Problem Set 4

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [4].
Please see the references section for detailed citation information.

We use documentation in [**logicNot**], [**proofs**], [**bars**], [**packageClash**], [**curlyBrace**], [**cases**] to write the LaTeXsource code for this document.

2 Time to failure

The first group of problems Orloff and Bloom have for us involve some random variables that follow an exponential distribution.
The exponential distribution they give us to work with has probability density function (pdf):

$$f\left(x\right) = \lambda e^{-\lambda x}, x \geq 0. \tag{1}$$

2.1 $P\left(X \geq x\right)$

We know how to calculate $P\left(X < x\right)$ as a definite integral [1], therefore we will find $P\left(X < x\right)$, and our final result will be to find $P\left(X \geq x\right) = 1 - P\left(X < x\right)$.

In order to calculate this probability, we will do a change of variable similar to the technique Orloff and Bloom show in section 3.4 of [2].
We change the variable in the pdf $f(x)$ to u; therefore we rewrite the pdf as $f(u)$:

$$f(u) = \lambda e^{-\lambda u}. \tag{2}$$

We use this definition, the fact that f is defined for $x \geq 0$, and the definition of probability of continuous random variables [**readingx5b**] to write this equation:

$$P(X < x) = \int_0^x \lambda e^{-\lambda u}\, du. \tag{3}$$

We substitute the integral on the right hand side of the previous equation with its antiderivative to get:

$$P(X < x) = -e^{-\lambda u}\Big|_{u=0}^{x}. \tag{4}$$

We evaluate the the antiderivative at the limits of integration:

$$P(X < x) = -e^{-\lambda x} - -e^{-\lambda 0}. \tag{5}$$

Now we simplify the previous equation:

$$P(X < x) = -e^{-\lambda x} + 1. \tag{6}$$

Now, we apply the identity:

$$P(X \geq x) = 1 - P(X <). \tag{7}$$

Therefore

$$P(X \geq x) = 1 - \left(-e^{-\lambda x} + 1\right). \tag{8}$$

The previous equation simplifies to:

$$P(X \geq x) = e^{-\lambda x}. \tag{9}$$

subsectionCDF of Minimum of two exponential random variables
In this section Orloff and Bloom ask us to find the cumulative distribution function (CDF) of two independent random variables X_1, and X_2 that both follow an exponential distribution, and that both have mean $\frac{1}{\lambda}$.
In [3] Orloff and Bloom state that the mean of a random variable that has probability mass function (pmf) $\lambda e^{-\lambda x}$ is $\frac{1}{\lambda}$.
Therefore X_1, and X_2 both have pmf's $\lambda e^{-\lambda x}$.
For this problem, Orloff and Bloom let $T = \min(X_1, X_2)$.
They ask us for the cdf of T.
The cdf of T is a function $F(t) = P(T < t)$.

In the previous section, we found that for a random varialbe X that has pdf $\lambda e^{-\lambda x}$,

$$P(X \geq x) = e^{-\lambda x}. \tag{10}$$

We use the definition of T to write the equation:

$$P(T \geq t) = P(\min(X_1, X_2) \geq t). \tag{11}$$

$T = \min(X_1, X_2)$, so $T \geq t$ if, and only if, $X_1 \geq t$, and $X_2 \geq t$.

Lemma 1. *If two events A and B have a biconditional relation, then*

$$P(A) = P(B). \tag{12}$$

Proof. The proof is by contradiction. Assume events A and B are biconditionally related, but $P(A) \neq P(B)$. Then there would be unequal chances of events A and B occuring, which means that one event would occur while the other does not. But A and B are biconditionally related, so event A occurs when, and only when, event B occurs. This is a contradiction, so $P(A) = P(B)$. $\square$

Therefore

$$P(T \geq t) = P(X_1 \geq t, X_2 \geq t). \tag{13}$$

X_1 and X_2 are independent events. In [2] Orloff and Bloom state that random variables X_1 and Y_1 are independent if and only if:

$$P(X_1, X_2) = F_{X_1}(x_1) F_{X_2}(x_2). \tag{14}$$

$F(X_1, X_2)$ is the cdf of X_1, and X_2. $F_{X_1}(x_1)$, and $F_{X_2}(x_2)$ are the marginal cumulative distribution functions of X_1, and X_2.
We know that X_1, and Y_1 are exponentially distributed random variables with mean $\frac{1}{\lambda}$. The answer we find in the previous section implies that the cdf of X_1 is $F_{X_1}(x_1) = e^{-\lambda x_1}$, and the cdf of X_2 is $F_{X_2}(x_2) = e^{-\lambda x_2}$.
Since X_1, and X_2 are independent,

$$F(X_1, X_2) = e^{-\lambda x_1} e^{-\lambda x_2}. \tag{15}$$

We add the exponents of the base e to simplify the previous equation to:

$$F(X_1, X_2) = e^{-\lambda(x_1 + x_2)}. \tag{16}$$

We are finding the cdf of T.
Consider $F(X_1 < t, X_2 < t)$. We use the previous equation to write:

$$F(X_1 < t, X_2 < t) = e^{-\lambda 2t}. \tag{17}$$

References

[1] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf` (Spring 2014).

[2] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf` (Spring 2014).

[3] Jeremy Orloff and Jonathan Bloom. *Gallery of Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5c.pdf` (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *Joint Distributions, Independence Covariance and Correlation 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class7slides.pdf` (Spring 2014).