FOR ONLINE PUBLICATION
APPENDIX FOR "CAN TOLLING HELP EVERYONE? ESTIMATING THE AGGREGATE AND DISTRIBUTIONAL CONSEQUENCES OF CONGESTION PRICING"

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Appendix A. Details for Section 3

This portion of the appendix shows that the production possibilities frontier of the bottleneck model is similar to one estimated from data, as well being similar to common models used in the transportation engineering literature.

The production possibilities frontier (PPF) of the bottleneck model is shown in Figure A.1. The solid line is the PPF, while the dotted line shows speed-flow combinations that are possible even though they are not on the PPF. The PPF is horizontal up to s^* because up till the point which the bottleneck is binding there is no congestion. Once the bottleneck is binding, throughput falls to s and travel times climb as the queue grows. Since travel time is simply total distance divided by average speed, this means average speed is falling. For different queue lengths there will be different average speeds, all of which have throughput of s. Thus the dotted line is vertical.

Notice that there is a single point which maximizes speed and throughput, labeled on Figure A.1 as the optimal point. The bottleneck model does not have the traditional trade-off between throughput and speed, where increasing one requires decreasing the other (cf. Pigou, 1912, Knight, 1924, Walters, 1961).

Figure A.2 shows the PPF for a bottleneck on I-805N in San Diego. It was created using data from Cassidy and Rudjanakanoknad (2005), who used video recordings of morning

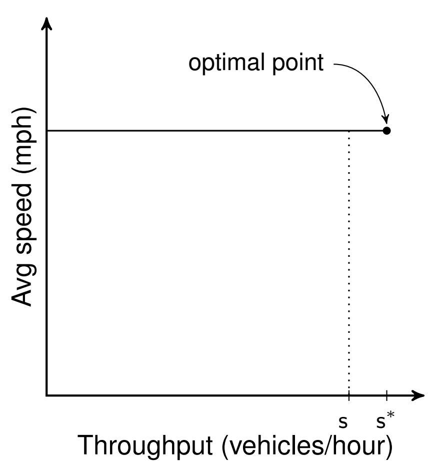


Figure A.1. Production possibility frontier for bottleneck model

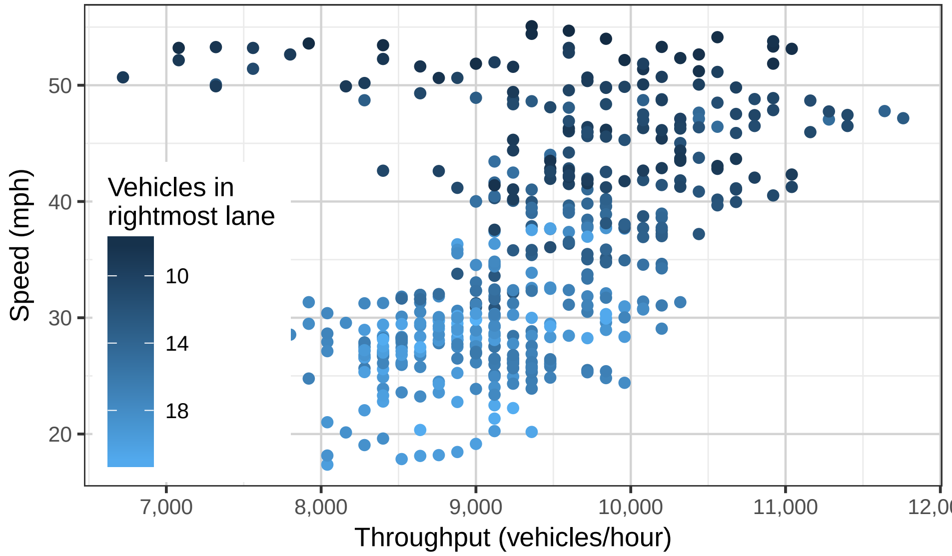


FIGURE A.2. Production possibilities frontier implied by observations from I-805N at Palm Avenue on September 17, 2002, 6:08–6:28 AM and September 18, 2002, 6:10-6:30 AM I take the 30 second moving average of 5 second level data. Data from Cassidy and Rudjanakanoknad (2005).

traffic to extract second-by-second throughput for each lane at four locations, each 120 meters apart; as well as to measure how long it took to traverse the entire 360 meters, which was measured every five seconds. From the video they are able to verify that vehicle flows through the bottleneck are not constrained by traffic further downstream.

Notice that the PPF bends backwards, that we can have a throughput of 9,000 vehicles per hour at either 28 miles per hour, or 50, and that we can even have a throughput of 10,500 vehicles per hour at a speed of fifty miles per hour. This PPF bends backwards because throughput falls when a queue forms at the bottleneck. As a measure of whether a queue has formed we can look at the number of vehicles in the rightmost lane, which is represented in Figure A.2 by the color of each dot. The dots are lightest when there are a large number of vehicles in the rightmost lane, and those points invariably are those with low speed and throughput.

The PPF of the bottleneck model, shown in Figure A.1, is very similar to that estimated from the bottleneck in San Diego, shown in Figure A.2. The primary difference is that the PPF of the bottleneck model does not curve backward, rather throughput falls by a constant amount once a queue forms.

However, the time series of throughput supports the claim that throughput falls by a constant amount, as shown in Panel A of figure 3 in Cassidy and Rudjanakanoknad (2005). In that plot, they show the detrended cumulative count of vehicles passing four locations on the road. The queue can be seen by the vertical distance between the lines. Once the queue grows a modest amount, throughput falls and stays relatively constant.

The bottleneck model is also similar to other models commonly used by transportation engineers. It corresponds to the much beloved hydrodynamic theory of traffic flow (Lighthill and Whitham, 1955, Richards, 1956) when the left side of the density-flow curve is linear, as well as the widely used cell transmission model (Daganzo, 1994). The key difference is that the bottleneck model does not consider what happens within the queue or how long the queue grows, while the transportation engineers' models deal explicitly with those issues at the cost of greater complexity. The models all make the same predictions for travel times (and so average speed) as a function of the history of departures and so for the questions I address in this paper the simplifications the bottleneck model makes are innocuous.

Appendix B. Details for Section 4

In this appendix I derive equilibrium trip costs. First, I summarize some useful results from Hall (2018). Then I use these results to help find equilibrium when the entire road is free, and show how the derivation and answers changes when finding equilibrium when the entire road is priced. Finally, I solve for equilibrium when value pricing, and show this equilibrium is unique. I also prove that the equilibrium trip prices, travel times, and tolls are unique.

B.1. Summarizing existing results.

To solve for equilibrium I use existing results from Hall (2018) on equilibrium arrival rates, assigning agents to arrival times, and backing out equilibrium travel times and tolls given the assignment of agents to arrival time. This subsection summarizes the results and gives the intuition for them.

Lemma B.1 (Arrival rates)**.** *On a free route the equilibrium arrival rate is $\lambda_{free} \cdot s$ for all of rush hour, and on a tolled route the equilibrium arrival rate is $\lambda_{toll} \cdot s^*$ for all of rush hour.*

The intuition for this result for a free route is that on a free route, in order to induce agents to arrive early or late, agents must receive a compensating differential in the form of lower travel times when they arrive early or late. The only way to have non-zero (variable) travel time is for there to be queuing, and so there will always be at least some congestion on the free route during rush hour, except at the very start and end, a zero measure set. Once a queue forms, throughput falls and the arrival rate on the free route is $\lambda_{\text{free}} \cdot s$ for all of rush hour.

The intuition for the tolled route is that in the bottleneck model there is no benefit to reducing the arrival rate below its maximum, $\lambda_{\text{toll}} \cdot s^*$, and doing so creates unnecessary schedule delay. Allowing more than $\lambda_{\text{toll}} \cdot s^*$ vehicles to depart generates queuing, which wastes time and decreases throughput.

Lemma B.2 (Assigning agents to arrival times)**.** *On a free route, those with a high δ arrive closer to their desired arrival times. More formally, if type i is more inflexible than type j (i.e., $\delta_i > \delta_j$) then if an agent from type i with desired arrival time t^* arrives at t on a free route then no agent from type j arrives between t^* and t on a free route.*

On a priced route, those with a high β arrive closer to their desired arrival times. More formally, if $\beta_i > \beta_j$ then if an agent from type i with desired arrival time t^* arrives at t on the priced route then no agent from type j arrives between t^* and t on the priced route.

The intuition for this is that the most desirable arrival times are allocated to those who are willing to pay the most for them. For a free route the currency used is travel time. This means those who are very inflexible arrive closer to their desired arrival time because an agent's inflexibility is his willingness to pay in travel time to reduce schedule delay, that is, his willingness to pay in travel time to arrive closer to his desired arrival time. For a priced route the currency used to allocate arrival times is money. This means those with a high β arrive closer to their desired arrival time because an agent's β is his willingness to pay in money to reduce schedule delay.

Lemma B.3 (Travel times and tolls)**.**

$$\{t, \text{free}\} \in \sigma(i, t^*) \Rightarrow \begin{cases} \frac{dT}{dt}(t) = \alpha_i^{-1} \frac{dD_i}{dt}(t) & \text{if } t \neq t^*, \\ -\xi\delta_i \leq \frac{dT}{dt}(t^*) \leq \delta_i & \text{if } t = t^*. \end{cases}$$

$$\{t, \text{toll}\} \in \sigma(i, t^*) \Rightarrow \begin{cases} \frac{d\tau}{dt}(t) = \frac{dD_i}{dt}(t) & \text{if } t \neq t^*, \\ -\xi\beta_i \leq \frac{d\tau}{dt}(t^*) \leq \beta_i & \text{if } t = t^*. \end{cases}$$

This result is simply the familiar requirement that the marginal rate of substitution must be tangent to the budget line, unless at a corner solution.

To finish defining $T(t)$ we add the initial condition that the travel time at the start of rush hour is zero.

To finish defining the toll schedule I assume the toll is zero when the road is uncongested and so is zero at the start of rush hour. Allowing negative tolls is an effective way to "spend" the revenue raised by congestion pricing to improve congestion pricing's distributional impacts; by ruling out negative tolls we make it harder to generate a Pareto improvement.

B.2. **Solving for equilibrium trip costs on a free route.** I find equilibrium trip costs for every agent by first assigning agents to arrival times, then using Lemma B.3 to find travel times, and finally combining agent's travel time and schedule delay to find their trip costs.

First we need to assign agents to arrival times, and do so using the following algorithm that uses the results from Section B.1. Define $t_i^{\max}$ as the time such that the agent of type i with this desired arrival time is indifferent between arriving early or late. Any agent from type i who has desired arrival time $t^* < t_i^{\max}$ strictly prefers to arrive early or on-time, and similarly if $t^* > t_i^{\max}$ then they strictly prefer to arrive late or on-time.[1] I use the superscript "max" for two reasons: first, the agent from type i with desired arrival time $t_i^{\max}$ will have the largest trip cost of any agent of type i; second, the peak of rush hour, $t^{\max}$, occurs at one or more types $t_i^{\max}$.

Given these definitions and Lemma B.2 we can assign agents to arrival times on a free route as follows. First, assume we know $t^{\max}$ and $t_i^{\max}$ for all $i \in \mathcal{G}$. Then starting at $t^{\max}$ and working our way backward, assign to each arrival time t the most inflexible agents of those who want to arrive early or on-time at t and are not yet assigned an arrival time until we have filled the available capacity. Likewise start at $t^{\max}$ and work forward, assigning the most inflexible agents who want to arrive late or on-time at t. Break ties by allowing those with an earlier desired arrival time to arrive earlier.

Let $n(\alpha, \delta, t)$ be the distribution of agent preferences. If we only consider those who want to arrive on-time at t, then the problem of finding the least inflexible agent to arrive at t amounts to finding the $\hat{\delta}$ such that

$$\int_{\hat{\delta}}^{1} \int_{0}^{\infty} n(\alpha, \delta, t)\, d\alpha d\delta = s. \tag{B.1}$$

Because desired arrival times are uniformly distributed and independent of value of time and inflexibility the solution to this equation is the same for all $t \in [t_s, t_e]$. If we assign all agents with $\delta \geq \hat{\delta}$ to arrive on-time then we will fill the road to capacity during $[t_s, t_e]$ and avoid violating Lemma B.2. At each time $t \in [t_s, t_e]$ the set of agents who want to arrive early or late at t only contains agents with $\delta < \hat{\delta}$ and so the s most inflexible agents of those who want to arrive at t are those who want to arrive exactly on-time and have $\delta \geq \hat{\delta}$. This means that the marginal inflexibility is $\hat{\delta}$ for $t \in [t_s, t_e]$.

We can use the result that $\hat{\delta}$ and $\hat{\beta}$ are constant during $[t_s, t_e]$ to help prove that rush hour has just a single peak, which we do in the following proposition.

Proposition B.1 (Rush hour has single peak)**.** *If the support of the distribution of types is a connected set, and there is a single δ or β that is marginal over the support of the distribution of desired arrivals then rush hour has a single peak.*[2]

Proof. The intuition is that were there multiple peaks then there will be an agent who arrives early or late who would prefer to arrive in the valley between the peaks to their actual arrival time. To see this visually, draw the indifference curve of the agent who desired arrival time is in the valley, and whose inflexibility is just below $\hat{\delta}$ or $\hat{\beta}$; they will prefer arriving in the valley to early or late.

[1]This relies on $T(t)$ having a single local maximum, which I prove shortly in Proposition B.1.

[2]This also holds for a finite number of types, rather than a continuum, but requires an additional assumption. Simply let $\tilde{\delta} = \hat{\delta}$, and as long as the mass of agents with $\delta \geq \hat{\delta}$ does not exactly equal s, then there is an agent with inflexibility $\hat{\delta}$ who does not get to arrive on-time. Everything is the same except that some inequalities become equalities.

Assume by way of contradiction that there is more than one peak on a free route. In an abuse of notation, label two consecutive peaks as $t_1^{\max}$ and $t_2^{\max}$, with $t_1^{\max} < t_2^{\max}$.

A peak must occur within the support of the distribution of desired arrival times, since for travel times to be increasing before a peak requires there to be agents arriving early before the peak, and for travel times to be decreasing after a peak requires there to be agents arriving late after the peak. Thus there must be agents who wish to arrive at or before the peak, but who actually arrive late, and agents who wish to arrive at or after the peak, but who actually arrive early.

Since the peaks are in the support and the support is a connected set, then the local mimima between peaks must also be in the support of the distribution of desired arrival times. Let $t^{\min}$ be the local minimum that occurs between $t_1^{\max}$ and $t_2^{\max}$.

If the distribution of types is continuous, consider an agent with desired arrival time $t^{\min}$ and $\tilde{\delta} \in \left(\hat{\delta} - \epsilon, \hat{\delta}\right)$ for an arbitrarily small $\epsilon > 0$. Since no agent with a $\delta < \hat{\delta}$ arrives in the support of the distribution of desired arrival times, this agent will either arrive before t_s or after t_e. First let's consider whether the agent can arrive at a $t' < t_s$, and note that this means

$$\begin{aligned} & p\left(t', \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right) \leq p\left(t_1^{\max}, \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right) \\ & \Rightarrow T\left(t'\right) + \tilde{\delta}\left(t^{\min} - t'\right) \leq T\left(t_1^{\max}\right) + \tilde{\delta}\left(t^{\min} - t_1^{\max}\right) \\ & \Rightarrow T\left(t'\right) + \tilde{\delta}\left(t^{\max} - t'\right) \leq T\left(t_1^{\max}\right). \end{aligned} \tag{B.2}$$

Since an agent with inflexibility $\hat{\delta}$ arrives on-time,

$$\begin{aligned} & p\left(t_1^{\max}, \text{free}; \alpha, \hat{\delta}, t_1^{\max}\right) \leq p\left(t', \text{free}; \alpha, \hat{\delta}, t_1^{\max}\right) \\ & \Rightarrow T\left(t_1^{\max}\right) \leq T\left(t'\right) + \hat{\delta}\left(t_1^{\max} - t'\right). \end{aligned} \tag{B.3}$$

And so by (B.2) and (B.3)

$$T\left(t'\right) + \tilde{\delta}\left(t^{\max} - t'\right) \leq T\left(t_1^{\max}\right) \leq T\left(t'\right) + \hat{\delta}\left(t_1^{\max} - t'\right)$$

for $\tilde{\delta}$ arbitrarily close to $\hat{\delta}$, and so

$$T\left(t_1^{\max}\right) = T\left(t'\right) + \hat{\delta}\left(t_1^{\max} - t'\right). \tag{B.4}$$

This implies

$$\begin{aligned} \frac{p\left(t', \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right)}{\tilde{\alpha}} &= T\left(t'\right) + \tilde{\delta}\left(t^{\min} - t'\right) \\ &= T\left(t_1^{\max}\right) + \left(\tilde{\delta} - \hat{\delta}\right)\left(t_1^{\max} - t'\right) + \tilde{\delta}\left(t^{\min} - t_1^{\max}\right) \\ &> T\left(t_1^{\max}\right) \end{aligned} \tag{B.5}$$

since $\tilde{\delta}$ can be arbitrarily close to $\hat{\delta}$, while $t^{\min}$ is not arbitrarily close to $t_1^{\max}$.

Since the marginal type during $[t_s, t_e]$ is $\hat{\delta}$,

$$T\left(t^{\min}\right) = T\left(t_1^{\max}\right) - \xi\hat{\delta}\left(t^{\min} - t_1^{\max}\right). \tag{B.6}$$

However, combining (B.4) with (B.6) yields

$$\frac{p\left(t^{\min}, \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right)}{\tilde{\alpha}} = T\left(t^{\min}\right) = T\left(t_1^{\max}\right) - \xi\hat{\delta}\left(t^{\min} - t_1^{\max}\right) < T\left(t_1^{\max}\right),$$

which by (B.5) implies

$$\frac{p\left(t^{\min}, \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right)}{\tilde{\alpha}} < \frac{p\left(t', \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right)}{\tilde{\alpha}}$$
$$\Rightarrow p\left(t^{\min}, \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right) < p\left(t', \text{free}; \tilde{\alpha}, \tilde{\delta}, t^{\min}\right),$$

and so an agent with inflexibility $\tilde{\delta}$ and desired arrival time $t^{\min}$ will choose to arrive at $t^{\min}$ instead of t'; and so this agent cannot arrive early.

We can likewise show that for a $t' > t_e$, the agent will prefer an agent with inflexibility $\tilde{\delta}$ and desired arrival time $t^{\min}$ will choose to arrive at $t^{\min}$ instead of t'; and so this agent will not arrive late.

As a result, in equilibrium there cannot be multiple peaks.

The proof for a priced route is follows the same steps but using a $\tilde{\beta}$ arbitrarily close to $\hat{\beta}$. □

We now want to continue assigning agents to arrival times using our algorithm. We have filled in all of the arrival times in $[t_s, t_e]$ and so now turn to those before t_s and after t_e. All of the remaining agents to be assigned are either early or late, and for arrivals before t_s we can just work backwards assigning the most inflexible remaining agents who want to be early to each arrival time. Likewise, for arrivals after t_e we can just work forwards assigning the most inflexible remaining agents who want to be late to each arrival time.

With a few definitions we can write down when an agent arrives. First, it will be simpler to write $t_i^{\max}$ as $t^{\max}(\alpha_i, \delta_i)$. The mass of agents who arrive prior to the peak of rush hour is the sum of the mass who arrive early and the mass who arrives on time prior to the peak:

$$m_e = \int_0^{\hat{\delta}} \int_0^{\infty} \int_{t_s}^{t^{\max}(\alpha,\delta)} n\left(\alpha, \delta, t^*\right)\, dt^* d\alpha d\delta + \int_{\hat{\delta}}^{1} \int_0^{\infty} \int_{t_s}^{t^{\max}} n\left(\alpha, \delta, t^*\right)\, dt^* d\alpha d\delta.$$

The mass of agents arriving after the peak $m_l = 1 - m_e$, where the subscripts e and l stand for early and late. The cumulative distribution function of inflexibility for those arriving before the peak is

$$N_{\delta,e}(\delta) = m_e^{-1} \begin{cases} \int_0^{\delta} \int_0^{\infty} \int_{t_s}^{t^{\max}(\alpha,\delta')} n\left(\alpha, \delta', t^*\right)\, dt^* d\alpha d\delta' & \delta \leq \hat{\delta}, \\ N_{\delta,e}\left(\hat{\delta}\right) + \int_{\hat{\delta}}^{\delta} \int_0^{\infty} \int_{t_s}^{t^{\max}} n\left(\alpha, \delta', t^*\right)\, dt^* d\alpha d\delta' & \delta > \hat{\delta}. \end{cases}$$

By Lemma B.2 an agent with inflexibility $\delta < \hat{\delta}$ arrives before everyone who is more inflexible and after those who are less inflexible. This implies an agent whom arrives before the peak and has inflexibility δ and desired arrival time t^* arrives at

$$A_e(\delta, t^*) = \begin{cases} t^{\max} - [1 - N_{\delta,e}(\delta)]\, m_e/s & \delta < \hat{\delta}, \\ t^* & \delta \geq \hat{\delta}. \end{cases} \tag{B.7}$$

The top line (B.7) is the peak of rush hour minus the amount of time it will take everyone who is more inflexible to arrive. This implies the start of rush hour is at

$$t_{01} = t^{\max} - m_e/s. \tag{B.8}$$

We can use Lemma B.3 to derive travel times, but first we need to know the marginal type arriving at each time. The marginal type is the type whose preferences determine the slope of $T(t)$. We can then integrate over the slope at each arrival time before t to find the travel time at t.

We can find the marginal type at each arrival time by finding the least inflexible agent to arrive at each arrival time. For $t \in [t_s, t_e]$ the marginal type has inflexibility $\hat{\delta}$. For $t \in [t_{01}, t_s)$ we can find the marginal type by inverting $A_e(\delta, t^*)$ for $\delta < \hat{\delta}$. Doing so and using (B.8) to simplify, we find the marginal type is at t is

$$B_e(t) = \begin{cases} N_{\delta,e}^{-1}\left(\frac{t - t_{01}}{t^{\max} - t_{01}}\right) & t_{01} \leq t < t_s, \\ \hat{\delta} & t_s \leq t \leq t^{\max}. \end{cases}$$

We can then impose the initial condition that travel time at the start of rush hour is zero and integrate over the marginal type at each arrival time to find the travel time for $t \leq t^{\max}$:

$$\begin{aligned} T(t) &= \int_{t_{01}}^{t} B_e(t')\, dt' \\ &= \begin{cases} 0 & t < t_{01}, \\ \int_{t_{01}}^{t} N_{\delta,e}^{-1}\left(\frac{t' - t_{01}}{t^{\max} - t_{01}}\right) dt & t_{01} \leq t \leq t_s, \\ T(t_s) + (t - t_s)\,\hat{\delta} & t_s < t \leq t^{\max}. \end{cases} \end{aligned} \tag{B.9}$$

We can rewrite the middle line in a way that will make it easier to interpret later. Let's do a change of variables with $\delta' = N_{\delta,e}^{-1}([t' - t_{01}] / [t^{\max} - t_{01}])$, so that $(t' - t_{01}) / (t^{\max} - t_{01}) = N_{\delta,e}(\delta)$ and $dt = (t^{\max} - t_{01})\, n_{\delta,e}(\delta)\, d\delta$, where $n_{\delta,e}(\delta)$ is the probability density function of δ. Doing so gives us

$$T(t) = (t^{\max} - t_{01}) \int_0^{N_{\delta,e}^{-1}\left(\frac{t - t_{01}}{t^{\max} - t_{01}}\right)} \delta' n_{\delta,e}(\delta')\, d\delta' \quad \text{if } t_{01} \leq t \leq t_s. \tag{B.10}$$

The amount of schedule delay an agent has is the difference between his desired arrival time and when he actually arrives,

$$C_e(\delta, t^*) = t^* - A_e(\delta, t^*).$$

Combining the travel time costs and schedule delay costs gives us an agent's trip cost:

$$\bar{p}_e(\alpha, \delta, t^*) = \alpha T \circ A_e(\delta, t^*) + \alpha\delta C_e(\delta, t^*)$$

for $\delta < \hat{\delta}$

$$= \alpha\left(t^{\max} - t_{01}\right)\int_0^{\delta} \delta' n_{\delta,e}\left(\delta'\right)\, d\delta' + \alpha\delta\left\{t^* - \left[t^{\max} - \frac{m_e}{s}\left(1 - N_{\delta,e}(\delta)\right)\right]\right\}$$

$$= \alpha\frac{m_e}{s}\left[\int_0^{\delta} \delta' n_{\delta,e}\left(\delta'\right)\, d\delta' + \delta\left(1 - N_{\delta,e}(\delta)\right)\right] + \alpha\delta\left(t^* - t^{\max}\right)$$

$$= \alpha\frac{m_e}{s}\left[\int_0^{1} \min\left\{\delta', \delta\right\} n_{\delta,e}\left(\delta'\right)\, d\delta'\right] + \alpha\delta\left(t^* - t^{\max}\right) \tag{B.11}$$

for $\delta \geq \hat{\delta}$

$$= \alpha\left[\left(t^{\max} - t_{01}\right)\int_0^{\hat{\delta}} \delta' n_{\delta,e}\left(\delta'\right)\, d\delta' + \left(t^* - t_s\right)\hat{\delta}\right]$$

substituting $t_s = t^{\max} - \left[1 - N_{\delta,e}\left(\hat{\delta}\right)\right] m_e/s$ yields

$$= \alpha\left[\frac{m_e}{s}\int_0^{\hat{\delta}} \delta' n_{\delta,e}\left(\delta'\right)\, d\delta' + \left(t^* - t^{\max}\right)\hat{\delta} + \frac{m_e}{s}\left[1 - N_{\delta,e}\left(\hat{\delta}\right)\right]\hat{\delta}\right]$$

$$= \alpha\frac{m_e}{s}\left[\int_0^{1} \min\left\{\delta', \hat{\delta}\right\} n_{\delta,e}\left(\delta'\right)\, d\delta'\right] + \alpha\hat{\delta}\left(t^* - t^{\max}\right). \tag{B.12}$$

We can summarize (B.11) and (B.12) as

$$\bar{p}_e(\alpha, \delta, t^*) = \alpha\frac{m_e}{s}\left[\int_0^{1} \min\left\{\delta', \delta, \hat{\delta}\right\} n_{\delta,e}\left(\delta'\right)\, d\delta'\right] + \alpha\min\left\{\delta, \hat{\delta}\right\}\left(t^* - t^{\max}\right).$$

Repeating all of these steps for late arrivals give us

$$T(t) = \begin{cases} T(t_e) + (t_e - t)\,\xi\hat{\delta} & t_e < t \leq t^{\max} \\ \left(t_{10} - t^{\max}\right)\int_0^{N_{\delta,l}^{-1}\left(\frac{t_{10} - t}{t_{10} - t^{\max}}\right)} \xi\delta' n_{\delta,l}\left(\delta'\right)\, d\delta' & t^{\max} \leq t \leq t_{10} \\ 0 & t_{10} < t \end{cases} \tag{B.13}$$

where we are using the initial condition that travel times are zero at the end of rush hour, and

$$\bar{p}_l(\alpha, \delta, t^*) = \alpha\frac{m_l}{s}\xi\left[\int_0^{1} \min\left\{\delta', \delta, \hat{\delta}\right\} n_{\delta,l}\left(\delta'\right)\, d\delta'\right] - \alpha\xi\min\left\{\delta, \hat{\delta}\right\}\left(t^* - t^{\max}\right).$$

Everything we have done so far in this section has taken $t^{\max}(\alpha,\delta)$ and $t^{\max}$ as given. We can find $t^{\max}(\alpha,\delta)$ by using its definition: the desired arrival time for each type $\{\alpha,\delta\}$ that is indifferent between arriving early or late. Recall that $t^{\max}(\alpha,\delta)$ is only defined for $\delta \leq \hat{\delta}$ since those with $\delta > \hat{\delta}$ will arrive on time.[3] This gives us the following functional equation:

$$\bar{p}_e\left(\alpha,\delta,t^{\max}(\alpha,\delta)\right) = \bar{p}_l\left(\alpha,\delta,t^{\max}(\alpha,\delta)\right)$$

$$\Rightarrow t^{\max}(\alpha,\delta) = t^{\max} + (\delta+\xi\delta)^{-1}\int_0^1 \min\left\{\delta',\delta\right\}\left[\xi\frac{m_l}{s}n_{\delta,l}\left(\delta'\right) - \frac{m_e}{s}n_{\delta,e}\left(\delta'\right)\right] d\delta' \tag{B.14}$$

First, note (B.14) is not defined for $\delta = 0$, since these agents are completely flexible and pay no schedule delay costs. All of these agents are indifferent between arriving at any time travel times are zero, and so they are all indifferent between arriving early or late. Second, notice this equation doesn't depend on α, and so $t^{\max}(\alpha_1,\delta) = t^{\max}(\alpha_2,\delta)\,\forall\alpha_1,\alpha_2$, and so I can drop the dependence on α; this allows us to simplify (B.14) to

$$t^{\max}(\delta) = t^{\max} + (1+\xi)^{-1}\int_0^1 \min\left\{\frac{\delta'}{\delta},1\right\}\left[\xi\frac{m_l}{s}n_{\delta,l}\left(\delta'\right) - \frac{m_e}{s}n_{\delta,e}\left(\delta'\right)\right] d\delta' \tag{B.15}$$

Because the distribution of desired arrival times is independent of α and δ, and is uniform on $[t_s,t_e]$,

$$n(\alpha,\delta,t^*) = \begin{cases} (t_e-t_s)^{-1}\, n_{\alpha,\delta}(\alpha,\delta) & \text{if } t^* \in [t_s,t_e] \\ 0 & \text{otherwise} \end{cases}$$

and so, defining $v^{\max}(\delta) = \min\left\{\max\left\{t^{\max}(\delta),t_s\right\},t_e\right\}$ to avoid notational clutter,

$$\begin{aligned} n_{\delta,e}(\delta) &= m_e^{-1}\int_0^\infty \int_{t_s}^{t^{\max}(\delta)} (t_e-t_s)^{-1} n_{\alpha,\delta}(\alpha,\delta)\, dt^* d\alpha \\ &= m_e^{-1}\int_0^\infty \frac{v^{\max}(\delta)-t_s}{t_e-t_s} n_{\alpha,\delta}(\alpha,\delta)\, d\alpha \\ &= m_e^{-1}\frac{v^{\max}(\delta)-t_s}{t_e-t_s} n_\delta(\delta). \end{aligned}$$

Similarly,

$$n_{\delta,l}(\delta) = m_l^{-1}\frac{t_e - v^{\max}(\delta)}{t_e-t_s} n_\delta(\delta).$$

Substituting these into (B.15) yields

$$\begin{aligned} t^{\max}(\delta) &= t^{\max} + [s(1+\xi)]^{-1}\int_0^1 \min\left\{\frac{\delta'}{\delta},1\right\}\left[\xi\frac{t_e - v^{\max}(\delta')}{t_e-t_s}n_\delta\left(\delta'\right) - \frac{v^{\max}(\delta')-t_s}{t_e-t_s}n_\delta\left(\delta'\right)\right] d\delta' \\ &= t^{\max} + [s(1+\xi)]^{-1}\int_0^1 \min\left\{\frac{\delta'}{\delta},1\right\}\left[\frac{t_s+\xi t_e-(1+\xi)\,v^{\max}(\delta')}{t_e-t_s}n_\delta\left(\delta'\right)\right] d\delta' \end{aligned}$$

[3] The types with inflexibility $\hat{\delta}$ will arrive on-time and be indifferent between arriving a little earlier or later, depending on whether they are arriving before or after the peak. The agent with inflexibility $\hat{\delta}$ who arrives exactly at the peak will be indifferent between arriving early or late.

(B.16)

$$= t^{\max} - \left[s\left(t_e - t_s\right)\right]^{-1} \int_0^1 \min\left\{\frac{\delta'}{\delta}, 1\right\} \left(v^{\max}\left(\delta'\right) - \frac{t_s + \xi t_e}{1+\xi}\right) n_\delta\left(\delta'\right)\, d\delta'.$$

And so

$$\frac{dt^{\max}\left(\delta\right)}{d\delta} = \left[s\left(t_e - t_s\right)\right]^{-1} \int_0^\delta \frac{\delta'}{\delta^2} \left(v^{\max}\left(\delta'\right) - \frac{t_s + \xi t_e}{1+\xi}\right) n_\delta\left(\delta'\right)\, d\delta'.$$

Note that for all $\delta > 0$

$$\begin{aligned} t^{\max}\left(\delta'\right) &= \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow v^{\max}\left(\delta'\right) &= \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow \frac{dt^{\max}\left(\delta\right)}{d\delta} &= 0 \\ \Rightarrow t^{\max}\left(\delta\right) &= \frac{t_s + \xi t_e}{1+\xi}, \end{aligned}$$

and

$$\begin{aligned} t^{\max}\left(\delta'\right) &< \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow v^{\max}\left(\delta'\right) &< \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow \frac{dt^{\max}\left(\delta\right)}{d\delta} &< 0 \\ \Rightarrow t^{\max}\left(\delta\right) &< \frac{t_s + \xi t_e}{1+\xi}, \end{aligned}$$

and likewise

$$\begin{aligned} t^{\max}\left(\delta'\right) &> \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow v^{\max}\left(\delta'\right) &> \frac{t_s + \xi t_e}{1+\xi} \forall \delta' < \delta \\ \Rightarrow \frac{dt^{\max}\left(\delta\right)}{d\delta} &> 0 \\ \Rightarrow t^{\max}\left(\delta\right) &> \frac{t_s + \xi t_e}{1+\xi}. \end{aligned}$$

Since schedule delay costs are continuous, travel times are continuous, and so writing the definition of $t^{\max}\left(\delta\right)$ using (1) allows us to use the implicit function theorem to show $t^{\max}\left(\delta\right)$ is continuous. Thus, while $t^{\max}\left(\delta\right)$ is not defined at zero, there is an open interval $\mathcal{A} = \left(0, 2\epsilon\right)$ for $\epsilon > 0$ where if $t^{\max}\left(\epsilon\right) < \frac{t_s+\xi t_e}{1+\xi}$, then $t^{\max}\left(a\right) < \frac{t_s+\xi t_e}{1+\xi}$ for all $a \in \mathcal{A}$, or if $t^{\max}\left(\epsilon\right) > \frac{t_s+\xi t_e}{1+\xi}$, then $t^{\max}\left(a\right) > \frac{t_s+\xi t_e}{1+\xi}$ for all $a \in \mathcal{A}$. And so if $t^{\max}\left(\delta\right) < \frac{t_s+\xi t_e}{1+\xi}$, then it is always less; if it is more, it is always more; if it is equal, it is always equal.

We can find $t^{\max}$ by imposing the condition that travel times at at $t^{\max}$ are the same when calculated from the start or end of rush hour:

$$\lim_{t \to t^{\max}+} T(t) = \lim_{t \to t^{\max}-} T(t). \tag{B.17}$$

Since an agent with inflexibility $\hat{\delta}$ arrives at all arrival times near $t^{\max}$, has no schedule delay, and endures a travel time of $T(t^{\max})$, (B.17) is equivalent to

$$\alpha^{-1} \bar{p}_e\left(\alpha, \hat{\delta}, t^{\max}\right) = \alpha^{-1} \bar{p}_l\left(\alpha, \hat{\delta}, t^{\max}\right),$$

which is equivalent to (B.14) evaluated at $\hat{\delta}$. This allows us to simply (B.17), using (B.16), to

$$0 = \int_0^1 \min\left\{\frac{\delta'}{\hat{\delta}}, 1\right\} \left(v^{\max}(\delta') - \frac{t_s + \xi t_e}{1+\xi}\right) n_\delta(\delta')\ d\delta'.$$

This means that $t^{\max}(\delta)$ cannot always be less than $\frac{t_s+\xi t_e}{1+\xi}$, nor can $t^{\max}(\delta)$ always be greater than $\frac{t_s+\xi t_e}{1+\xi}$, and so the unique solution is

$$t^{\max}(\delta) = \frac{t_s + \xi t_e}{1+\xi}.$$

Plugging all of these results into our formulas for equilibrium trip cost we find

$$\begin{aligned} \bar{p}_{\text{free}}(\alpha, \delta, t^*) = {} & \alpha \frac{1}{s} \frac{\xi}{1+\xi} \left[\int_0^1 \min\left\{\delta', \delta, \hat{\delta}\right\} n_\delta(\delta')\ d\delta'\right] \\ & - (t^{\max} - t^*)\, \alpha \min\left\{\delta, \hat{\delta}\right\} \begin{cases} 1 & t^* \le t^{\max} \\ -\xi & t^* > t^{\max} \end{cases}. \end{aligned} \tag{B.18}$$

Proposition B.2 (Uniqueness)**.** *Travel times and trip prices on a free route are unique.*

Proof. Before starting on the proof, note that if there was a finite number of types, uniqueness would come from being able to write down a system of linear equations of full rank that defines equilibrium.

With a continuum of types, the proof of uniqueness is as follows. By Lemma B.2, the ordering of agents who are early is unique, as is the ordering of those who are late, up to ties among those with the same inflexibility, and we have just shown that $t^{\max}(\alpha, \delta)$ is unique, and so the set of agents who are early and the set of agents who are late are unique. Since by Lemma B.3 travel times are uniquely determined by the inflexibility of the marginal agent arriving at each time, this means travel times are unique. Thus trip prices must also be unique. □

B.2.1. *Finding equation for travel time for the sake of later empirics.* For the sake of estimating the marginal distribution of inflexibility using GMM, we can use what we have found to write travel times in terms of exogenous variables. Substituting in for m_e we found in the previous paragraph, B.8, and B.10 into B.9 gives us

$$\text{(B.19)} \qquad T(t) = \begin{cases} 0 & t < t_{01}, \\ \left(\frac{\xi}{1+\xi}\frac{1}{s}\right)\int_0^{N_\delta^{-1}\left(\frac{t-t_{01}}{t^{\max}-t_{01}}\right)} \delta' n_\delta(\delta')\ d\delta' & t_{01} \le t < t_s, \\ T(t_s) + (t - t_s)\hat{\delta} & t_s \le t \le t^{\max}, \\ T(t_s + (t_e - t)\xi) & t > t^{\max}, \end{cases}$$

where

$$t_{01} = \frac{t_s + \xi t_e}{1+\xi} - \frac{\xi}{1+\xi}\frac{1}{s},$$
$$t^{\max} = \frac{t_s + \xi t_e}{1+\xi}.$$

Also note that by integrating (B.1) for arrival times between t_s and t_e, we can define $\hat{\delta}$ as the solution to

$$\text{(B.20)} \qquad 1 - N_\delta\left(\hat{\delta}\right) = s\left(t_e - t_s\right).$$

Also for the sake of our later estimation, note that if the distribution of inflexibility is a mixture of the distribution of inflexibility of two categories of agents, one broadly flexible and the other broadly inflexible, and ϕ is the fraction of agents who are flexible and N_δ^f is the CDF of the distribution of inflexibility of the flexible category and N_δ^i is the CDF of the distribution of inflexibility of the inflexible category, then when the fraction of agents who are flexible is greater than the fraction of agents who are able to arrive on-time, $\phi > s\left(t_e - t_s\right)$, we can simplify (B.20) as

$$\begin{aligned} 1 - N_\delta\left(\hat{\delta}\right) &= s\left(t_e - t_s\right) \\ \Leftrightarrow 1 - \phi \cdot N_\delta^f\left(\hat{\delta}\right) - (1-\phi)\, N_\delta^i\left(\hat{\delta}\right) &= s\left(t_e - t_s\right) \\ \Leftrightarrow 1 - \phi \cdot N_\delta^f\left(\hat{\delta}\right) &= s\left(t_e - t_s\right). \end{aligned}$$

B.3. **Solving for equilibrium trip costs on a priced route.** To find the equilibrium trip costs when the entire road is priced we follow the same steps as when it is free. The only difference is that we replace δ with β. So in the algorithm we use agents' β to order them rather than δ, and instead of finding $\hat{\delta}$ we find $\hat{\beta}$, and instead of finding the marginal distribution of δ, we find the marginal distribution of $\beta = \alpha\delta$, etc. Doing so yields

$$\text{(B.21)} \qquad \bar{p}_{\text{toll}}\left(\alpha, \delta, t^*\right) = \frac{1}{s^*}\frac{\xi}{1+\xi}\left[\int_0^\infty \min\left\{\beta', \alpha\delta, \hat{\beta}\right\} n_\beta\left(\beta'\right)\ d\beta'\right] - \left(t^{\max} - t^*\right)\min\left\{\alpha\delta, \hat{\beta}\right\}\begin{cases} 1 & t^* \le t^{\max} \\ -\xi & t^* > t^{\max} \end{cases}.$$

We have a closed form solution for each agents trip cost on a completely free or priced route, up to the possible need to solve the integrals numerically.

B.4. **Equilibrium when value pricing .** In contrast to the case when the entire road is either free or priced, I must solve for the value pricing equilibrium numerically instead of analytically. To do so, I first assign agents to routes and then solve for the equilibrium on each route. Solving numerically requires me to use several approximations, which I choose so I can use the closed-form solutions for trip prices on a completely free or priced highway, (2) and (3), to find equilibrium trip prices on a route given the agents who are on it.

The assignment of agents to routes is made simpler by the following lemma, which allows us to divide the space of agents' preference parameters into those on the free route and those on the priced route using a continuous function.

Lemma B.4. *For a given flexibility and desired arrival time there is a value of time, $\hat{\alpha}(\delta, t^*)$ such that all agents with a higher value of time travel on the priced route and all agents with a lower value of time travel on the free route. Furthermore, $\hat{\alpha}$ is a continuous function if $T(t)$ and $\tau(t)$ are continuous.*

It is unlikely that after conditioning on route choice the distribution of desired arrival times will be uniform and independent of α and δ. This means that $\hat{\delta}$ and $\hat{\beta}$ need not be constant over $[t_s, t_e]$; however, in practice they are nearly constant and so I approximate them with a constant. The largest approximation error in $\hat{\delta}$ and $\hat{\beta}$ ranges from 0.2 to 2.9% across my main three specifications.[4] Making this approximation allows me to apply (2) and (3) to each route individually, adjusting for route capacity and the distribution of agents on the route. Given the small size of the approximation error and how much it helps in solving for the equilibrium, this approximation seems reasonable.

I adjust (B.18) and (B.21) for route capacity and the distribution of agents on the route, the trip costs for the free and priced routes as follows

(B.22)

$$\bar{p}_{\text{free}}(\alpha, \delta, t^*) = \alpha \frac{1}{(1-\lambda)s} \frac{\xi}{1+\xi} \left[\int_{t_s}^{t_e} \int_0^1 \int_0^{\hat{\alpha}(\delta', t')} \min\left\{\delta', \delta, \hat{\delta}\right\} n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' \right] - \alpha \min\left\{\delta, \hat{\delta}\right\} \left(t^{\max} - t^*\right) \begin{cases} 1 & t^* \leq t^{\max} \\ -\xi & t^* > t^{\max} \end{cases},$$

(B.23)

$$\bar{p}_{\text{toll}}(\alpha, \delta, t^*) = \frac{1}{\lambda s^*} \frac{\xi}{1+\xi} \left[\int_{t_s}^{t_e} \int_0^1 \int_{\hat{\alpha}(\delta', t')}^{\infty} \min\left\{\alpha'\delta', \alpha\delta, \hat{\beta}\right\} n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' \right]$$

[4] Specifically, given the equilibrium I have found, I find the marginal δ and β for each $t \in [t_s, t_e]$ and compare it to $\hat{\delta}$ and $\hat{\beta}$.

$$- \min\left\{\alpha\delta, \hat{\beta}\right\}\left(t^{\max} - t^*\right)\begin{cases}1 & t^* \leq t^{\max} \\ -\xi & t^* > t^{\max}\end{cases}.$$

To convert (B.22) and (B.23) to a form like (4); we just need to multiply and divide the first term by the mass of agents on that route. The mass of agents divided by the route's capacity gives the length of rush hour on that route, and dividing the integral by the mass of agents on that route turns it back into a censored mean, this time using the distribution of agents on the route. Note that the intuition of (4) also applies to trip costs when value pricing, with the adjustment that the censored mean is taken on the distribution of agents on the route in question.

Given the approximation of $\hat{\delta}$ and $\hat{\beta}$ over $[t_s, t_e]$ I can further simplify $\hat{\alpha}$ using the next lemma, which shows that $\hat{\alpha}$ is often flat in one dimension.

Lemma B.5. *All agents of a type that is not inframarginal regardless of which route they are on, travel on the same route or are indifferent between both routes, that is,*

$$\delta < \hat{\delta} \text{ and } \alpha\delta < \hat{\beta} \Rightarrow \frac{\partial\hat{\alpha}\left(\delta, t^*\right)}{\partial t^*} = 0.$$

Similarly, all agents who are inframarginal regardless of which route they are on and who have the same value of time and desired arrival time, travel on the same route or are indifferent between both routes, that is,

$$\delta > \hat{\delta} \text{ and } \alpha\delta > \hat{\beta} \Rightarrow \frac{\partial\hat{\alpha}\left(\delta, t^*\right)}{\partial \delta} = 0.$$

The intuition for the first claim is that when an agent is not inframarginal his desired arrival time does not determine his actual arrival time, but only whether he is early or late, and so his trip cost differs from the other agents of his type only by the adjustment for desired arrival time. This adjustment is the same on both routes and so cancels out when looking at the difference between trip costs on either route. Thus if one route is preferred by one agent of a type, it must be preferred by all agents of that type.

The proof of the second claim is that if an agent is inframarginal regardless of which route he chooses, then he arrives on-time regardless of the route he chooses. This means his cost on the free route is $\alpha T\left(t^*\right)$ and his cost on the priced route is $\tau\left(t^*\right)$, and he chooses whichever route has the lowest cost. This holds for any agent who is inframarginal regardless of which route he chooses, and who has the same value of time and desired arrival time, and so all of these agents make the same choice.

Based on Lemma B.5, I approximate $\hat{\alpha}\left(\delta, t^*\right)$ as

$$\hat{\alpha}\left(\delta, t^*\right) = \begin{cases}\hat{\alpha}_M\left(\delta\right) & \delta < \hat{\delta} \\ \hat{\alpha}_I\left(t^*\right) & \delta \geq \hat{\delta}\end{cases}, \tag{B.24}$$

where $\hat{\alpha}_M(\delta)$ and $\hat{\alpha}_I(t^*)$ are solved for using Chebyshev collocation. This approximation performs significantly better than the two dimensional Chebyshev approximation of $\hat{\alpha}(\delta, t^*)$: in my main specifications the approximation error[5] is less than a tenth of a cent using (B.24) with tenth degree Chebyshev polynomials, for twenty nodes total, while the approximation error is nearly a dollar using the tensor product of two tenth degree Chebyshev polynomials, for one hundred nodes in total.[6]

To solve for the equilibrium I find the type that is the marginal agent during $[t_s, t_e]$ on each route ($\hat{\delta}$ on the free route and $\hat{\beta}$ on the priced route), as well as the function $\hat{\alpha}(\delta, t^*)$ that separates the space of agent preferences by which route they are on, by solving the following set of equations numerically:

$$\int_{t_s}^{t_e}\int_{\hat{\delta}}^{1}\int_{0}^{\hat{\alpha}(\delta,t^*)} n(\alpha,\delta,t)\, d\alpha d\delta dt = (1-\lambda)\, s\,(t_e - t_s),$$

$$\int_{t_s}^{t_e}\int_{0}^{1}\int_{\max\left\{\hat{\alpha}(\delta,t^*),\delta^{-1}\hat{\beta}\right\}}^{\infty} n(\alpha,\delta,t)\, d\alpha d\delta dt = \lambda s^*(t_e - t_s),$$

$$\bar{p}_{\text{free}}\left(\hat{\alpha}(\delta,t^*),\delta,t^*\right) = \bar{p}_{\text{toll}}\left(\hat{\alpha}(\delta,t^*),\delta,t^*\right) \quad \text{for all } \{\delta, t^*\} \in \mathcal{C},$$

where $\mathcal{C}$ is the set of Chebyshev collocation nodes. In Appendix B.5 I show there is a unique solution to this set of equations.

B.5. **Proving uniqueness when value pricing.** What follows is an alternative algorithm for finding $\hat{\alpha}(\delta, t^*)$, and which allows me to show that $\hat{\alpha}(\delta, t^*)$ is unique. The key idea for the alternative algorithm is that when we know $\hat{\alpha}(\delta, t^*)$ up to a given δ, and we know the total mass of agents on each route, then we know everything we need to solve for $\hat{\alpha}(\delta, t^*)$. Thus from a starting guess for $\hat{\alpha}(\delta, t^*)$ at the smallest δ, we can solve $\hat{\alpha}(\delta, t^*)$ for all δ and t^*, and so just need to find the starting guess that assigns the correct mass of agents to each route.

Let m_f be the mass of agents on the free route, and m_t be the mass of agents on the toll route, so

$$m_f = \int_{t_s}^{t_e}\int_{0}^{1}\int_{0}^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right) d\alpha' d\delta' dt'$$

$$m_t = 1 - m_f = \int_{t_s}^{t_e}\int_{0}^{1}\int_{\hat{\alpha}(\delta',t')}^{\infty} n\left(\alpha',\delta',t'\right) d\alpha' d\delta' dt'.$$

The function $\hat{\alpha}(\delta, t^*)$ is defined as the solution to

$$\bar{p}_{\text{free}}\left(\hat{\alpha}(\delta,t^*),\delta,t^*\right) = \bar{p}_{\text{toll}}\left(\hat{\alpha}(\delta,t^*),\delta,t^*\right) \tag{B.25}$$

[5] Measured as the largest welfare loss from traveling on the route assigned by $\hat{\alpha}(\delta, t^*)$ instead of the route that actually minimizes trip cost.

[6] Using (B.24) has worse asymptotic properties than the two dimensional Chebyshev approximation of $\hat{\alpha}(\delta, t^*)$, as it will not converge to the true $\hat{\alpha}(\delta, t^*)$ over the small area where $\left(\delta - \hat{\delta}\right)\left(\hat{\alpha}(\delta, t^*)\,\delta - \hat{\beta}\right) < 0$, regardless of the degree of the Chebyshev polynomial.

using (B.22) and (B.23) for trip prices.

Let the smallest δ with a positive mass of agents be $\underline{\delta}$. I assume $\underline{\delta}$ is positive, but arbitrarily close to zero . This implies

$$\begin{aligned}
\bar{p}_{\text{free}}\left(\alpha,\underline{\delta},t^*\right) &= \alpha\frac{1}{(1-\lambda)s}\frac{\xi}{1+\xi}\delta\left[\int_{t_s}^{t_e}\int_0^1\int_0^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right] \\
&\qquad - \alpha\delta\left(t^{\max}-t^*\right)\begin{cases}1 & t^*\le t^{\max}\\ -\xi & t^* > t^{\max}\end{cases} \\
&= \frac{1}{\lambda s^*}\frac{\xi}{1+\xi}\alpha\delta\left[\int_{t_s}^{t_e}\int_0^1\int_{\hat{\alpha}(\delta',t')}^{\infty} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right] \\
&\qquad - \alpha\delta\left(t^{\max}-t^*\right)\begin{cases}1 & t^*\le t^{\max}\\ -\xi & t^* > t^{\max}\end{cases} = \bar{p}_{\text{toll}}\left(\alpha,\delta,t^*\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{1}{(1-\lambda)s}\left[\int_{t_s}^{t_e}\int_0^1\int_0^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right] \\
&\qquad\qquad = \frac{1}{\lambda s^*}\left[\int_{t_s}^{t_e}\int_0^1\int_{\hat{\alpha}(\delta',t')}^{\infty} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right] \\
&\Rightarrow \frac{1}{(1-\lambda)s}m_f = \frac{1}{\lambda s^*}m_t \\
&\Rightarrow m_f = \frac{(1-\lambda)s}{(1-\lambda)s+\lambda s^*}.
\end{aligned}$$

Thus the mass of agents on each route is independent of the distribution of agent preferences. Also note that $\hat{\alpha}\left(\underline{\delta},t^*\right)$ is defined by the requirement that the mass on each route be as above.

We can use this to simplify an agents trip price on a free route as follows.

$$\begin{aligned}
\bar{p}_{\text{free}}\left(\alpha,\delta,t^*\right) &= \alpha\frac{1}{(1-\lambda)s}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^1\int_0^{\hat{\alpha}(\delta',t')} \min\left\{\delta',\delta,\hat{\delta}\right\} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right] \\
&\qquad - \alpha\min\left\{\delta,\hat{\delta}\right\}\left(t^{\max}-t^*\right)\begin{cases}1 & t^*\le t^{\max}\\ -\xi & t^* > t^{\max}\end{cases} \\
&= \alpha\frac{1}{(1-\lambda)s}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^{\min\left\{\delta,\hat{\delta}\right\}}\int_0^{\hat{\alpha}(\delta',t')} \delta' n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right. \\
&\qquad \left. + \min\left\{\delta,\hat{\delta}\right\}\left(\int_{t_s}^{t_e}\int_{\min\left\{\delta,\hat{\delta}\right\}}^1\int_0^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \alpha \min\left\{\delta, \hat{\delta}\right\}\left(t^{\max} - t^*\right)\begin{cases}1 & t^* \leq t^{\max} \\ -\xi & t^* > t^{\max}\end{cases} \\
&= \alpha \frac{1}{(1-\lambda) s} \frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e} \int_0^{\min\left\{\delta, \hat{\delta}\right\}} \int_0^{\hat{\alpha}(\delta', t')} \delta' n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt'\right. \\
&\quad \left. + \min\left\{\delta, \hat{\delta}\right\}\left(m_f - \int_{t_s}^{t_e} \int_0^{\min\left\{\delta, \hat{\delta}\right\}} \int_0^{\hat{\alpha}(\delta', t')} n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt'\right)\right] \\
&- \alpha \min\left\{\delta, \hat{\delta}\right\}\left(t^{\max} - t^*\right)\begin{cases}1 & t^* \leq t^{\max} \\ -\xi & t^* > t^{\max}\end{cases}.
\end{aligned} \tag{B.26}
$$

The value of this simplification is that it only depends on $\hat{\alpha}(\delta', t')$ for $\delta' < \delta$.

To repeat the process for a priced route we need a few results first.

Lemma B.6. *If there are two agents, one more inflexible than the other, but with the same β and desired arrival time, and if there is a positive mass of agents on the free route who are more flexible than the more inflexible agent, then if the flexible agent uses the free route, the inflexible agent will strictly prefer to use the free route; and if the inflexible agent uses the priced route, the flexible agent will strictly prefer to use the priced route.*

Proof. Index the flexible agent with a 1 and the inflexible agent with a 2.

Assume, by way of contradiction, that an agent with preferences $\{\alpha_1, \delta_1, t^*\}$ arrives at t_f on the free route and an agent with preferences $\{\alpha_2, \delta_2, t^*\}$ arrives at t_t on the tolled route, and that $\delta_1 < \delta_2$ and $\alpha_1\delta_1 = \alpha_2\delta_2$. This implies

$$p\left(t_t, \text{toll}; \alpha_1, \delta_1, t^*\right) \geq p\left(t_f, \text{free}; \alpha_1, \delta_1, t^*\right), \tag{B.27}$$

$$p\left(t_t, \text{toll}; \alpha_2, \delta_2, t^*\right) \leq p\left(t_f, \text{free}; \alpha_2, \delta_2, t^*\right). \tag{B.28}$$

Note that t_f is the start or end or rush hour if and only if $T\left(t_f\right) = 0$. Since there is a positive mass of agents with inflexibility less than δ_2 on the free route, by lemma B.2 the inflexible agent will not arrive at the exact start or end of rush hour on a free route, and so there exists a t_f' such that

$$p\left(t_f', \text{free}; \alpha_2, \delta_2, t^*\right) < p\left(t_f, \text{free}; \alpha_2, \delta_2, t^*\right),$$

and so if $T\left(t_f\right) = 0$, the the inequality in (B.28) will be strict.

Substituting in the definitions for trip prices gives us

$$\text{(B.27)} \Leftrightarrow \tau\left(t_t\right) + \alpha_1\delta_1\left(t^* - t_t\right)\begin{cases}1 & t_t < t^* \\ -\xi & t_t \geq t^*\end{cases} \geq \alpha_1 T\left(t_f\right) + \alpha_1\delta_1\left(t^* - t_f\right)\begin{cases}1 & t_f < t^* \\ -\xi & t_f \geq t^*\end{cases},$$

$$\text{(B.28)} \Leftrightarrow \tau\left(t_t\right) + \alpha_2\delta_2\left(t^* - t_t\right)\begin{cases}1 & t_t < t^* \\ -\xi & t_t \geq t^*\end{cases} \leq \alpha_2 T\left(t_f\right) + \alpha_2\delta_2\left(t^* - t_f\right)\begin{cases}1 & t_f < t^* \\ -\xi & t_f \geq t^*\end{cases};$$

with the inequality in the second equation strict if $T\left(t_f\right) = 0$.

Since $\alpha_1\delta_1 = \alpha_2\delta_2$, taking the difference of these two equations yields

$$0 \geq \left(\alpha_1 - \alpha_2\right) T\left(t_f\right)$$

with the inequality strict if $T\left(t_f\right) = 0$. If $T\left(t_f\right) = 0$ then we have a contradiction, and if $T\left(t_f\right) > 0$ then $\alpha_2 \geq \alpha_1$. However, $\delta_1 < \delta_2$ and $\alpha_1\delta_1 = \alpha_2\delta_2$ implies $\alpha_2 < \alpha_1$, and so this too is a contradiction.

$\square$

The condition that there be positive mass of agents on the free route who are more flexible than the more inflexible agent will be met as long as the distribution of agents has full support over the space of agent preferences, or if there are a finite number of types and the flexible type has positive mass.

Lemma B.7. *If there is a $\breve{\delta}$ which solves $\alpha\delta = \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta}$ for a given $\alpha\delta$, then it is unique and $\hat{\alpha}\left(\delta', t'\right) > \min\left\{\alpha\delta, \hat{\beta}\right\}/\delta'$ for $\delta' > \breve{\delta}$ and $\hat{\alpha}\left(\delta', t'\right) < \min\left\{\alpha\delta, \hat{\beta}\right\}/\delta'$ for $\delta' < \breve{\delta}$.*

Proof. By way of contradiction, assume there are more than one solution to $\alpha\delta = \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta}$. Without loss of generality, let $\breve{\delta}_1 < \breve{\delta}_2$ be two such solutions. This means that for a given β (i.e., a given $\alpha\delta$) there are two levels of δ where agents are willing to travel on both routes. However, by Lemma B.6, if an agent with $\breve{\delta}_1$ is willing to travel on the free route, than any agent with the same β but a higher δ strictly prefers to to travel on the free route. This is a contradiction, and so if there is a solution to $\alpha\delta = \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta}$, it must be unique.

Furthermore, by Lemma B.6, if $\delta' > \breve{\delta}$, then any agent with the same β is on the free route, and so the β that is indifferent between the two routes at δ' must be larger than the β that is indifferent at $\breve{\delta}$, and so $\hat{\alpha}\left(\delta', t'\right)\delta' > \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta} = \alpha\delta \Rightarrow \hat{\alpha}\left(\delta', t'\right) > \min\left\{\alpha\delta, \hat{\beta}\right\}/\delta'$.

Likewise, by Lemma B.6, if $\delta' < \breve{\delta}$, then any agent with the same β is on the priced route, and so the β that is indifferent between the two routes at δ' must be smaller than the β that is indifferent at $\breve{\delta}$, and so $\hat{\alpha}\left(\delta', t'\right)\delta' < \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta} = \alpha\delta \Rightarrow \hat{\alpha}\left(\delta', t'\right) < \min\left\{\alpha\delta, \hat{\beta}\right\}/\delta'$. $\square$

Given Lemma B.7, we can now simplify the equation for trip price on a priced route. Define $\breve{\delta}$ as the solution to $\alpha\delta = \hat{\alpha}\left(\breve{\delta}\right)\breve{\delta}$ for a given $\alpha\delta$.

$$\begin{aligned}
\bar{p}_{\text{toll}}\left(\alpha, \delta, t^*\right) &= \frac{1}{\lambda s^*}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^1\int_{\hat{\alpha}(\delta',t')}^{\infty} \min\left\{\alpha'\delta', \alpha\delta, \hat{\beta}\right\} n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt'\right] \\
&\qquad - \min\left\{\alpha\delta, \hat{\beta}\right\}\left(t^{\max} - t^*\right)\begin{cases} 1 & t^* \leq t^{\max} \\ -\xi & t^* > t^{\max}\end{cases} \\
&= \frac{1}{\lambda s^*}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^{\breve{\delta}}\int_{\hat{\alpha}(\delta',t')}^{\min\left\{\alpha\delta,\hat{\beta}\right\}/\delta'} \alpha'\delta' n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt'\right. \\
&\qquad + \min\left\{\alpha\delta, \hat{\beta}\right\}\int_{t_s}^{t_e}\int_0^{\breve{\delta}}\int_{\min\left\{\alpha\delta,\hat{\beta}\right\}/\delta'}^{\infty} n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt'
\end{aligned}$$

$$+\min\left\{\alpha\delta,\hat{\beta}\right\}\int_{t_s}^{t_e}\int_{\breve{\delta}}^{1}\int_{\hat{\alpha}(\delta',t')}^{\infty} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\Bigg] \tag{B.29}$$

$$-\min\left\{\alpha\delta,\hat{\beta}\right\}\left(t^{\max}-t^*\right)\begin{cases}1 & t^*\leq t^{\max}\\ -\xi & t^*>t^{\max}\end{cases}$$

$$=\frac{1}{\lambda s^*}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_{0}^{\breve{\delta}}\int_{\hat{\alpha}(\delta',t')}^{\min\left\{\alpha\delta,\hat{\beta}\right\}/\delta'}\alpha'\delta' n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right. \tag{B.30}$$

$$\left.+\min\left\{\alpha\delta,\hat{\beta}\right\}\left(m_t-\int_{t_s}^{t_e}\int_{0}^{\breve{\delta}}\int_{\hat{\alpha}(\delta',t')}^{\min\left\{\alpha\delta,\hat{\beta}\right\}/\delta'} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)\right]$$

$$-\min\left\{\alpha\delta,\hat{\beta}\right\}\left(t^{\max}-t^*\right)\begin{cases}1 & t^*\leq t^{\max}\\ -\xi & t^*>t^{\max}\end{cases}$$

Lemma B.7 helps by keeping line (B.29) simple. Once again we can simplify an agent's trip price so that it only depends on $\hat{\alpha}\left(\delta',t'\right)$ for $\delta'<\delta$.

The values of $\hat{\delta}$ and $\hat{\beta}$ can likewise be defined in terms of $\hat{\alpha}\left(\delta',t'\right)$ for $\delta'<\delta$. Let $\breve{\delta}_1$ be the solution to $\hat{\beta}=\hat{\alpha}\left(\breve{\delta}\right)\breve{\delta}$, then $\hat{\delta}$ and $\hat{\beta}$ are the solutions to

$$m_f-\int_{t_s}^{t_e}\int_{0}^{\hat{\delta}}\int_{0}^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt' = (1-\lambda)\,s\left(t_e-t_s\right), \tag{B.31}$$

$$m_t-\int_{t_s}^{t_e}\int_{0}^{\breve{\delta}_1}\int_{\hat{\alpha}(\delta',t')}^{\hat{\beta}/\delta'} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt' = \lambda s^*\left(t_e-t_s\right). \tag{B.32}$$

Given (B.26) and (B.30), $\hat{\alpha}\left(\delta,t^*\right)$ is uniquely defined given a starting value, $\hat{\alpha}\left(\underline{\delta}\right)$. The starting value is pinned down by the requirement that the mass of agents on each route equals the mass that should be on the route. Since the mass of agents is normalized to one, we just need to check for one route, say the free route.

$$\int_{t_s}^{t_e}\int_{0}^{1}\int_{0}^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt' = \frac{(1-\lambda)\,s}{(1-\lambda)\,s+\lambda s^*}. \tag{B.33}$$

The next step is to show that there is a unique solution. Define

$$f\left(\alpha\left(\delta,t^*\right),\delta\right)=\left[\bar{p}_{\text{free}}\left(\alpha,\delta,t^*\right)-\bar{p}_{\text{toll}}\left(\alpha,\delta,t^*\right)\right]\frac{\lambda_s}{\hat{\alpha}\left(\delta,t^*\right)}= \tag{B.34}$$

$$=\frac{\lambda s^*}{(1-\lambda)\,s}\left[\int_{t_s}^{t_e}\int_{0}^{\delta}\int_{0}^{\hat{\alpha}(\delta',t')}\delta' n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right.$$

$$\left.+\delta\left(m_f-\int_{t_s}^{t_e}\int_{0}^{\delta}\int_{0}^{\hat{\alpha}(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)\right]$$

$$-\frac{1}{\hat{\alpha}\left(\delta,t^*\right)}\int_{t_s}^{t_e}\int_{0}^{\delta}\int_{\hat{\alpha}(\delta',t')}^{\hat{\alpha}(\delta,t^*)\delta/\delta'}\alpha'\delta' n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'$$

$$-\delta\left(m_t-\int_{t_s}^{t_e}\int_{0}^{\delta}\int_{\hat{\alpha}(\delta',t')}^{\hat{\alpha}(\delta,t^*)\delta/\delta'} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)$$

where we use the fact that for an agent who is indifferent between both routes (i.e., lies on $\hat{\alpha}$), $\breve{\delta} = \delta$. Note that f that $\hat{\alpha}(\delta)$ is the solution to $f(\hat{\alpha}(\delta), \delta) = 0$.

Lemma B.8. *If $\hat{\alpha}_1(\delta', t^*) \geq \hat{\alpha}_2(\delta', t^*)$ for all $\delta' < \delta$, then $\bar{p}_{free}(\alpha, \delta, t^* \,|\hat{\alpha}_1(\delta')) \leq \bar{p}_{free}(\alpha, \delta, t^* \,|\hat{\alpha}_2(\delta'))$, with the inequality strict if*

$$\int_{t_s}^{t_e} \int_0^{\delta} \int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')} n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' > 0. \tag{B.35}$$

Proof. The intuition for this result is if in determining $\hat{\alpha}$ prior to δ we have assigned more agents to the free route, then the censored mean of δ on the free route is lower, and so the price on the free route for an agent with inflexibility δ is lower.

This intuition continues when we consider $\hat{\delta}$. Since we have assigned more agents to the free route at a low δ, the δ at which all remaining agents must be inframarginal is lower, and so $\hat{\delta}$ is lower; similarly.

Let $\hat{\delta}_i$ be the solution (B.31) under $\hat{\alpha}_i$ for $i \in \{1, 2\}$. The left-hand side of (B.31) is decreasing $\hat{\delta}$ and $\hat{\alpha}$, so that if $\delta \geq \hat{\delta}_1$ and $\hat{\alpha}_1(\delta', t^*) \geq \hat{\alpha}_2(\delta', t^*)$ for all $\delta' < \delta$, then $\hat{\delta}_1 \leq \hat{\delta}_2$, with the inequality strict if (B.35) holds. Note that it is not possible for $\hat{\delta}_2 \leq \delta < \hat{\delta}_1$.

Next, note that substituting in for $t^* = t_s + (t^* - t_s)$ or $t^* = t_e + (t^* - t_e)$, using $t^{\max} = (t_s + \xi t_e) / (1 + \xi)$, and defining $\tilde{m}_f = m_f - (1 - \lambda) s (t_e - t_s)$, we can simplify the trip price as follows.

$$\begin{aligned}
\bar{p}_{\text{free}}\left(\alpha, \delta, t \,\middle|\hat{\alpha}\left(\delta'\right)\right) = \alpha \frac{1}{(1-\lambda) s} \frac{\xi}{1+\xi} \Bigg[& \int_{t_s}^{t_e} \int_0^{\min\{\delta, \hat{\delta}\}} \int_0^{\hat{\alpha}(\delta',t')} \delta' n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' \\
& + \min\left\{\delta, \hat{\delta}\right\} \left(\tilde{m}_f - \int_{t_s}^{t_e} \int_0^{\min\{\delta, \hat{\delta}\}} \int_0^{\hat{\alpha}(\delta',t')} n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' \right) \Bigg] \\
& + \alpha \min\left\{\delta, \hat{\delta}\right\} \begin{cases} t^* - t_s & t_s < t^* \leq t^{\max}, \\ \xi\left(t_e - t^*\right) & t^{\max} < t^* \leq t_e. \end{cases}
\end{aligned}$$

To complete the proof I must show

$$\bar{p}_{\text{free}}\left(\alpha, \delta, t^* \,\middle|\hat{\alpha}_1\left(\delta'\right)\right) - \bar{p}_{\text{free}}\left(\alpha, \delta, t^* \,\middle|\hat{\alpha}_2\left(\delta'\right)\right) \leq 0 \tag{B.36}$$

with the inequality strict if (B.35) holds. I do this in three cases, $\delta \leq \min\left\{\hat{\delta}_1, \hat{\delta}_2\right\}$, $\hat{\delta}_1 \leq \delta \leq \hat{\delta}_2$, and $\hat{\delta}_1 \leq \hat{\delta}_2 \leq \delta$.

(1) If $\delta \leq \min\left\{\hat{\delta}_1, \hat{\delta}_2\right\}$ then (B.36) simplifies to

$$\begin{aligned}
&\bar{p}_{\text{free}}\left(\alpha, \delta, t^* \,\middle|\hat{\alpha}_1\left(\delta'\right)\right) - \bar{p}_{\text{free}}\left(\alpha, \delta, t^* \,\middle|\hat{\alpha}_2\left(\delta'\right)\right) = \\
&\qquad \alpha \frac{1}{(1-\lambda) s} \frac{\xi}{1+\xi} \left[\int_{t_s}^{t_e} \int_0^{\delta} \int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')} \left(\delta' - \delta\right) n\left(\alpha', \delta', t'\right) \, d\alpha' d\delta' dt' \right] \leq 0.
\end{aligned}$$

The inequality is strict if (B.35) holds.

(2) If $\hat{\delta}_1 \le \delta \le \hat{\delta}_2$ then (B.36) simplifies to

$$\bar{p}_{\text{free}}\left(\alpha,\delta,t^* \left|\hat{\alpha}_1\left(\delta'\right)\right.\right) - \bar{p}_{\text{free}}\left(\alpha,\delta,t^* \left|\hat{\alpha}_2\left(\delta'\right)\right.\right) =$$

$$\alpha \frac{1}{(1-\lambda)\,s}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^{\hat{\delta}_1}\int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')}\left(\delta'-\delta\right)n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right.$$
$$+\left(\hat{\delta}_1-\delta\right)\int_{t_s}^{t_e}\int_{\hat{\delta}_1}^{\hat{\delta}_2}\int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'$$
$$+\int_{t_s}^{t_e}\int_{\hat{\delta}_1}^{\hat{\delta}_2}\int_0^{\hat{\alpha}_2(\delta',t')}\left(\hat{\delta}_1-\delta'\right)n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'$$
$$\left.+\left(\hat{\delta}_1-\delta\right)\left(\tilde{m}_f-\int_{t_s}^{t_e}\int_0^{\delta}\int_0^{\hat{\alpha}_2(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)\right]$$
$$+\alpha\left(\hat{\delta}_1-\delta\right)\begin{cases} t^*-t_s & t_s<t^*\le t^{\max} \\ \xi\left(t_e-t^*\right) & t^{\max}<t^*\le t_e \end{cases} \le 0$$

since each line is non-positive. The inequality is strict if (B.35) holds.

(3) If $\hat{\delta}_1 \le \hat{\delta}_2 \le \delta$ then (B.36) simplifies to

$$\bar{p}_{\text{free}}\left(\alpha,\delta,t^* \left|\hat{\alpha}_1\left(\delta'\right)\right.\right) - \bar{p}_{\text{free}}\left(\alpha,\delta,t^* \left|\hat{\alpha}_2\left(\delta'\right)\right.\right) =$$

$$\alpha \frac{1}{(1-\lambda)\,s}\frac{\xi}{1+\xi}\left[\int_{t_s}^{t_e}\int_0^{\hat{\delta}_1}\int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')}\left(\delta'-\hat{\delta}_2\right)n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right.$$
$$+\left(\hat{\delta}_1-\hat{\delta}_2\right)\int_{t_s}^{t_e}\int_{\hat{\delta}_1}^{\hat{\delta}_2}\int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'$$
$$+\int_{t_s}^{t_e}\int_{\hat{\delta}_1}^{\hat{\delta}_2}\int_0^{\hat{\alpha}_2(\delta',t')}\left(\hat{\delta}_1-\delta'\right)n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'$$
$$\left.+\left(\hat{\delta}_1-\hat{\delta}_2\right)\left(\tilde{m}_f-\int_{t_s}^{t_e}\int_0^{\delta}\int_0^{\hat{\alpha}_2(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt'\right)\right]$$
$$+\alpha\left(\hat{\delta}_1-\hat{\delta}_2\right)\begin{cases} t^*-t_s & t_s<t^*\le t^{\max} \\ \xi\left(t_e-t^*\right) & t^{\max}<t^*\le t_e \end{cases} \le 0$$

since each line is non-positive. The inequality is strict if (B.35) holds.

□

Lemma B.9. *If $\hat{\alpha}_1\left(\delta',t^*\right) \ge \hat{\alpha}_2\left(\delta',t^*\right)$ for all $\delta'<\delta$, then $\bar{p}_{toll}\left(\alpha,\delta,t^*\left|\hat{\alpha}_1\left(\delta'\right)\right.\right) \ge \bar{p}_{toll}\left(\alpha,\delta,t^*\left|\hat{\alpha}_2\left(\delta'\right)\right.\right)$, with the inequality strict if*

$$\int_{t_s}^{t_e}\int_0^{\delta}\int_{\hat{\alpha}_2(\delta',t')}^{\hat{\alpha}_1(\delta',t')} n\left(\alpha',\delta',t'\right)\,d\alpha' d\delta' dt' > 0. \tag{B.37}$$

Proof. The proof is essentially the same as for Lemma B.8, but now for a priced route.

The intuition is likewise similar. If in determining $\hat{\alpha}$ prior to δ we have assigned more agents to the free route, then we have assigned fewer agents to the priced route, and so the censored mean of β on the priced route is higher, and so the price on the priced route for an agent with $\beta = \alpha\delta$ is higher.

This intuition continues when we consider $\hat{\beta}$. Since we have assigned fewer agents to the priced route at low values of β, the β at which all remaining agents must be inframarginal is lower, and so $\hat{\beta}$ is lower. □

Lemma B.10. *If $\hat{\alpha}_1(\delta') > \hat{\alpha}_2(\delta')$ for all $\delta' < \delta$, then $\hat{\alpha}_1(\delta) \geq \hat{\alpha}_2(\delta)$, with the inequality strict if $\int_{t_s}^{t_e}\int_0^{\delta}\int_{\hat{\alpha}_2(\delta')}^{\hat{\alpha}_1(\delta')} n(\alpha', \delta', t')\, d\alpha' d\delta' dt' > 0$.*

Proof. By Lemma B.8 the price on the free route is lower under $\hat{\alpha}_1$ and by Lemma B.9 the price on the priced route is higher under $\hat{\alpha}_1$, and so

$$f_1(a, \delta) \leq f_2(a, \delta)$$

with the inequality strict if $\int_{t_s}^{t_e}\int_0^{\delta}\int_{\hat{\alpha}_2(\delta')}^{\hat{\alpha}_1(\delta')} n(\alpha', \delta', t')\, d\alpha' d\delta' dt' > 0$.

Since

$$\frac{\partial f(a, \delta)}{\partial a} = a^{-2}\int_{t_s}^{t_e}\int_0^{\delta}\int_{\hat{\alpha}(\delta')}^{\hat{\alpha}(\delta)\delta/\delta'} \alpha'\delta' n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt' > 0$$

the a_1 which solves $f_1(a_1, \delta) = 0$ must be weakly larger than a_2 which solves $f_2(a_2, \delta) = 0$, and so $\hat{\alpha}_1(\delta) \geq \hat{\alpha}_2(\delta)$, with both inequalities strict if

$$\int_{t_s}^{t_e}\int_0^{\delta}\int_{\hat{\alpha}_2(\delta')}^{\hat{\alpha}_1(\delta')} n\left(\alpha', \delta', t'\right)\, d\alpha' d\delta' dt' > 0.$$

□

Lemma B.10 means that the left side of (B.33) is strictly increasing in the initial guess, as the right side is a constant there cannot be multiple initial guesses which solve (B.33).

Appendix C. Details for Section 5

C.1. **Empirical results using typical-trip measure of flexibility.** This section reproduces the results of Section 5 using the typical-trip definition of flexibility.

Using the typical-trip definition of flexibility I find that the distribution of the value of time is almost identical for the flexible and inflexible agents. Figure C.1 plots the distribution of income by flexible/inflexible, and Table C.1 reports the estimates for the median and interquartile range. The point estimates for the median value of time are nearly identical, and the point estimates for the interquartile range are similar. I test whether flexible agents are richer than inflexible agents by computing the Goodman and Kruskal's rank correlation between flexibility and income (reported in the last row), and find the correlation to be small, negative, and statistically insignificant.

This contrasts with using the specific-trip definition of flexibility results reported in Table 2, where the flexible were richer than the inflexible. The difference in the correlation between value of time and inflexibility for the two definitions of inflexibility is likely due to a subtle difference in their definitions. The typical-trip question asks whether a respondent

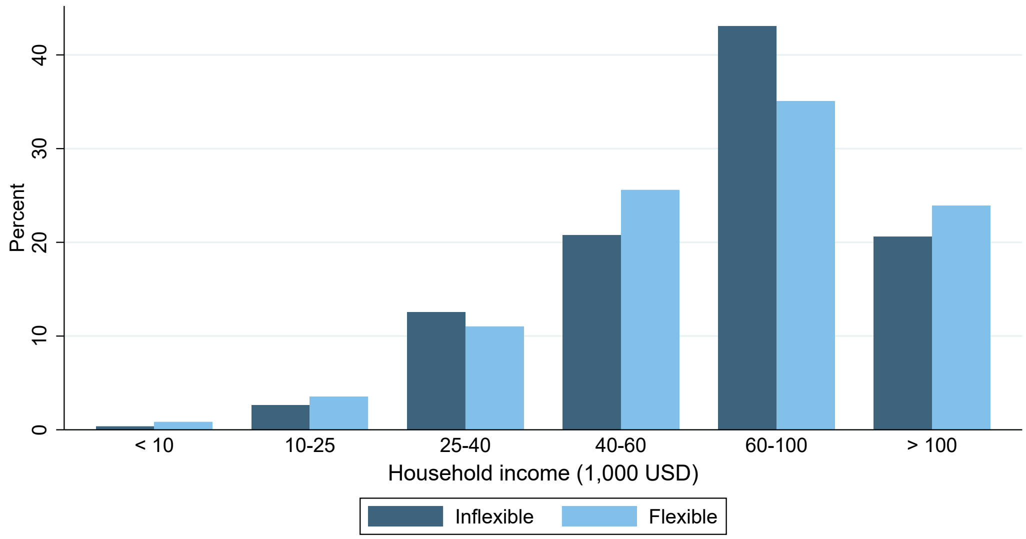


FIGURE C.1. Distribution of household income by flexible/inflexible

Notes: Data from SR-91 Impact Study. Dollars are in 1998 dollars.

TABLE C.1. Distribution of value of time for morning highway users

	Flexible	Inflexible
Median	22.46	22.51
	(0.49)	(0.54)
Interquartile range	18.19	16.04
	(0.96)	(0.83)
N	707	534
Rank correlation†	-0.015	
	(0.043)	

Notes: Bootstrapped standard errors in parentheses. I convert household income to value of time using a formula from the USDOT (Belenky, 2011), adjust dollar amounts to 2012 dollars using the CPI, and then fit the categorical data to a log-normal distribution using maximum likelihood.

† Goodman and Kruskal's γ between income and flexibility.

takes an action that reveals their flexibility, while the specific-trip questions asks if they could choose their arrival time but not whether other constraints in their life prevented them from taking advantage of that flexibility. This means we can interpret the correlations from the two definitions of flexibility as saying that while higher income people have more flexible trips, they are not actually any more flexible than the poor due to other factors in their lives. These results are consistent with the intuition that better paid jobs tend to be

TABLE C.2. Remaining parameter estimates using typical-trip definition of flexibility

Model	GMM
Length of desired arrivals (hours) $(t_e - t_s)$	4.69
	(0.22)
First desired arrival time (hours) (t_s)	5.482
	(0.063)
Length of rush hour on free route (hours) $(1/s)$	7.67
	(0.40)
Maximum inflexibility of flexible agents $\left(\bar{\delta}\right)$	0.342
	(0.042)
Ratio of schedule delay costs late to early (ξ)	0.392
	(0.033)
Free flow travel time (minutes) $\left(T_f\right)$	36.63
	(0.90)

Note: Bootstrapped standard errors in parentheses. The estimate in first row comes from fitting the largest and smallest observations of the trimmed sample of the inflexible agents' desired arrival times to the expected value of their order statistics (N = 248). The last five rows report the GMM estimates (N = 250).

more flexible, but that better paid workers tend to be older and have more constraints in their personal lives, such as needing to take care of their children.

Table C.2 reproduces Table 4 using the typical-trip definition of flexibility, and finds similar estimates for most of the parameters. The biggest difference is in the maximum inflexibility of flexible agents $\left(\tilde{\delta}\right)$, but this is primarily due to the typical-trip definition having a larger share of drivers who are flexible, which means that for the same mass of all agents to have a given flexibility the maximum inflexibility of flexible agents must be larger.[7]

C.2. **Evidence that value of time and desired arrival time are independent.** In this subsection I test whether it is reasonable to assume that value of time and desired arrival time are independently distributed. Recall from Section 5.3 that we only observe the desired arrival time at work of inflexible agents.[8] As in the rest of the paper, I use household income as a proxy for value of time.

Figure C.2 redraws Figure C.11a for three categories of income; and shows that the distribution of desired arrival times does not vary significantly by income. The largest

[7]That is, if ϕ_i is the fraction of drivers who are flexible under definition i, and $\tilde{\delta}_i$ is the maximum inflexibility of flexible drivers under definition i, then the mass of agents with inflexibility $\delta < \min\left\{\tilde{\delta}_1, \tilde{\delta}_2\right\}$ under definition i is $\phi_i/\tilde{\delta}_i$. For the mass at a given δ to be the same given that $\phi_1 > \phi_2$, we need $\tilde{\delta}_1 > \tilde{\delta}_2$.

[8]A flexible agent's choice of when to arrive at work only gives us bounds on when he want to arrive, if he is arriving before the peak then we know he is arriving early at work and if he is arriving after the peak we know he is arriving late at work.

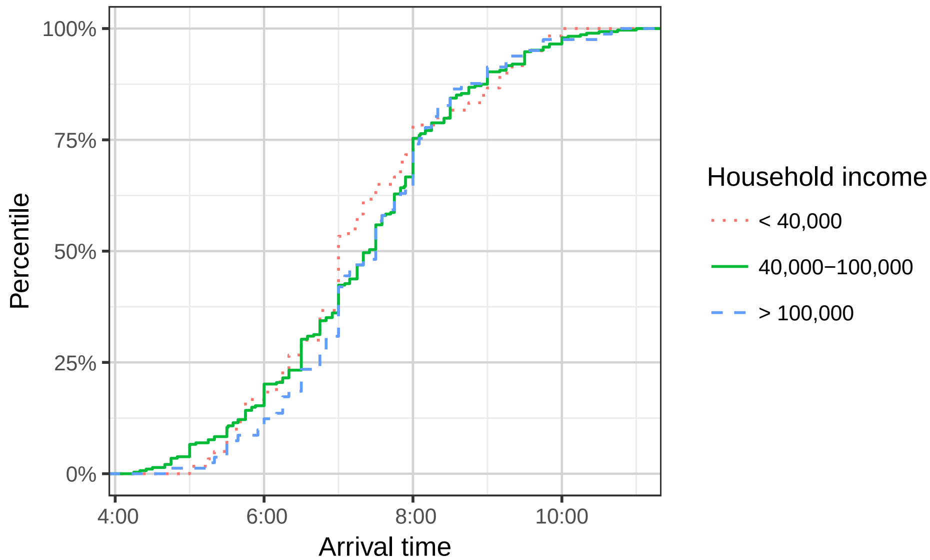


FIGURE C.2. Cumulative distribution function of arrival times for agents who cannot choose when they arrive at their destination and arrive before noon, split by income. Data from SR-91 Impact Study.

difference between the CDFs occurs at 7:00 when there is a larger jump in the CDF for those with incomes below 40,000 dollars than for those with incomes above 40,000 dollars; this difference between the CDFs goes away at 8:00 when there is a larger jump in the CDF for those with incomes above 40,000 dollars than for those with incomes below 40,000 dollars.

I test for whether the distribution of desired arrival times are the same for each income group using the two-sample Kolmogorov-Smirnov test. The null hypothesis is that the data for each income group comes from the same distribution as the data from the rest of the income groups. Table C.3 reports the results of this test for each income group as well as for each possible way of splitting the population into rich and poor; in every case I fail to reject the null hypothesis at conventional significance levels.

Finally, we can also test whether the average arrival time differs by income groups by regressing household income on arrival times for those who cannot choose when they arrive at work. Table C.4 shows that the differences in average arrival time are rarely statistically significant and never economically significant; the largest difference occurs in the NHTS data where those with incomes above 60,000 dollars arrive twenty minutes later than those with incomes between below 60,000.

C.3. **Evidence that the inflexibility and value of time are independent within the flexible category.** In this subsection I test whether it is reasonable to assume that value of time and inflexibility are independently distributed within the flexible category. As I do

TABLE C.3. Kolmogorov-Smirnov tests for whether distribution of desired arrival times vary with household income

Household income	Test statistic	P-value	Fraction of observations in income category
$<$ 10,000	0.65	0.79	0.002
10,000–25,000	0.31	0.15	0.033
25,000–40,000	0.10	0.85	0.105
40,000–60,000	0.04	1.00	0.238
60,000–100,000	0.06	0.83	0.434
$>$ 100,000	0.08	0.84	0.189
$<$ 25,000	0.26	0.29	0.035
$<$ 40,000	0.12	0.41	0.140
$<$ 60,000	0.07	0.62	0.378
$<$ 100,000	0.08	0.84	0.811

Source: Sullivan (1999)
Notes: N = 429. Sample is limited to those individuals who cannot choose their arrival time, and so are inflexible, who arrive before noon, and for whom we know their income. Each row reports the results from testing whether the distribution of arrival times for that income group is different from the distribution of arrival times for all other income groups. Specifically, the null hypothesis is that both sets of data come from the same distribution.

not observe inflexibility for those in the inflexible category, I am unable to test whether it value of time and inflexibility are independent within the inflexible category.

By Lemma B.2 we know that among agents who are not arriving exactly on-time, the order of arrival before the peak, as well as the order of arrivals after the peak, is determined by their inflexibility. Specifically, more inflexible agents arrive closer to the peak than less inflexible agents. While I cannot compare an agent who arrives after the peak to one who arrives before the peak without knowing how the cost of being early compares to the cost of being late, among agents who are early (and among agents who are late), the ranking of their inflexibility and the ranking of how close to the peak they arrive is the same.

As long as an individual's value of time is a strictly increasing function of his household income, then the ranking of an individual's value of time will be the same as the ranking of their household income. While this is not strictly true, there are poor people with a high value of time, and rich people with a low value of time, value of time and household income are highly correlated and I am using household income as a proxy for value of time. To estimate the distribution of value of time I use a US DOT formula from Belenky (2011, p. 12) to convert household income into an value of time. This formula is a monotonically increasing function, and so the ranking of agents' value of time and household income is the same.

Table C.4. Relationship between income and desired arrival time

	SR-91 IS	NHTS
Household income		
Missing	12	13.0
	(15)	(6.9)
< 25,000	-18	-21.9***
	(22)	(5.2)
25,000–60,000	-7	-22.2***
	(11)	(3.9)
60,000–100,000	-6	1.8
	(11)	(3.2)
Constant	440***	475.4***
	(9)	(8.2)
MSA fixed effects	N/A	Yes
N	413	71,595

Source: Sullivan (1999) and U.S. Department of Transportation (2009).
Notes: Standard errors in parentheses. Omitted category is those with household incomes over 100,000 dollars. I only observed desired arrival times for inflexible workers; it is their actual arrival time (see Section 5.4 for more details). Therefore, the sample is limited to those who are inflexible according to whether they can chose their arrival time at work. For SR-91 IS the dependent variable is actual arrival time, for NHTS the dependent variable is typical arrival time. NHTS sample is also restricted to those who commute to work via the interstate and live in an MSA with a population more than three million. NHTS data is weighted using individual weights and standard errors for NHTS are calculated using jackknife-2 replicate weights.
* $p < .05$
** $p < .01$
*** $p < .001$

Because these rankings are the same, I can estimate the rank correlation between inflexibility and value of time for those agents who do not arrive at their desired arrival time by estimating the rank correlation between how far from the peak an agent arrives, $-|\text{arrival time} - \text{peak}|$, and the agent's household income.

Table C.5 reports these rank correlations for early and late arrivals, and for a variety of different definitions of the peak of rush hour. The estimates of the rank correlation are small, statistically insignificant, and do not have a consistent sign. While my power is low, these estimates lend support to the assumption that inflexibility and value of time are independent within the flexible category.

TABLE C.5. Correlation between inflexibility and value of time within the category of flexible drivers

	Peak at 6:45 a.m. Correlation	Peak at 6:45 a.m. p-value	Peak at 7:00 a.m. Correlation	Peak at 7:00 a.m. p-value	Peak at 7:15 a.m. Correlation	Peak at 7:15 a.m. p-value
	Early arrivals					
Kendall's τ-b	0.0515	0.532	-0.00297	0.971	-0.0423	0.563
Spearman's ρ	0.0711	0.498	0.000896	0.993	-0.0498	0.596
N	93		109		116	
	Late arrivals					
Kendall's τ-b	0.0115	0.831	0.00101	0.986	0.0381	0.504
Spearman's ρ	0.0128	0.854	-0.000189	0.998	0.0467	0.526
N	209		202		185	

Source: Sullivan (1999).

Notes: This table reports the rank correlation between $-|\text{arrival time} - \text{peak}|$ and household income. By Lemma B.2 I know $-|\text{arrival time} - \text{peak}|$ is a monotone increasing function of the inflexibility of an agent who does not arrive at his desired arrival time. Because I am using a US DOT formula to convert household income to value of time, household income is a monotone increasing function of value of time. This means the rank correlation between the rank correlation between $-|\text{arrival time} - \text{peak}|$ and household income is the same as the rank correlation between inflexibility and value of time.

C.4. **Discussion of assumption of homogeneous commutes.** A simplifying assumption my model makes is that everyone has the same commute.[9] In this subsection I (1) document the heterogeneity in trip length and in where trips start and end, (2) estimate the correlation between trip length and key preference parameters of my model, and (3) discuss how the simplifying assumption of a homogeneous commute affects my results. I find that this assumption makes it harder to generate a Pareto improvement, and limits the share of the lanes that can be priced, but also biases the total welfare gains upwards.

First, trip lengths are heterogeneous. Figure C.3 plots the distribution of trip lengths of people using SR-91, showing that trip length varies and has a mean of 34.38 miles. Figure C.4a shows the specific origin-destination pairs for drivers using SR-91 from the SR-91 Impact Study, and Figure C.4b shows where those living in Corona work, according to the LEHD Origin-Destination Employment Statistics for 2015 (Census Bureau, 2017).

Next, I estimate how trip length varies with the key preference parameters of the model, starting with income. As Muth (1969) explains, there are reasons to expect the relationship

[9] For examples of papers that consider a network, see de Palma et al. (2005), who simulate equilibrium in a large network with bottleneck congestion on each route, or Arnott and DePalma (2011) and Osawa et al. (2018), who consider a single corridor connecting the central business district (where all jobs are located) to a variety of locations where people live. There is also a literature which models congestion within a dense network built of recent findings on a macroscopic fundamental diagram of traffic flow which allow for heterogeneous distances under the assumption that congestion is the same everywhere (Arnott, 2013, Fosgerau, 2015).

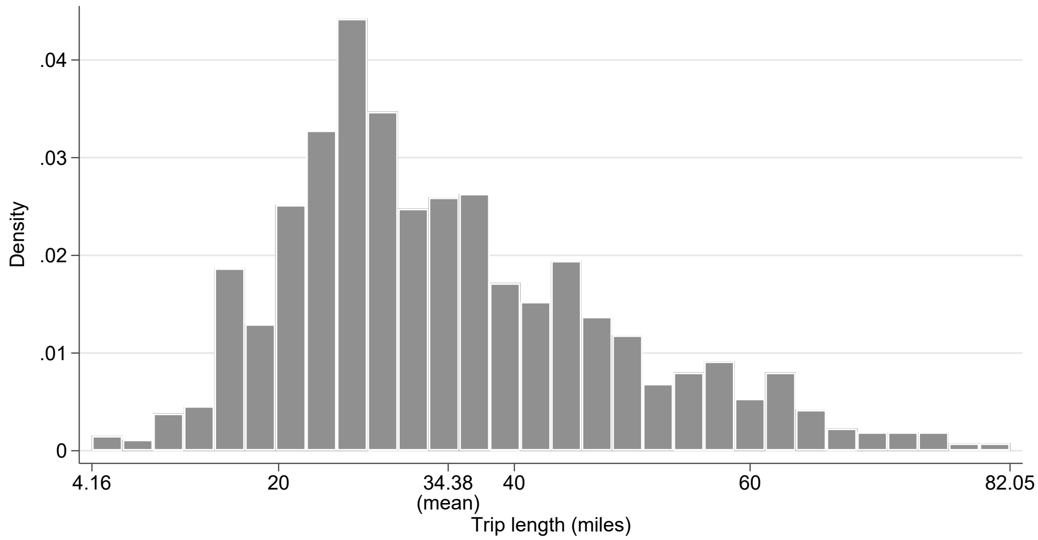


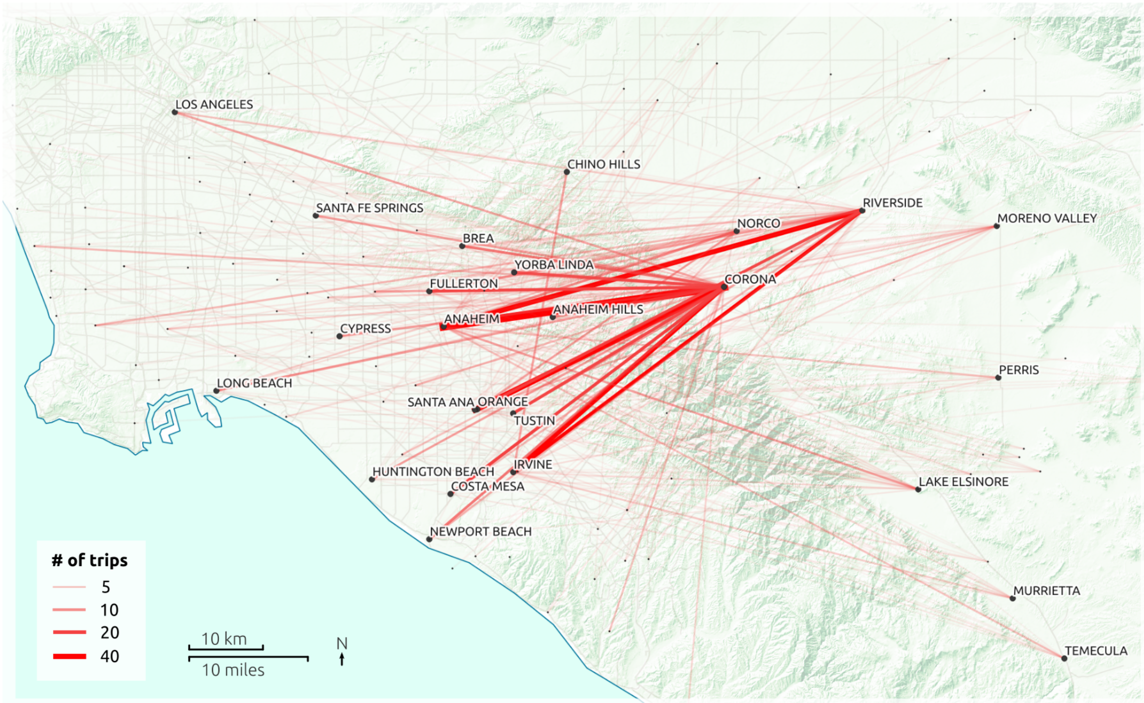
FIGURE C.3. Distribution of weekday trip length for those using SR-91
Notes: Data from SR-91 Impact Study. Length is the minimum distance from the origin to the destination, and may not be the actual distance traveled.

to be either positive or negative. Higher-income households have higher values of time, and so are willing to pay more to live closer to work, but they also have higher housing consumption, and want to live further so the price per unit of housing is lower. Figure C.5 is a box plot showing that the distribution of trip length on SR-91 is consistent across different incomes. Using an F-test, I fail to reject the hypothesis that the mean trip length is identical for all income categories.

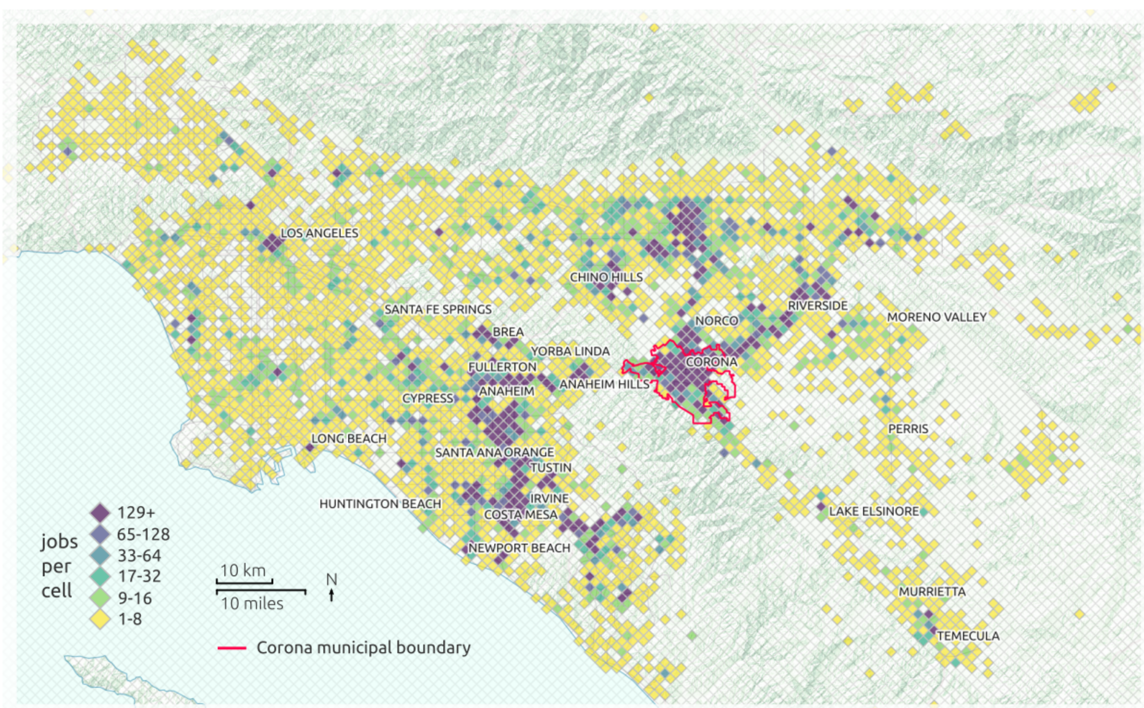
The pattern of trip length being independent of income also holds in the NHTS sample of commutes in large MSAs via an interstate. Figure C.6 shows commute length is fairly constant across all income levels. Figure C.7 plots estimates of the mean commute length by household income, showing these differences are statistically significant.

Consistent with the prior literature (e.g., Wheaton, 1977, Brueckner et al., 1999, Glaeser et al., 2008), I find that without limiting the sample to those who commute via an interstate, I find that commute length is increasing in income. Figure C.8 shows the average trip length by income for all workers in large MSAs. It shows that commute lengths almost double as incomes increase.

Next I estimate how trip length differs by whether a driver is flexible or not. Figure C.9 shows the distribution of trip length for flexible and inflexible drivers for SR-91 (Panel A) and for large MSAs (Panel B). On SR-91, flexible drivers have commutes that are on average 3 miles longer than inflexible drivers (a difference which is statistically significant), while in the NHTS there is essentially no difference in commute length between the flexible and inflexible.



(A) City-level flows using SR-91



(B) Workplaces of those who live in Corona, CA

FIGURE C.4. Trip origins and destinations

Notes: Panel (A) plots the origins and destinations for weekday trips that use SR-91 from the SR-91 Impact Study. Panel (B) plots the workplaces of those who live in Corona using LEHD Origin-Destination Employment Statistics for 2015 (Census Bureau, 2017).

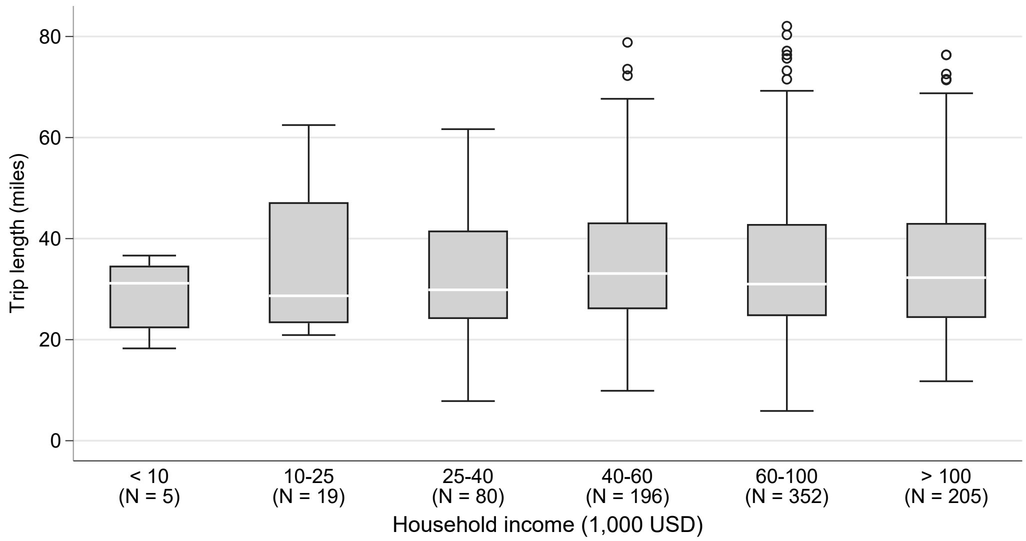


FIGURE C.5. Weekday trip length by household income

Notes: Length is the minimum distance from the origin to the destination, and may not be the actual distance traveled. Household income measured in 1998 dollars. Data from SR-91 Impact Study.

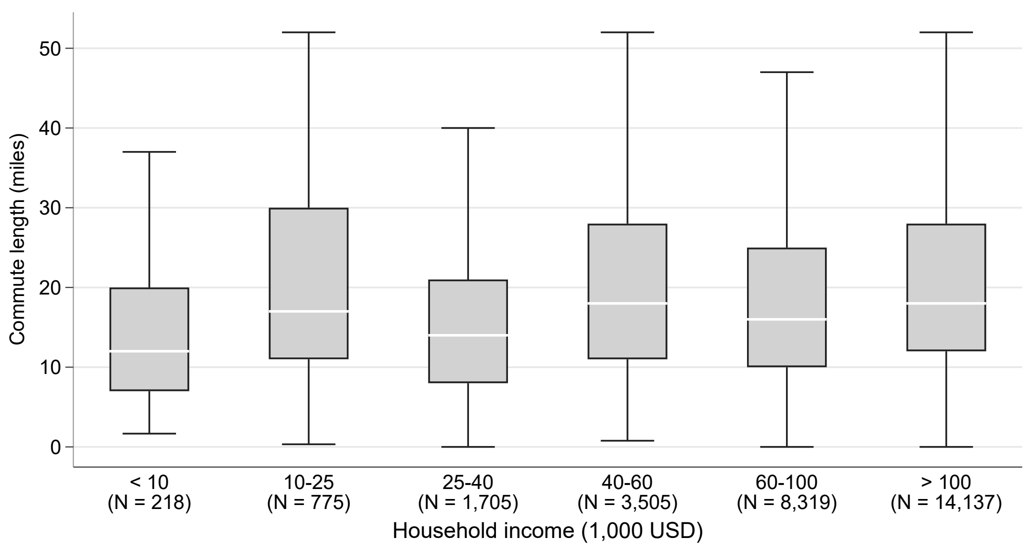


FIGURE C.6. Interstate commute length for large MSAs by household income

Notes: This box plot shows the distribution of commute lengths for those living in a metropolitan statistical area with a population greater than three million who use an interstate as part of their commute. Outliers are not displayed. Household income measured in 2009 dollars. Data from 2009 NHTS and is weighted using individual weights.

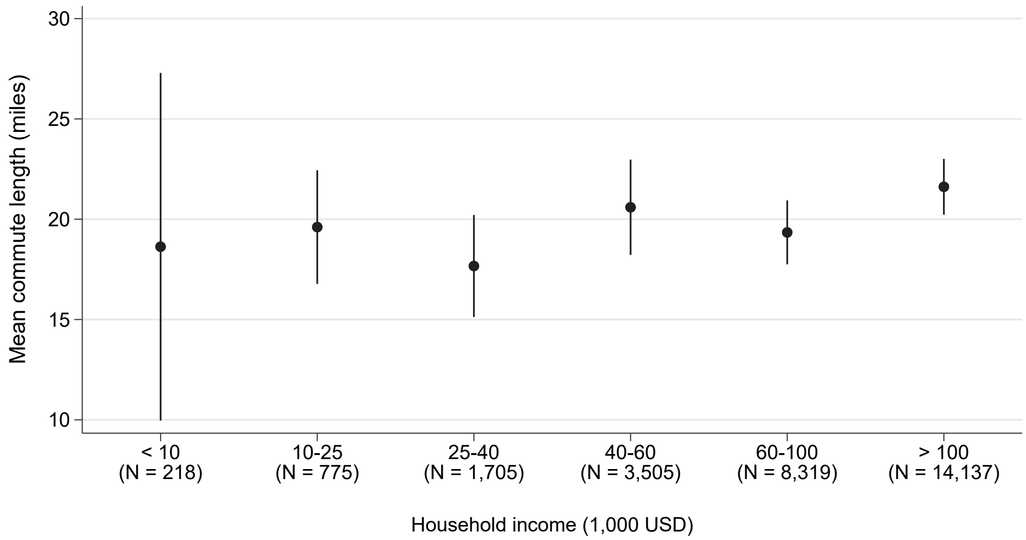


FIGURE C.7. Mean interstate commute length for large MSAs

Notes: Mean commute length for those living in a metropolitan statistical area with a population greater than three million who use an interstate as part of their commute. Bars plot 95 percent confidence intervals. All income intervals are closed on the left and open on the right. Household income measured in 2009 dollars. Data from 2009 NHTS and is weighted using individual weights. Standard errors calculated using jackknife-2 replicate weights.

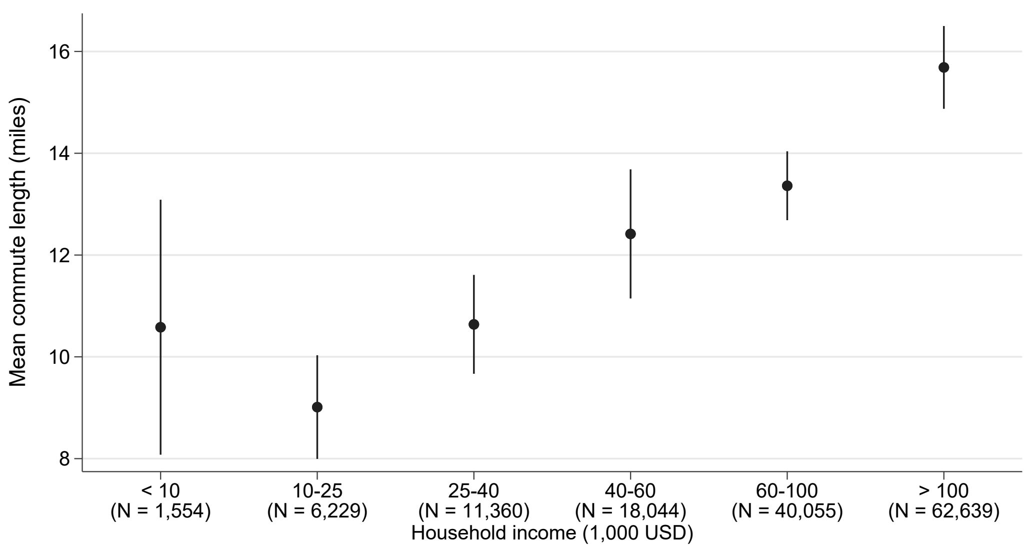
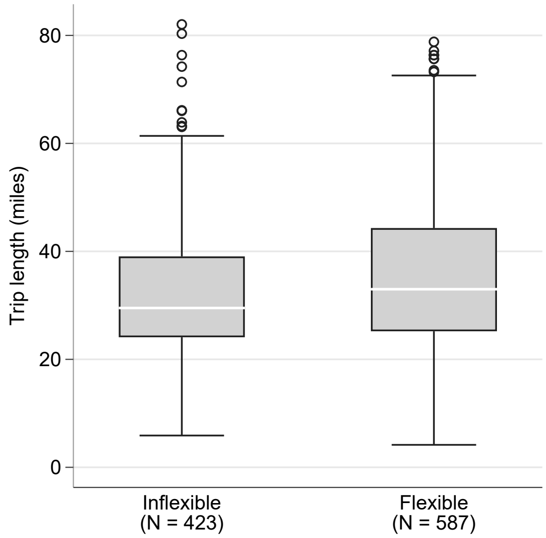
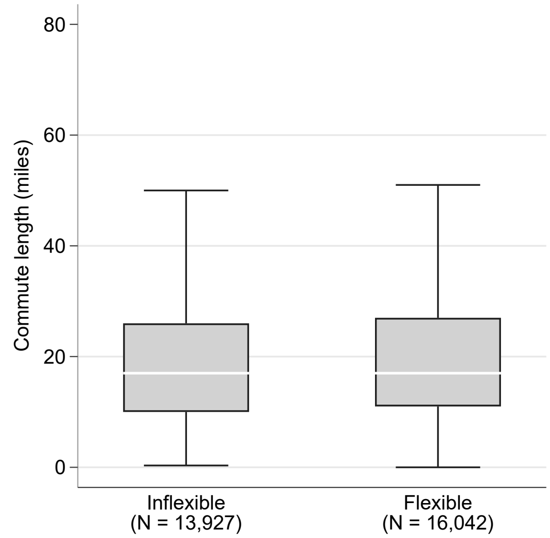


FIGURE C.8. Mean commute length for large MSAs

Notes: Mean commute length for those living in a metropolitan statistical area with a population greater than three million. Bars plot 95 percent confidence intervals. All income intervals are closed on the left and open on the right. Household income measured in 2009 dollars. Data from 2009 NHTS and is weighted using individual weights. Standard errors calculated using jackknife-2 replicate weights.



(A) SR-91



(B) Large MSAs

FIGURE C.9. Trip length by inflexible/flexible

Notes: Panel (A) data is for weekday trips from the SR-91 Impact Study. Length is the minimum distance from the origin to the destination, and may not be the actual distance traveled. Panel (B) data is for commutes which use the interstate in MSAs with a population greater than three million from the 2009 NHTS and is weighted using individual weights. Panel (B) does not show the outliers.

Given an equilibrium, by assuming everyone has the same trip length I overestimate the trip cost for those with short trips and underestimate the trip cost for those with long trips. Given that household income is independent of trip length, the average affect by household income is correct. Since flexible drivers have longer trips, I underestimate the cost for the flexible and overestimate the cost for the inflexible. Since the flexible have an average trip length about 5% longer than the unconditional average, I understate their costs by about 5%. Likewise for the inflexible, I overstate their costs by about 5%. Given that we are primarily interested in the changes in trip cost, and that this does not affect the direction of the change, this means it does not affect the headline results. However, since in my counterfactuals the change in trip cost is greater for the flexible than the unflexible, this suggests I overstate the total welfare gains by a small amount.

This raises the question of how accounting for trip length would change the equilibrium. Within the model, where there is a single congestible point (the bottleneck), additional distance before or after the bottleneck adds a constant to drivers trip cost, and so doesn't affect any decisions.[10] However, if we consider a setting where every location is congestible, the for a given parameter value for their inflexiblity, as a driver's trip length shortens, they choose to arrive closer to their desired arrival time. Just as it isn't worth driving across

[10]See Section C.5 for a derivation of this.

town to save 30% on a pencil, but is to save 30% on a new car; it isn't worth leaving early to reduce travel time on a very short trip, but is for a longer trip. For a short enough trip, even a very flexible driver will arrive exactly on-time.

For the purpose of my estimates and counterfactuals, a flexible driver who arrives on-time because they have a short trip looks like they are inflexible. For a flexible poor driver, they look like they are inflexible and poor, and so I overstate the number of drivers who are hurt. Furthermore, by increasing the number of inflexible poor, I reduce my estimate for the share of lanes that can be tolled while generating a Pareto improvement. For a flexible rich driver, I count them as an inflexible rich driver and overstate the benefits to them. Based on Figure 5, this overstating of the gains is minimal unless the driver was among the most flexible.

Furthermore, inasmuch as the flexible have longer trips, then I am underestimating how much travel time they save by leaving early or late, and so I am underestimating how inflexible they are (cf. Lemma B.3). By overstating the heterogeneity in inflexibility, I make it harder to generate a Pareto improvement.

C.5. **Discussion of assumption that commutes do not include surface streets.** A related issue to that of heterogeneous trip length (as discussed in Section C.4) is that trips do not start and end on the highway, but rather a portion of the trip is on surface streets. Most surface streets are difficult to price, and it is even more difficult to price just a portion of their lanes. This means a portion of a driver's commute will be unpriced. In this section I show that (1) if travel times on surface streets are not time varying, then considering surface streets would not affect my results, (2) that travel times on surface streets are 94–99% of the variation in travel times, and so assuming travel times on surface streets are not time varying likely has only a small affect on my results, and (3) that doing so likely biases my results towards being harder to generate a Pareto improvement.

Considering surface streets does not affect my results if travel times on surface streets are not time varying, even if different drivers face different lengths of trips on the surface streets. This is because the travel time cost on surface streets becomes a constant, which does not affect traveler choices and which differences out when calculating trip costs. To see this formally, write trip cost in terms of *departure* times (subscripts d for departure), and allow travel time to be the sum of travel on surface streets prior to the highway, $T_i^{s,1}$, travel time on the highway, which depends on when the driver reaches the start of the highway, $T_d(t_d + T_i^{s,1})$, and travel time on surface streets after the highway, $T_i^{s,2}$. This yields

$$p(t_d, r; i, t^*) = \alpha T_i^{s,1} + \alpha T_{d,r}(t_d + T_i^{s,1}) + \alpha T_i^{s,2} + \tau_{d,r}(t_d + T_i^{s,1}) \\ + D_i(t^* - t_d - [T_i^{s,1} + T_d(t_d + T_i^{s,1}) + T_i^{s,2}]).$$

Re-arranging terms gives us

$$p(t_d, r; i, t^*) = \underbrace{\alpha(T_i^{s,1} + T_i^{s,2})}_{\text{constant}} + \alpha\, \underbrace{T_d(t_d + T_i^{s,1})}_{T(t)} + \underbrace{\tau_{d,r}(t_d + T_i^{s,1})}_{\tau_r(t)} + D_i(\underbrace{[t^* - T_i^{s,2}]}_{t^*} - \underbrace{[t_d + T_i^{s,1} + T_d(t_d + T_i^{s,1}])}_{t}),$$

which is equivalent to (1), with the exceptions that, as discussed in Section 5.4, desired arrival time is defined in terms of the desired arrival time at the end of the highway, $t^* - T_i^{s,2}$, arrival time is likewise defined in terms of the arrival time at the end of the highway, and that we have added a constant to the trip cost. This constant does not affect any choices, and cancels out when estimating the change in trip cost, so does not affect my results.

This raises the question of how much do travel times on surface streets vary during the morning peak. To estimate this, I use data from Uber Movement 2019 on the average speed on road segments in New York City during the fourth quarter of 2018.[11] These speeds are constructed from Uber trip data. I limit the sample to those road segments for which data is available for all hours between 4 and 10 AM.[12] This leaves me with data on 73,679 road segments.

Figure C.10 shows how travel times on surface streets evolve over the day, and for comparison, plots how travel times on SR-91 evolve over the day. Travel times are normalized by dividing by the travel time at 4 AM. On SR-91 travel times during the peak hour are 58% higher than at 4 AM, while for the typical NYC surface street they are 17% higher. Thus, while travel times do vary on surface streets, they vary significantly less than they do on highways.

Furthermore, for areas with dense networks of highways, such as Los Angeles, most of the trip is on the highway. For a random sample of 30 trips in the SR 91 Impact Study, on average 92% of the trip is on a highway (as measured in miles).[13] Thus, even if the average free flow speed on a surface street is a third that on the highway, then highways account for 94% of the change in travel time from 4 to 7 AM, and 99% of the change from 7 to 10

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[11]Uber has not released data for Los Angeles, and this is the only time period the data for which the data is available.

[12]This data includes some limited-access highways, however, by plotting median travel times I minimize the bias due to including highways. A crude approach for limiting highways is limiting the data to those segments with a median speed at 4 AM above 45 miles per hour. Doing so changes the plotted data in Figure C.10 by less than one percent.

[13]Respondents reported the nearest cross-streets to their home and workplace. I used Google Maps in June 2019 to get directions between these cross streets at the departure time reported in the survey, and then calculated the share on surface streets vs. limited-access highways. Highways account for 79% of the travel time Travel times thus are from almost 20 years later than the survey was done, so likely have some error. The 95% confidence intervals, calculated by bootstrapping, are [.90, .93] and [.76, .82] respectively.

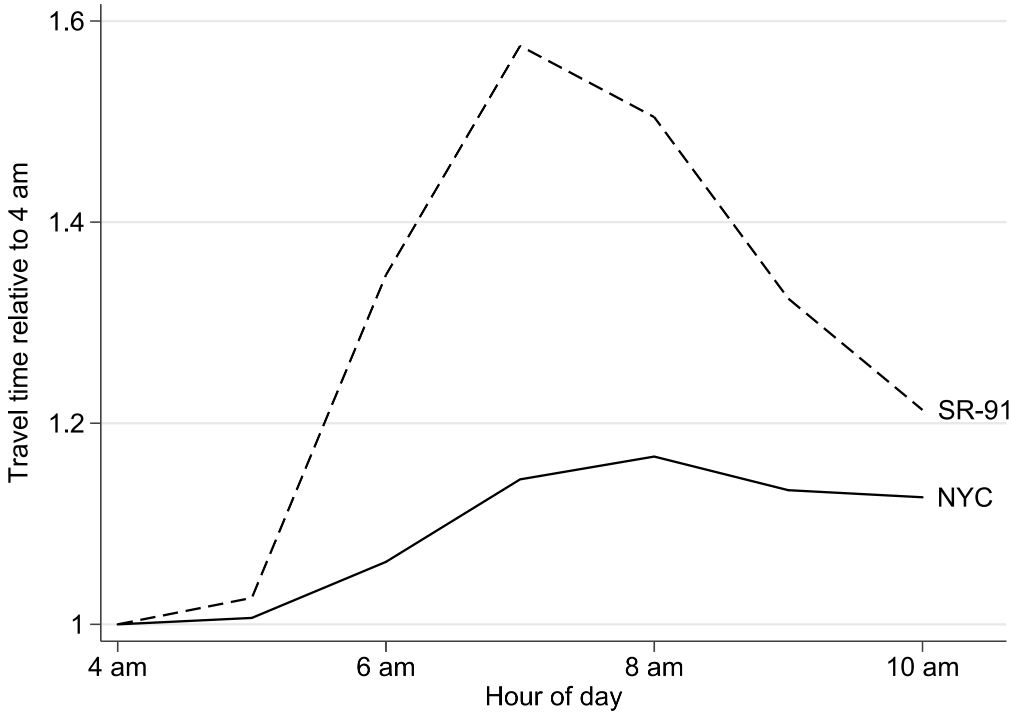


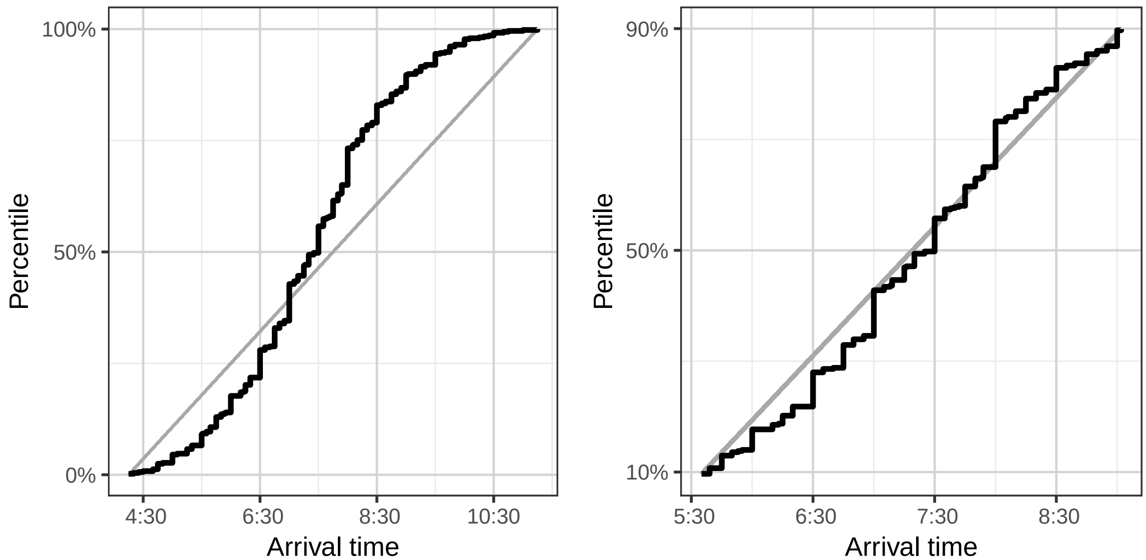
FIGURE C.10. Travel times on New York City (NYC) surface streets and California SR-91

Notes: The solid line plots the median ratio of the median travel time at a given hour over the median travel time at 4 AM across the 73,679 NYC road segments for which data is available for 4–10 AM using data from Uber Movement (2019). The dotted line plots the ratio of the mean travel time to the mean travel time at 4 AM on California SR-91 from the center of Corona to the junction of SR-91 and I-605 using data from PeMS. This is the same data as in Figure 3, but aggregated to the hourly level to be more comparable to the NYC data.

AM.[14] As such, not allowing for time-varying travel times on surface streets is likely to be innocuous.

Finally, not allowing for time-varying travel times on surface streets makes it harder to generate a Pareto improvement. This is because the main barrier to generating a Pareto improvement is that, once there is a tolled option, flexible rich drivers decide to start traveling at the peak which displaces inflexible poor drivers and leaves them worse off. However, in the presence of untolled surface streets that have time-varying travel times, traveling at the peak on the tolled route still requires paying additional travel time. This additional travel time makes switching to the peak less attractive for the flexible rich. Thus, my model overstates the willingness of the flexible rich to switch to the peak, exaggerating the amount of displacement, and making it harder to generate a Pareto improvement.

[14]Specifically, I use the results Figure C.10 to estimate how travel times on surface streets and highways vary over time, the estimates reported in this paragraph of the share of the trip on the highway, and the assumption that free flow speeds on surface streets are a third that on the highway.



(A) All data (B) 10th–90th percentiles

FIGURE C.11. Empirical cumulative distribution function of arrival time for agents who cannot choose when they arrive at their destination and arrive before noon. Data from SR-91 Impact Study.

C.6. **Evidence that assuming desired arrival times are uniformly distributed is reasonable.** In this subsection, I provide evidence that it is reasonable to approximate the distribution of desired arrival times with a uniform distribution and explain why I do not expect this assumption to affect my results.

To test whether the distribution of desired arrival times is uniformly distributed, I compare the cumulative distribution function (CDF) of a uniform distribution to the empirical CDF of desired arrival times for the inflexible agents using SR-91. I do so in Figure C.11a. If the distribution were uniform then the empirical CDF would lie along the 45 degree line; it is clear that the distribution of desired arrival times is not uniform. However, when we remove the first and last ten percent of road users to arrive, as in Figure C.11b, then the distribution is close to being uniform. A remaining difference is that the empirical CDF is not as smooth as that of a uniform distribution, but, as discussed in the Section 5.4, this is exactly what we expected to find.

This same pattern shows up using data from the NHTS for the Los Angeles and New York City metropolitan statistical areas, as well as the entire sample. Figure C.12 shows that while the distribution of arrival times is not uniform, once we remove the first and last ten percent of road users to arrive, then the distribution is close to being uniform.

While Figure C.12b shows a tendency for drivers to arrive later than would be expected if the distribution was uniform, this is largely driven by the fact that in every MSA people arrive at work between 8:00 and 9:00, while only in some do many people arrive between 6:00 and 7:00.

Truncating the extreme deciles is relatively innocuous. By doing so I am ignoring agents whose desired arrival times are to arrive extremely early or late. Some of these agents are arriving outside of rush hour, and so they are not relevant for my analysis. The rest are among those who are least harmed by congestion pricing; they are already traveling at undesirable times, and so will not be displaced. In particular, in my model, if an agent who desires to arrive at the very peak of rush hour is better off, then all agents of his type are also better off. All of this suggests that should congestion pricing help those who want to arrive at the peak of rush hour, then it also helps these agents I am ignoring.

C.7. **Proof that estimator of length of desired arrivals is unbiased.**

Proposition C.1. *An unbiased estimator of the maximum and minimum of a truncated uniform distribution is*

$$\hat{a} = \frac{m \cdot X_{(n)} - n \cdot X_{(m)}}{m-n}$$
$$\hat{b} = \frac{(N-n+1) \cdot X_{(m)} - (N-m+1) \cdot X_{(n)}}{m-n},$$

where there are N observations, where the $N-m+1$ observations are discarded and the $N-n-1$ observations are discarded, so that m is the highest remaining order statistic and n is the lowest remaining order statistic.

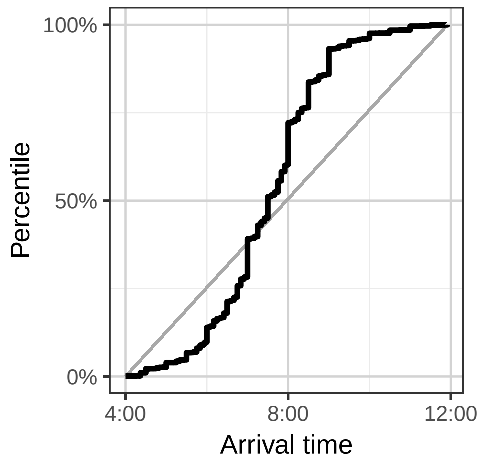
And an unbiased estimator of the length of desired arrivals of

$$\widehat{LDA} = \frac{N+1}{m-n}\left(X_{(m)} - X_{(n)}\right).$$

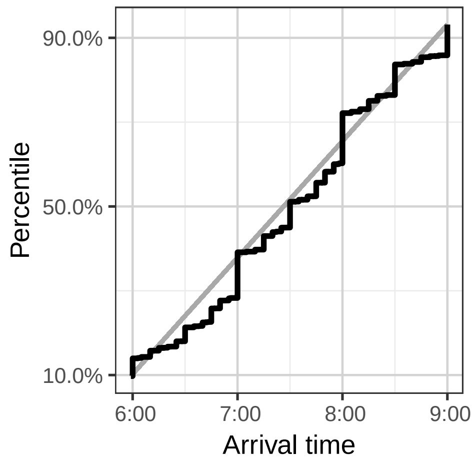
Proof. First, we show that $\hat{a}$ and $\hat{b}$ are unbiased estimators.

$$\begin{aligned}
E(\hat{a}) &= \frac{m \cdot E\left(X_{(n)}\right) - n \cdot E\left(X_{(m)}\right)}{m-n} \\
&= \frac{m\left[a + (b-a)\frac{n}{N+1}\right] - n\left[a + (b-a)\frac{m}{N+1}\right]}{m-n} \\
&= \frac{(m-n)\,a}{m-n} \\
&= a.
\end{aligned}$$

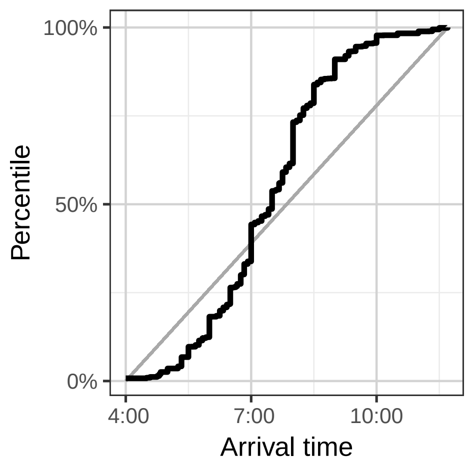
$$\begin{aligned}
E\left(\hat{b}\right) &= \frac{(N-n+1) \cdot E\left(X_{(m)}\right) - (N-m+1) \cdot E\left(X_{(n)}\right)}{m-n} \\
&= \frac{(N-n+1) \cdot \left[a + (b-a)\frac{m}{N+1}\right] - (N-m+1) \cdot \left[a + (b-a)\frac{n}{N+1}\right]}{m-n} \\
&= \frac{(m-n)\,b}{m-n}
\end{aligned}$$



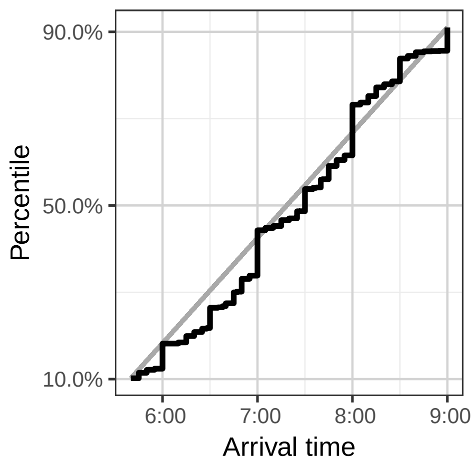
(A) United States: All data



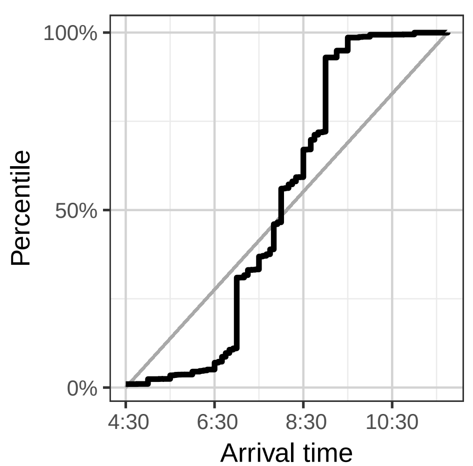
(B) United States: 10th–90th percentiles



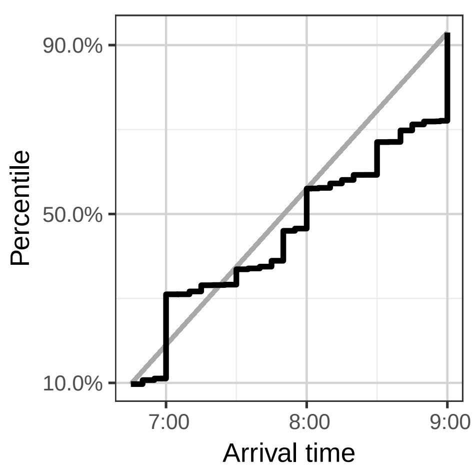
(C) Los Angeles MSA: All data



(D) Los Angeles MSA: 10th–90th percentiles



(E) New York City MSA: All data



(F) New York City MSA: 10th–90th percentiles

FIGURE C.12. Empirical cumulative distribution function of typical arrival time for workers who commute via the interstate, cannot choose when they arrive at work, and arrive before noon. Data from NHTS and weighted using individual weights.

$$= b.$$

As a result, $\widehat{LDA}$ is an unbiased estimator of the length of desired arrivals because

$$E\left(\widehat{LDA}\right) = E\left(\hat{b} - \hat{a}\right) = b - a = LDA.$$

□

C.8. **Nonparametric estimators.** In this subsection I explain how I use the predicted travel times from nonparametrically fitting the model to the data, $\hat{T}_i$, to back out estimates of $t_s, t_e - t_s, 1/s, \xi$, and T_f. For the sake of simplicity, when dealing with slopes I will ignore the denominator, which always cancels out.[15]

When desired arrival times are uniformly distributed, the marginal type arriving is constant from the first desired arrival time until the peak of rush hour, and so the first desired arrival time is given by

$$\hat{i}_s = \min i \text{ s.t. } \left|\left(\hat{T}_{t+1} - \hat{T}_t\right) - \left(\hat{T}_{i+1} - \hat{T}_i\right)\right| < 10^{-5} \;\forall\; i^{\max} > t > i.$$

Then $\hat{t}_s = \hat{i}_s \cdot 5/60 + 4$ converts the index into the hour of the first desired arrival time.

Similarly, when desired arrival times are uniformly distributed, the marginal type is constant from the peak of rush hour until the last desired arrival time, and so the last desired arrival time is given by

$$\begin{aligned}
\hat{i}_e &= \max i \text{ s.t. } \left|\left(\hat{T}_i - \hat{T}_{i-1}\right) - \left(\hat{T}_t - \hat{T}_{t-1}\right)\right| < 10^{-5} \;\forall\; i > t > i^{\max} + 1, \\
\hat{t}_e &= \hat{i}_e \cdot 5/60 + 4.
\end{aligned}$$

I can then estimate the length of desired arrivals, measured in hours, as $\widehat{t_e - t_s} = \left(\hat{t}_e - \hat{t}_s\right)$.

The free flow travel time, T^f, is the travel time experienced by someone traveling outside of rush hour, and estimated as $\hat{T}^f = \hat{T}_1$. Travel times are constant until the start of rush hour, which is the first time travel times are above $\hat{T}^f$:

$$\begin{aligned}
\hat{i}_{01} &= \min i \text{ s.t. } \hat{T}_i - \hat{T}^f > 10^{-5}, \\
\hat{t}_{01} &= \hat{i}_{01} \cdot 5/60 + 4.
\end{aligned}$$

To estimate when rush hour ends I need to expand the time of day for which I use data. For the GMM estimation, I limit myself to data between 4–10 in the morning, now I will use data between 4 AM and 2 PM This will allows me to observe the end of rush hour explicitly, and in some bootstrap draws, also allows me to observe the last desired arrival time. Extending the time window does not otherwise effect my nonparametric estimates.

Using the extended time window creates a problem. Due to the effects of those who wish to arrive in the middle of the day and the evening rush hour, travel times on SR-91

[15]Put differently, I will change the unit of time to be a five minute window, so the denominator is one.

do not return to free flow levels in between the morning and afternoon peaks. This is one of the reasons I limit myself to the data between 4–10 in the morning when estimating preferences using GMM. It also means I cannot directly use the definition of rush hour as the time when travel times are above their free flow level. Instead, I allow the middle of the day to have its own free flow speed, T^f_{after}, which I estimate as being 3.08 minutes longer than $\hat{T}^f$. I then define the end of rush hour as the last time travel times are above this

$$\hat{i}_{10} = \max i \text{ s.t. } \hat{T}_i - \hat{T}^f_{\text{after}} > 10^{-5},$$
$$\hat{t}_{10} = \hat{i}_{10} \cdot 5/60 + 4.$$

Then I can estimate the length of rush hour on a free route as

$$\widehat{1/s} = \hat{t}_{10} - \hat{t}_{01}.$$

I estimate the ratio of the schedule delay cost of being late to early by estimating the ratio of the schedule delay cost while late to the schedule delay cost while early for the marginal type arriving during the peak, which is the slope immediately after the peak divided by the slope immediately before the peak:

$$\hat{\xi} = \frac{\hat{T}_{i^{\max}} - \hat{T}_{i^{\max}-1}}{\hat{T}_{i^{\max}+2} - \hat{T}_{i^{\max}+1}}.$$

Note that for many of the parameters there are multiple ways of nonparametrically estimating them, especially when we are willing to impose more assumptions. This, of course, is why we have more moments than parameters when doing GMM.

C.9. **Why I cannot nonparametrically estimate N_δ.** There are several difficulties with nonparametrically estimating the distribution on inflexibility.

As before, I only observe the inflexibility of agents who are not inframarginal; but unlike before, I no longer know the density of agents who are inframarginal at any given arrival time. This means that I observe the range of values which β/α takes, but not the density at each value. To know the density at each value I need to know the distribution of desired arrival times.

In the relaxed model I no longer impose that $\gamma_i = \xi\beta_i$ for all types i, and so I know the marginal β/α for early arrival times and the marginal γ/α for late arrival times, but not the joint distribution of β/α and γ/α. That said, I can estimate a lower bound for β/α for those arriving late and a lower bound for γ/α for those arriving early from the fact that they preferred to arrive late (early) rather than early (late).

In addition, given what is unobserved, it would be impossible to solve for counterfactuals without also making similar assumptions about the joint distribution on β and γ which would give me the same set of results.

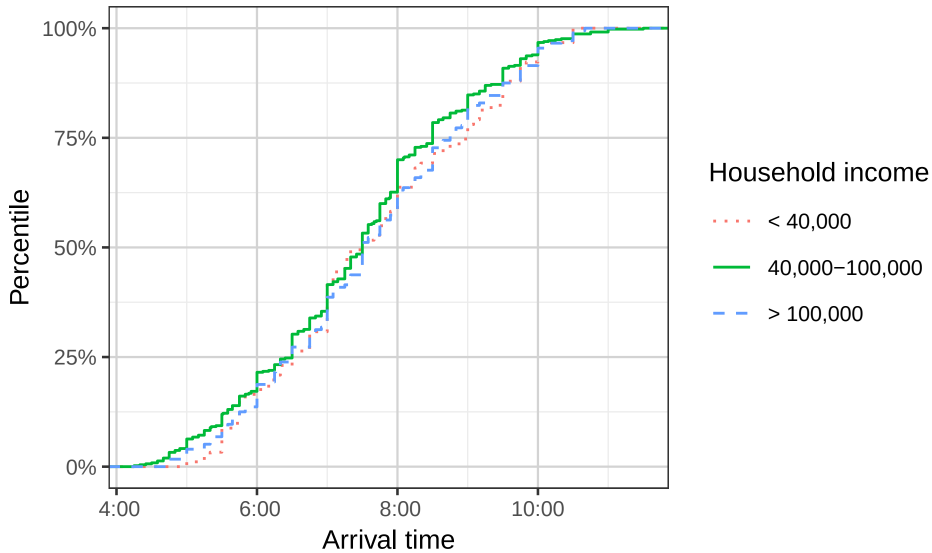


FIGURE C.13. Cumulative distribution function of arrival times for agents who cannot choose when they arrive at their destination and arrive before noon, split by income. Data from SR-91 Impact Study.

C.10. **Additional tests of model fit.** My model predicts that more flexible drivers arrive further from the peak. As such, another way to estimate the correlation between value of time (as proxied for by household income) and flexibility is to look at how the distribution of arrival times differs by household income. Figure C.13 plots the distribution of arrival times for three income categories, showing they are quite similar.

I formally test for differences in the distribution of arrival times by household income using the two-sample Kolmogorov-Smirnov test. The null hypothesis is that the data for each income group comes from the same distribution as the data from the rest of the income groups. Table C.6 reports the results of this test for each income group as well as for each possible way of splitting the population into rich and poor; in every case I fail to reject the null hypothesis at conventional significance levels.

This evidence suggests that value of time and flexibility are uncorrelated. This is consistent with the typical-trip measure of flexibility (reported in the appendix), where value of time and flexibility are uncorrelated (see Table C.1), and is inconsistent with the specific-trip measure of flexibility (reported in the main text), which suggest that value of time and flexibility are positively correlated (see Table 2). Given that the specific-trip measure gives more conservative results, I present those estimates in the main text.

TABLE C.6. Kolmogorov-Smirnov tests for whether distribution of arrival times vary with household income

Household income	Test statistic	P-value	Fraction of observations in income category
$<$ 10,000	0.60	0.23	0.004
10,000–25,000	0.29	0.16	0.021
25,000–40,000	0.06	0.96	0.100
40,000–60,000	0.05	0.93	0.224
60,000–100,000	0.06	0.48	0.409
$>$ 100,000	0.06	0.70	0.242
$<$ 25,000	0.17	0.66	0.025
$<$ 40,000	0.06	0.90	0.125
$<$ 60,000	0.03	1.00	0.349
$<$ 100,000	0.06	0.70	0.758

Source: Sullivan (1999)
Notes: N = 727. Sample is limited to those individuals arrive before noon, and for whom we know their income. Each row reports the results from testing whether the distribution of arrival times for that income group is different from the distribution of arrival times for all other income groups. Specifically, the null hypothesis is that both sets of data come from the same distribution.

APPENDIX D. DETAILS FOR SECTION 6

This appendix contains a description of how I extrapolate my results to other cities, as well as a number of tables and figures reporting sensitivity tests. These tests are discussed in the main body of the paper.

D.1. **Extrapolating to other cities.** To extrapolate my results to other cities I adjust for the severity of congestion in each city to find the social and private welfare gains per person-mile of travel, and then determine the number of person-miles exposed to congestion (both total and for the typical trip). The product of these gives the total social and private welfare gains from pricing, both total and for the typical trip. The results of doing so are in Tables D.1 and D.2.

To adjust for the severity of congestion I find the length of rush hour in each city that generates the Travel Time Index (TTI) and Commuter Stress Index (CSI) reported in Schrank et al. (2012). A city's TTI is the ratio of mean travel time between 6–10 AM and 3–7 PM to the free flow travel time, while a city's CSI is the the same ratio for travel in the peak direction, and so for each MSA m I solve for the scaling of the length of rush hour ζ_m such that

$$TTI_m = 49^{-1} \sum_i \frac{T_f + T\left(t_i; \tilde{\delta}, \xi, \zeta_m \cdot s, t_s, \phi, t_e - t_s\right)}{T_f} \quad \text{for } t_i \in \{6{:}00, 6{:}05, \ldots, 10{:}00\} ,$$

where $T\left(t_i; \tilde{\delta}, \xi, s, t_s, \phi, t_e - t_s\right)$ is defined by (B.19) in Appendix B.

I solve for equilibrium for each ζ_m to find the social and private welfare gains per person-mile of travel for each MSA. My key assumption is that the joint distribution of agent preferences is identical in every city; only the length of rush hour differs. I use the TTI welfare gains for evaluating the total welfare gain, and the CSI welfare gains for evaluating the per peak driver welfare gains.

To find the miles traveled on urban highways during the peak period I first need to define the peak period, which I do by solving for the start and end of the morning and evening peak given the length of rush hour. This gives me the following equations for $k \in \{AM, PM\}$:

$$t_{01,m,k} = t_k^{\max} - \zeta_m s^{-1} \frac{\xi}{1+\xi},$$
$$t_{10,m,k} = t_k^{\max} + \zeta_m s^{-1} \frac{1}{1+\xi};$$

where $t_{AM}^{\max} = (t_s + \xi t_e) / (1+\xi)$ (and so comes from my empirical estimates in Section 5) and $t_{PM}^{\max}$ is chosen so that when rush hour is four hours long it lasts from 3–7 PM I combine these results with the estimates of the share of daily traffic that occurs each hour from Margiotta et al. (1994) to estimate the fraction of average daily vehicle miles traveled that occurs during the peak. Multiplying this fraction by the total average daily highway miles traveled from Schrank et al. (2012) (who calculate this using data from the Highway Performance Monitoring System) gives me the miles exposed to peak period congestion. I then multiply by the average vehicle occupancy for cities of similar size, calculated from U.S. Department of Transportation (2009), to find person-miles exposed to peak period congestion.

To find the miles traveled in the typical trip I adjust the total peak period miles traveled for the fraction of miles traveled by trucks (7% according to Schrank et al. (2012)), and divide by the number of peak period trips, which I estimate using U.S. Department of Transportation (2009). Because the sample size for many MSAs is small, I calculate the length of the typical trip for four MSA population categories: more than 3 million, between 1–3 million, between 0.5–1 million, and between 250,000–500,000.

Table D.1. Total welfare gains (millions of dollars)

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
Akron OH	72	94	64	51
Albany NY	120	100	100	68
Albuquerque NM	57	96	53	47
Allentown-Bethlehem PA-NJ	91	73	75	50
Anchorage AK	25	20	21	13
Atlanta GA	1300	850	1000	560
Austin TX	410	230	310	150
Bakersfield CA	25	37	23	19
Baltimore MD	620	410	480	270
Baton Rouge LA	110	77	88	51
Beaumont TX	30	51	28	25
Birmingham AL	200	150	160	100
Boise ID	9.5	29	10	10
Boston MA-NH-RI	1300	790	990	510
Boulder CO	13	9.7	10	6.5
Bridgeport-Stamford CT-NY	310	190	240	120
Brownsville TX	15	12	13	8.1
Buffalo NY	120	95	97	65
Cape Coral FL	27	24	23	16
Charleston-North Charleston SC	59	51	50	35
Charlotte NC-SC	260	180	200	120
Chicago IL-IN	1700	1100	1300	720
Cincinnati OH-KY-IN	390	280	310	190
Cleveland OH	300	250	250	170
Colorado Springs CO	63	72	54	42
Columbia SC	68	100	62	52
Columbus OH	300	230	240	160
Corpus Christi TX	17	52	19	19
Dallas-Fort Worth-Arlington TX	1800	1100	1400	730
Dayton OH	84	130	77	65
Denver-Aurora CO	610	370	460	240
Detroit MI	650	500	520	340
El Paso TX-NM	130	88	98	59
Eugene OR	14	31	14	14
Fresno CA	32	71	32	32
Grand Rapids MI	52	100	50	47
Greensboro NC	46	78	43	38
Hartford CT	220	170	180	120
Honolulu HI	240	130	180	84
Houston TX	1600	980	1200	640
Indianapolis IN	240	190	200	130
Indio-Cathedral City-Palm Springs CA	8.6	19	8.7	8.7

Table D.1. (continued)

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
Jackson MS	47	79	44	39
Jacksonville FL	170	150	150	110
Kansas City MO-KS	280	320	240	190
Knoxville TN	91	76	76	52
Lancaster-Palmdale CA	8.2	18	8.2	8.2
Laredo TX	8.3	7.4	7.1	5.2
Las Vegas NV	240	170	190	120
Little Rock AR	55	140	57	57
Los Angeles-Long Beach-Santa Ana CA	5700	2900	4200	1900
Louisville KY-IN	220	170	180	120
Madison WI	39	58	35	30
McAllen TX	54	44	44	30
Memphis TN-MS-AR	150	120	130	81
Miami FL	1200	760	910	500
Milwaukee WI	170	150	150	100
Minneapolis-St. Paul MN	640	450	500	300
Nashville-Davidson TN	350	230	270	150
New Haven CT	140	110	120	77
New Orleans LA	95	69	75	46
New York-Newark NY-NJ-CT	4300	2400	3200	1600
Oklahoma City OK	170	140	140	100
Omaha NE-IA	59	87	54	45
Orlando FL	270	190	210	130
Oxnard CA	73	120	68	60
Pensacola FL-AL	16	23	14	12
Philadelphia PA-NJ-DE-MD	980	610	750	400
Phoenix-Mesa AZ	590	450	470	300
Pittsburgh PA	280	180	210	120
Portland OR-WA	430	260	320	170
Poughkeepsie-Newburgh NY	64	84	57	46
Providence RI-MA	190	160	160	110
Provo-Orem UT	57	51	49	36
Raleigh-Durham NC	180	160	150	110
Richmond VA	140	200	120	100
Riverside-San Bernardino CA	520	340	400	230
Rochester NY	85	98	74	57
Sacramento CA	310	220	250	150
Salem OR	23	20	20	14
Salt Lake City UT	120	110	100	74
San Antonio TX	360	270	290	180
San Diego CA	740	570	600	380
San Francisco-Oakland CA	1200	850	970	570

Table D.1. (continued)

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
San Jose CA	410	270	320	180
San Juan PR	320	200	240	130
Sarasota-Bradenton FL	32	41	28	23
Seattle WA	950	590	720	390
Spokane WA-ID	29	38	26	21
Springfield MA-CT	76	87	66	51
St. Louis MO-IL	420	380	360	270
Stockton CA	36	61	34	30
Tampa-St. Petersburg FL	290	210	230	140
Toledo OH-MI	54	62	47	36
Tucson AZ	64	53	53	36
Tulsa OK	96	120	85	68
Virginia Beach VA	260	190	210	130
Washington DC-VA-MD	1400	800	1100	520
Wichita KS	42	82	41	38
Winston-Salem NC	53	79	48	41
Worcester MA-CT	75	86	66	50
Total	39300	26600	30400	17300
Small Area Average	34	51	31	26
Medium Area Average	91	79	77	55
Large Area Average	310	220	250	150
Very Large Area Average	1600	1000	1200	660

Notes: See Appendix D for details of calculations. Population groups are defined as follows: small: 250,000–500,000; medium: 0.5–1 million; large: 1–3 million; very large: >3 million.

TABLE D.2. Welfare gains per typical commuter

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
Akron OH	640	730	550	420
Albany NY	970	680	770	460
Albuquerque NM	590	770	530	420
Allentown-Bethlehem PA-NJ	970	680	770	460
Anchorage AK	400	270	310	180
Atlanta GA	1400	740	1000	490
Austin TX	1200	580	850	390
Bakersfield CA	640	730	550	420
Baltimore MD	940	560	710	360
Baton Rouge LA	1200	720	880	470
Beaumont TX	240	280	210	160
Birmingham AL	1000	700	810	460
Boise ID	220	290	200	160
Boston MA-NH-RI	1400	750	1000	490
Boulder CO	350	260	280	170
Bridgeport-Stamford CT-NY	1200	730	910	480
Brownsville TX	340	250	270	170
Buffalo NY	670	500	530	330
Cape Coral FL	350	260	280	170
Charleston-North Charleston SC	850	650	690	440
Charlotte NC-SC	870	540	660	350
Chicago IL-IN	1300	730	960	480
Cincinnati OH-KY-IN	780	520	610	350
Cleveland OH	700	500	550	340
Colorado Springs CO	770	630	630	430
Columbia SC	260	230	220	160
Columbus OH	750	520	590	340
Corpus Christi TX	210	310	190	160
Dallas-Fort Worth-Arlington TX	1400	740	1000	490
Dayton OH	640	730	550	420
Denver-Aurora CO	1100	580	790	380
Detroit MI	980	670	760	440
El Paso TX-NM	1000	690	790	460
Eugene OR	170	330	170	160
Fresno CA	400	900	400	400
Grand Rapids MI	590	770	530	420
Greensboro NC	240	280	210	160
Hartford CT	970	680	770	460
Honolulu HI	3900	2600	3200	2000
Houston TX	1400	750	1000	490
Indianapolis IN	750	520	590	340
Indio-Cathedral City-Palm Springs CA	540	810	500	420

Table D.2. (continued)

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
Jackson MS	220	290	200	160
Jacksonville FL	670	500	530	330
Kansas City MO-KS	540	470	460	320
Knoxville TN	890	660	710	450
Lancaster-Palmdale CA	540	810	500	420
Laredo TX	280	240	230	160
Las Vegas NV	920	550	690	360
Little Rock AR	140	350	140	140
Los Angeles-Long Beach-Santa Ana CA	1500	760	1100	510
Louisville KY-IN	750	520	590	340
Madison WI	220	290	200	160
McAllen TX	930	670	740	450
Memphis TN-MS-AR	780	520	610	350
Miami FL	1400	750	1000	490
Milwaukee WI	670	500	530	330
Minneapolis-St. Paul MN	940	560	710	360
Nashville-Davidson TN	920	550	690	360
New Haven CT	930	670	740	450
New Orleans LA	750	520	590	340
New York-Newark NY-NJ-CT	1600	750	1100	510
Oklahoma City OK	850	650	690	440
Omaha NE-IA	640	730	550	420
Orlando FL	890	550	680	360
Oxnard CA	540	810	500	420
Pensacola FL-AL	290	240	240	170
Philadelphia PA-NJ-DE-MD	1400	750	1000	490
Phoenix-Mesa AZ	1200	710	880	460
Pittsburgh PA	970	560	730	370
Portland OR-WA	1100	580	810	380
Poughkeepsie-Newburgh NY	720	630	610	430
Providence RI-MA	670	500	530	330
Provo-Orem UT	280	240	230	160
Raleigh-Durham NC	730	510	570	340
Richmond VA	680	610	580	430
Riverside-San Bernardino CA	780	520	610	350
Rochester NY	810	650	660	440
Sacramento CA	750	520	590	340
Salem OR	310	250	250	170
Salt Lake City UT	610	480	500	330
San Antonio TX	870	540	660	350
San Diego CA	1200	710	880	460
San Francisco-Oakland CA	980	670	760	440

Table D.2. (continued)

Fraction of lanes priced	1		0.5	
MSA	Social	Private	Social	Private
San Jose CA	810	530	630	350
San Juan PR	1100	580	790	380
Sarasota-Bradenton FL	810	650	660	440
Seattle WA	1400	740	1000	490
Spokane WA-ID	240	280	210	160
Springfield MA-CT	810	650	660	440
St. Louis MO-IL	610	480	500	330
Stockton CA	240	280	210	160
Tampa-St. Petersburg FL	920	550	690	360
Toledo OH-MI	770	630	630	430
Tucson AZ	970	680	770	460
Tulsa OK	770	630	630	430
Virginia Beach VA	920	550	690	360
Washington DC-VA-MD	1500	760	1100	510
Wichita KS	590	770	530	420
Winston-Salem NC	260	230	220	160
Worcester MA-CT	280	240	230	160
Average	1100	660	850	440
Small Area Average	230	280	210	160
Medium Area Average	810	650	660	440
Large Area Average	840	540	640	350
Very Large Area Average	1300	740	980	480

Notes: See Appendix D for details of calculations. Typical driver travels in the peak direction, two trips per day, 250 working days per year, and has a trip the length of the average highway miles per trip for the MSA size category. Population groups are defined as follows: small: 250,000–500,000; medium: 0.5–1 million; large: 1–3 million; very large: >3 million.

Table D.3. Average annual welfare effects of congestion pricing when using typical-trip definition of flexibility

Size of throughput drop (%)	10		17.5		25	
Fraction of lanes priced	1	0.25	1	0.5	1	0.5
Largest welfare loss ($)[†]	2320	0	1690	0	1090	0.0
	(900)	(170)	(570)	(73)	(400)	(0.0075)
	[0.0]	[0.69]	[0.0]	[0.91]	[0.01]	[1.0]
Welfare gains ($)						
Social	2350	1090	2450	1820	2520	1990
	(270)	(140)	(270)	(220)	(280)	(240)
Private	1070	392	1470	910	1860	1190
	(510)	(98)	(370)	(180)	(330)	(210)
Reduction in travel time (hours)	79.0	20.8	79.0	44.6	79.0	50.4
	(8.8)	(3.7)	(8.8)	(6.3)	(8.8)	(7.1)
Reduction in travel time costs ($)	2060	940	2060	1530	2060	1640
	(240)	(150)	(240)	(210)	(240)	(230)
Reduction in schedule delay (hours)	114.2	30.9	200	109.5	286	163.2
	(6.9)	(1.9)	(12)	(6.6)	(17)	(9.9)
Reduction in schedule delay costs ($)	289	150	386	284	464	351
	(40)	(49)	(65)	(65)	(87)	(75)
Tolls Paid ($)	1280	694	970	910	670	800
	(450)	(77)	(310)	(110)	(240)	(110)

Notes: Bootstrapped standard errors in parentheses. The fraction of bootstrapping iterations for which pricing a given fraction of the road yields a Pareto improvement is in brackets. I assume two trips per working day and 250 working days per year. Social welfare gains are the sum of the reduction in travel time costs and the reduction in schedule delay costs; and they do not include the value of saving gasoline or reducing pollution. Private welfare gains are social welfare gains minus the private cost of the tolls paid. Numbers in the table do not add up exactly due to rounding.
[†] The largest welfare loss is not an average, but the maximum annual welfare loss.

D.2. **Sensitivity tests.** This subsection contains the sensitivity tests discussed in the main text of the paper.

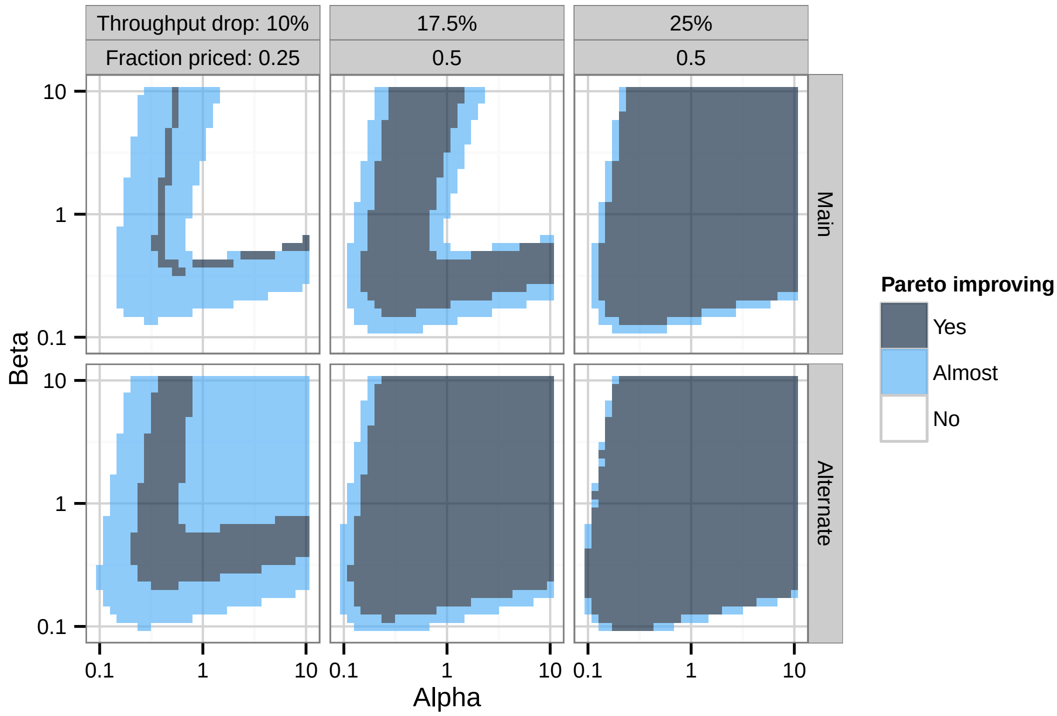


Figure D.1. Sensitivity of obtaining a Pareto improvement to the distribution of inflexibility of those in the inflexible category. The axis are the parameters of the beta distribution assumed for the distribution of inflexibility of the agents in the inflexible category. Pricing *almost* yields a Pareto improvement if the maximum harm is less than fifty cents per trip.

Table D.4. Definitions of alternate distributions of inflexibility for agents in the inflexible category

	α	β
Strictly decreasing	0.5	5
Uniform	1	1
Symmetric bimodal	0.5	0.5
Symmetric unimodal	5	5
Unimodal and skewed left	5	2
Unimodal and skewed right	2	5

Notes: These are the parameters of the beta distribution I am assuming for the distribution of inflexibility of those agents in the inflexible category; and so α is the first parameter of the distribution and β is the second parameter.

TABLE D.5. Sensitivity of main results to the distribution of inflexiblility for agents in the inflexible category

Size of throughput drop (%)	10		17.5		25	
Fraction of lanes priced	1	0.25	1	0.5	1	0.5
	Largest welfare loss					
Specific-trip definition of flexibility						
Strictly decreasing	2281	0	1841	0	1038	0
Uniform	2955	581	2258	229	1585	0
Symmetric bimodal	2748	36	2129	0	1519	0
Symmetric unimodal	3209	2333	2391	873	1627	0
Unimodal and skewed left	3417	2757	2448	851	1631	0
Unimodal and skewed right	2804	787	2191	346	1567	0
Typical-trip definition of flexibility						
Strictly decreasing	2167	0	1637	0	1086	0
Uniform	2335	51	1607	0	1119	0
Symmetric bimodal	2283	0	1639	0	1104	0
Symmetric unimodal	2360	51	1726	0	1120	0
Unimodal and skewed left	2374	51	1730	0	1040	0
Unimodal and skewed right	2313	51	1713	0	1118	0
	Average annual social welfare gains					
Specific-trip definition of flexibility						
Strictly decreasing	2327	1006	2425	1737	2514	1906
Uniform	2431	975	2557	1776	2664	1969
Symmetric bimodal	2361	1013	2478	1762	2578	1930
Symmetric unimodal	2416	758	2551	1729	2662	1969
Unimodal and skewed left	2422	708	2562	1741	2675	1982
Unimodal and skewed right	2438	948	2559	1766	2664	1969
Typical-trip definition of flexibility						
Strictly decreasing	2358	1093	2451	1821	2528	1991
Uniform	2462	1081	2563	1861	2645	2037
Symmetric bimodal	2403	1080	2500	1838	2580	2006
Symmetric unimodal	2463	1083	2564	1863	2647	2039
Unimodal and skewed left	2461	1082	2563	1862	2646	2037
Unimodal and skewed right	2464	1083	2564	1862	2646	2037
	Average annual private welfare gains					
Specific-trip definition of flexibility						
Strictly decreasing	1090	304	1385	761	1692	1019
Uniform	824	199	1263	691	1703	993
Symmetric bimodal	876	292	1273	749	1667	1000
Symmetric unimodal	683	-164	1190	551	1677	993
Unimodal and skewed left	585	-256	1165	560	1684	1006
Unimodal and skewed right	905	156	1302	663	1713	993
Typical-trip definition of flexibility						
Strictly decreasing	1163	400	1504	915	1863	1195
Uniform	1148	338	1546	881	1943	1165
Symmetric bimodal	1135	364	1514	902	1897	1170
Symmetric unimodal	1137	342	1544	882	1944	1166
Unimodal and skewed left	1128	340	1540	881	1943	1163
Unimodal and skewed right	1162	341	1550	881	1945	1165

Notes: See Table D.4 for the definitions of each alternate distribution.

Appendix E. Omitted proofs

E.1. **Additional notation used in omitted proofs.** It will be helpful for many of the proofs below to introduce some additional notation.

We can define an agent's indifference curve over arrival time, travel time, and tolls using (1). Since on a free route there will be no toll, we can define the indifference curve on a free route as

$$\check{T}\left(t;i,t^{*},p\right)=\alpha_{i}^{-1}\left[p\left(t,\text{free};i,t^{*}\right)-D_{i}\left(t-t^{*}\right)\right]. \tag{E.1}$$

Similarly, since there is to congestion related travel time on a priced route, we can define the indifference curve on a priced route as

$$\check{\tau}\left(t;i,t^{*},p\right)=p\left(t,\text{toll};i,t^{*}\right)-D_{i}\left(t-t^{*}\right). \tag{E.2}$$

Further, define

$$s_r=\begin{cases} s & r=\text{free},\\ s^{*} & r=\text{toll}.\end{cases}$$

I define Type 1 as rich and Type 2 as poor (i.e., $\alpha_1>\alpha_2$).

E.2. **Proof of Lemma B.2.**

Proof. I start by proving that on a free route, those with a high δ arrive closer to their desired arrival times. In an abuse of notation, let $t'\in[t,t^{*}]$ mean that if $t<t^{*}$, $t'\in[t,t^{*}]$, and if $t>t^{*},t'\in[t^{*},t]$.

First notice that for $t'\in[t,t^{*}]$

$$\alpha_{i}^{-1}D_{i}\left(t^{*}-t\right)=\alpha_{i}^{-1}D_{i}\left(t^{*}-t'\right)+\frac{\beta_{i}}{\alpha_{i}}\left(t'-t\right)\begin{cases}1 & t\leq t^{*}\\ -\xi & t>t^{*}\end{cases}. \tag{E.3}$$

and similarly,

$$\alpha_{j}^{-1}D_{j}\left(\tilde{t}^{*}-t\right)\leq\alpha_{j}^{-1}D_{j}\left(\tilde{t}^{*}-t'\right)+\frac{\beta_{j}}{\alpha_{j}}\left(t'-t\right)\begin{cases}1 & t\leq t^{*}\\ -\xi & t>t^{*}\end{cases}, \tag{E.4}$$

which holds with equality if $t\geq t'\geq\tilde{t}^{*}$ or $t\leq t'\leq\tilde{t}^{*}$. Since $\beta_j/\alpha_j<\beta_i/\alpha_i$, (E.3) and (E.4) imply

$$\alpha_{i}^{-1}\left[D_{i}\left(t^{*}-t\right)-D_{i}\left(t^{*}-t'\right)\right]>\alpha_{j}^{-1}\left[D_{j}\left(\tilde{t}^{*}-t\right)-D_{j}\left(\tilde{t}^{*}-t'\right)\right]\forall t'\in\left(t,t^{*}\right]. \tag{E.5}$$

This simply says type i agents face greater schedule delay costs, measured in travel time, for arriving at t rather than t', than do type j agents.

If an agent from type i with desired arrival time t^{*} arrives at t then

$$p\left(t,\text{free};i,t^{*}\right)\leq p\left(t',\text{free};i,t^{*}\right)\forall t'$$

$$\Leftrightarrow \alpha_i^{-1} p\left(t, \text{free}; i, t^*\right) \leq \alpha_i^{-1} p\left(t', \text{free}; i, t^*\right) \forall t'$$
$$\Leftrightarrow T\left(t\right) + \alpha_i^{-1} D_i\left(t^* - t\right) \leq T\left(t'\right) + \alpha_i^{-1} D_i\left(t^* - t'\right) \forall t'$$
$$\Leftrightarrow \alpha_i^{-1}\left[D_i\left(t^* - t\right) - D_i\left(t^* - t'\right)\right] \leq T\left(t'\right) - T\left(t\right) \forall t' \in \left[t, t^*\right] \forall t'$$

by (E.5) this implies

$$\alpha_j^{-1}\left[D_j\left(\tilde{t}^* - t\right) - D_j\left(\tilde{t}^* - t'\right)\right] < T\left(t'\right) - T\left(t\right) \forall t' \in \left(t, t^*\right]$$
$$\Rightarrow p\left(t, \text{free}; j, \tilde{t}^*\right) < p\left(t', \text{free}; j, \tilde{t}^*\right) \forall t' \in \left(t, t^*\right]$$

and so all agents of type j prefer to arrive at t rather than $t' \in (t, t^*]$ and thus will not arrive at t'.

Next I show that on a priced route, those with a high β arrive closer to their desired arrival time. This proof is the same as above, but replacing β/α with β, and T with τ. In an abuse of notation, let $t' \in [t, t^*]$ mean that if $t < t^*$, $t' \in [t, t^*]$, and if $t > t^*, t' \in [t^*, t]$.

First notice that for $t' \in [t, t^*]$

$$D_i\left(t^* - t\right) = D_i\left(t^* - t'\right) + \beta_i\left(t' - t\right)\begin{cases} 1 & t \leq t^* \\ -\xi & t > t^* \end{cases}, \tag{E.6}$$

and similarly,

$$D_j\left(\tilde{t}^* - t\right) \leq D_j\left(\tilde{t}^* - t'\right) + \beta_j\left(t' - t\right)\begin{cases} 1 & t \leq t^* \\ -\xi & t > t^* \end{cases}, \tag{E.7}$$

which holds with equality if $t \geq t' \geq \tilde{t}^*$ or $t \leq t' \leq \tilde{t}^*$. Since $\beta_j < \beta_i$, (E.6) and (E.7) imply

$$D_i\left(t^* - t\right) - D_i\left(t^* - t'\right) > D_j\left(\tilde{t}^* - t\right) - D_j\left(\tilde{t}^* - t'\right) \forall t' \in \left(t, t^*\right]. \tag{E.8}$$

This simply says type i agents face greater schedule delay costs, measured in dollars, for arriving at t rather than t', than do type j agents.

If an agent of type i with desired arrival time t^* arrives at t then

$$p\left(t, \text{toll}; i, t^*\right) \leq p\left(t', \text{toll}; i, t^*\right) \forall t'$$
$$\Leftrightarrow \tau\left(t\right) + D_i\left(t^* - t\right) \leq \tau\left(t'\right) + D_i\left(t^* - t'\right) \forall t'$$
$$\Leftrightarrow D_i\left(t^* - t\right) - D_i\left(t^* - t'\right) \leq \tau\left(t'\right) - \tau\left(t\right) \forall t' \in \left[t, t^*\right] \forall t'$$

by (E.8) this implies

$$D_j\left(\tilde{t}^* - t\right) - D_j\left(\tilde{t}^* - t'\right) < \tau\left(t'\right) - \tau\left(t\right) \forall t' \in \left(t, t^*\right]$$
$$\Rightarrow p\left(t, \text{toll}; j, \tilde{t}^*\right) < p\left(t', \text{toll}; j, \tilde{t}^*\right) \forall t' \in \left(t, t^*\right]$$

and so all agents of type j prefer to arrive at t rather than $t' \in (t, t^*]$ and thus will not arrive at t'.

$\square$

E.3. **Proof of Lemma B.3.**

Proof. I first derive the result for travel times.

$$\begin{aligned}
\left\{t', \text{free}\right\} \in \sigma\left(i, t^*\right) \text{ and } t' \neq t^* &\Rightarrow t' \in \arg\min_t p\left(t, \text{free}\right) \\
&\Rightarrow \frac{dp}{dt}\left(t', \text{free}\right) = 0 \\
&\Leftrightarrow \alpha_i^{-1}\frac{dD_i}{dt}\left(t'\right) = \frac{dT}{dt}\left(t'\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\{t^*, \text{free}\right\} \in \sigma\left(i, t^*\right) &\Rightarrow t^* \in \arg\min_t p\left(t, \text{free}\right) \\
&\Rightarrow \frac{dp}{dt^+}\left(t^*, \text{free}\right) \leq 0 \text{ and } \frac{dp}{dt^-}\left(t^*, \text{free}\right) \geq 0 \\
&\Leftrightarrow -\frac{\gamma_i}{\alpha_i} \leq \frac{dT}{dt}\left(t^*\right) \leq \frac{\beta_i}{\alpha_i}.
\end{aligned}$$

Next I derive the result for tolls.

$$\begin{aligned}
\left\{t', \text{toll}\right\} \in \sigma\left(i, t^*\right) \text{ and } t' \neq t^* &\Rightarrow t' \in \arg\min_t p\left(t, \text{toll}\right) \\
&\Rightarrow \frac{dp}{dt}\left(t', \text{toll}\right) = 0 \\
&\Leftrightarrow \frac{dD_i}{dt}\left(t'\right) = \frac{dT}{dt}\left(t'\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\{t^*, \text{toll}\right\} \in \sigma\left(i, t^*\right) &\Rightarrow t^* \in \arg\min_t p\left(t, \text{toll}\right) \\
&\Rightarrow \frac{dp}{dt^+}\left(t^*, \text{toll}\right) \leq 0 \text{ and } \frac{dp}{dt^-}\left(t^*, \text{toll}\right) \geq 0 \\
&\Leftrightarrow -\gamma_i \leq \frac{dT}{dt}\left(t^*\right) \leq \beta_i.
\end{aligned}$$

$\square$

E.4. **Proof of Lemma B.4.**

Proof. Consider two agents, one of whom has a higher value of time than the other, so $\alpha_1 > \alpha_2$, but whom have the same desired arrival time, so $t_1^* = t_2^*$, and inflexibility, so $\beta_1/\alpha_1 = \beta_2/\alpha_2$.

Note that

$$p\left(t, \text{free}; 1, t^*\right)/\alpha_1 = p\left(t, \text{free}; 2, t^*\right)/\alpha_2,$$

$$p\left(t,\text{toll};1,t^*\right)/\alpha_1 < p\left(t,\text{toll};2,t^*\right)/\alpha_2$$

for all t because

$$\begin{aligned}
&\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2}\\
\Leftrightarrow\; &T\left(t\right) + \frac{D_1\left(t-t^*\right)}{\alpha_1} = T\left(t\right) + \frac{D_2\left(t-t^*\right)}{\alpha_2}\\
\Leftrightarrow\; &\frac{p\left(t,\text{free};1,t^*\right)}{\alpha_1} = \frac{p\left(t,\text{free};2,t^*\right)}{\alpha_2}
\end{aligned}\tag{E.9}$$

and

$$\begin{aligned}
&\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2}\\
\Rightarrow\; &\frac{\tau\left(t\right) + D_1\left(t-t^*\right)}{\alpha_1} < \frac{\tau\left(t\right) + D_2\left(t-t^*\right)}{\alpha_2}\\
\Leftrightarrow\; &\frac{p\left(t,\text{toll};1,t^*\right)}{\alpha_1} < \frac{p\left(t,\text{toll};2,t^*\right)}{\alpha_2}.
\end{aligned}\tag{E.10}$$

If type 1 arrives at t' on the free route then

$$\begin{aligned}
&p\left(t',\text{free};1,t^*\right) \leq p\left(t,\text{toll};1,t^*\right)\forall t\\
\Leftrightarrow\; &\frac{p\left(t',\text{free};1,t^*\right)}{\alpha_1} \leq \frac{p\left(t,\text{toll};1,t^*\right)}{\alpha_1}\forall t.
\end{aligned}$$

Substituting (E.9) on the left side and (E.10) on the right side and canceling α_2 yields

$$p\left(t',\text{free};2,t^*\right) < p\left(t,\text{toll};2,t^*\right)\forall t,$$

implying that agent 2 strictly prefers arriving at t' on the free route over arriving at any time on the priced route and hence does not travel on the priced route.

Similarly, if type 2 arrives at t' on the priced route then

$$\begin{aligned}
&p\left(t',\text{toll};2,t^*\right) \leq p\left(t,\text{free};2,t^*\right)\forall t\\
\Leftrightarrow\; &\frac{p\left(t',\text{toll};2,t^*\right)}{\alpha_2} \leq \frac{p\left(t,\text{free};2,t^*\right)}{\alpha_2}\forall t
\end{aligned}$$

Substituting (E.9) on the right side and (E.10) on the left side and canceling α_i yields

$$p\left(t',\text{toll};1,t^*\right) < p\left(t,\text{free};1,t^*\right)\forall t,$$

implying that agent 1 strictly prefers arriving at t' on the priced route over arriving at any time of the free route and hence does not travel on the free route.

Thus, should there exist an $\hat{\alpha}\left(\delta,t^*\right)$ that is indifferent between traveling on either route, then all agents with the same desired arrival time and inflexibility but $\alpha > \hat{\alpha}\left(\delta,t^*\right)$ will travel on the priced route while all agents with the same desired arrival time and inflexibility but $\alpha < \hat{\alpha}\left(\delta,t^*\right)$ will travel on the free route.

Next I show that such an $\hat{\alpha}(\delta, t^*)$ exists and that it is a continuous function. First, notice that $\hat{\alpha}(\delta, t^*)$ is implicitly defined by

$$f(\alpha, \delta, t^*) = \min_t \alpha^{-1} p(t, \text{free}; \alpha, \delta, t^*) - \min_t \alpha^{-1} p(t, \text{toll}; \alpha, \delta, t^*) = 0.$$

Second, I show that f is continuous. Since T, τ, and D are continuous, $p(t, r; \alpha, \delta, t^*)$ is the sum of continuous functions, and so is continuous for all types and routes. Since the minimum of a continuous function is continuous, f is the sum of continuous functions, and so is continuous.

Third, I show a solution always exists. As shown in (E.9), $p(t, \text{free}; \alpha_1, \delta, t^*) / \alpha_1 = p(t, \text{free}; \alpha_2, \delta, t^*) / \alpha_2 \forall t$ and so $\min_t \alpha^{-1} p(t, \text{free}; \alpha, \delta, t^*)$ is constant as we vary α. Thus the first term in f is constant as we vary α. Furthermore, as long as $T(t^*) > 0$ the first term will be positive and have a maximum at the schedule delay costs of arriving at the start or end of rush hour. Next, note that

$$\min_t \alpha^{-1} p(t, \text{toll}; \alpha, \delta, t^*) = \min_t \alpha^{-1} \tau(t) + (t - t^*) \begin{cases} -\delta & t \leq t^* \\ \xi\delta & t > t^* \end{cases}.$$

The second term of this is constant and non-negative as we vary α, while by choosing α I can make the first term any positive number as long as $\tau(t) > 0$. Letting $\alpha \to \infty$ and $t = t^*$ we can get the second term of f arbitrarily close to zero. Similarly, we can let $\alpha \to 0$ so that the optimal time to arrive is at the start or end of rush hour, getting the second term of f equal to the schedule delay costs of doing so. Thus by the intermediate value theorem, there is an α such that the first and second terms of f are equal and so a solution to $f(\alpha, \delta, t^*) = 0$ always exists.

Fourth, I show f is one-to-one is α. If $\alpha_1 > \alpha_2$ then by (E.9) and (E.10)

$$\min_t p(t, \text{free}; \alpha_1, \delta, t^*) = \min_t p(t, \text{free}; \alpha_1, \delta, t^*),$$
$$\min_t p(t, \text{toll}; \alpha_1, \delta, t^*) < \min_t p(t, \text{toll}; \alpha_1, \delta, t^*),$$

and so

$$f(\alpha_1, \delta, t^*) > f(\alpha_2, \delta, t^*).$$

If $\alpha_1 \neq \alpha_2$ then $f(\alpha_1, \delta, t^*) \neq f(\alpha_2, \delta, t^*)$ and so f is one-to-one in α.

Putting all of these steps together allow us to use Kumagai (1980) to conclude $\hat{\alpha}$ exists and is continuous. □

E.5. **Proof of Lemma B.5.**

Proof. The first claim is most easily proved by noting that the difference in cost between the free route and priced route (given by (B.22)-(B.23)) does not depend on t^* when $\delta < \hat{\delta}$ and $\alpha\delta < \hat{\beta}$.

The proof for the second claim is in the text of the paper, and is repeated here for convenience. If an agent is inframarginal regardless of which route he chooses, then he arrives on-time regardless of the route he chooses. This means his cost on the free route is $\alpha T(t^*)$ and his cost on the priced route is $\tau(t^*)$, and he will chose whichever route has the lowest cost. This holds for any agent who is inframarginal regardless of which route he chooses and who has the same value of time and desired arrival time, and so all of these agents will make the same choice. □

References

Arnott, Richard and Elijah DePalma. 2011. “The Corridor Problem: Preliminary Results on the No-Toll Equilibrium.” *Transportation Research Part B: Methodological* 45:743–768. doi:10.1016/j.trb.2011.01.004.

Arnott, Richard J. 2013. “A Bathtub Model of Downtown Traffic Congestion.” *Journal of Urban Economics* 76:110–121. doi:10.1016/j.jue.2013.01.001.

Belenky, Peter. 2011. “Revised Departmental Guidance on Valuation of Travel Time in Economic Analysis.” U.S. Department of Transportation. Washington, D.C.

Brueckner, Jan K., Jacques-François Thisse, and Yves Zenou. 1999. “Why Is Central Paris Rich and Downtown Detroit Poor?: An Amenity-Based Theory.” *European Economic Review* 43:91–107. doi:10.1016/S0014-2921(98)00019-1.

Cassidy, Michael J. and Jittichai Rudjanakanoknad. 2005. “Increasing the Capacity of an Isolated Merge by Metering Its On-Ramp.” *Transportation Research Part B: Methodological* 39:896–913. doi:10.1016/j.trb.2004.12.001.

Census Bureau. 2017. “LEHD Origin-Destination Employment Statistics.” `https://lehd.ces.census.gov/data/`.

Daganzo, Carlos F. 1994. “The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent with the Hydrodynamic Theory.” *Transportation Research Part B: Methodological* 28:269–287. doi:10.1016/0191-2615(94)90002-7.

de Palma, André, Moez Kilani, and Robin Lindsey. 2005. “Congestion Pricing on a Road Network: A Study Using the Dynamic Equilibrium Simulator METROPOLIS.” *Transportation Research Part A: Policy and Practice* 39:588–611. doi:10.1016/j.tra.2005.02.018.

Fosgerau, Mogens. 2015. “Congestion in the Bathtub.” *Economics of Transportation* 4:241–255. doi:10.1016/j.ecotra.2015.08.001.

Glaeser, Edward L., Matthew E. Kahn, and Jordan Rappaport. 2008. “Why Do the Poor Live in Cities? The Role of Public Transportation.” *Journal of Urban Economics* 63:1–24. doi:10.1016/j.jue.2006.12.004.

Hall, Jonathan D. 2018. “Pareto Improvements from Lexus Lanes: The Effects of Pricing a Portion of the Lanes on Congested Highways.” *Journal of Public Economics* 158:113–125. doi:10.1016/j.jpubeco.2018.01.003.

Knight, Frank. H. 1924. "Some Fallacies in the Interpretation of Social Cost." *Quarterly Journal of Economics* 38:582–606. doi:10.2307/1884592.

Kumagai, Saki. 1980. "An Implicit Function Theorem: Comment." *Journal of Optimization Theory and Applications* 31:285–288. doi:10.1007/BF00934117.

Lighthill, M. J. and G. B. Whitham. 1955. "On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads." *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 229:317–345. doi:10.1098/rspa.1955.0089.

Margiotta, Richard, Harry Cohen, Robert Morris, Jeffrey Trombly, and Andrew Dixson. 1994. "Roadway Usage Patterns: Urban Case Studies." Volpe National Transportation Systems Center and Federal Highway Administration.

Muth, Richard F. 1969. *Cities and Housing : The Spatial Pattern of Urban Residential Land Use*. Chicago: University of Chicago Press.

Osawa, Minoru, Haoran Fu, and Takashi Akamatsu. 2018. "First-Best Dynamic Assignment of Commuters with Endogenous Heterogeneities in a Corridor Network." *Transportation Research Part B: Methodological* 117:811–831. doi:10.1016/j.trb.2017.09.003.

Pigou, Arthur Cecil. 1912. *Wealth and Welfare*. London: Macmillan and co., limited.

Richards, P. I. 1956. "Shock Waves on the Highway." *Operations Research* 4:42–51.

Schrank, David, Bill Eisele, and Tim Lomax. 2012. "2012 Urban Mobility Report." Texas A&M Transportation Institute. College Station, Texas.

Sullivan, Edward. 1999. *State Route 91 Impact Study Datasets*. San Luis Obispo, California: California Polytechnic State University. `http://ceenve3.civeng.calpoly.edu/sullivan/sr91/`.

Uber Technologies, Inc. 2019. "Uber Movement." `https://movement.uber.com`.

U.S. Department of Transportation. 2009. *2009 National Household Travel Survey*. Washington, D.C.. `http://nhts.ornl.gov`.

Walters, Alan A. 1961. "The Theory and Measurement of Private and Social Cost of Highway Congestion." *Econometrica* 29:676–699. doi:10.2307/1911814.

Wheaton, William C. 1977. "Income and Urban Residence: An Analysis of Consumer Demand for Location." *American Economic Review* 67:620–631. `https://www.jstor.org/stable/1813394`.