Problem Set 4

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [4].
Please see the references section for detailed citation information.
The material for the course is licensed under the terms at `http://ocw.mit.edu/terms`.
We use documentation in [**logicNot**], [**proofs**], [**bars**], [**packageClash**], [**curlyBrace**], [**cases**] to write the LaTeXsource code for this document.

2 Time to failure

The first group of problems Orloff and Bloom have for us involve some random variables that follow an exponential distribution.
The exponential distribution they give us to work with has probability density function (pdf):

$$f(x) = \lambda e^{-\lambda x}, x \geq 0. \tag{1}$$

2.1 $P\left(X \geq x\right)$

We know how to calculate $P\left(X < x\right)$ as a definite integral [1], therefore we will find $P\left(X < x\right)$, and our final result will be to find $P\left(X \geq x\right) = 1 - P\left(X < x\right)$.
In order to calculate this probability, we will do a change of variable similar to the technique Orloff and Bloom show in section 3.4 of [2].
We change the variable in the pdf $f\left(x\right)$ to u; therefore we rewrite the pdf as $f\left(u\right)$:

$$f\left(u\right) = \lambda e^{-\lambda u}. \tag{2}$$

We use this definition, the fact that f is defined for $x \geq 0$, and the definition of probability of continuous random variables [1] to write this equation:

$$P\left(X < x\right) = \int_0^x \lambda e^{-\lambda u}\, du. \tag{3}$$

We substitute the integral on the right hand side of the previous equation with its antiderivative to get:

$$P\left(X < x\right) = -e^{-\lambda u}\bigg|_{u=0}^{x}. \tag{4}$$

We evaluate the the antiderivative at the limits of integration:

$$P\left(X < x\right) = -e^{-\lambda x} - -e^{-\lambda 0}. \quad (5)$$

Now we simplify the previous equation:

$$P\left(X < x\right) = -e^{-\lambda x} + 1. \quad (6)$$

Now, we apply the identity:

$$P\left(X \geq x\right) = 1 - P\left(X <\right). \quad (7)$$

Therefore

$$P\left(X \geq x\right) = 1 - \left(-e^{-\lambda x} + 1\right). \quad (8)$$

The previous equation simplifies to:

$$P\left(X \geq x\right) = e^{-\lambda x}. \quad (9)$$

2.2 CDF of Minimum of two exponential random variables

In this section Orloff and Bloom ask us to find the cumulative distribution function (CDF) of two independent random variables X_1, and X_2 that both follow an exponential distribution, and that both have mean $\frac{1}{\lambda}$.
In [3] Orloff and Bloom state that the mean of a random variable that has probability mass function (pmf) $\lambda e^{-\lambda x}$ is $\frac{1}{\lambda}$.

Therefore X_1, and X_2 both have pmf's $\lambda e^{-\lambda x}$.
For this problem, Orloff and Bloom let $T = \min(X_1, X_2)$.
They ask us for the cdf of T.
The cdf of T is a function $F(t) = P(T < t)$.
In the previous section, we found that for a random varialbe X that has pdf $\lambda e^{-\lambda x}$,

$$P(X \geq x) = e^{-\lambda x}. \tag{10}$$

We use the definition of T to write the equation:

$$P(T \geq t) = P(\min(X_1, X_2) \geq t). \tag{11}$$

$T = \min(X_1, X_2)$, so $T \geq t$ if, and only if, $X_1 \geq t$, and $X_2 \geq t$.

Lemma 1. *If two events A and B have a biconditional relation, then*

$$P(A) = P(B). \tag{12}$$

Proof. The proof is by contradiction. Assume events A and B are biconditionally related, but $P(A) \neq P(B)$. Then there would be unequal chances of events A and B occuring, which means that one event would occur while the other does not. But A and B are biconditionally related, so event A occurs when, and only when, event B occurs. This is a contradiction, so $P(A) = P(B)$. □

Therefore

$$P(T \geq t) = P(X_1 \geq t, X_2 \geq t). \tag{13}$$

X_1 and X_2 are independent events. In [2] Orloff and Bloom state that random variables X_1 and Y_1 are independent if and only if:

$$P(X_1, X_2) = F_{X_1}(x_1) F_{X_2}(x_2). \tag{14}$$

$F(X_1, X_2)$ is the cdf of X_1, and X_2. $F_{X_1}(x_1)$, and $F_{X_2}(x_2)$ are the marginal cumulative distribution functions of X_1, and X_2.
We know that X_1, and Y_1 are exponentially distributed random variables with mean $\frac{1}{\lambda}$. The answer we find in the previous section implies that the cdf of X_1 is $F_{X_1}(x_1) = e^{-\lambda x_1}$, and the cdf of X_2 is $F_{X_2}(x_2) = e^{-\lambda x_2}$.
Since X_1, and X_2 are independent,

$$F(X_1, X_2) = e^{-\lambda x_1} e^{-\lambda x_2}. \tag{15}$$

We add the exponents of the base e to simplify the previous equation to:

$$F(X_1, X_2) = e^{-\lambda(x_1 + x_2)}. \tag{16}$$

We are finding the cdf of T.

Consider $F(X_1 < t, X_2 < t)$. We use the previous equation to write:

$$F(X_1 < t, X_2 < t) = e^{-\lambda 2t}. \tag{17}$$

Since $P(T < t) = P(X_1 < t, X_2 < t)$:

$$F(T < t, X_2 < t) = e^{-\lambda 2t}. \tag{18}$$

2.3 Three lightbulbs

In this section we answer a question about three lightbulbs, $B1$, $B2$, and $B3$, where each lighbulb's lifetime is an exponential random variable with mean values $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$, respectively. The unit for each mean value is years.
Furthermore, Orloff and Bloom state that the lifetimes of the lightbulbs are independent.
Let X_1, X_2, and X_3 be the random variables equal to the lifetimes of B_1, B_2, and B_3, respectively.
Since the mean value of X_1 is $\frac{1}{2}$, and X_1 follows an exponential distribution, $X_1 \sim 2e^{-2t}$.
Similarly, $X_2 \sim 3e^{-3t}$, and $X_3 \sim 5e^{-5t}$
We apply logic similar to what we use in the previous question to state that the cdf of a random variable $T = \min(X_1, X_2, X_3)$ is

$$F(T < t) = e^{-1(2+3+5)t} = e^{-10t}. \tag{19}$$

The expected value of a random variable with cdf e^{10t} is $\frac{1}{10}$ year.

3 Aching Joints

We deal with the joint distribution of two continuous random variables X, and Y. The probability density function for X, and Y is $f(x) = c\left(x^2 + xy\right)$. Furthermore, f is defined on $[0,1] \times [0,1]$.

3.1 Value of c

The first thing Orloff and Bloom ask us for is the value of c in f as defined above.
In order for $f(x, y)$ to be a probabilty distribution function (PDF),

$$c \int_0^1 \int_0^1 x^2 + xy \, dx \, dy = 1. \tag{20}$$

This is true, if and only if:

$$c \int_0^1 \frac{x^3}{3} + \frac{x^2 y}{2} \, dy \bigg|_{x=0}^{1} = 1. \tag{21}$$

We evaluate the anti-derivative over the interval indicated in the equation above to obtain:

$$c \int_0^1 \frac{1}{3} + \frac{y}{2}\, dy = 1. \tag{22}$$

We now replace the integral of the function of y above with its anti-derivative:

$$c\frac{y}{3} + \frac{y^2}{4}\Bigg|_{y=0}^{1} = 1. \tag{23}$$

And, we now evaluate the anti-derivative over the interval indicated in the equation above:

$$c\frac{1}{3} + \frac{1}{4} = 1. \tag{24}$$

Now, we simplify the previos equation:

$$c\frac{7}{12} = 1. \tag{25}$$

And, we solve for c:

$$c = \frac{12}{7}. \tag{26}$$

3.2 Marginal cumulative distribution, and probability density functions

Orloff and Bloom are asking us to find four functions

- the marginal CDF $F_Y(y)$,
- the marginal CDF $F_X(x)$.
- the marginal PDF $f_Y(y)$, and
- the marginal PDF $f_X(x)$.

The definition of marginal CDF dictates that we evaluate the CDF at the upper limit of the variable that we are not finding the marginal CDF for.
Therefore,

$$F_Y(y) = F(1, y). \tag{27}$$

In order to find the marginal CDF's, we need to find the anti-derivative of the PDF:

$$F(x, y) = \int \int \frac{12}{7} xy + x^2 \, dx \, dy. \tag{28}$$

First we find the anti-derivative with respect to x:

$$F(x, y) = \int \frac{12}{7} fracx^2y2 + fracx^2 2 \, dy. \tag{29}$$

Next, we find the anti-derivative with respect to y:

$$F(x, y) = \frac{12}{7} \frac{x^2 y^2}{2} + \frac{x^2 y}{2}. \tag{30}$$

Now that we have the CDF, we can partially evaluate it to obtain the marginal CDF's.

$$F_Y(y) = F(1, y). \tag{31}$$

We replace the right hand side of the equation above with the CDF we found, and replace x with the value 1 to find the CDF for y.

$$F_Y(y) = \frac{12}{7}\frac{y^2}{2} + \frac{2y}{2}. \tag{32}$$

$$F_X(x) = F(x, 1). \tag{33}$$

We replace the right hand side of the equation above with the CDF we found, and replace y with the value 1 to find the CDF for X.

$$F_Y(y) = \frac{12}{7}\left(\frac{x^2}{2} + \frac{x^2}{2}\right). \tag{34}$$

The expression above simplifies to:

$$F_Y(y) = \frac{12x^2}{7}. \tag{35}$$

Now we move on to deriving the marginal PDF's. We use the definition of marginal PDF from [2].

First we tackle the marginal PDF $f_X(x)$.

$$f_X(x) = \int_0^1 \frac{12}{7}\left(x^2 + xy+\right)\, dy \tag{36}$$

We replace the function with its anti-derivative:

$$f_X(x) = \frac{12}{7}\left(x^2 y + \frac{xy^2}{2}+\right)\Bigg|_{y=0}^{1}. \tag{37}$$

Now we evaluate the anti-derivative for the interval $[0, 1]$.

$$f_X(x) = \frac{12}{7}\left(x^2 + \frac{x^2}{2}+\right). \tag{38}$$

This is the marginal PMF $f_X(x)$.
We do a similar integration to derive the marginal PDF $f_Y(y)$.

$$f_Y(y) = \frac{12}{7}\left(\int_0^1 x^2 + xy\, dx\right). \tag{39}$$

We replace the integral in the equation above with its anti-derivative.

$$f_Y(y) = \frac{12}{7}\left(\frac{x^3}{3} + fracx^2y2\Bigg|_0^1\right). \tag{40}$$

We evaluate the anti-derivative over the interval $[0, 1]$.

$$f_Y(y) = \frac{12}{7}\left(\frac{1}{3} + fracy2\right). \tag{41}$$

This is the marginal PDF for $f_Y(y)$.

3.3 $E(X)$, and Var(X)

In order to calculate $E(X)$, we integrate the product of the marginal PDF $f_X(x)$, and x.

$$E(X) = int_0^1 f_X(x)\, dx. \tag{42}$$

References

[1] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf (Spring 2014).

[2] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf (Spring 2014).

[3] Jeremy Orloff and Jonathan Bloom. *Gallery of Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5c.pdf (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *Joint Distributions, Independence Covariance and Correlation 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class7slides.pdf (Spring 2014).