MIT Introduction to Statistics 18.05 Reading 7a Questions

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [3].
Please see the references section for detailed citation information.

We use documentation in to write LaTeXsource code for this document.

2 Joint pdf

The first question Orloff and Bloom ask is what the value of the constant c is, where $f(x, y)$ is a pdf. They give further details on f:

- f is defined on $[0, 1] \times [0, 1]$, and
- $f(x, y) = cxy$.

In order for f to be a pdf:

$$\int_0^1 \int_0^1 cxy \, dy \, dx = 1. \tag{1}$$

Equation 1 comes from the definition and properties of a joint pdf that Orloff and Bloom state in [2].
We use properties of double integrals from [1], and methods of integration in [5] to replace the integral on the left hand side of 1 with its anti-derivative evaluated over the region $[0, 1] \times [0, 1]$

$$c \frac{x^2}{2} \bigg|_{x=0}^{1} \frac{y^2}{2} \bigg|_{y=0}^{1} = 1. \tag{2}$$

We evaluate the left hand side of 2 to get

$$c \frac{1}{4} = 1. \tag{3}$$

Therefore $c = 4$.

3 Joint probability table

Orloff and Bloom give us this table for this question:

X/Y	1	2	3	4
1	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
3	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Note: the index of a square in a table is the ordered pair (i, j) such that the square is in the i^{th} row and j^{th} column of the table.
Note: the table we are dealing with in these questions is a joint probability table that Orloff and Bloom introduce, but do not define precisely, in [2].
For an informal definition of a joint probability table, see [4].
They ask us what is the probability that $X \leq 2$ and $Y \leq 2$.
We follow the method of example 3 in [2] to compute this probability.
We describe the event A that $X \leq 2$ and $Y \leq 2$ with the set of ordered pairs $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
The probability of A is the sum of the probabilities of each of the squares with indicies that are elements of A. Therefore

$$P(A) = \frac{1}{24} + \frac{1}{24} + \frac{1}{12} + \frac{1}{12} = \frac{6}{24} = \frac{1}{4}. \tag{4}$$

4 Marginal probability

Orloff and Bloom ask us what is the marginal probability that $X = 1$, where X is the random variable in the table above.
The marginal probability that $X = 1$ is the sum of all the squares in the table above where the first index of the square has value 1.
For this problem this probability m is

$$m = \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} = \frac{4}{24} = \frac{1}{6}. \tag{5}$$

5 Independence

The last question Orloff and Bloom have for us is about the independence of X and Y. They ask us whether or not these random variables are independent.
Orloff and Bloom give a definition for the independence of jointly-distributed random variables in [2]. The definition states that two jointly distributed random variables are independent if their joint cumulative distribution function (cdf) is the product of the marginal cdf.
In the discrete case, such as the one we have for this problem, Orloff and Bloom go on to state in [2] that this definition of independence is equivalent to the condition that the joint probability mass function is equal to the product of the marginal probability mass function. When they write, "the joint probability mass function," in [2], it is clear that they mean the joint probability mass function of each square. So, in order to verify that two discrete, jointly-distributed random variables are independent, we must check that the probability in each square of the joint probabiltiy table is the product of the marginal probabilities for the row and column that the square is in.
We agument the table Orloff and Bloom give us for these questions with the marginal probabilities, then inspect the probabilities within it to find out whether or not X and Y are independent.

X/Y	1	2	3	4	
1	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{6}$
2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
3	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

We are grateful to Orloff and Bloom for giving us a table with repeating values in every row, so we need only check that the products of the marginal probabilities are equal to the joint probabilities in one column.

$$\frac{1}{24} = \frac{1}{4} \times \frac{1}{6} \tag{6}$$

$$\frac{1}{12} = \frac{1}{4} \times \frac{1}{3} \tag{7}$$

$$\frac{1}{8} = \frac{1}{4} \times \frac{1}{2} \tag{8}$$

The equations above hold for the joint probability in any square that is on the same row in the table Orloff and Bloom give for this problem, so every square has a value that is equal to the product of its respective marginal probabilites. Hence, the random variables X, and Y, of the joint distribution that the table above describes are independent.

References

[1] Paul Dawkins. *Paul's Online Math Notes*. Available at `http://tutorial.math.lamar.edu/Classes/CalcIII/DIGeneralRegion.aspx` (2017).

[2] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf` (Spring 2014).

[3] Jeremy Orloff and Jonathan Bloom. *Reading Questions 7a*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/reading-questions-7a/` (Spring 2014).

[4] StatTrack.com. *Statistics and Probability Dictionary*. Available at `http://stattrek.com/statistics/dictionary.aspx?definition=joint_probability_distribution` (2017).

[5] Unknown. *Double integrals*. Available at `http://www.stankova.net/statistics_2012/double_integration.pdf` (Accessed 4/11/2017).