Answers To Questions in Conditional Probability, Independence, Bayes Theorem 18.05

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Contents

1 References and License

The material for the course is licensed under the terms at `http://ocw.mit.edu/terms`.
In this document, we are answering the questions in [1].
We use documentation in for properly writing the LaTeXsource code for this document.

2 Probability At Least 3 Heads, Given First Toss Tails

We are tossing a coin four times. Therefore we define the sample space

$$\Omega = \{(x_1, x_2, x_3, x_4), x_1, x_2, x_3, x_4 \in \{H, T\}\} \quad (1)$$

$|\Omega| = 16$
We assume all outcomes are equally likely.
A is the event that at we toss heads at least 3 times.

B is the event that we toss tails the first time.
We use the definition of conditional probability to cacluate $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ Provided } P(B) \neq 0 \tag{2}$$

$P(A) = \frac{5}{16}$ since there are 5 elements of Ω that represent the event that we toss heads at least three times, and we assume all outcomes are equally likely.
These are:(T, H, H, H), (H, T, H, H), (H, H, T, H), (H, H, H, T), (H, H, H, H).
The elements of B are (T, T, T, T), (T, T, T, H), (T, T, H, T), (T, T, H, H), (T, H, T, T), (T, H, T, H), (T, H, H, T), (T, H, H, H).
By inspection $A \cap B$ is the element (T, H, H, H).
We substitute values into 2 to get

$$P(A \mid B) = \frac{\frac{1}{16}}{\frac{8}{16}} = \frac{1}{8} \tag{3}$$

3 Probability First Toss Tails, Given At Least 3 Heads

We use 2 and definitions of the sets Ω, A, $A \cap B$, and B that we define in section 1. In addition we assume all outcomes are equally likely.
We use 2 to get

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}, \text{ Provided } P(A) \neq 0 \tag{4}$$

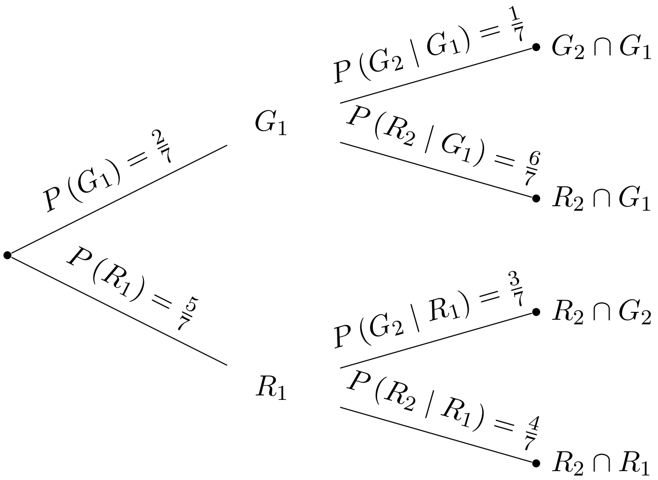
The $\cap$ operator is commutative, so $P(A \cap B) = P(B \cap A)$, and we discover $P(A \cap B)$ in section 1.
Therefore,

$$P(B \mid A) = \frac{\frac{1}{16}}{\frac{5}{16}} = \frac{1}{5} \tag{5}$$

4 Probability Urn

4.1 Probability Second Ball Red

Orloff and Bloom ask the following question in [1], "What is the probability the second ball is red?"
We are given the probability tree below:



We use the law of total probability to write an equation for $P(R_2)$.

$$P(R_2) = P(R_1 \cap R_2) + P(G_1 \cap R_2) \tag{6}$$

Now we can use the definition of conditional probability to rewrite 6:

$$P(R_2) = P(R_2 \mid R_1)\, P(R_1) + P(R_2 \mid G_1)\, P(R_1) \tag{7}$$

Orloff and Bloom give us the probabilities in 7 in the probability tree above, so we can use them to compute $P(R_2)$.

$$P(R_2) = \left(\frac{4}{7}\right)\left(\frac{5}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{2}{7}\right) = \frac{20}{49} + \frac{12}{49} = \frac{32}{49} \approx 0.653 \tag{8}$$

4.2 Probability First Ball Red, Given second Ball Red

To answer the question, "What is the probability the first ball was red given the second ball was red?" [1]
This question is asking for $P(R_1 \mid R_2)$.
Since we know $P(R_2 \mid R_1)$, we apply Bayes Theorem to compute $P(R_1 \mid R_2)$.

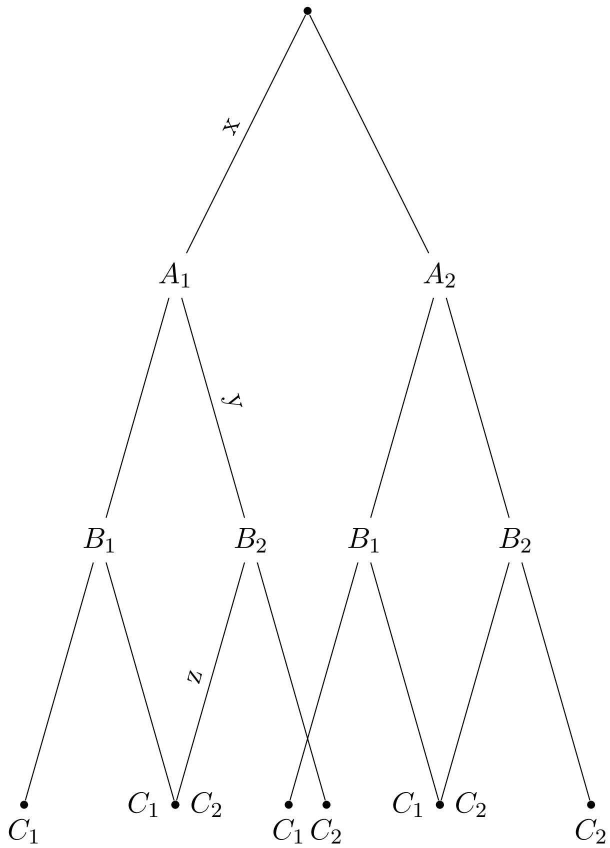
$$P(R_1 \mid R_2) = \frac{P(R_2 \mid R_1)\, P(R_1)}{P(R_2)} \tag{9}$$

Bloom and Orloff give us the values for all of the probabilities in the right hand side of 10 in the probability tree above.
Therefore

$$P(R_1 \mid R_2) = \frac{\left(\frac{4}{7}\right)\left(\frac{5}{7}\right)}{\frac{32}{49}} = \left(\frac{20}{49}\right)\left(\frac{49}{32}\right) = \frac{20}{32} \approx 0.625 \tag{10}$$

5 Concept Questions on Probability Trees

In this section, we answer questions about the probability tree that Orloff and Bloom give in [1].



The edge labeled, "x," in the figure above represents $P(A_1)$.
The edge labeled, "y," in the figure above represets $P(B_2 \mid A_1)$.
The edge labeled, "z," in the figure above represents $P(C_1 \mid A_1 \cap B_2)$. We read section 5.1, titled, "Shorthand vs. precise trees," in [1] in order to understand what the edge labeled, "z," represents. This section explains that nodes having a distance of two or more edges to the root of the tree represent the probabilities of intersections of sets of outcomes.

References

[1] Jeremy Orloff and Jonathan Bloom. *Conditional Probability, Independence, Bayes Theorem 18.05*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class3slides.pdf` (Spring 2014).