dx\,dy$ *over the triangle bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$.

Solution:

The given region of integration R can be expressed as $0 \le x \le 1$, $0 \le y \le 1 - x$,

where the first integration is to be performed w.r.t. y regarding x as a constant.

$$\therefore \iint_R e^{x+3y}\,dx\,dy = \int_0^1\int_0^{1-x} e^{2x+3y}\,dx\,dy$$

$$= \int_0^1\left[\frac{e^{2x+3y}}{3}\right]_0^{1-x}dx = \frac{1}{3}\int_0^1[e^{3-x}-e^{2x}]\,dx$$

$$= \frac{1}{3}\left[-e^{3-x}-\frac{e^{2x}}{3}\right]_0^1 = -\frac{1}{3}[(e^2-e^3)+\frac{1}{2}(e^2-e^0)]$$

$$= -\frac{1}{3}[-e^2(e-1)+\frac{1}{2}(e+1)(e-1)] = \frac{1}{3}(e-1)[e^2-\frac{1}{2}(e+1)]$$

$$= \frac{1}{6}(e-1)(2e^2-e-1) = \frac{1}{6}(e-1)\{(e-1)(2e+1)\}$$

$$= \frac{1}{6}(e-1)^2(2e+1).$$

Example 32:

Evaluate $\iint(x^2+y^2)dx\,dy$ *over the region in the positive quadrant for which* $x + y \le 1$.

Solution:

The region of integration R is the area bounded by the coordinate axes and the straight line $x + y = 1$. Therefore, the region R is bounded by $y = 0$, $y = 1 = x$ and $x = 0$, $x = 1$.

Therefore $\iint_R (x^2+y^2)dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)dx\,dy$,

the first integration to be performed w.r.t. y regarding x as constant

$$= \int_0^1\left[x^2y+\frac{y^2}{3}\right]_0^{1-x}dx = \int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{3\times 4}\right]_0^1 = \left[\frac{1}{3}-\frac{1}{4}+\frac{1}{12}\right] = \frac{1}{6}.$$

Example 33:

Evaluate $\iint_A (x^2+y^2)dx\,dy$, *where A is the region bound by* $x = 0$, $y = 0$, $x + y = 1$.

Solution:

Do your self.

The region A is bounded by $y = 0$, $y = 1 - x$ and $x = 0$, $x = 1$.

Example 34:

Evaluate $\iint xy(x+y)dx\,dy$ *over the area between* $y = x^2$ *and* $y = x$.

Solution:

Draw the given curves $y = x^2$ and $y = x$ in the same figuare. The two curves intersect at the points whose abscissae are given by $x^2 = x$ or $x(x - 1) = 0$ *i.e.*, $x = 0$ or 1. When $0 < x < 1$, we have $x > x^2$. So the area of integration can be considered as lying between the curves $y = x^2$, $y = x$, $x = 0$ and $x = 0$ and $x = 1$.

Therefore the required integral

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)dx\,dy = \int_0^1\left[\int_{x^2}^{x}(x^2y+xy^2)xy\right]dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^{x}dx = \int_0^1\left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \int_0^1\left[\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx = \left[\frac{x^5}{6}-\frac{x^7}{14}-\frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = \frac{28-12-7}{168} = \frac{9}{168} = \frac{3}{56}.$$

Example 35:

Find by double integration the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:

Solving $y = 4x - x^2$ and $y = x$ for x, we have $4x - x^2$ or $x^2 - 3x = 0$ or $x(x - 3) = 0$ *i.e.*, $x = 0$ or 3.

This the curves $y = 4x - x^2$ and $y = x$ intersect at the points where $x = 0$ and $x = 3$. When $0 < x < 3$, we have $4x - x^2 > x$.

So the required area can be considered as lying between the curves $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

Therefore the required area $= \int_{x=0}^{3}\int_{y=x}^{4x-x^2}dx\,dy$

$$= \int_0^3[y]_x^{4x-x^2}dx = \int_0^3(4x-x^2-x)dx = \int_0^3(3x-x^2)dx$$

$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3 = \frac{27}{2}-\frac{27}{3} = 27\cdot\frac{1}{6} = \frac{9}{2}.$$

Example 36:

Prove by the method of double integration that the area lying between the parabolas $y^2 = 4ax$ *and* $x^2 = 4ay$ *is* $\frac{16}{3}a^2$

Solution:

Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissase are given by $(x^2/4a)^2 = 4ax$ *i.e.*, $x(x^3 - 64a^3) = 0$ *i.e.*, $x = 0$ and $x^3 = 64a^3$. Thus the two parabolas intersect at the points where $x = 0$ and $x = 4a$.

Now the area of a small element situated at any point (x,y) = dx dy.

$\therefore$ the required area $= \int_{x=0}^{4a}\int_{y=x^2/4a}^{\sqrt{(4ax)}}dx\,dy = \int_0^{4a}[y]_{x^2/4a}^{\sqrt{(4ax)}}dx$

$$= \int_0^{4a}\left[2\sqrt{a}.x^{1/2}-\frac{1}{4a}.x^2\right]dx = \left[2\sqrt{a}.x^{3/2}.\frac{2}{3}-\frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}.(4a)^{3/2}-\frac{1}{12a}.64a^3 = \frac{32}{3}a^2-\frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Example 37:

Evaluate $\iint y\,dx\,dy$ *over the area between the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.

Solution:

The two parabolas intersect at the points whose abscissae area given by $\left(\frac{1}{4}x^2\right)^2 = 4x$ or $x(x^3 - 64) = 0$ *i.e.*, $x = 0$ or 4. When $0 < x < 4$, we have $2\sqrt{x} > \frac{1}{4}x^2$. Therefore the given region of integration can be expressed as $0 \le x \le 4$, $\frac{1}{4}x^2 \le y \le 2\sqrt{x}$.

$\therefore$ the required integral $= \int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}}y\,dx\,dy$

$$= \int_0^4\left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}}dx = \int_0^4\left[2x-\frac{x^4}{32}\right]dx = \left[\frac{2x^2}{2}-\frac{x^5}{32\times 2}\right]_0^4$$

$$= 16-\frac{32}{5} = \frac{48}{5}.$$

Example 38:

When the region of integration A is the triangle given by $y = 0$, $y = x$ *and* $x = 1$, *show that*

$$\iint_A \sqrt{(4x^2-y^2}(dx\,dy) = \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

Solution:

In the diagram draw the straight lines $y = 0$, $y = x$ and $x = 1$. Then we odserve that the region of integration A can be expressed as $0 \le y \le x$, $0 \le x \le 1$.

$$\therefore \iint_A\sqrt{(4x^2-y^2)}dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x}\sqrt{(4x^2-y^2)}dx\,dy$$

$$= \int_0^1\left[\frac{y}{2}\sqrt{(4x^2-y^2)}+2x^2\sin^{-1}\frac{y}{2x}\right]_{y=0}^{x}dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^1\left[\frac{x}{2}\sqrt{(4x^2-y^2)}+2x^2\sin^{-1}\frac{1}{2}-0\right]dx$$

$$= \int_0^1\left[\frac{\sqrt{3}}{2}x^2+\frac{\pi}{3}x^2\right]dx = \left[\frac{\sqrt{3}}{2}.\frac{x^3}{3}+\frac{\pi}{3}.\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right).$$

Example 39:

Evaluate $\iint\frac{xy}{\sqrt{(1-y^2)}}dx\,dy$ *over the positive quadrant of the circle* $x^2 + y^2 = 1$.

Solution:

Here the region of integration R is she area of the circle $x^2 + y^2$ = lying in the positive quadrant. This region of integration R can be expressed as $0 \le x \le \sqrt{(1 - y^2)}$, $0 \le y \le 1$.

$$\therefore \iint_R\frac{dy}{\sqrt{(1-y^2)}}dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{(1-y^2)}}\frac{xy}{\sqrt{(1-y^2)}}dx\,dy \quad \int_0^1\frac{y}{\sqrt{(1-y^2)}}\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{(1-y^2)}}dy,$$

integrating w.r.t. x treating y as constant

$$= \frac{1}{2}\int_0^1 y\sqrt{(1-y^2)}dy = \frac{1}{2}\int_0^1-\frac{1}{2}.(1-y^2)^{1/2}(-2y)dy$$

$= -\frac{1}{4}.\frac{2}{3}\left[(1-y^2)^{3/2}\right]_0^1$, by power formula

$$= \frac{1}{6}.$$

Example 40:

Evaluate the double integral $\int_0^a\int_0^{\sqrt{(a^2-x^2)}}x^2y\,dx\,dy.$

Mention the region of integration involved in this double integral.

Solution:

The given integral

$$I = \int_{x=0}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}} x^2 y \, dx \, dy$$

$$= \int_0^a x^2 \left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{(a^2-x^2)}} dx, \text{integrating w.r.t. y treating x as constant}$$

$$= \frac{1}{2}\int_0^a x^2 (a^2 - x^2) dx = \frac{1}{2}\int_0^a (a^2x^2 - x^4) dx$$

$$= \frac{1}{2}\left[a^2 \frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{1}{15}a^5.$$

From the limits of integration it is obvious that the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 = a^2$ between the lines $x = 0$, $x = a$ and lying above the line $y = 0$ *i.e.*, the axis of x. Thus the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

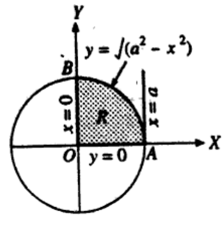

Fig. 1.5

Example 41:

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r \, d\theta \, dr$.

Solution:

Here the limits of r are variable and those of θ are constant. Therefore first integration shall be performed w.r.t. r tegarding θ as a constant. We have

$$\int_0^{\pi}\int_0^{a\sin\theta} r \, d\theta \, dr = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} a^2 \sin^2\theta \, d\theta$$

$$= \frac{a^2}{2}.2\int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{a^2}{2}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Example 42:

Evaluate $\int_0^{\pi/2}\int_0^{a\cos\theta} r \sin\theta \, d\theta \, dr$.

Solution:

$$\text{We have } \int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta \, d\theta \, dr = \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta,$$

integrating first w.r.t. r regarding θ as a constant

$$= \frac{1}{2}\int_0^{\pi/2} \sin\theta . a^2 \cos^2\theta \, d\theta = \frac{a^2}{2}\int_0^{\pi} \sin\theta \cos^2\theta \, d\theta$$

$$= \frac{1}{2}a^2 . \frac{1.1}{3.1} = \frac{1}{6}a^2.$$

Example 43:

Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr$.

Solution:

$$\text{We have } \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{a} \cos\theta . a^2 (1+\cos\theta)^3 d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} \cos\theta(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta) d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} [\cos\theta + 3\cos^2\theta + \cos^4\theta] d\theta$$

$$= 2.\frac{a^2}{3}\int_0^{\pi/2} [3\cos^2\theta + \cos^4\theta] d\theta$$

$$\left[\because \int_0^{\pi} \cos^n\theta \, d\theta = 0 \text{ or } 2\int_0^{\pi/2} \cos^n\theta \, d\theta \text{ according as n is odd or even}\right]$$

$$= \frac{2a^3}{3}\left[3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3.1}{4.2}\cdot\frac{\pi}{2}\right] = \frac{2a^2}{3}\cdot\frac{3\pi}{4}\left[1 + \frac{1}{4}\right]$$

$$= \frac{2a^3}{3}\cdot\frac{3\pi}{4}\cdot\frac{5}{4} = \frac{5\pi a^3}{8}.$$

Example 44:

Evaluate the triple integral of the function $f(x, y, z) = x^2$ over the region V enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.

Solution:

The given region V is bounded by the co-ordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = a$. To cover the region V, let the values of x, y lie within the triangle bounded by x-axis, the y-axis and the line $(x + y = a, z = 0)$. Then for any point (x, y, 0) within this triangle, z varies from $z = a - x - y$ in the region V.

But the values of x and y vary within the triangle formed in the xy-plane. Therefore x varies from 0 to a and for any intermediary value of x, y varies from 0 to $a - x$.

Therefore, the region of integration V can be expressed as $0 \le x \le a$, $0 \le y \le a - x$, $0 \le z \le a - x - y$.

Hence the required triple integral

$$= \int_0^a\int_0^{a-x}\int_0^{a-x-y} x^2 dx \, dy \, dz$$

Example 45:

Find the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Solution:

Here the region of integration V to cover the volume of the tetrahedron can be expressed as $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 1 - x - y$.

Therefore the required volume of the tetrahedron

$$= \iiint_V dx \, dy \, dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dx \, dy \, dz$$

$$= \int_0^1\int_0^{1-x} [z]_0^{1-x-y} dx \, dy = \int_0^1\int_0^{1-x} (1 - x - y) dx \, dy$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2 dx = \frac{1}{2}\left[\frac{(1-x)^3}{3.(-1)}\right]_0^1 = -\frac{1}{6}[0 - 1] = \frac{1}{6}.$$

Example 46:

Find the volume of the tetrahedron bounded by the plane $x/a + y/b + z/c = 1$ and the coordinate planes.

Solution:

Here the region of integration V to cover the volume of the given tetrahedron can be depressed as $0 \le x \le a$, $0 \le y \le b(1 - x/a)$, $0 \le z \le c(1 - x/a - y/b)$.

Therefore the required volume of the tetrahedron

$$= \iiint_V dx \, dy \, dz = \int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} dx \, dy \, dz.$$

$$\text{The required volume} = \frac{abc}{6}.$$

Example 47:

Find the volume of a sphere of radius a by triple integral.

Solution:

Referred to centre as origin the equation of a sphere of radius a is $x^2 + y^2 + z^2 = a^2$. ...(1)

The sphere (1) symmetrical in all the eight octants.

∴ volume of the sphere (1) = 8. (the volume of the part of the sphere lying in the positive octant).

Now for the region consisting of the volume of the sphere (1) lying in the positive octant, we have $0 \le x \le a$, $0 \le y \le \sqrt{(a^2 - x^2)}$, $0 \le z \le \sqrt{(a^2 - x^2 - y^2)}$.

∴ the required volume of a sphere of radius a

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}}\int_{z=0}^{\sqrt{(a^2-x^2-y^2)}} dx \, dy \, dz$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}} [z]_{z=0}^{\sqrt{(a^2-x^2-y^2)}} dx \, dy$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}} \sqrt{(a^2 - x^2 - y^2)} dx \, dy$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{\{(a^2 - x^2) - y^2\}} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{(a^2 - x^2)}}\right]_{y=0}^{\sqrt{(a^2-x^2)}} dx$$

$$= 8\int_0^a \left[0 + \frac{a^2 - x^2}{2}\cdot\frac{\pi}{2} - 0 - 0\right] dx$$

$$= 8\cdot\frac{\pi}{4}\int_0^a (a^2 - x^2) dx = 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi\left[a^3 - \frac{a^3}{3}\right] = \frac{4}{3}\pi a^3.$$

Example 48:

Evaluate $\iiint (x + y + z) dx \, dy \, dz$ *over the tetrahedron $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.*

Solution:

The region of integrating V for the given tetrahedron can be expressed as $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 1 - x - y$.

Hence the required triple integral $= \iiint_V (x+y+z)\,dx\,dy\,dz$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x}\left[(x+y)z+\frac{z^2}{2}\right]_0^{1-x-y} dx\ dy$$

$$= \int_0^1\int_0^{1-x}\left[(x+y)(1-x-y)+\frac{(1-x-y)^2}{2}\right]dx\,dy$$

$$= \int_0^1\int_0^{1-x}(1-x-y)\left(x+y+\frac{1-x-y}{2}\right)dx\,dy$$

$$= \int_0^1\int_0^{1-x}\frac{1}{2}(1-x-y)(1+x+y)\,dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x}[1-(x+y)^2]\,dx\,dy = \frac{1}{2}\int_0^1\left[y-\frac{(x+y)^3}{3}\right]_0^{1-x}dx$$

$$= \frac{1}{2}\int_0^1\left(1-x-\frac{1}{3}+\frac{x^3}{3}\right)dx = \frac{1}{2}\int_0^1\left(\frac{2}{3}-x+\frac{x^3}{3}\right)dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x-\frac{x^2}{2}+\frac{x^4}{3\times 4}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3}-\frac{1}{2}+\frac{1}{12}\right] = \frac{1}{2}\cdot\frac{1}{4}\cdot = \frac{1}{8}.$$

Example 49:

Evaluate the integral $\iiint xyz\,dx\,dy\,dz$ *over the volume enclosed by three coordinate planes and the plane* $x + y + z = 1$.

Solution:

The region of integration V enclosed by the three coordinate planes and the plane $x + y + z = 1$ can be expressed as $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 1 - x - y$.

$\therefore$ the required triple integral $\iiint_V xyz\,dx\,dy\,dz$

$$= \int_{x=0}^1\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xyz\,dx\,dy\,dz.$$

Example 50:

Evaluate $\iiint \dfrac{dx\,dy\,dz}{(x+y+z+1)^3}$ *over the reagin* $x \ge 0,\ y \ge 0,\ z \ge 0,\ x + y + z \le 1$.

Solution:

The given region of integration R can be expressed as $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 1 - x - y$.

Hence the required triple integral

$$= \iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}(x+y+z+1)^{-3}dz\right]dx\,dy$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y}dx\,dy$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dx\,dy$$

$$= -\frac{1}{2}\int_0^1\left[\frac{y}{4}+\frac{1}{(x+y+1)}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{2}-\frac{1}{(x+1)}\right]dx$$

$$= -\frac{1}{2}\left[\frac{(1-x)^2}{2\times 4\times(-1)}+\frac{1}{2}x-\log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\left\{0+\frac{1}{2}-\log 2\right\}-\left\{-\frac{1}{8}+0-0\right\}\right] = \frac{1}{2}\left[\frac{1}{2}-\log 2+\frac{1}{8}\right]$$

$$= -\frac{1}{2}\left[\frac{5}{8}-\log 2\right] = \frac{1}{2}\left[\log 2-\frac{5}{8}\right].$$

Example 51:

Evaluate $\iiint xyz\,dx\,dy\,dz$ *over the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution:

Hence the region of integration can be expressed as $-a \le x \le a$, $-b\sqrt{\{1-(x^2/a^2)\}} \le y \le b\sqrt{\{1-(x^2/a^2)\}}$ and $-c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}} \le z \le c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}$.

$\therefore$ the required triple integral

$$= \int_{-a}^{a}\int_{-b\sqrt{\{1-(x^2/a^2)\}}}^{b\sqrt{\{1-(x^2/a^2)\}}}\left[\int_{-c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}}^{c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}}(xy).z\,dz\right]dx\,dy$$

$= 0$, [$\because$ z is an odd function of z and xy is treated as constant while integrating w.r.t. z]

Example 51:

Evaluate $\iiint z^2\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Solution:

Here the region of integration can be expressed as $-1 \le x \le 1$, $-\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$, $-\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}$.

$\therefore$ the required triple integral

$$= \int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} z^2\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\left[\frac{z^3}{3}\right]_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}}dx\,dy$$

$$= \frac{1}{3}\int_{-1}^{1}\left[\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}2(1-x^2-y^2)^{3/2}dy\right]dx$$

$$= \frac{2}{3}\int_{-1}^{1}\left[\int_{-\pi/2}^{\pi/2}[(1-x^2)\cos^2\theta]^{3/2}.\sqrt{(1-x^2)}.\cos\theta\,d\theta\right]dx$$

[putting $y = \sqrt{(1-x^2)}\sin\theta$ so that $dy = \sqrt{(1-x^2)}\cos\theta\,d\theta$; also when $y = 0$, $\theta = 0$ and when $y = \sqrt{(1-x^2)}$, $\theta = \pi/2$]

$$= \frac{2}{3}\int_{-1}^{1}\left[2.\int_0^{\pi/2}(1-x^2)^2\cos^4\theta\,d\theta\right]dx$$

$$= \frac{4}{3}\int_{-1}^{1}(1-x^2)^2\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}dx = \frac{\pi}{2}\int_{-1}^{1}(1-x^2)^2dx$$

$$= \frac{\pi}{4}\cdot 2\int_0^1(1-2x^2+x^4)dx = \frac{\pi}{2}\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_0^1$$

$$= \frac{\pi}{2}\left[1-\frac{2}{3}+\frac{1}{5}\right] = \frac{\pi}{2}\cdot\frac{8}{15} = \frac{4\pi}{15}.$$

Example 52:

Evaluate $\iiint (z^5+z)\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Solution:

The given region of integration can be expressed as $-1 \le x \le 1$, $-\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$, $-\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}$.

Hence the required triple integral

$$= \int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\left[\int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}}(z^5+z)\,dz\right]dx\,dy$$

$= 0$, [$\because$ $(z^5 + z)$ is an odd function of z].

Example 53:

Evaluate $\iiint_R u^2v^2w\,du\,dv\,dw$, *where R is the region* $u^2 + v^2 \le 1$, $0 \le w \le 1$.

Solution:

Here the limits of integration to cover the region R can be taken as $-1 \le u \le 1$, $-\sqrt{(1-u^2)} \le v \le \sqrt{(1-u^2)}$, $0 \le w \le 1$, where the first integration is to be performed with respect to v.

$$\therefore \iiint_R u^2v^2w\,du\,dv\,dw = \int_0^1\int_{-1}^{1}\int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}}u^2v^2w\,dw\,du\,dv$$

$$= \int_0^1\int_{-1}^{1}u^2w\left[\int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}}v^2dv\right]dw\,du,$$

because the first integration is to be performed w.r.t. v regarding u and w as constants

$$= \int_0^1\int_{-1}^{1}\left[2u^2w\int_0^{\sqrt{(1-u^2)}}v^2dv\right]dw\,du$$

because v^2 is an even function of v

$$= \int_0^1\int_{-1}^{1}2u^2w\left[\frac{v^3}{3}\right]_0^{\sqrt{(1-u^2)}}dw\,du$$

$$= \frac{2}{3}\int_0^1\int_{-1}^{1}wu^2(1-u^2)^{3/2}dw\,du$$

$$= \frac{2}{3}\int_0^1\left[w.2\int_0^1 u^2(1-u^2)^{3/2}\,du\right]dw$$

$$= \frac{4}{3}\int_0^1 w\left[\int_0^{\pi/2}\sin^2\theta.\cos^3.\cos\theta\,d\theta\right]dw,\ \text{putting } u = \sin\theta$$

$$= \frac{4}{3}\int_0^1 w\left[\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta\right]dw = \frac{4}{3}\int_0^1 w\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2}dw$$

$$= \frac{\pi}{24}\int_0^1 w\,dw = \frac{\pi}{24}\left[\frac{w^2}{2}\right]_0^1 = \frac{\pi}{48}[1-0] = \frac{\pi}{48}.$$

Example 54:

Change the order of integration in the double integral $\int_0^a\int_0^x f(x,y)\,dx\,dy$.

Solution:

In the given integral the limits if integration are given by the straight lines y = 0, y = x = 0 and x = a. Draw these lines bounding the region of integration in the same figure. We observe the the region of integration is the area ONM.

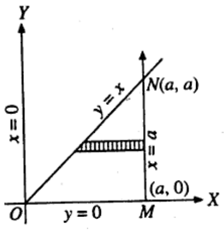

Fig. 1.6

In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as constant and then w.r.t. x.

To reverse the order of integration, we have to integrate first w.r.t. x regarding y as constant and then w.r.t. y. This is done by dividing the area ONM into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line ON (*i.e.*, y = x) and terminating on the line MN (*i.e.*, x = a). Thus for this region ONM, x varies from y to a and y varies from 0 to a.

Hence the changing the order of integration, we have

$$\int_0^a\int_0^x f(x,y)\,dx\,dy = \int_0^a\int_y^a f(x,y)\,dx\,dy.$$

Example 55:

Prove that $\int_a^b dx\int_a^x f(x,y)\,dy = \int_a^b dy\int_y^b f(x,y)\,dx.$

Solution:

Let $I = \int_a^b dx\int_a^x f(x,y)\,dy$.

We are required to change the order of integration in the integral I. In the integral I the limits of integration of y are given by the straight lines y = a and y = x. Also the limits of integration of x are given by the straight lines x = a and x = b. Draw the straight lines y = a, y = x, x = a and x = b, bounding the region of integration, in the same figure. We observe that the region of integration is the area of the triangle ABC.

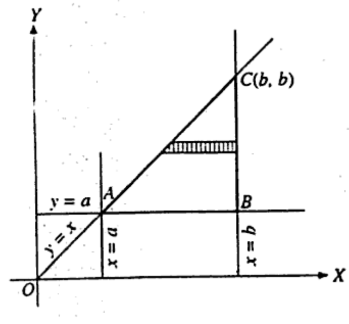

Fig. 1.7

In the integral I we are required to integrate first w.r.t. y and then w.r.t. x. To reverse the order of integration we have to integrate first w.r.t. x and then w.r.t. y. This is done by dividing the area ABC into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line AC (*i.e.*, y = x) and terminating on the line BC (*i.e.*, x = b). Thus for the region ABC, x-varies from y to b and y varies from a to b. Hence by changing the order of integration, we have

$$\int_a^b dx\int_a^x f(x,y)\,dy = \int_a^b dy\int_y^b f(x,y)\,dx.$$

Example 56:

Change the order of integration in $\int_0^1\int_x^{x(2-x)} f(x,y)\,dx\,dy$.

Solution:

In the given integral the limits of integration of y are given by y = x, which is a straight line passing through the origin, and y = x (2 – x) or y = 2x – x² or (x – 1)² = – (y – 1) which is a parabola with vertex (1, 1) a and passing through the origin.

Again the limits of integration of x are given by x = 0 *i.e.*, the y-axis and x = 1 which is a straight line parallels to the y-axis at a distance 1 from the origin.

We draw the curves y = x, (x – 1)² = – (y – 1), x = 0 and x = 1, giving the limits of integration, in the same figure. We observe that the region of integration is the area OLBMO.

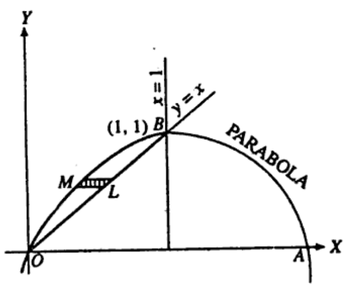

Fig. 1.8

In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as a constant and then w.r.t. x.

If we want to reverse the order of integration, we have to fist integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covering the area of integration OLBMO by drawing the straight lines y = constant *i.e.*, by dividing this area into strips parallel to the x-axis.

So divide the region OLBMO into strips parallel to the x-axis starting from the arc OMB of the parabola and terminating on the line OLB.

For the point B, x = 1. Putting x = 1 in the equation of the line y = x, we get y = 1. So the y-coordinate of the point B is also 1.

For the region OMBLO, the lower limit of x is the value of x found interns of y from the equation (x – 1)² = 1 – y and the upper limit of x is the value of

x found in terms of y from the equation y = x. From the equation (x – 1)² = 1 – y, we get x – 1 = ± √(1 – y) or x = 1 ± √(1 – y). Since in the region OMBLO, x takes values less than 1, therefore we take x = 1 – √(1 – y).

Thus in the region OMBLO, x varies form 1 – √(1 – y) to y and y varies from 0 to a.

Hence by changing the order of integration, we have the given integral

$$= \int_0^1\int_{1-\sqrt{(1-y)}}^{y} f(x,y)\,dy\,dx.$$

Example 57:

Change the order of integration in $\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} f(x,y)\,dx\,dy$.

Solution:

In the given integral the limits of integration of are given by y = 0 (*i.e.*, the x-axis) and y = √(2ax – x²) *i.e.*, y² = 2ax – x² *i.e.*, (x – a)² + y² = a² which is a circle with centre (a, 0) and raidius a. Again the limits of integration of x are given by the straight lines x = 0 (*i.e.*, the y-axis) and x = 2a.

Draw the curves (x + a)² + y² = a², y = 0, x = 0 and x = 2a, bounding the region of integration, in the same figure. From figure we observe that the area of integration is OMNO.

In the given integral we are required to integrate first w.r.t y regarding x as a constant ans then w.r.t. x.

To reverse the order of integration, divide the area OMBO into strips parallel to the x-axis. These strips will have their extremities on the portions ON and NM of the circle.

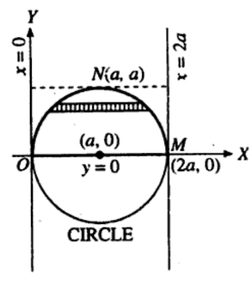

Fig. 1.9

Solving the equation of circle (x – a)² + y² = a² for x, we get (x – a)² = a² – y² *i.e.*, x – a = ± √(a² – y²) *i.e.*, x = a ± √(a² – y²).

So for the region OMNO, x varies from a – √(a² – y²) to a + √(a² – y²) and y varies from 0 to a.

Therefore, changing the order of integration, the given double integral transforms to $\int_0^a\int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx.$

Example 58:

Change the order of integration in the integral

$$\int_0^3 \int_0^{\sqrt{(4-y)}} (x+y)\,dy\,dx.$$

Solution:

In the given integral the limits of integration of x are given by the straight line x= 1 and the curve $x = \sqrt{(4-y)}$ *i.e.*, $x^2 = 4 - y$ *i.e.*, $x^2 = -(y-4)$ which is a parabola, symmetrical about the y-axis, with vertex at the point (0, 4) and existing in the region $y \le 4$. Again the limits of integration of y are given by the straight lines y = 0 (*i.e.*, the x-axis) and y = 3.

We draw the curves $x = 1$, $x^2 = -(y-4)$, $y = 0$ and $y = 3$, giving the limits of integration in the same figure. Putting x = 1 in the equation $x^2 = -(y-4)$, we get y = 3. Thus the straight line y = 3 passes through the point of intersection C of x = 1 and $x^2 = -(y-4)$. Also at the point of intersection B of the parabola $x^2 = -(y-4)$ and the x-axis (*i.e.*, the line y = 0), we have x = 2. We observe that the region of integration is the area ABCA.

In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y.

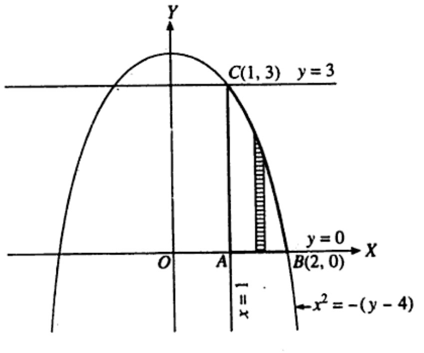

Fig. 1.10

If we want to change the order of integration, we have to first integrate w.r.t. y regarding x as a constant and then we integrate w.r.t. x. This is done

by covering the area ABCA by strips drawn parallel to the y-axis. These strips start from the line AB (*i.e.*, y = 0) and terminate on the arc BC of the parabola $x^2 = 4 - y$. Therefore, for the region ABCA, y varies from 0 to $4 - x^2$ and x varies from 1 to 2. Hence by changing the order of integration, we have the given integral

$$= \int_1^2 \int_0^{4-x^2} (x+y)\,dx\,dy$$

Example 59:

Change the order of integration $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} dy\,dx.$

Solution:

In the given integral the limits of integration of x are given by $x = a - \sqrt{(a^2 - y^2)}$ and $x = a + \sqrt{(a^2 - y^2)}$ anzd those of y are given by y = 0 and y = a.

When $x = a - \sqrt{(a^2 - y^2)}$

or $x = a + \sqrt{(a^2 - y^2)}$,

we have $(x-a)^2 = a^2 - y^2$ or $(x-a)^2 + y^2 = a^2$

or $y^2 = 2ax - x^2$ which is a circle with center (a, 0) and radius a.

To reverse the order of integration, we divide the area OMNO into strips parallel to the y-axis. These strips will have their extremities on the x-axis and on the circular arc given by $y = \sqrt{(2ax - x^2)}$. Also x will go from 0 to 2a.

Hence, changing the order of integration, the given double integral transforms to $\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} dx\,dy.$

Example 60:

Change the order of integration in the double integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ *and hence find its value.*

Solution:

In the given integral the limits of integration area given by the lines y = x, y = ∞, x = 0 and x = ∞. Therefore the region of integration is bounded by x = 0, y = x and, an infinite boundary. In the given integral the limits of integration of y are variable while those of x are constant. Thus, we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by first integrating w.r.t. y along strip drawn parallel to the y-axis and then integrating w.r.t. x along all such strips so drawn as to cover the whole region of integration.

If we want to reverse the order of integration, we have the to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by dividing this area into strips parallel to the x-axis. So we take strips parallel to the x-axis. Starting from the line x = 0 and terminating on the line y = x. Now the limits for x are 0 to y and the limits for x are 0 to y and the limits for y are 0 to ∞.

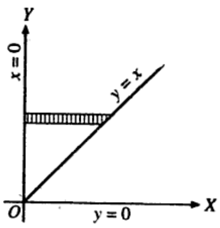

Fig. 1.11

Hence by changing the order of integration, we have

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dy\,dx$$

$$= \int_0^\infty \frac{e^{-y}}{y}[x]_0^y\,dy = \int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy = \int_0^\infty e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = 1.$$

Example 61:

Transform $\int_0^a \int_0^{a-x} f(x,y)\,dx\,dy$, *by substitution x + y = u, y = uv.*

Solution:

We have $\iint f(x,y)\,dx\,dy = \iint F(u,v)\,u\,du\,dv.$

Now in the given integral, the region of integration is bounded by the lines y = 0, y = a − x, x = 0 and x = a.

Put $x = u - y = u - uv = u(1-v)$ and $y = uv$.

Then in the uv-plane the four straight lines become uv = 0, uv = a − u (1 − v), u (1 − v) = 0 and u (1 − v) = a, giving v = 0, v = 1, u = 0 and u = a.

Hence for the given region, v varies from 0 to 1 and u varies from 0 to a.

Therefore, by changing the variables, the given double integral transforms to $\int_0^a \int_0^1 F(u,v)\,u\,du\,dv.$

Example 62:

By using the transformation x + y = u, y = uv, show that

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dx\,dy = \frac{1}{2}(e-1).$$

Solution:

We have dx dy = u du dv.

Here the region of integration is bounded by the lines

y = 0, y = 1 − x, x = 0 and x = 1.

Changing these equations to new variables u and v by using the relations $x = u - y = u - uv = u(1-v)$ and $y = uv$, we have uv = = 0, uv = 1 − u (1 − v), u (1 − v) = 0 and u (1 − v) = 1, giving v = 0, = 1, u = 0 and u = 1.

Hence for the given region v varies from 0 to 1 and u varies from 0 to 1.

Further $e^{y/(x+y)} = e^{uv/u} = e^v$. $[\because x + y = u,\ y = uv]$

Therefore, changing the variables to u, v, the given integral becomes

$$= \int_0^1 \int_0^1 e^v\,.u\,du\,dv = \int_0^1 \left[e^v\right]_0^1 u\,du = \int_0^1 (e^1 - e^0)\,u\,du$$

$$= (e-1)\int_0^1 u\,du = (e-1).\left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}(e-1)$$

Example 63:

Change the order of integration in the integral

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

Solution:

In the given integral the limits of integration of y are given by y = x tan α which is a straight line passing through the origin and

$y = \sqrt{(a^2 - x^2)}$

i.e., $y^2 = a^2 - x^2$

i.e., $x^2 + y^2 = a^2$

which is a circle of radius a with centre at the origin (0, 0).

Again the limits of integration of x are given by x = 0 *i.e.*, the y-axis and x = a cos α which is a straight line parallel to the y-axis at a distance a cos from the origin.

We draw the curve y = x tan α, $x^2 + y^2 = a^2$, x = 0 and x = a cos α, giving the limits of integration, in the same figure. We observe that the region of integration is the area OMNO.

In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of integration OMNO by drawing the straight lines x = constant *i.e.*, by dividing this area into strips parallel to the y-axis.

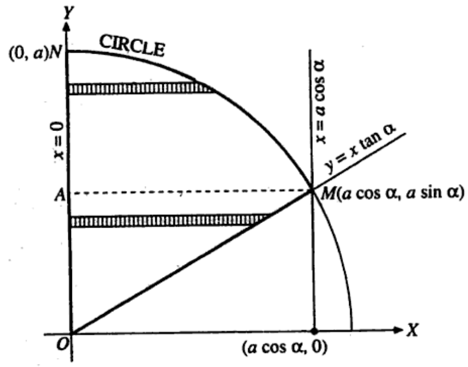

Fig. 1.12

If we want to reverse the order of integration, we have to first integrate with respect to x regarding yas constant and then we integrate w.r.t. y. This is done by covering the area of integration OMNO by drawing the straight lines y = constant *i.e.*, by dividing this area into strips parallel to the x-axis.

Now if we take strips parallel to the x-axis starting from the line x = 0; some of these strips end on the line OM while the others end on the arc MN of the circle $x^2 + y^2 = a^2$. So we draw the line of demarcation MA dividing the area OMNO into two portions OMA and AMN.

For the point M, x = a cos α. Putting x = a cos α in the equation of the line y = x tan α, we get y = a sin α. So the y-coordinate of the point M is a sin α and the equation of the line of demarcation MA is y = a sin α.

For the region OMA, x varies from 0 to y cot α and y varies from 0 to a sin α.

For the region AMN, x varies from 0 to $\sqrt{(a^2 - y^2)}$ and y varies from a sin α to a.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dy\,dx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx.$$

Example 64:

Change the order of integration in the integral $\int_0^a\int_0^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$

Solution:

In the given integral the limits of integration of y are given by the straight line y = 0 (*i.e.*, the x-axis) and the curve $y = \sqrt{(a^2 - x^2)}$ *i.e.*, $y^2 = a^2 - x^2$ *i.e.*, $x^2 + y^2 = a^2$ which is circle with centre at the origin and radius a. Again the limits of integration of x are given by the lines x = 0 and x = a.

We draw the curves y = 0, $x^2 + y^2 = a^2$, x = 0 and x = a, giving the limits of integration, in the same very figure and we observe that the region of integration is the area OAB of the quadrant of the circle $x^2 + y^2 = a^2$.

To change the order of integration in the given integral, we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covring the area OAB by strips drawn parallel to the x-axis. These strips start from the line OB (*i.e.*, x = 0) and terminate on the arc AB of the circle $x^2 + y^2 = a^2$. So on these strips x varies from 0 to $\sqrt{(a^2 - y^2)}$. Also to cover the area OAB, y varies from 0 to a. Hence by changing the order of integration, we have the given integral

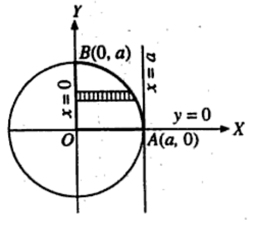

Fig. 1.13

$$= \int_0^a\int_0^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

Example 65:

Change the order of integration in $\int_0^a\int_{mx}^{lx} f(x,y)\,dx\,dy.$

Solution:

Here the area of integration is bounded by the straight lines y = mx, y = *l*x, x = 0 and x = a. Drawing all these lines in one figure, we observe that area of integration is OABO.

To reverse the order of integration, cover this area OABO by strips parallel to the axis of x. Draw the straight line AN parallel to the x-axis and thus

divide the area OABO into two portions OAN and NBA according to the character of the strips.

For the point A, x = a.

Putting x = a in the equation of the line y = mx,

we get y = ma.

Also for the point B, x = a;

therefore putting x = a in the equation of the line y = *l*x, we get y = *l*a.

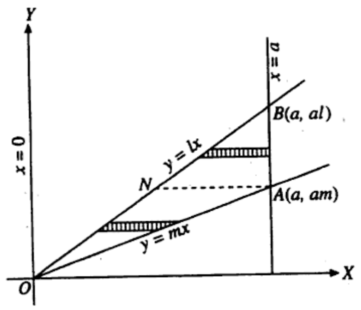

Fig. 1.14

Now for the area ONA, x varies from the line y = lx to y = mx *i.e.*, x varies from y/l to y/m and y varies from 0 to am. Again for the area NBA, x varies from the line NB (y = lx) to the line x = a *i.e.*, x varies from y/l to a and y varies from am to al.

Therefore, by changing the order of integration the given integral transforms to

$$\int_0^{am}\int_{y/l}^{y/m} f(x,y)\,dy\,dx + \int_{am}^{al}\int_{y/l}^{a} f(x,y)\,dy\,dx.$$

Example 66:

Change the order of integration in $\int_0^{2a}\int_{x^2/4a}^{3a-x} f(x,y)\,dx\,dy.$

Solution:

In the given integral the limits of integration are given by $x^2/4a = y$ *i.e.*, $x^2 = 4ay$, (which is a parabola passing through the origin), and the lines

y = 3a – x, x = 0, and x = 2a. Drawing these curves in one figure we observe that the region of integration is the area OABMO.

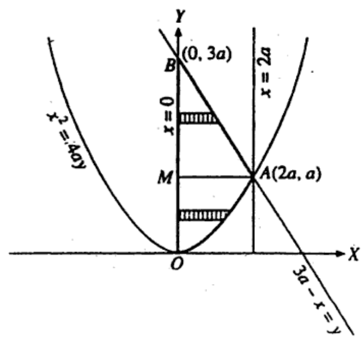

Fig. 1.15

To change the order of integration, first we divide the region of integration into two portions OAM and MAB, by drawing the line AM parallel to the x-axis. Now to reverse the order of integration, cover the whole region OABMO by strips parallel to the x-axis starting from the line x = 0. Some of these strips end on the arc OA while others end on the line AB.

For the point A, we have x = 2a. Putting x = 2a in the equation of the line y = 3a – x, we get y = a.

For the region OAM , x varies form 0 to $\sqrt{(4ay)}$ and y varies form 0 to a. Again for the region MAB, x varies form 0 to 3a – y and y varies from a to 3a.

Hence the transformed integral is given by

$$\int_0^a\int_0^{\sqrt{(4ay)}} f(x,y)\,dy\,dx + \int_a^{3a}\int_0^{3a-y} f(x,y)\,dy\,dx.$$

Example 67:

Change the order of integration $\int_0^a\int_{x^2/a}^{2a-x} xy\,dx\,dy.$

Solution:

In the given integral the limits of integration are given by $x^2/a = y$ *i.e.*, $x^2 = ay$ (which is a parabola passing through the origin), and the straight lines y = 2a – x, x = 0 and x = a.

Here the coordinates of A are (a, a) and those of B are (0, 2a).

The transformed integral is given by

$$\int_0^a\int_0^{\sqrt{(ay)}} xy\,dy\,dx + \int_a^{2a}\int_0^{2a-y} xy\,dy\,dx.$$

Example 68:

Change the order of integration in the double integral

$$\int_0^a\int_0^{b/(b+x)} f(x,y)dx\,dy.$$

Solution:

In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = b/(b + x)$ *i.e.*, $y (b + x) = b$ which is a rectangular hyperbola having for its asymptotes the straight lines $y = 0$ and $x = -b$. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$. We draw the curves $y (b + x) = b$, $y = 0$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area OMABO.

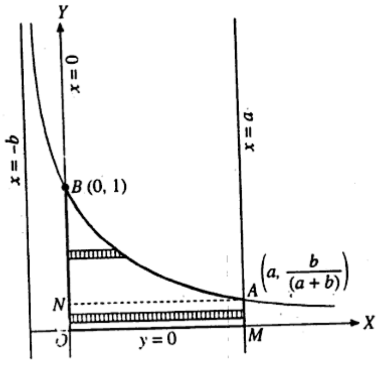

Fig. 1.16

In the given integral we are required to integrate first w.r.t. y and then w.r.t x. To change the order ofintegration, we have to first integrate w.r.t.

regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration OMABO by drawing the straight lines y = constant *i.e.*, by dividing this area into strips parallel to the x-axis.

Now, if we take strips parallel to the x-axis originating from the line $x = 0$, some of these strips terminate on the line AM while the others terminate on the arc AB. So according to the character of the strips we divide the region of integration into two portions namely NOMA and NAB, by drawing the line AN parallel to the axis of x.

For the point B, $x = 0$. Putting $x = 0$ in the equation $y (b + x) = b$, we get $y = 1$. So the coordinates of the point B are (1, 1).

Similarly, putting $x = a$ in the equation $y (b + x) = b$, we get $y = b/(a + b)$ and thus the coordinates of the point Aare $(a, b/(a + b))$.

For the area NBA, x varies form 0 to $b (1 - y)/y$ and y varies from $b/(a + b)$ to 1.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{b/(a+b)}\int_0^a f(x,y)dy\,dx + \int_{b/(a+b)}^1\int_0^{b(1-y)/y} f(x,y)dy\,dx.$$

Example 69:

Evaluate $\iint xy(x^2+y^2)^{3/2}\,dx\,dy$

over the positive quadrant of the circle $x^2 + y^2 = 1$.

Solution:

Changing to polers by putting $x = r\cos\theta$,

$y = r\sin\theta$, we have $J = r$

so that $dxdy = J\,d\theta\,dr = r\,d\theta\,dr$.

The given region of integration is the area lying in the positive quadrant of the circle $x^2 + y^2 = 1$.

Changing to polar coordinates, this region of integration is covered when r varies 0 to 1 and θ varies from 0 to $\pi/2$.

∴ the required integral

$$\iint xy(x^2+y^2)^{3/2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 r\cos\theta.r\sin\theta.(r^2)^{3/2}.r\,d\theta\,dr$$

$$= \int_0^{\pi/2}\int_0^1 r^6\sin\theta\cos\theta\,d\theta\,dr = \int_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^1 \sin\theta\cos\,d\theta$$

$$= \frac{1}{7}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta d\theta = \frac{1}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = -\frac{1}{28}[-1-1] = \frac{1}{14}.$$

Example 70:

Evaluate $\iint\sqrt{(a^2-x^2-y^2)}\,dx\,dy$

over the semi-circle $x^2 + y^2 = ax$ *in the positive quadrant.*

Solution:

Here the region of integration is a semi-circle. Therefore, for the sake of convenience, changing to polar by putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = ax$, we have $r^2\cos^2\theta + r^2\sin^2\theta = a\,r\cos\theta$ or $r^2(\sin^2\theta + \cos^2\theta) = a\,r\cos\theta$

or $\quad r = a\cos\theta$.

The equation $r = a\cos\theta$ represents a circle passing through the pole and diameter through the pole along the initial line.

For the given region r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\pi/2$.

$$\therefore \iint\sqrt{(a^2-x^2-y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{a\cos\theta}\sqrt{(a^2-r^2)}.r\,d\theta\,dr, \quad [\because x^2 + y^2 = r^2 \text{ and } dx\,dy = r\,d\theta\,dr]$$

$$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}.(-2r)dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta.$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^2\sin^3\theta - a^3)d\theta = -\frac{a^3}{3}\left[\frac{2}{3.1}-\frac{\pi}{2}\right]$$

$$= \frac{1}{3}a^3\left(\frac{1}{3}\pi - \frac{2}{3}\right).$$

Example 71:

Changed the order of integration in $\int_0^a\int_x^{a^2/x} f(x,y)dx\,dy$.

Solution:

In the given integral the limits of integration of y are given by $y = x$ which is a straight line passing through the origin equally inclined to both the axes and $y = a^2/x$ or $xy = a^2$ which is a rectangular hyperbola. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$.

We draw the curves $y = x$, $xy = a^2$, $x = 0$, $x = 0$, and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area LMOY... extended upto infinity on the above side.

In the given integral we are required to integrate first w.r.t.; y and then w.r.t. x. If we want to change the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y This is done by covering the area of integration by strips parallel to the x-axis.

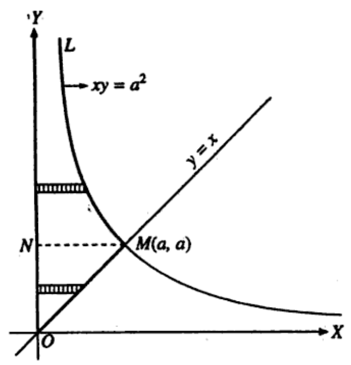

Fig. 1.17

Now if we take strips parallel to the x-axis starting from the line $x = 0$,some of these strips end on the in OM while the others end on the arc ML of the rectangular hyperbola. So we divide the region of integration into two portions, the triangle OMN and the area YNML which extends upto infinity, by drawing the line MN parallel to the axis of x.

For the point M, $x = a$. Putting $x = a$ in the equation of the line $y = x$ or the rectangular hyperbola $xy = a^2$, we get $y = a$.

So the y-coordinate of the point M is a and the equation of the line of demarcation MN is $y = a$.

For the area OMN, x varies from 0 to y and y varies from 0 to a.

For the area YMNL..., x varies from 0 to a^2/y and y varies from a to ∞.

Hence by changing the order of integration, we have the given integral

$$= \int_0^a \int_0^y f(x,y)\,dy\,dx + \int_a^\infty \int_0^{a^2/y} f(x,y)\,dy\,dx.$$

Example 72:

Change the order of integration in

$\int_0^a \int_{(b/a)\sqrt{(a^2-x^2)}}^b f(x,y)\,dx\,dy$, *where $c < a$.*

Solution:

In the given integral the limits of integration of y are given by

$y = \frac{b}{a}\sqrt{(a^2-x^2)}$ *i.e.*, $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

which is an ellipse with centre (0, 0) and the straight line $y = b$.

Again the limits of integration of x are given by the straight lines $x = c$ and $x = a$.

Draw the ellipse $x^2/a^2 + y^2/y^2 = 1$ and the straight lines $y = b$, $x = c$ and $x = a$, bounding the region of integration, in the same figure. We observe that the region of integration is the area ABECA. In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y and then w.r.t. x.

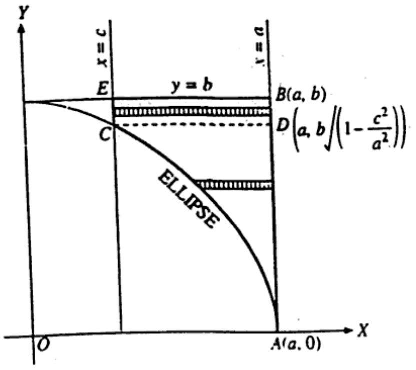

Fig. 1.18

In order to integrate in the reverse order, divide whole the area into strips parallel to the x-axis originating either form the EC (*i.e.*, $x = c$) or form the arc

AC of the ellipse and terminating on the line BA (*i.e.*, $x = a$). While integrating we must first obviously divide the region of integration ABEA into two portions AD and ECDB according to the character of the strips. For the point C, $x = c$. Putting $x = c$ in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $y = b\sqrt{\{1-(c^2/a^2)\}}$ which is the y-coordinate of the point C. The equation of the line of demarcation CD is thus $y = b\sqrt{\{1-(c^2/a^2)\}}$.

For the area CAD, x varies from $a\sqrt{\{1-(y^2/b^2)\}}$ to a and y varies from 0 to $b\sqrt{\{1-(c^2/a^2)\}}$.

For the area ECDB, x varies from c to a and y varies form $b\sqrt{\{1-(c^2/a^2)\}}$ to b.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{b\sqrt{\{1-(c^2/a^2)\}}} \int_{a\sqrt{\{1-(y^2/b^2)\}}}^a f(x,y)\,dy\,dx + \int_{b\sqrt{\{1-(c^2/a^2)}}^a \int_c^a f(x,y)\,dy\,dx.$$

Example 73:

Change the order of integration in the double integral

$$\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x,y)\,dx\,dy.$$

Solution:

The given double integral transforms to

$$\int_0^{a/2} \int_{y^2/a}^{\frac{1}{2}[a-\sqrt{(a^2-4y^2)}]} f(x,y)\,dy\,dx$$

$$= \int_0^{a/2} \int_{\frac{1}{2}[a+\sqrt{(a^2-4y^2)}]}^a f(x,y)\,dy\,dx + \int_{a/2}^a \int_{y^2/a}^a f(x,y)\,dy\,dx.$$

Example 74:

Change the order of integration in $\int_0^{a/2} \int_{x^2/a}^{x-(x^2/a)} f(x,y)\,dx\,dy.$

Solution:

In the given integral the limits of integration of y are given by $y = x^2/a$ *i.e.*, $x^2 = ay$ which is a parabola with vertex (0, 0) and $x - x^2/a = y$ *i.e.*, $ax - x^2 = ay$ *i.e.*, $\left(x-\frac{1}{2}a\right)^2 = -a\left(y-\frac{1}{4}a\right)$ which is also a parabola with vertex $\left(\frac{1}{2}a, \frac{1}{4}a\right)$.

The points of intercession of the two parabolas are (0, 0) and $\left(\frac{1}{2}a, \frac{1}{4}a\right)$.

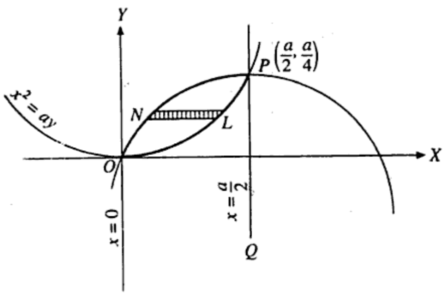

Fig. 1.19

Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = a/2$ which is a straight line parallel to the y-axis at a distance a/2 from the origin.

Draw the two parabolas $x^2 = ay$ and $\left(x-\frac{1}{2}a\right)^2 = -a\left(y-\frac{1}{4}a\right)$ intersecting at O (0, 0) and P $\left(\frac{1}{2}a, \frac{1}{4}a\right)$ along with the lines $x = 0$ and $x = a/2$ in the same figure. We observe that the region of integration is ONPLO. In the given integral we are required to integrate first w.r.t. y ($\because$ the limits of integration of y are variable) and then w.r.t. x. To reverse the order of integration, draw strips parallel to the x-axis originating from the arc ONP of the parabola $ax - x^2 = ay$ and terminating on the arc OLP of the parables $x^2 = ay$. Then for the region ONPLO, the limits of integration for x are given by $ax - x^2 = ay$ and $x^2 = ay$. Solving $ay = ax - x^2$ *i.e.*, $x^2 - ax + ay = 0$ for x, we get

$$x = \frac{1}{2}[a \pm \sqrt{(a^2-4ay)}]$$

or $$x = \frac{1}{2}[a - \sqrt{(a^2-4ay)}],$$

rejecting the +ive sign since x cannot be greater than $\frac{1}{2}$ a in the region ONPLO.

Thus the limits of x are $x = \frac{1}{2}[a-\sqrt{(a^2-4ay)}]$ and $x = \sqrt{(ay)}$.

Clearly for this region y varies from 0 to $\frac{1}{4}$a.

Hence by changing the order of integration, we have

$$\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy = \int_0^{a/4} \int_{\frac{1}{2}[a-\sqrt{(a^2-4ay)}]}^{\sqrt{(ay)}} f(x,y)\,dy\,dx.$$

Example 75:

Change the order of integration in $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\,dx\,dy$ ***and hence find its value.***

Solution:

In the given integral the limits of integration of y are given by $y = \sqrt{x}$ and $y = 1$.

When $y = \sqrt{x}$, we have $y^2 = x$ which is a parabola with vertex at (0, 0) and the axis of x as its axis. Also the limits of integration of x are given by $x = 0$ and $x = 1$

The region of integration is the area OABO.

To reverse the order of integration, we divide the area OABO into strips parallel to the x-axis.

Changing the order of integration, the given double integral I transforms to

$$I = \int_{y=0}^1 \int_{x=0}^{y^2} e^{x/y}\,dy\,dx.$$

We have $$I = \int_{y=0}^1 \left[ye^{x/y}\right]_{x=0}^{y^2} dy$$

$$= \int_0^1 (ye^y - y)\,dy$$

$$= \left[ye^y - e^y - \frac{y^2}{2}\right]_0^1$$

$$= e - e - \frac{1}{2} + 1 = \frac{1}{2}.$$

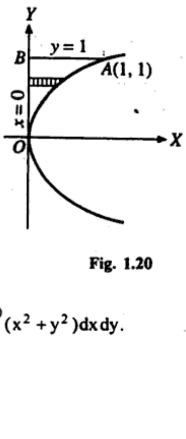

Fig. 1.20

Example 76:

Change the order of integration $\int_0^a \int_{x/a}^{\sqrt{(x/a)}} (x^2+y^2)\,dx\,dy.$

Solution:

In the given integral the limits of integration of y are given by $y = x/a$ and $y = \sqrt{(x/a)}$. $y = x/a$ is a straight line passing through the origin.

When $y = \sqrt{(x/a)}$, we have $y^2 = x/a$ which is a parabola with vertex at (0, 0) and x-axis as its axis.

The straight line $y = x/a$ meets $y^2 = x/a$ at the points A(a, 1).

The limits of integration of x are given by $x = 0$ and $x = a$.

Thus, the region of integration is the area OABO.

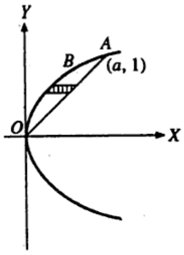

Fig. 1.21

To reverse the order of integration, we divide the area OABO into strips parallel to x-axis.

Changing the ordure of integration, the given double integral I transforms to

$$I = \int_{y=0}^{1}\int_{x=ay^2}^{ay}(x^2+y^2)\,dy\,dx.$$

Example 77:

Change the order of integration in $\int_0^a\int_{\sqrt{(a^2-x^2)}}^{x+2a} f(x,y)\,dx\,dy$.

Solution:

Here the area of integration is bounded by the curves $y = \sqrt{(a^2 - x^2)}$

i.e., $x^2 + y^2 = a^2$

which is a circle with centre (0, 0) and radius a, $y = x + 2a$ which is a straight line passing through (0, 2a), $x = 0$ *i.e.*, the y-axis and the line $x = a$ which a line parallels to the y-axis at a distance a from the origin.

We draw the curves $x^2 + y^2 = a^2$, $y = x + 2a$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area MLANM.

To reverse the order of integration, cover this area of integration MLANM by strips parallel to the x-axis. Draw the lines MC and NB parallels to the x-axis so that the region of integration MLANM is divided into three portions MLC, NMCB and NAB.

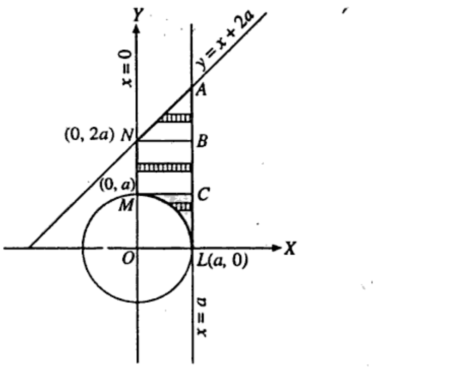

Fig. 1.22

For the region MLC, x varies from the arc ML of the circle $x^2 + y^2 = a^2$ to the line $x = a$ *i.e.*, x varies from $\sqrt{(a^2 - y^2)}$ to a and y varies from 0 to a.

For the region NMCB, x varies from 0 to a and y varies from a to 2a.

For the region NBA, x varies from $y - 2a$ to a and y varies from 2a to 3a.

Therefore, changing the order of integration, the given integral transforms to

$$\int_0^a\int_{\sqrt{(a^2-x^2)}}^{a} f(x,y)\,dx\,dy + \int_a^{2a}\int_0^a f(x,y)\,dy\,dx + \int_{2a}^{3a}\int_{y-2a}^{a} f(x,y)\,dy\,dx.$$

Example 78:

Change the order of integration in the double integral

$$\int_0^{ab/\sqrt{(a^2+b^2)}}\int_0^{(a/b)\sqrt{(b^2-y^2)}} f(x,y)\,dy\,dx.$$

Solution:

In the given integral the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = (a/b)\sqrt{(b^2 - y^2)}$ *i.e.*, $x^2/a^2 + y^2/b^2 = 1$ which is an ellipse with centre as origin.

Again the limits ofintegration of y are given by $y = 0$ *i.e.*, the x-axis and $y = ab/\sqrt{(a^2 + b^2)}$ which is a straight line parallel to the x-axis at a distance $ab/\sqrt{(a^2 + b^2)}$ from the origin.

We draw the curves $x = 0$, $x^2/a^2 + y^2/b^2 = 1$, $y = 0$

and $y = ab/\sqrt{(a^2 + b^2)}$, giving the limits of integration, in the same figure. We observe that the region of integration is the area OPBAO.

In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t x regarding y as a constant and then we integrate w.r.t. y.

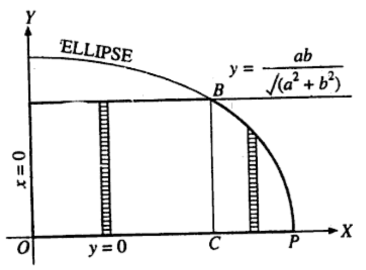

Fig. 1.23

If we want to reveres the order of integration, we have to first integrate w.r.t. y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of intgration OPBAO by strips parallel to the y-axis. Now if we take strips parallel to the y-axis starting from the line $y = 0$, some of these strips and on the line AB while the others end on the arc BP of the ellipse. So we draw the line of demarcation BC dividing the area OPBAO into two portions OCBA and BCP. For the point B, $y = ab/\sqrt{(a^2 + b^2)}$. Putting this value of y in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $x = ab/\sqrt{(a^2+b^2)}$. For the region OCBA, y varies from 0 to $(ab)/\sqrt{(a^2 + b^2)}$ and x varies from 0 to $ab/\sqrt{(a^2 + b^2)}$ to a.

Hence the given integral transforms to

$$\int_0^{ab/\sqrt{(a^2+b^2)}}\int_0^{ab/\sqrt{(a^2+b^2)}} f(x,y)\,dy\,dx$$

$$+ \int_{ab/\sqrt{(a^2+b^2)}}^{0}\int_a^{(b/a)\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

Example 79:

Change the order of integration in $\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,d\theta\,dr$.

Solution:

Here the region of integration is bounded by the polar curve $r = 0$ (the pole), $r = 2a\cos\theta$ (a circle of diameter 2a passing through the pole), $\theta = 0$ (the initial line) and $\theta = \pi/2$ (a line through the pole perpendicular to initial line).

We draw the curves $r = 0$, $r = 2a\cos\theta$, $\theta = 0$ and $\theta = \pi/2$, giving the limits of integration, in the same figure.

We observe that the region of integration is the are of the semi-circle OMPO.

In the given integral the limits of integration of r area variable while those of θ are constant. Thus, we have to first integrate with respect to r regarding q as a constant and then we integrate w.r.t. θ.

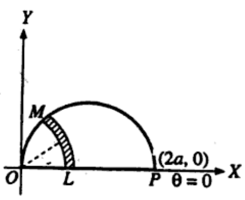

Fig. 1.24

If we want to reverse the order of integration, we have to first integrate with respect to θ regarding r as constant and then we integrate w.r.t. r. This is done by covering the area of integration OMPO by circular arcs with centre as pole. On these arcs θ varies and r remains constant. Thus, for the area OMPO, for a fixed value of r, θ varies from the initial line (*i.e.*, $\theta = 0$ to a point on the arc OMP of the circle $r = 2a\cos\theta$ *i.e.*, to a point for which $\theta = \cos^{-1}(r/2a)$ and r varies from 0 to 2a.

Hence by chaining the order of integration, we have

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,d\theta\,dr = \int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)\,dr\,d\theta.$$

Example 80:

Transform $\iint f(x,y)\,dx\,dy$ by the substitution $x + y = u$, $y = uv$.

Solution:

We have $x + y = u$ and $y = uv$. ...(1)

From these, we have $x = u - y = u - uv$

and $y = uv$. ...(2)

$\therefore \dfrac{\partial x}{\partial u} = 1 - v$,

$\dfrac{\partial x}{\partial v} = -u$, $\dfrac{\partial y}{\partial u} = v$

and $\frac{\partial y}{\partial v} = u.$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u.$$

$\therefore$ dx dy = J du dv = u du dv.

Hence the given integral transforms to

$$\iint F(u,v)\,u\,du\,dv.$$

Example 81:

Transform $\iint f(x,y)\,dx\,dy$ *to polar coordinates.*

Solution:

We have x = r cos θ, y = r sin θ.

$$\text{Now} \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore$ dx dy = Jdθ dr = r dθ dr.

Hence the given integral transforms to $\iint F(r,\theta)\,r\,d\theta\,dr$.

Example 82:

By using transformation x + y = u, y = uv, prove that

$$\iint \{xy(1-x-y)\}^{1/2}\,dx\,dy$$

taken over the area of the triangle bounded by the lines x = 0, y = 0, x + y = 1 is 2π/105.

Solution:

We have dx dy = udu dv; u varies from 0 to 1 and also v varies from 0 to 1.

Now $\{xy(1-x-y)\}^{1/2} = [xy\{1-(x+y)\}]^{1/2}$

$= [u(1-v).uv.(1-u)]^{1/2}$ $\quad [\because x = u(1-v),\ y = uv]$

$= u(1-u)^{1/2}.\ v^{1/2}(1-v)^{1/2}.$

Hence the given double integral transforms to

$$\int_0^1\int_0^1 u(1-u)^{1/2}.v^{1/2}(1-v)^{1/2}.u\,du\,dv$$

$$= \left[\int_0^1 u^2(1-u)^{1/2}\,du\right]\cdot\left[\int_0^1 v^{1/2}(1-v)^{1/2}\,dv\right]$$

$$= \left[\int_0^1 u^{3-1}(1-u)^{3/2-1}\,du\right]\cdot\left[\int_0^1 v^{3/2-1}(1-v)^{3/2-1}\,dv\right]$$

$$= B\left(3,\frac{3}{2}\right).B\left(\frac{3}{2},\frac{3}{2}\right), \text{ [by the def. of Beta function]}$$

$$= \frac{\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(3+\frac{3}{2}\right)}\cdot\frac{\Gamma\frac{3}{2}.\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{2\left[\frac{1}{2}\sqrt{\pi}\right]^3}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}.2} = \frac{2\pi}{105}.$$

Example 83:

Evaluate $\iint (x^2+y^2)^{7/2}\,dx\,dy$ *over the circle* $x^2 + y^2 = 1$.

Solution:

Here the region of integration is a circle. Therefore we shall change the given double integral to polar coordinates by putting x = r cos θ and y = r sin θ. We have

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore$ dx dy = J dθ dr = r dθ dr.

Clearly, the region of integration is the circle $x^2 + y^2 = 1$ *i.e.*, the circle with centre (0, 0) and radius 1.

Changing to polar coordinates, the region of integration is covered when r varies from 0 to 1 and θ varies from 0 to 2π.

$$\therefore \iint_{x^2+y^2\le 1}(x^2+y^2)^{7/2}\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2)^{7/2}\,J\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 r^7.r.d\theta\,dr = \int_0^{2\pi}\int_0^1 r^8\,d\theta\,dr = \int_0^{2\pi}\left[\frac{r^9}{9}\right]_0^1 d\theta$$

$$= \frac{1}{9}\int_0^{2\pi} d\theta = \frac{1}{9}[\theta]_0^{2\pi} = \frac{2}{9}\pi.$$

Example 84:

Evaluated $\iint e^{-(x^2+y^2)}\,dx\,dy$ *over the circle* $x^2 + y^2 = a^2$.

Solution:

Changing to polar coordinates, the equation $x^2 + y^2 = a^2$ transforms to $r^2\cos^2\theta + r^2\sin^2\theta = a^2$ *i.e.*, r = a.

Hence for the given region r varies from 0 to a and θ varies from 0 to 2π.

Also dx dy = r dθ dr.

$\therefore$ the required integral

$$\iint e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{a^2} e^{-t}.\frac{1}{2}\,d\theta\,dt,$$

putting $r^2 = t$ so that 2r dr = dt

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{e^{-t}}{-1}\right]_0^{a^2} d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta$$

$$= -\frac{1}{2}(e^{-a^2}-1)[\theta]_0^{2\pi} = \frac{1}{2}(1-e^{-a^2}).2\pi = \pi(1-e^{-a^2}).$$

Example 85:

Evaluate the following double integrals:

(i) $\int_0^a\int_0^b (x^2+y^2)\,dx\,dy;$

(ii) $\int_1^a\int_1^b \frac{dx\,dy}{xy};$

(iii) $\int_1^2\int_0^x \frac{dx\,dy}{x^2+y^2};$

(iv) $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx;$

(v) $\int_0^1\int_0^{x^2} e^{y/x}\,dx\,dy;$

(vi) $\int_1^2\int_0^{3y} y\,dy\,dx.$

(vii) $\int_0^2\int_0^{2x-4} \frac{2y-1}{x+1}\,dx\,dy$

Solution:

(i) We have

$$\int_0^a\int_0^b (x^2+y^2)\,dx\,dy = \int_0^a\left[x^2y + \frac{y^3}{3}\right]_{y=0}^b dx,$$

(integration w.r.t. y treating x as constant)

$$= \int_0^a\left[bx^2 + \frac{b^3}{3}\right]dx = \left[b\frac{x^3}{3} + \frac{b^3}{3}x\right]_0^a = \frac{ba^3}{3} + \frac{b^3a}{3}$$

$$= \frac{1}{3}(ab)(a^2+b^2).$$

(ii) $\int_1^a\int_1^b \frac{dx\,dy}{xy} = \int_1^a \frac{1}{x}[\log y]_{y=1}^b\,dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_1^a \frac{(\log b - \log 1)}{x}\,dx$$

$$= \log b\int_1^a \frac{1}{x}\,dx = (\log b)[\log x]_1^a = (\log b)\ (\log a - \log 1)$$

$= (\log b).(\log a).$

(iii) $\int_1^2\int_0^x \frac{dx\,dy}{x^2+y^2} = \int_1^2\left[\int_0^x \frac{dy}{x^2+y^2}\right]dx$

$$= \int_1^2\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{y=0}^x dx \text{ (integrating w.r.t. y treating x as constant)}$$

$$= \int_1^2\left[\frac{1}{x}(\tan^{-1}1 - \tan^{-1}0)\right]dx = \frac{\pi}{4}\int_1^2\frac{dx}{x} = \frac{\pi}{4}[\log x]_1^2$$

$$= \frac{1}{4}\pi[\log 2 - \log 1] = \frac{1}{4}\pi\log 2.$$

(iv) $\int_0^{\pi/2}\int_{\pi/2}^{\pi}\cos(x+y)\,dy\,dx = \int_0^{\pi/2}\left[\int_{\pi/2}^{\pi}\cos(x+y)\,dx\right]dy$

$$= \int_0^{\pi/2}[\sin(x+y)]_{x=\pi/2}^{\pi}\,dy,$$

(integrating w.r.t x treating y as constant)

$$= \int_0^{\pi/2}\left[\sin(\pi+y) - \sin\left(\frac{1}{2}\pi + y\right)\right]dy$$

$$= \int_0^{\pi/2} (-\sin y - \cos y)dy$$

$$= [\cos y - \sin y]_0^{\pi/2} = (0 - 1) - (1 - 0) = -2.$$

(v) $\int_0^1\int_0^{x^2} e^{y/x} dx\, dy = \int_0^1 \left[x e^{y/x}\right]_{y=0}^{x^2} dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_0^1 [xe^{x^2/x} - xe^{0/x}]dx = \int_0^1 (xe^x - x)dx$$

$$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx - \left[\frac{x^2}{2}\right]_0^1$$

$$= e - \left[e^x\right]_0^1 - \frac{1}{2} = e - (e - 1) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(vi) We have

$$\int_1^2\int_0^{3y} dy\, dx = \int_1^2 y[x]_0^{3y} dy,$$

integrating w.r.t. x regarding y as a constant

$$= \int_1^2 y[3y - 0]dy = 3\int_1^2 y^2 dy = 3\left[\frac{y^3}{3}\right]_1^2 = \left[y^3\right]_1^2 = 8 - 1 = 7.$$

(vii) The given integral

$$I = \int_{x=0}^2\int_{y=0}^{2x-4} \frac{2y-1}{x+1} dx\, dy$$

$$= \int_0^2 \frac{1}{x+1}\left[y^2 - y\right]_{y=0}^{2x-4} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^2 \frac{1}{x+1}[(2x-4)^2 - (2x-4)]dx = \int_0^2 \frac{4x^2 - 18x + 20}{x+1} dx$$

$$= \int_0^2 \left[4x - 22 + \frac{42}{x+1}\right]dx, \quad \text{dividing the Nr. by the Dr.}$$

$$= [2x^2 - 22x + 42\log(x+1)]_0^2$$

$$= 8 - 44 + 42 \log 3 = -36 + 42 \log 3.$$

Example 86:

Evaluate

(i) $\int_0^3\int_1^2 xy(1 + x + y)dx\, dy.$

(ii) $\int_0^1\int_0^{\sqrt{(1+x^2)}} \frac{dx\, dy}{1 + x^2 + y^2}.$

(iii) $\int_0^2\int_0^{\sqrt{(4+x^2)}} \frac{dx\, dy}{4 + x^2 + y^2}$

(iv) $\int_0^1\int_0^{\sqrt{(1-y^2)}} 4y\, dy\, dx.$

(v) $\int_0^1\int_x^{\sqrt{x}} (x^2 - y^2)dx\, dy.$

(vi) $\int_2^3\int_0^{y-1} \frac{dy\, dx}{y}.$

Solution:

(i) $\int_0^3\int_1^2 xy(1 + x + y)\, dx\, dy$

$$= \int_0^3 \left[x \cdot \frac{y^2}{2} + x^2 \cdot \frac{y^2}{2} + x \cdot \frac{y^3}{3}\right]_{y=1}^2 dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^3 \left[\frac{x}{2}(4-1) + \frac{x^2}{2}(4-1) + \frac{x}{3}(8-1)\right]dx$$

$$= \int_0^3 \left[\left(\frac{3}{2} + \frac{7}{3}\right)x + \frac{3}{2}x^2\right]dx = \left[\frac{23}{6}\cdot\frac{x^2}{2} + \frac{3}{2}\cdot\frac{x^3}{3}\right]_0^3$$

$$= \frac{23}{6}\cdot\frac{9}{2} + \frac{27}{2} = \frac{123}{4} = 30\ \frac{3}{4}.$$

(ii) $\int_0^1\int_0^{\sqrt{(1+x^2)}} \frac{dxdy}{1 + x^2 + y^2}$

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(1+x^2)}}\right]_{y=0}^{\sqrt{(1+x^2)}} dx$$

(integrating w.r.t.y treating x as constant)

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}[\tan^{-1} 1 - \tan^{-1} 0]dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{(1+x^2)}}$$

$$= \frac{\pi}{4}\left[\log\{x + \sqrt{(1+x^2)}\}\right]_0^1 = \frac{\pi}{4}\log(1 + \sqrt{2}).$$

(iii) The given integral

$$I = \int_{x=0}^2\int_{y=0}^{\sqrt{(4+x^2)}} \frac{dx\, dy}{(4+x^2) + y^2}$$

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(4+x^2)}}\right]_{y=0}^{\sqrt{(4+x^2)}} dx,$$

(integrating w.r.t y treating x as constant)

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}}[\tan^{-1} 1 - \tan^{-1} 0]dx = \frac{\pi}{4}\int_0^2 \frac{dx}{\sqrt{(4+x^2)}}$$

$$= \frac{\pi}{4}\left[\log\{x + \sqrt{(4+x^2)}\}\right]_0^2 = \frac{\pi}{4}[\log(2 + 2\sqrt{2}) - \log 2]$$

$$= \frac{\pi}{4}\log\frac{2 + 2\sqrt{3}}{2} = \frac{\pi}{4}\log(1 + \sqrt{2}).$$

(iv) The given integral

$$I = \int_{y=0}^1\int_{x=0}^{\sqrt{(1-y^2)}} 4y\, dy\, dx$$

$$= \int_0^1 4y[x]_{x=0}^{\sqrt{(1-y^2)}} dy, \text{ integrating w.r.t. x treating y as constant}$$

$$= 4\int_0^1 y\sqrt{(1-y^2)}dy = 4\int_0^1 \left(-\frac{1}{2}\right)\cdot(1-y^2)^{1/2}(-2y)dy$$

$$= -2\cdot\frac{2}{3}\left[(1-y^2)^{3/2}\right]_0^1, \quad \text{by power formula}$$

$$= -\frac{4}{3}[0 - 1] = \frac{4}{3}.$$

(v) The given integral

$$I = \int_{x=0}^1\int_{y=x}^{\sqrt{x}} (x^2 + y^2)dx\, dy$$

$$= \int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_{y=x}^{\sqrt{x}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^1 \left[x^2\sqrt{x} + \frac{1}{3}x\sqrt{x} - x^3 - \frac{1}{3}x^3\right]dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{1}{3}x^{3/2} - \frac{4}{3}x^3\right]dx$$

$$= \left[\frac{2}{7}x^{7/2} + \frac{1}{3}\cdot\frac{2}{5}x^{5/2} - \frac{1}{3}x^4\right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30 + 14 - 35}{105} = \frac{3}{35}.$$

(vi) The given integral

$$I = \int_{y=2}^3\int_{x=0}^{y-1} \frac{dy\, dx}{y}$$

$$= \int_2^3 \frac{1}{y}[x]_{x=0}^{y-1} dy, \text{ integrating w.r.t. x treating y as constant}$$

$$= \int_2^3 \frac{y-1}{y} dy = \int_2^3 \left(1 - \frac{1}{y}\right)dy = [y - \log y]_2^3$$

$$= 3 - \log 3 - 2 + \log 2 = 1 - \log\frac{3}{2}.$$

Example 87:

Evaluate

(i) $\int_0^a\int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2 - x^2 - y^2)}dy\, dx.$

(ii) $\int_0^a\int_0^{\sqrt{(a^2-y^2)}} (a^2 - x^2 - y^2)dy\, dx.$

(iii) $\int_0^a\int_0^{\sqrt{(a^2-y^2)}} (x + y)dx\, dy.$

Solution:

(i) Here the variable limits are those of x and so the first integration must be performed w.r.t. x taking y as constant.

$$\therefore \int_0^a\int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2 - x^2 - y^2)}dy\, dx$$

$$= \int_0^a\left[\int_0^{\sqrt{(a^2-y^2)}} \sqrt{\{(a^2 - y^2) - x^2\}}dx\right]dy$$

$$= \int_0^a \left[\frac{x\sqrt{(a^2-y^2-x^2)}}{2} + \frac{(a^2-y^2)}{2}\sin^{-1}\frac{x}{\sqrt{(a^2-y^2)}} \right]_{x=0}^{\sqrt{(a^2-y^2)}} dy$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^a \left[0+\frac{a^2-y^2}{2}\cdot\frac{\pi}{2}\right]dy = \frac{\pi}{4}\left[a^2y-\frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3-\frac{a^3}{3}\right] = \frac{1}{6}\pi a^3.$$

(ii) The given integral

$$I = \int_{y=0}^a \int_{x=0}^{\sqrt{(a^2-y^2)}} [(a^2-y^2)-x^2]dy\,dx$$

$$= \int_0^a \left[(a^2-y^2)x - \frac{1}{3} x^3\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy,$$

Integrating w.r.t. x treating y as constant

$$=\int_0^a\left[(a^2-y^2)^{3/2}-\frac{1}{3}(a^2-y^2)^{3/2}\right]dy$$

$$=\frac{2}{3}\int_0^a(a^2-y^2)^{3/2}dy$$

$$=\frac{2}{3}\int_0^{\pi/2}a^3\cos^3\theta.\ a\cos\theta\,d\theta,$$

putting $y = a\sin\theta$
so that $dy = a\cos\theta\, d\theta$

$$=\frac{2}{3} a^4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{2}{3} a^4\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2},$$ (by Walli's formula)

$= \pi a4/8.$

(iii) The given integral

$$I=\int_{x=0}^a\int_{y=0}^{\sqrt{(a^2-x^2)}}(x+y)dx\,dy$$

$$=\int_0^a\left[xy+\frac{1}{2}y^2\right]_{y=0}^{\sqrt{(a^2-x^2)}}dx,$$

integrating w.r.t. y treating x as constant

$$=\int_0^a\left[x\sqrt{(a^2-x^2)}+\frac{1}{2}(a^2-x^2)\right]dx$$

$$=\int_0^a\left[-\frac{1}{2}(a^2-x^2)^{1/2}(-2x)+\frac{1}{2}(a^2-x^2)\right]dx$$

$$=\left[-\frac{1}{2}\cdot\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a+\frac{1}{2}\left[a^2x-\frac{1}{3}x^3\right]_0^a,$$ (by power formula)

$$=0+\frac{1}{3}a^3+\frac{1}{2}\left[a^3-\frac{1}{3}a^3\right]=\frac{2}{3}a^3.$$

Example 88:

Evaluate $\int_0^2\int_0^{\sqrt{(2x-x^2)}} x\,dx\,dy.$

Solution:

Here the variable limits are those of y and so the first integration must be performed w.r.t. y regading x as constant.

$$\therefore \int_0^2\int_0^{\sqrt{(2x-x^2)}} x\,dx\,dy = \int_0^2 x[y]_0^{\sqrt{(2x-x^2)}}dx$$

$$= \int_0^2 x\sqrt{(2x-x^2)}dx = \int_0^2 x\sqrt{\{1-(1-x)^2\}}dx.$$

Now put $(1-x) = t$ so that $-dx = dt$.

Also when $x = 0$, $t = 1$ and when $x = 2$, $t = -1$.

$\therefore$ the required integral $= \int_{-1}^1 (1-t)\sqrt{(1-t^2)}dt$

$$= \int_{-1}^1\sqrt{(1-t^2)}dt - \int_{-1}^1 t\sqrt{(1-t^2)}dt$$

$$= 2\int_0^1\sqrt{(1-t^2)}dt - 0,$$

the second integral vanishes because the integrand is an odd function of t

$$= 2\left[\frac{t}{2}\sqrt{(1-t^2)} + \frac{1}{2}\sin^{-1} t\right]_0^1$$

$$= 2\left[0 + \frac{1}{2}\cdot\frac{1}{2}\cdot\pi\right] = \frac{1}{2}\pi.$$

Example 89:

Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{\{(1-x^2)(1-y^2)\}}}.$

Solution:

We have $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{\{(1-x^2)(1-y^2)\}}}$

$$= \int_{y=0}^1 \frac{1}{\sqrt{(1-y^2)}}\left[\int_{x=0}^1 \frac{1}{\sqrt{(1-x^2)}}dx\right]dy$$

$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}}\left[\sin^{-1}x\right]_0^1 dy,$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^1 \frac{\pi}{2\sqrt{(1-y^2)}}dy = \frac{\pi}{2}\left[\sin^{-1}y\right]_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}.$$

Example 90:

Show that $\int_1^2\int_3^4(xy+e^y)dx\,dy = \int_3^4\int_1^2(xy+e^y)dx\,dy.$

Solution:

Integral on the L.H.S. $\int_1^2\left[\int_3^4(xy+e^y)dy\right]dx$

$$=\int_1^2\left[\frac{xy^2}{2}+e^y\right]_3^4 dx = \int_1^2\left[8x+e^4-\frac{9}{2}x-e^3\right]dx$$

$$=\int_1^2\left[\frac{7}{2}x+e^4-e^3\right]dx = \left[\frac{7x^2}{4}+(e^4-e^3)x\right]_1^2$$

$$= 7 + 2(e^4-e^3) - \frac{7}{4} - (e^4-e^3) = \frac{21}{4} + e^4 - e^3.$$

And the integral on the R.H.S.

$$=\int_3^4\left[\int_1^2(xy+e^y)dx\right]dy$$

$$=\int_3^4\left[\frac{yx^2}{2}+xe^y\right]_1^2 dy = \int_3^4\left[2y+2e^y-\frac{y}{2}-e^y\right]dy$$

$$=\int_3^4\left[\frac{3y}{2}+e^y\right]dy = \left[\frac{3y^2}{4}+e^y\right]_3^4$$

$$= 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3.$$ Hence the result.

Example 91:

Show that $\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2\int_0^{x/2} y\,dy\,dx.$

Solution:

We have

$$\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2\left[y\int_0^{y/2}dx\right]dy = \int_1^2 y[x]_0^{Y/2}dy,$$

(integrating w.r.t. x treating y as a constant)

$$= \int_1^2 y\left[\frac{y}{2}-0\right]dy = \frac{1}{2}\int_1^2 y^2dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_1^2 = \frac{1}{6}[8-1] = \frac{7}{6}$$ ...(1)

Again $\int_1^2\int_1^{x/2} x\,dx\,dy = \int_1^2 x\left[\int_0^{x/2}dy\right]dx = \int_1^2 x[y]_0^{x/2}dx,$

(integrating w.r.t. y treating x as a constant)

$$= \int_1^2 x\left[\frac{x}{2}-0\right]dx = \frac{1}{2}\int_1^2 x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{6}(8-1) = \frac{7}{6}$$ ...(2)

Form (1) and (2), we see that

$$\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2\int_0^{x/2} x\,dx\,dy.$$

Example 92:

Evaluate $\iint x^2y^2\, dx\, dy$ *over the region* $x^2 + y^2 \le 1$.

Solution:

Let R denote the region $x^2 + y^2 \le 1$. Then R is the region in the xy-plane bounded by the circle $x^2 + y^2 = 1$. The limits of integration for this region can be expressed either as $-1 \le x \le 1$, $-\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$ or as $-\sqrt{(1-y^2)} \le x \le \sqrt{(1-y^2)}$, $-1 \le y \le 1$.

Because from the equation of the circle $x^2 + y^2 = 1$. we have $x^2=1-y^2$ so that $x = \pm\sqrt{(1-y^2)}$. Thus for a fixed value of y, x varies from $-\sqrt{(1-y^2)}$ to $\sqrt{(1-y^2)}$ in the area bounded by the circle $x^2 + y^2 = 1$. Also y varies from -1 to 1 to cover the whole area of the circle $x^2 + y^2 = 1$. Therefore, if the first integration is to be performed w.r.t. x regarding y as constant, then

$$\iint_R x^2y^2dx\,dy = \int_{y=-1}^{1}\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2y^2dx\,dy$$

$$= \int_{y=-1}^{1} y^2\left[\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2dx\right]dy$$

$$= \int_{-1}^{1} y^2\left[2\int_{x=0}^{\sqrt{(1-y^2)}} x^{2dx}\right]dy = \int_{-1}^{1} y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}} dy$$

$$= \int_{-1}^{1}\frac{2}{3}y^2(1-y^2)^{3/2}dy = 2.\frac{2}{3}\int_0^1 2y^2(1-y^2)^{3/2}dy.$$

Put $y = \sin\theta$

so that $dy = \cos\theta\, d\theta$;

when $y = 0$, $\theta = 0$

and when $y = 1$, $\theta = \pi/2$.

$$\therefore \iint_R x^2y^2dx\,dy = \frac{4}{3}\int_0^{\pi/2}\sin^2\theta(1-\sin^2\theta)^{3/2}.\cos\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta = \frac{4}{3}\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

Example 93:

Evaluate $\iint x^2y^3\,dx\,dy$ over the circle $x^2 + y^2 = a^2$.

Solution:

If the first integration is to be performed w.r.t. y regarding x as constant, then the region of integration R can be expressed as $- \le x \le a$, $-\sqrt{(a^2 - x^2)} \le y \le \sqrt{(a^2 - x^2)}$.

$$\therefore \quad \iint_R x^2y^3dx\,dy = \int_{x=-a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} x^2y^3dx\,dy = 0.$$

[$\because$ y^3 is an odd function of y]

Example 94:

Show that

$$\int_0^1 dx\int_0^1\frac{x-y}{(x+y)^3}dy \ne \int_0^1 dy\int_0^1\frac{x-y}{(x+y)^3}dx.$$

Find the values of the two integrals.

Solution:

The integral on the L.H.S.

$$= \int_0^1 dx\int_0^1\frac{2x-(x+y)}{(+y)^3}dy$$

$$= \int_0^1 dx\int_0^1\left\{\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right\}dy$$

$$= \int_0^1\left[\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right]_0^1 dx,$$

(integrating w.r.t. y regarding x as constant)

$$= \int_0^1\left[-\frac{x}{(1+x)^2} + \frac{1}{x} + \frac{1}{1+x} - \frac{1}{x}\right]dx$$

$$= \int_0^1\frac{dx}{(1+x)^2} = \left[\frac{-1}{1+x}\right]_0^1$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}.$$

And the integral on the R.H.S.

$$= \int_0^1 dy\int_0^1\frac{(x+y)-2y}{(x+y)^3}dx$$

$$= \int_0^1 dy\int_0^1\left\{\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right\}dx$$

$$= \int_0^1\left[\frac{-1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1\left[\frac{-1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y}\right]dy$$

$$= -\int_0^1\frac{dy}{(1+y)^2} = \left[\frac{1}{1+y}\right]_0^1$$

$= \frac{1}{2} - 1 = -\frac{1}{2}$. Thus, the two integrals are not equal.

Example 95:

Find by double integration the area of the region bounded by the circle $x^2 + y^2 = a^2$.

Solution:

The area of a small element situated at any point (x, y) is dx dy. To find the area bounded by the circle $x^2 + y^2 = a^2$, the region of integration R can be expressed as $-a \le y \le a$, $-\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)}$, where the first integration to be performed w.r.t. x regarding as constant.

$\therefore$ the required area

$$= \iint_R dx\,dy = \int_{y=-a}^{a}\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} 1.dx\,dy$$

$$= \int_{-a}^{a}\left[2\int_0^{\sqrt{(a^2-y^2)}} 1.dx\right]dy = 2\int_{-a}^{a}[x]_0^{\sqrt{(a^2-y^2)}}dy$$

$$= 2\int_{-a}^{a}\sqrt{(a^2-y^2)}dy = 2.2\int_0^a\sqrt{(a^2-y^2)}dy$$

$$= 4\left[\frac{y\sqrt{(a^2-y^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[0 + \frac{a^2}{2}\sin^{-1}1\right]$$

$$= 4.\frac{1}{2}a^2.\frac{1}{2}\pi = \pi a^2$$