Answers To Questions in Conditional Probability, Independence, Bayes Theorem 18.05

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1 References and License

In this document, we are answering the questions in [2].
We use documentation in [4] for properly writing the LaTeXsource code for this document.

2 Probability At Least 3 Heads, Given First Toss Tails

We are tossing a coin four times. Therefore we define the sample space

$$\Omega = \{(x_1, x_2, x_3, x_4), x_1, x_2, x_3, x_4 \in \{H, T\}\} \tag{1}$$

$|\Omega| = 16$
We assume all outcomes are equally likely.
A is the event that at we toss heads at least 3 times.
B is the event that we toss tails the first time.
We use the definition of conditional probability to cacluate $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ Provided } P(B) \neq 0 \tag{2}$$

$P(A) = \frac{5}{16}$ since there are 5 elements of Ω that represent the event that we toss heads at least three times, and we assume all outcomes are equally likely.
These are:(T, H, H, H), (H, T, H, H), (H, H, T, H), (H, H, H, T), (H, H, H, H).
The elements of B are (T, T, T, T), (T, T, T, H), (T, T, H, T), (T, T, H, H), (T, H, T, T), (T, H, T, H), (T, H, H, T), (T, H, H, H).
By inspection $A \cap B$ is the element (T, H, H, H).
We substitute values into 2 to get

$$P(A \mid B) = \frac{\frac{1}{16}}{\frac{8}{16}} = \frac{1}{8} \tag{3}$$

3 Probability First Toss Tails, Given At Least 3 Heads

We use 2 and definitions of the sets Ω, A, $A \cap B$, and B that we define in section 1. In addition we assume all outcomes are equally likely.
We use 2 to get

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}, \text{ Provided } P(A) \neq 0 \tag{4}$$

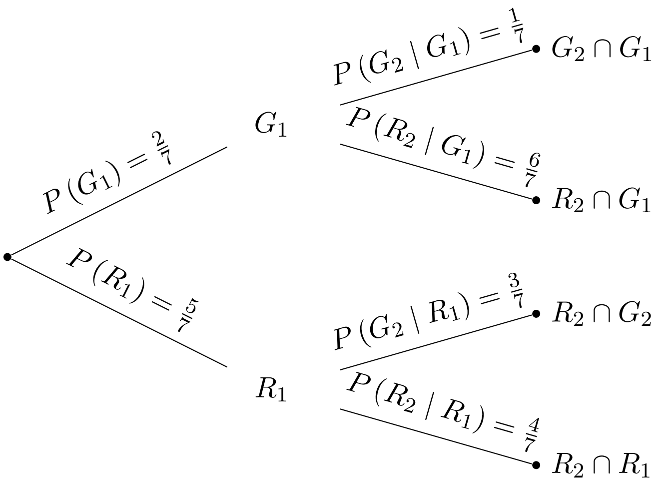
The $\cap$ operator is commutative, so $P(A \cap B) = P(B \cap A)$, and we discover $P(A \cap B)$ in section 1.
Therefore,

$$P(B \mid A) = \frac{\frac{1}{16}}{\frac{5}{16}} = \frac{1}{5} \tag{5}$$

4 Probability Urn

4.1 Probability Second Ball Red

Orloff and Bloom ask the following question in [2], "What is the probability the second ball is red?"
We are given the probability tree below:



We use the law of total probability to write an equation for $P(R_2)$.

$$P(R_2) = P(R_1 \cap R_2) + P(G_1 \cap R_2) \tag{6}$$

Now we can use the definition of conditional probability to rewrite 6:

$$P(R_2) = P(R_2 \mid R_1)\, P(R_1) + P(R_2 \mid G_1)\, P(R_1) \tag{7}$$

Orloff and Bloom give us the probabilities in 7 in the probability tree above, so we can use them to compute $P(R_2)$.

$$P(R_2) = \left(\frac{4}{7}\right)\left(\frac{5}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{2}{7}\right) = \frac{20}{49} + \frac{12}{49} = \frac{32}{49} \approx 0.653 \tag{8}$$

4.2 Probability First Ball Red, Given second Ball Red

To answer the question, "What is the probability the first ball was red given the second ball was red?" [2]
This question is asking for $P(R_1 \mid R_2)$.
Since we know $P(R_2 \mid R_1)$, we apply Bayes Theorem to compute $P(R_1 \mid R_2)$.

$$P(R_1 \mid R_2) = \frac{P(R_2 \mid R_1)\, P(R_1)}{P(R_2)} \tag{9}$$

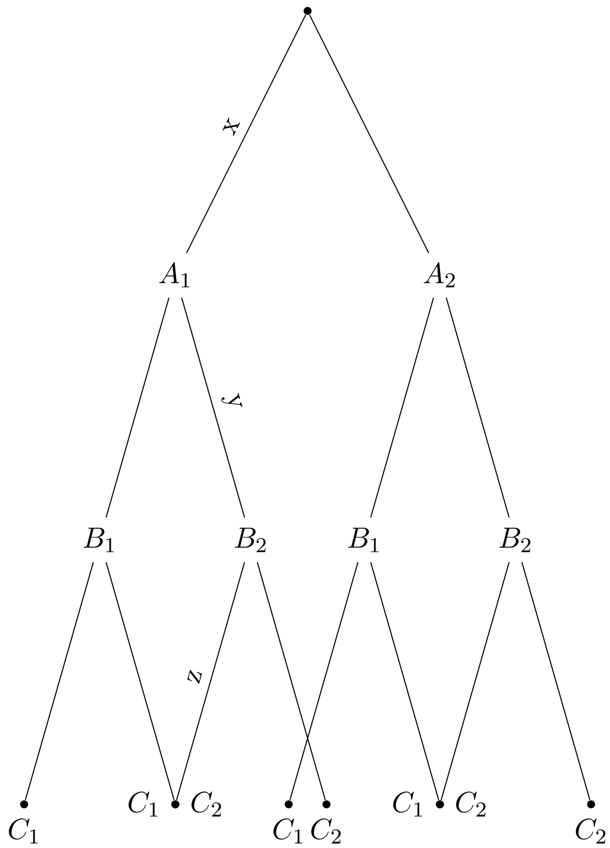
Bloom and Orloff give us the values for all of the probabilities in the right hand side of 10 in the probability tree above.

Therefore

$$P\left(R_1 \mid R_2\right) = \frac{\left(\frac{4}{7}\right)\left(\frac{5}{7}\right)}{\frac{32}{49}} = \left(\frac{20}{49}\right)\left(\frac{49}{32}\right) = \frac{20}{32} \approx 0.625 \tag{10}$$

5 Concept Questions on Probability Trees

In this section, we answer questions about the probability tree that Orloff and Bloom give in [2].



The edge labeled, "x," in the figure above represents $P\left(A_1\right)$.
The edge labeled, "y," in the figure above represets $P\left(B_2 \mid A_1\right)$.
The edge labeled, "z," in the figure above represents $P\left(C_1 \mid A_1 \cap B_2\right)$. We read section 5.1, titled, "Shorthand vs. precise trees," in [2] in order to understand what the edge labeled, "z," represents. This section explains that nodes having a distance of two or more edges to the root of the tree represent the probabilities of intersections of sets of outcomes.
The node labeled, "C_1," at the end of the path, A_1, B_2, C_2, represents the event, $A_1 \cap B_2 \cap C_1$. This is also clear when we read section 5.1, titled, "Shorthand vs. precise trees," in [2].

6 Monty Hall Problem

One should switch after Monty shows a goat.
We find a clear explanation of why in [3].
We reproduce the table in [3] that explains why one should always switch:

	Door 1	Door 2	Door 3	result
Game 1	Auto	Goat	Goat	Switch and you lose.
Game 2	Goat	Auto	Goat	Switch and you win.
Game 3	Goat	Goat	Auto	Switch and you win.
Game 4	Auto	Goat	Goat	Stay and you win.
Game 5	Goat	Auto	Goat	Stay and you lose.
Game 6	Goat	Goat	Auto	Stay and you lose.

This table illustrates the case where we always choose door number one, and the cars and goats are placed in any of the three possible ways to put the cars and goats behind doors one, two or three.
We can assume we always choose door number one without loss of generality because always choosing a different door results in a permutation in the values of the last column of the table above.
The first three rows of the table show the result of switching our choice of door after Monty shows one door, other than the one we choose, that has a goat behind it.
The last three rows of the table show the result of not switching our choice of door after Monty shows one door, other than the one we choose, that has a goat behind it.
We inspect this table to find that for any possible location of the car, two out of the three possible outcomes result in a win if we switch. On the other hand, for any possible location of the car, one out of three possible outcomes result in a win if we do not switch.
Therefore we should always switch.

7 Independent Events

We roll two dice.
We define the following events:

- A is the event that we roll a 3 with the first die.
- B is the event the sum of the values we roll for both dice is 6.

We assume that when we roll the two dice, for any die, rolling any value one through six is equally likely.
We show that A and B are not independent events.

A and B are independent events if, and only if,

$$P(A \cap B) = P(A) \cdot P(B) \tag{11}$$

There are 6 outcomes in A because the first die we roll must land on three, but the second die we roll can land on any of the six possible values. We count 6 sequences of integers where the first element of the sequence is a 3, and the second element has any value from one to six.
There are 36 possible sequences of outcomes when we roll two dice.
Therefore $P(A) = \frac{6}{36}$.
There are 5 outcomes in B. This is because there are 5 sequences of two integers with values from one to six that add to 6. They are: $(1,5), (2,4), (3,3), (4,2)$, and $(5,1)$.
There are 36 possible sequences of outcomes when we roll two dice.
Therefore $P(B) = 536$.
The only outcome in A that is also in B is where we roll a three with the first die, and a three with the second die.
There are 36 possible sequences of outcomes when we roll two dice.
Therefore $P(A \cap B) = \frac{1}{36} = \frac{6}{216}$.

$$\left(\frac{6}{36}\right)\left(\frac{5}{36}\right) = \frac{30}{216} \neq \frac{6}{216} \tag{12}$$

So A and B are not independent events.

8 Evil Squirrel Detector

The population size is $1,000,000$. There are 100 evil squirrels, and therefore $999,900$ non-evil squirrels. We have an alarm that detects whether or not a squirrel is evil that has a 0.99 true positive rate, and a 0.01 false positive rate.
We fill out the table for computing conditional probabilities for false positive, false negative, true positive, and true negative rates that Orloff and Bloom show in [2] and [1].
This is a reproduction of the table in [2]

	Evil	Good	Total
Alarm	99	9999	10098
No Alarm	1	989901	989902
Totals	100	999,900	1,000,000

In [2] Orloff and Bloom ask, given that the squirrel sets off the alarm, what is the probabiltiy that the squirrel is evil?
Let A be the event, "the squirrel sets off the alarm," and B be the event, "the squirrel is evil."

Then Orloff and Bloom are asking for $P(B \mid A)$
We are given the probability that the alarm will go off if the squirrel is evil.
In conditional probability notation: $P(A \mid B) = \frac{99}{100}$.
In order to compute $P(B \mid A)$ we apply Bayes' Theorem [1].

$$P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)} \tag{13}$$

From the table above we can compute the probability that a squirrel is evil.
This is $P(B) = \frac{100}{1{,}000{,}000}$.
Also from the table above, we can compute the probability $P(A)$ that the alarm will be set off. This is the total probability of the alarm getting set off for evil and good squirrels. In terms of A and B this total probability is:

$$P(A) = P(A \cap B) + P\left(A \cap B^{\complement}\right) = P(A \mid B) \cdot P(B) + P\left(A \mid B^{\complement}\right) \cdot P\left(B^{\complement}\right) \tag{14}$$

We use the numbers in the table above to calculate

$$P(A) = \left(\frac{99}{100}\right)\left(\frac{100}{1,000,000}\right) + \left(\frac{1}{100}\right)\left(\frac{999,900}{1,000,000}\right) = 0.010098 \tag{15}$$

Now we have all the probabilities on the right hand side of the equation of Bayes' Theroem, so we compute:

$$P(B \mid A) = \frac{\left(\frac{99}{100}\right)\left(\frac{100}{1{,}000{,}000}\right)}{0.010098} \approx 0.00980 \tag{16}$$

MIT should not employ the system because the conditional probability: given the alarm is set off, the probability that the squirrel is evil is very low.

References

[1] Jeremy Orloff and Jonathan Bloom. *Conditional Probability, Independence and Bayes Theorem Class 3, 18.05, Spring 2014*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading3.pdf` (Spring 2014).

[2] Jeremy Orloff and Jonathan Bloom. *Conditional Probability, Independence, Bayes Theorem 18.05*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class3slides.pdf` (Spring 2014).

[3] Marilyn vos Savant. *Game Show Problem*. Available at `http://marilynvossavant.com/game-show-problem/` (2006-2014).

[4] Stack Exchange User Michael Underwood. *What is the correct way of embedding text into math mode? (Micheal Underwood's answer)*. Available at `http://tex.stackexchange.com/questions/3415/what-is-the-correct-way-of-embedding-text-into-math-mode` (2010/9/23).