Mental Rotation as Bayesian Quadrature

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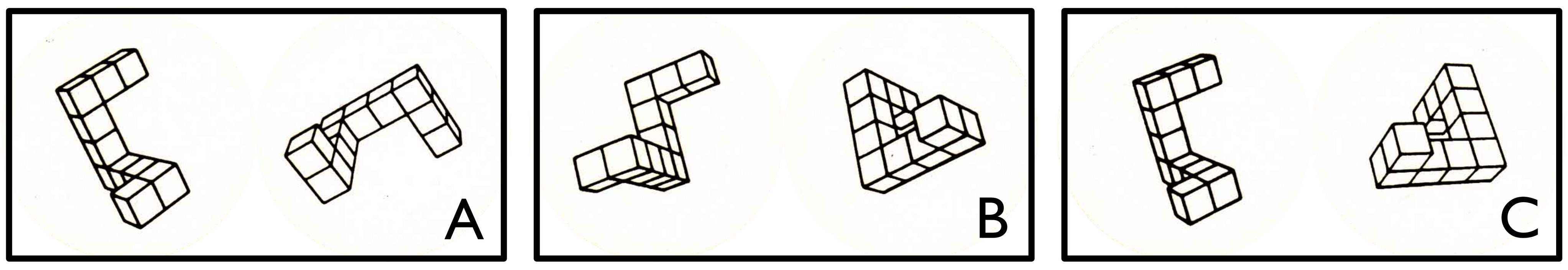
Introduction

How should an agent make best use of its computing resources?

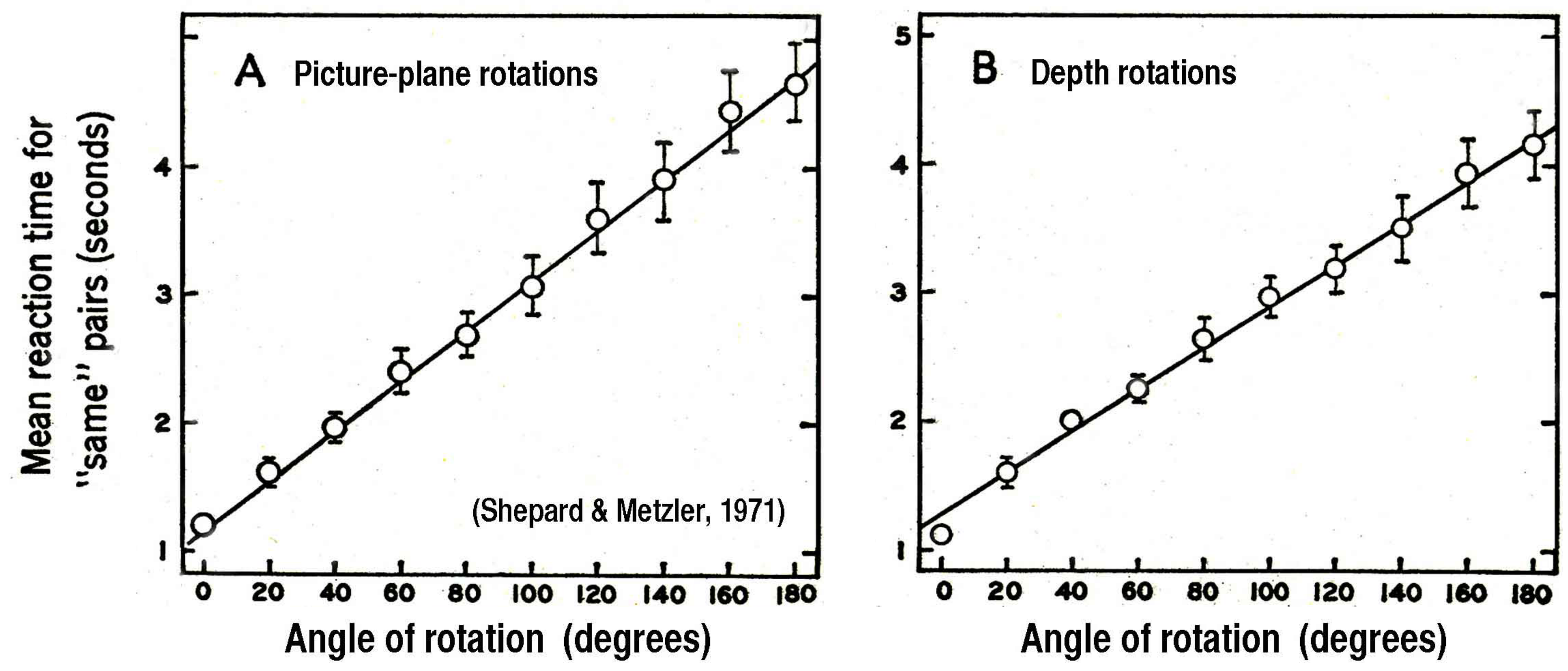
Mental rotation task: determine whether images in each pair depict the same object in multiple orientations, or different objects [1]:



How do people determine: (1) Which direction to rotate? (2) When to stop rotating?

Hypothesis: estimate the probability of one shape given the other using sequential rotations (samples), using active learning to choose the direction of rotation.

People seem to "mentally rotate" one image in the direction of least rotation until it is congruent with the other [1]:

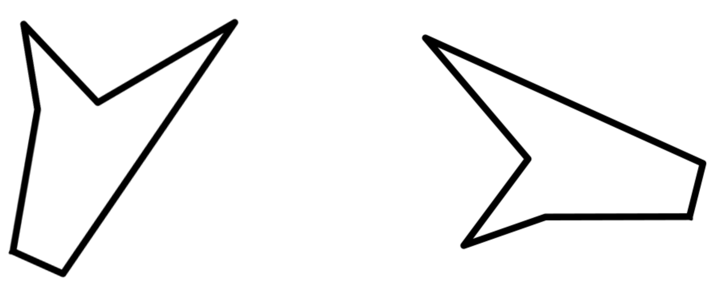


Computational-level model

Problem:

Determine if X_a is the same as X_b

X_a X_b



Hypotheses: $p(h \mid X_a, X_b) \propto p(X_a, X_b \mid h)p(h)$

h_0 : objects are different

$$p(X_a, X_b \mid h_0) = p(X_a)p(X_b)$$

h_1 : same object rotated by R

$$p(X_a, X_b \mid h_1) = \int_R p(X_a)p(X_b \mid X_a, R)p(R)\ \mathrm{d}R$$

Decision:

The hypotheses are equally likely *a priori*, so the priors cancel. So, we compare the hypotheses via their *likelihood ratio*:

$$\ell = \frac{\int_R p(X_b \mid X_a, R)p(R)\ \mathrm{d}R}{p(X_b)}$$

Choose h_0 when $\ell < 1$ Choose h_1 when $\ell > 1$

Algorithmic approximation

A *mental image* X_R is generated by performing many small rotations:

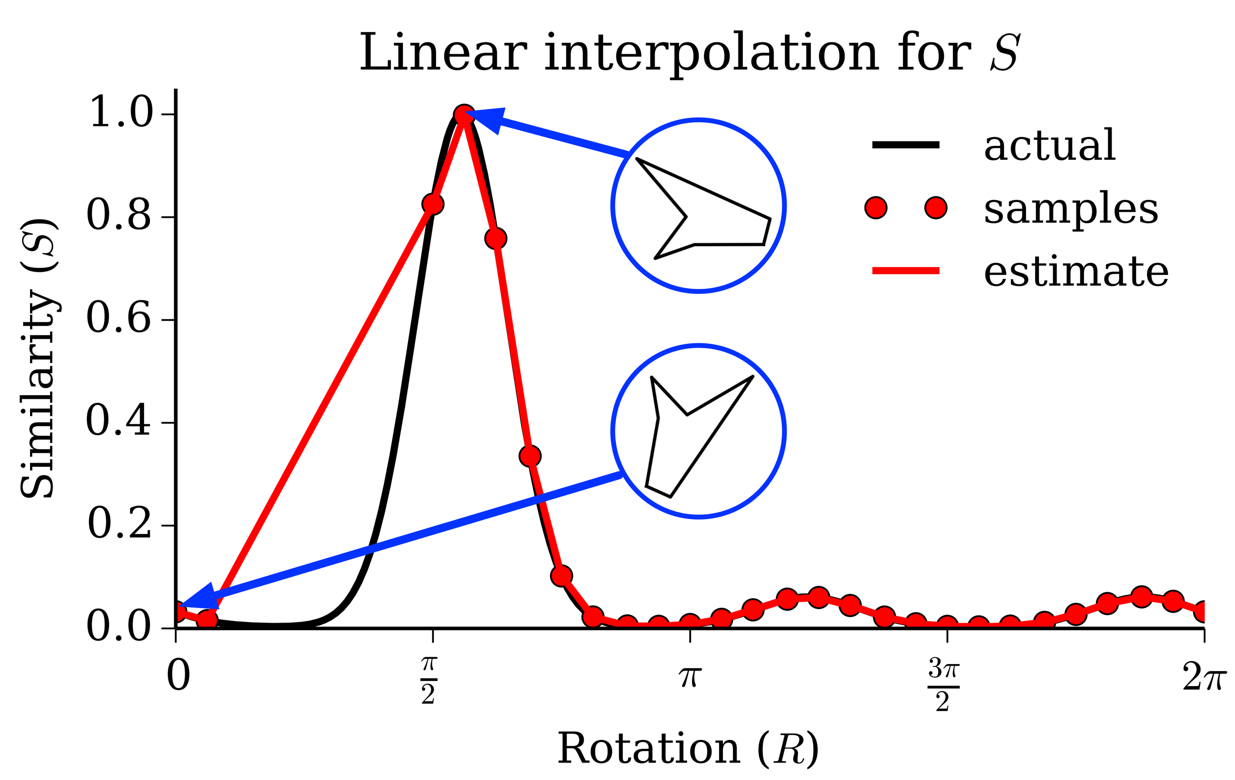
$$X_R = RX_a = \tau(X_{R-r}, r) = \tau(\tau(X_{R-2r}, r), r) = \ldots = \tau^{(\frac{R}{r})}(X_a, r)$$

We can approximate the likelihood using these small rotations:

$$\begin{aligned} p(X_a, X_b \mid h_1) &= \int_R \int_X p(X_b|X)p(X|X_a, R)p(X_a)p(R)\ \mathrm{d}X\ \mathrm{d}R \\ &= \int_R \int_X p(X_b|X)\delta(\tau^{(\frac{R}{r})}(X_a, r) - X)p(X_a)p(R)\ \mathrm{d}X\ \mathrm{d}R \\ &= p(X_a)\int_R p(X_b|X_R)p(R)\ \mathrm{d}R \approx p(X_a)\cdot Z \end{aligned}$$

where $Z = \int_R S(X_b, X_R)p(R)\ \mathrm{d}R$.

Naive: hill-climbing until maxima is reached, linearly interpolate to approximate S, then estimate Z with the trapezoidal rule.



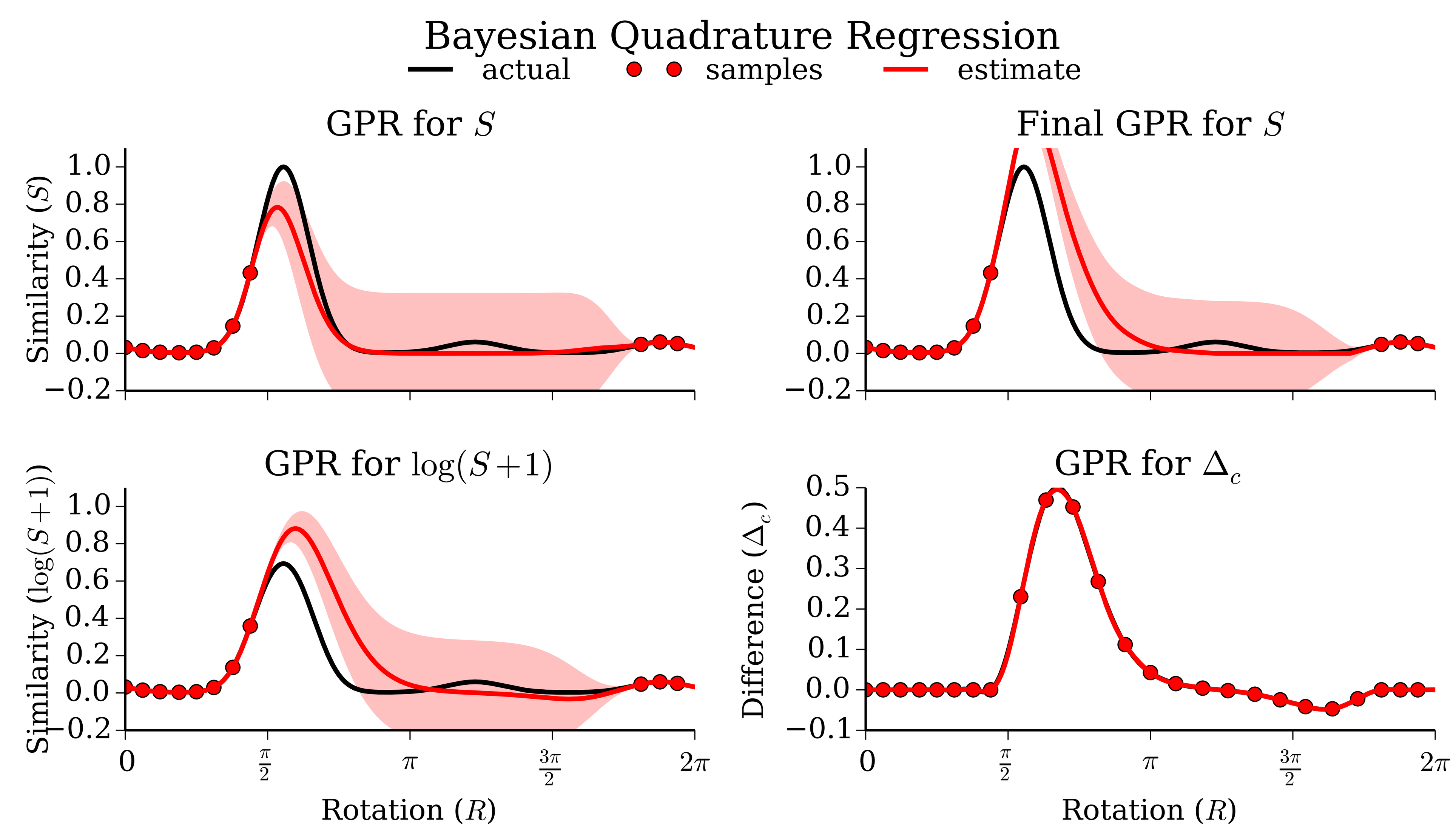
Bayesian Quadrature: one hill-climbing step, then Gaussian Process prior over $\log S$, then active sampling of rotations to reduce uncertainty, as in [2]:

$$\begin{aligned} \mu_Z &= \int_{\log S}\left(\int_R \exp(\log S(X_b, X_R))\,p(R)\ \mathrm{d}R\right)\mathcal{N}(\log S|\mu_{\log S}, \Sigma_{\log S})\ \mathrm{d}\log S \\ &\approx \int_R \mu_S(1+\mu_{\Delta_c})p(R)\ \mathrm{d}R, \text{ where } \Delta_c = \mu_{\log S} - \log \mu_S \end{aligned}$$

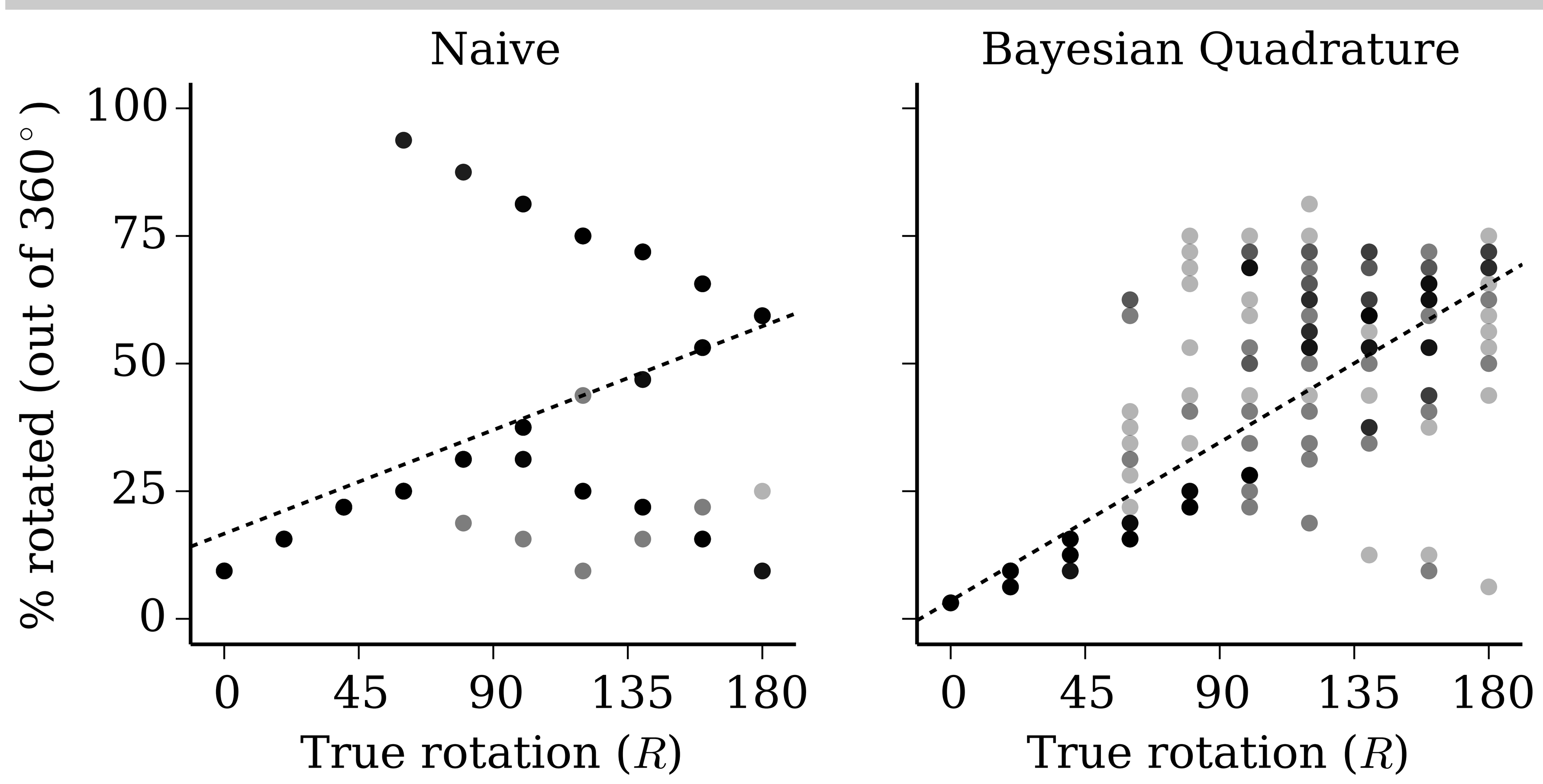
This gives us a distribution over ℓ:

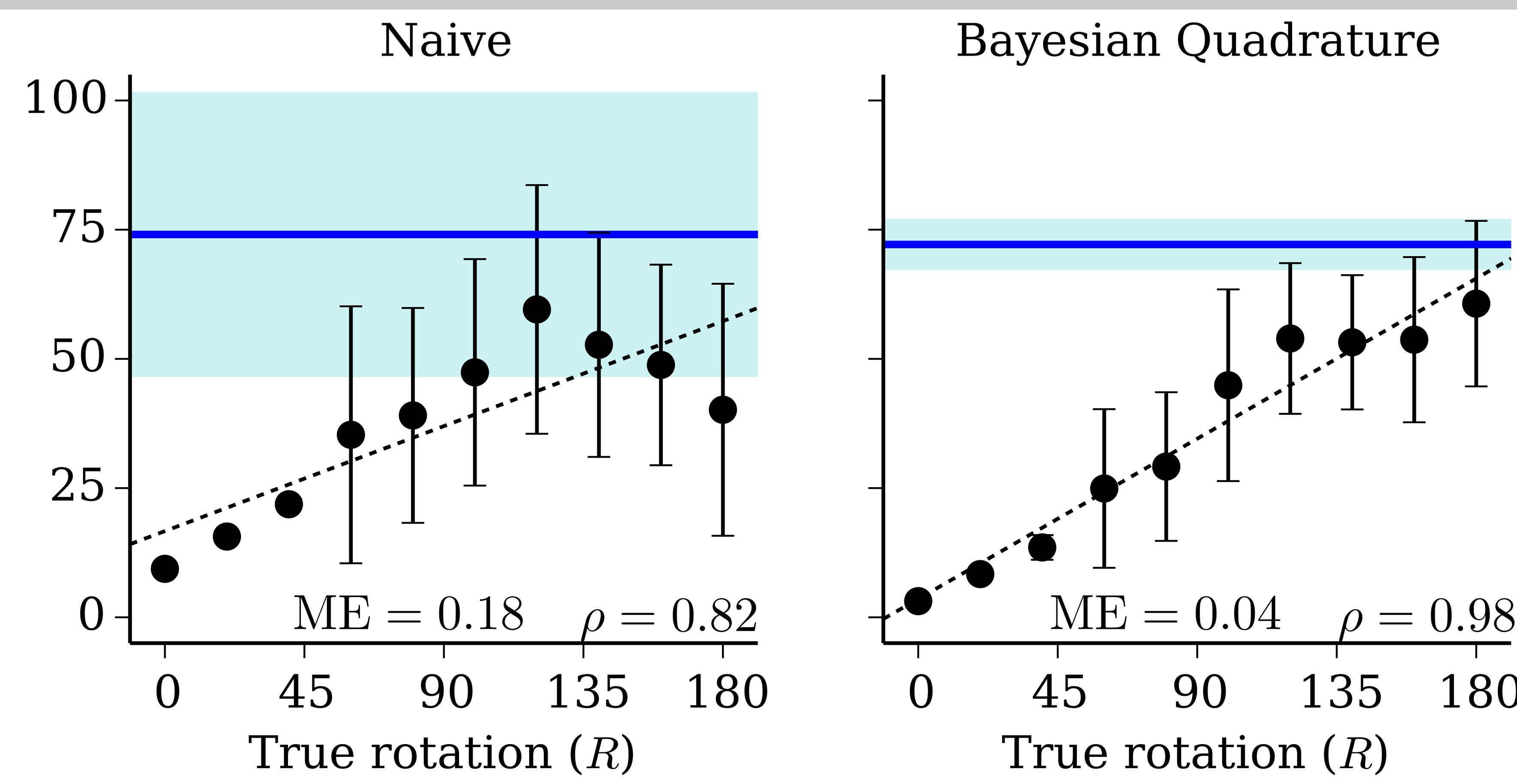
$$p(\ell) \approx \mathcal{N}(Z|\mu_Z, \sigma_Z)/p(X_b)$$

1. Choose h_0 when $p(\ell < 1) \geq 0.95$
2. Choose h_1 when $p(\ell > 1) \geq 0.95$
3. Otherwise, keep rotating



Results





Conclusions

Mental rotation task is nontrivial, and cannot be solved by a naive heuristic model

BQ model is qualitatively similar to classic results [1] and provides plausible answers:

1. Which direction? *Whichever lowers uncertainty.*
2. How long? *Until a certain threshold of confidence has been reached.*

References

1. Shepard & Metzler (1971). Mental Rotation of Three-Dimensional Objects. *Science* 171(3972), 701-703.
2. Osborne et al (2012). Active Learning of Model Evidence Using Bayesian Quatrature. *NIPS 2012*.

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