MIT Introduction to Statistics 18.05 Reading 4 - *Think* Questions

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
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The material for the course is licensed under the terms at `http://ocw.mit.edu/terms`.
We are answering the questions that Orloff and Bloom ask after the word, "think," in [3].
We use documentation in [5], [1], and [2] to write LaTeXsource code of this document.

2 The Probability Mass Function for $Z(i,j) = i + j$

We write the *pmf* for the events that we roll two dice and the sum of the values we roll is a particular value of a:

Value a	2	3	4	5	6	7	8	9	10	11	12
pmf $p(a)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Orloff and Bloom ask if this looks familiar. It does not look familiar to us at this time.

3 Properties of Cumulative Distribution Functions (cdf's)

3.1 cdf's are non-decreasing

Cdf's are non-decreasing because they are sums of probability mass function (pmf) values.
Orloff and Bloom define probability mass functions in [3], and they state that the value of a probability mass function p, for any input a is always greater than or equal to 0.
If we assume that for some cdf F that $F(b) < F(a), b > a$, that would mean that for some value $c, a < c \leq b$, $p(c) < 0$. Our assumption thus forces a contradiction of the definition of probability mass functions, so it must be wrong. Therefore cdf's are non-decreasing.

3.2 Cdf's approach 0 as $a \to -\infty$

Orloff and Bloom define random variables as functions X that map elements ω of a sample space Ω to elements of $\mathbb{R}$. We deonte a random variable as X. Orloff and Bloom define the mapping in symbols as $X : \Omega \to \mathbb{R}$.
Orloff and Bloom define a probability mass function as having the value 0 for values that the random variable X never takes.
We can order the values that X takes because they are elements of $\mathbb{R}$. There must be somme least value l that X takes. For any real number less than l, the probability mass function has value 0. Therefore the sums of probability mass functions $p(a)$ for $a < l$ will also be 0. We note that these sums satisfy the definition of a cumulative distribution functions $F(l)$. Therefore we conclude

$$\lim_{a \to -\infty} F(a) = 0 \tag{1}$$

3.3 Cdf's have values between 0 and 1

We show in the previous section that Cdf's have a minimum value of 0 for sufficiently small values of a.
Orloff and Bloom define probabilty mass functions $p(a)$ to be the probability of the event that a random variable X takes the value a.
In this section we define Ω to be the set of all events that a random variable takes on all of its possible values, and ω to be an element of Ω.

We claim that the elements ω are disjoint.
We justify this claim in a proof by contradiction. If some elements ω were not disjoint, then two events in Ω would have elements in common. This would mean that events where X takes on the same value a are considered different. This is absurd because we cannot distinguish the events. Therefore the elements of Ω are disjoint.
Since the elements of Ω are disjoint, the sum of the probability mass functions $P(X = a)$ are the sums of the probabilities of the unions of elements of Ω.
The sum of probabilities of all events in a sample space is one.
Therefore the maximum value of a Cdf is one.

3.4 Cdf's approach 1 as $a \to \infty$

We make a note that Orloff and Bloom define a cdf $F(a)$ as the sum of all pmf's $p(b)$ where b is any real number less than or equal to a.
In the previous section we showed that the sum of probability mass functions for all events that a random variable attains values is 1.
We note that as $a \to \infty$, in order to compute the cumulative distribution function $F(a)$ we are adding more probability mass functions $p(a)$ for events that our random variable takes the value a. At some point, we will include all possible values that X is defined to take, as a grows larger and larger. We will include all events in the sample space.
Hence, by our reasoning in the previous section:

$$\lim_{a\to\infty} F(a) = 1 \tag{2}$$

4 Binomial Distribution Probability Not Surprising

We equate tossing three or more heads out of five tosses with choosing 3 or more elements from a set of 5.
There are $\binom{5}{3} + \binom{5}{4} + \binom{5}{5}$ ways to choose these elements. This is equal to $10 + 5 + 1 = 16$.
There are 2^5 possible ways of tossing a coin, so a result of $\frac{1}{2}$ is not surprising.

5 Sum of Random Variables

Orloff and Bloom give two independent variables X, and Y.
Furthermore:

$$X\tilde{\ }\text{binomial}\left(n, \frac{1}{2},\right) \tag{3}$$

and

$$Y\tilde{}\text{binomial}\left(m, \frac{1}{2},\right) \tag{4}$$

X and Y are independent random variables so the probability of the union of events where X takes a value and Y takes a value is the sum of the probabilities of the individual events. We learn this in [4], section 3, "Some rules of probability."
We wish to know what $X + Y$ is.
Orloff and Bloom define random variables as functions with output values in the real numbers, so it makes sense to write $X + Y$ to mean the sum of the values of functions with output values in the real numbers. In this case, it is the sum of the number of times we toss heads in $m + n$ trials.
Therefore $X + Y$ is the count of the number of heads we toss for $m + n$ tosses, with probability of $\frac{1}{2}$.
Orloff and Bloom in [3], section 3 state that coin tosses are Bernoulli trials, with probability $\frac{1}{2}$.
$X + Y$ is the number of successes in $m + n$ independent Bernoulli trials, where the probability of success is $\frac{1}{2}$.
Therefore $X+Y$ is the same as the experiment that the binomial distribution models that Orloff and Bloom define in example 7 in section 3.2 of [3], if we substitute $m + n$ for the number of coin tosses.
Hence, $X + Y\tilde{}\text{binomial}\left(m + n, \frac{1}{2}\right)$

References

[1] StackExchange User Peter Grill. *LHow to make the limit (mathematics) sign? Answer*. Ed. by StakExchange User kiss my armpit. Available at `http://tex.stackexchange.com/questions/74969/how-to-make-the-limit-mathematics-sign`(2012/10/2).

[2] oeis.org. *List of LaTeX mathematical symbols*. Available at `https://oeis.org/wiki/List_of_LaTeX_mathematical_symbols`(2015/5/31).

[3] Jeremy Orloff and Jonathan Bloom. *Discrete Random Variables Class 4, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading4a.pdf` (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *Probability: Terminology and Examples Class 2, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading2.pdf` (Spring 2014).

[5] Scott Pakin. *The Comprehensive Latex Symbol List.* Available at `https://math.uoregon.edu/wp-content/uploads/2014/12/compsymb-1qyb3zd.pdf`(2002/10/8).