MIT Introduction to Statistics 18.05 Reading 6A Think Questions

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [2].
Please see the references section for detailed citation information.
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2 Questions about X

In this section we answer questions Orloff and Bloom ask in [3] regarding a random variable X.
Orloff and Bloom specify that X is defined on $[0, 1]$, and the pdf of X is cx^2.

2.1 Value of c

Orloff and Bloom ask us to calculate the value of c. We will use rules and properties for integration from [1] in order to calculate the value for c.

We know

$$\int_0^1 cx^2 \, dx = 1. \tag{1}$$

Therefore

$$c \int_0^1 x^2 \, dx = 1. \tag{2}$$

The anti-derivative of x^2 is $\frac{x^3}{3} + C$, so we can replace the integral in the equation above with:

$$c \left(\frac{x^3}{3} \Big|_0^1 \right) = 1. \tag{3}$$

We then evaluate the anti-derivative over the interval $[0, 1]$ to obtain:

$$c \left(\frac{1^3}{3} \right) = 1. \tag{4}$$

This implies $c = 3$.

2.2 Mean, variance, and standard deviation of X

2.2.1 Mean of X

We use the definition of mean value that Orloff and Bloom give in [3]. The mean value of X is

$$\mu = \int_0^1 x \left(3x^2\right) \, dx. \tag{5}$$

We multiply the terms in the polynomial in the integral above to get:

$$\mu = \int_0^1 \left(3x^3\right) \, dx. \tag{6}$$

We replace the integral above with its anti-derivative:

$$\mu = \frac{3x^4}{4} \Big|_0^1 . \tag{7}$$

We evaluate the anti-derivative over the closed interval $[0, 1]$ to find the value of the mean of X:

$$\mu = \frac{3}{5} - \frac{18}{16} + \frac{27}{48} \frac{3}{4}. \tag{8}$$

2.2.2 Variance of X

We use the definition of the variance of a continuous random variable in [3] to compute the variance of X.
The definition of Variance Orloff and Bloom give in [3]:

$$\mathrm{Var}\,(X) = E\left((X-\mu)^2\right). \tag{9}$$

We use the values for c and μ that we find above to find:

$$\mathrm{Var}\,(X) = int_0^1 x^2 3\left(x - \frac{3}{4}\right)^2 \, dx. \tag{10}$$

Now we multiply some of the factors in the polynomial in the integral above to get:

$$\mathrm{Var}\,(X) = int_0^1 x^2 3\left(x^2 - \frac{6x}{4} + \frac{9}{16}\right) dx. \tag{11}$$

We continue multiplying factors:

$$\mathrm{Var}\,(X) = int_0^1 3x^4 - \frac{18x^3}{4} + \frac{27x^2}{16}\, dx. \tag{12}$$

Now we replace the integral above with its anti-derivative:

$$\mathrm{Var}\,(X) = \frac{3x^4}{5} - \frac{18x^4}{16} + \frac{27x^3}{48}\Big|_0^1. \tag{13}$$

And, we evaluate the anti-derivative over the interval $[0,1]$:

$$\mathrm{Var}\,(X) = \frac{3}{5} - \frac{18}{16} + \frac{27}{48} = \frac{3}{5} - \frac{18}{16} + \frac{9}{16}. \tag{14}$$

Now we simplify the expression above further:

$$\mathrm{Var}\,(X) == \frac{3}{5} - \frac{18}{16} + \frac{9}{16} = \frac{3}{5} - \frac{9}{16} = \frac{48}{80} - \frac{45}{80} = \frac{3}{80}. \tag{15}$$

Therefore the variance of X is $\frac{3}{80}$.

2.3 Standard Deviation of X

The standard deviation of X is the square root of its variance [3]. Therefore the standard deviation of X is:

$$\sigma = \sqrt{\frac{3}{80}} \approx 0.194. \tag{16}$$

2.4 Median value of X

The median value of X is the 0.5 quantile of the cdf of X [3].
In the first part of this problem, we find that the pdf of X is $3x^2$.
Therefore we must solve the equation:

$$int_0^a 3x^2 \, dx = 0.5 \tag{17}$$

We replace the integral in the equation above with its anti-detivative:

$$\left. \frac{3x^3}{3} \right|_0^a = 0.5. \tag{18}$$

And, we evaluate the anti-derivative above over the interval $[0, a]$:

$$\frac{3a^2}{3} = 0.5. \tag{19}$$

Now it is a matter of doing some algebra to solve for a:

$$3a^2 = 0.5 \times 3. \tag{20}$$

This implies:

$$a^3 = 0.5. \tag{21}$$

Therefore, the median value of X is $\sqrt[3]{0.5} \approx 0.816$.

References

[1] Michael Dougherty. *Chapter 6 Basic Integration*. Available at `http://faculty.swosu.edu/michael.dougherty/book/chapter06.pdf` (2012/11/20).

[2] Jeremy Orloff and Jonathan Bloom. *Continuous Expectation and Variance, the Law of Large Numbers, and the Central Limit Theorem 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class6slides.pdf` (Spring 2014).

[3] Jeremy Orloff and Jonathan Bloom. *Expectation, Variance and Standard Deviation for Continuous Random Variables Class 6, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading6a.pdf` (Spring 2014).