MIT Introduction to Statistics 18.05 Problem Set 1

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
Please see the references section for detailed citation information.

2 Problem 1: Poker Hands

2.1 Two-Pair

We calculate the probability of the poker dealer dealing us a hand that is a two-pair hand. First we count the number of two-pair hands, then we divide the number of two-pair hands by the total number of hands to calculate the probability of the dealer dealing us a two pair hand.
The definition of a two-pair hand is, "Two cards have one rank, two cards have another rank, and the remaining card has a third rank. e.g.$\{2\heartsuit, 2\spadesuit, 2\clubsuit, 5\clubsuit, K\diamondsuit\}$." [4]
We take a combinations approach similar to the approach Orloff and Bloom take to calculate the probability of a one-pair hand in [6].
First we choose the ranks of the pairs. There are 13 ranks, so there are $\binom{13}{2}$ ways to choose the ranks of the pairs.
Next we choose the suits for the cards in the pairs. There are $\binom{4}{2}$ ways to select the suits for the cards in the first pair, and $\binom{4}{2}$ ways to select the suits for the cards in the second pair.
To complete the hand we select one card. We have 11 ranks to choose from for the fifth card, and $\binom{4}{1}$ ways to select its suit.
We apply the rule of product to count the number of two-pair hands:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 78 \times 6 \times 6 \times 11 \times 4 = 123552 \tag{1}$$

The number of all poker hands is the number of ways to select 5 items from a set of 52 items. Therefore the number of all poker hands is $\binom{52}{5} = 2598960$. Therefore the probability of a two-pair hand is $\frac{123552}{2598960} \approx 0.048$.

2.2 Three-of-a-Kind

Orloff and Bloom give the definition of a three-of-a-kind hand as, "Three cards have one rank and the remaining two cards have two other ranks. e.g. $\{2\heartsuit, 2\spadesuit, 2\clubsuit, 5\clubsuit, K\diamondsuit\}$." [4]
We use the same approach as above.
First we select the rank for the three cards that have the same rank.
There are 13 ranks, so there are $\binom{13}{1}$ ways to select this rank.
Next we select the suits for the three cards that have the same rank. There are 4 suits, and we choose one for each card, so there are $\binom{4}{3}$ ways to select the suits for the 3 cards.
We have $\binom{12}{2}$ ways to select the ranks for the fourth and fifth cards, and $\binom{4}{1}^2$ ways to select their suits.
Now we apply the rule of product to count the number of three-of-a-kind hands:

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 13 \times 4 \times 66 \times 16 = 54912 \tag{2}$$

Therefore the probability of a three-of-a-kind hand is $\frac{54912}{2598960} \approx 0.021$.

3 Problem 2: Non-transitive Dice

3.1 Probabilities and Ordering Dice

3.1.1 White vs. Green and Green vs. Red

We follow the method Orloff and Bloom use to calculate the probability that red beats white[6].
We write probability tables for white dice and green dice:

Green Die		
outcomes	1	4
probability	$\frac{1}{6}$	$\frac{5}{6}$

White Die		
outcomes	2	5
probability	$\frac{1}{2}$	$\frac{1}{2}$

Next we write the probability table for the product sample space of white and green dice:

		Green Die	
		1	4
White Die	2	$\frac{1}{12}$	$\frac{5}{12}$
	5	$\frac{1}{12}$	$\frac{5}{12}$

The pairs in the table above where the outcome for white is greater than the number for green correspond to outcomes in the product sample space where white wins. These are: $\{white = 2, green = 1\}$, $\{white = 5, green = 1\}$, and $\{white = 5, green = 4\}$.
We then add the corresponding probabilities for these outcomes where white wins to to calculate the probability that white wins:

$$\frac{1}{12} + \frac{1}{12} + \frac{5}{12} = \frac{7}{12} \approx 0.583 \tag{3}$$

3.1.2 Ordering the Dice

When we write one color beats another color, for example, "...red beats white..." we mean the outcome where the number we roll for the first color die is greater than the number we roll for the second color die. From [6], we know that there is a $\frac{7}{12}$ probability that red beats white. From the previous subsection, we know that there is a $\frac{7}{12}$ probability that white beats green.
Now we are required calculate the probability that green beats red.
We do the probability calculation as we do in the previous section.
We write the probability table for the red die:

Red Die		
outcomes	3	6
probability	$\frac{5}{6}$	$\frac{1}{6}$

For our convenience, we repeat the probability table for the green die:

Green Die		
outcomes	1	4
probability	$\frac{1}{6}$	$\frac{5}{6}$

Now we write the probability table for the product sample space of red and green dice:

		Green Die	
		1	4
Red Die	3	$\frac{5}{36}$	$\frac{25}{36}$
	6	$\frac{1}{36}$	$\frac{5}{36}$

The outcomes where red beats green are $\{red = 3, green = 1\}$, $\{red = 6, green = 1\}$, and $\{red = 6, green = 4\}$. We sum the corresponding probabilities in the above table to compute the probability that green beats red:

$$\frac{5}{36} + \frac{1}{36} + \frac{5}{36} = \frac{11}{36} \approx 0.306 \tag{4}$$

Now we are armed with enough information to answer the question of whether or not we can order the dice from best to worst.
From [6] we know the probability that red beats white is $\frac{7}{12}$. We have calculated here the probability that white beats green is $\frac{7}{12}$, and the probability that red beats green is $1 - \frac{11}{36} = \frac{25}{36}$. Therefore it is more likely that red will beat white, white will beat green, and green will beat red. Therefore we cannot arrange the dice in order from best to worst.
Note: we had to look at [5] to get the answer for the ordering question.

3.2 Rolling Two Dice

The authors of the problem set [4] ask us, "Suppose you roll two white dice against two red dice. What is the probability that the sum of the white dice is greater than the sum of the red dice?"

The authors also suggest we watch the video [3]. The material in the video is crucial to our ability to write this answer. The text of [4] also hints that we should follow the methods in [4], and use a probability tree to answer this question.

Note: we used the example in [2] to render the probability trees below.

We start with rolling the two white dice. The nodes linked to the root of the probability tree represent the possible combinations we can roll using two white dice. The edges linking the root node of the probability tree to the nodes representing the possible combinations we can roll for the two white dice are labeled with the probability of rolling a combination of dice.

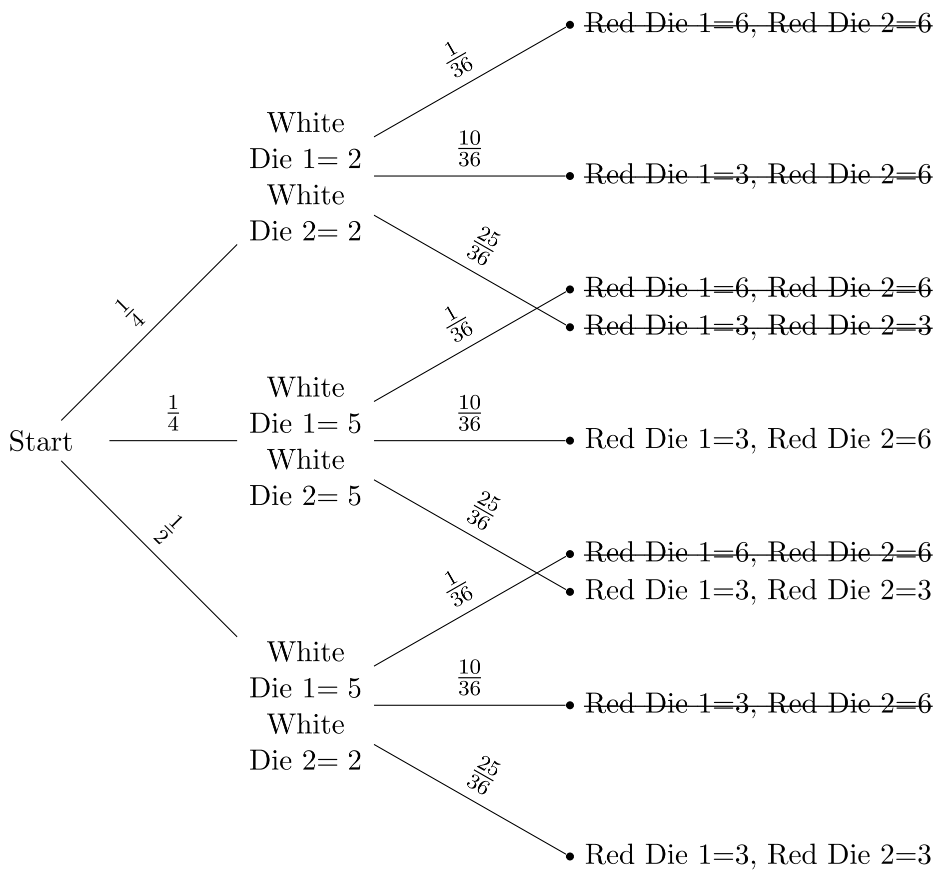
We use the multiplication rule for independent events to compute the probability of rolling a combination of white dice. The white die has three faces with a value of 5, and three faces with a value of 2, so we have a probability of $\frac{3}{6} = \frac{1}{2}$ for rolling either a 2 or a 5. Therefore there is a probability of $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of rolling two 5's or two 2's.

We use the multiplication and addition rules for independent events to compute the probability of rolling one 5 and one 2: there are two ways we can roll one 5 and one 2, so the probability of rolling one 5 and one 2 is:

$$\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{5}$$

We use similar reasoning for completing the probability tree.

We cross out nodes in the tree where the path through the tree to the leaf node represents an experiment where the sum of what we roll for the white dice is not greater than the sum of what we roll for the red dice.



We use the rule of product for independent probabilities to calculate the probability of each outcome where the sum of the value we roll for the white dice is greater than the sum of the value we roll for the red dice.

We multiply the probabilities on the edges of the probability tree above where the edge is on a path to a leaf node where the label on the leaf node is not crossed out.

Path 1: With the white dice we roll two 5's, and with the red dice we roll one 3 and one 6. The probability for this outcome is $\frac{1}{4} \times \frac{10}{36} = \frac{10}{144} \approx 0.069$.

Path 2: We roll two 5's with the white dice, and we roll two 3's with the red dice. The probability of this outcome is: $\frac{1}{4} \times \frac{25}{36} = \frac{25}{144} \approx 0.174$

Path 3: we roll one 5 and one 2 with the white dice, and with the red dice, we two 3's. The probability of this outcome is: $\frac{1}{2} \times \frac{25}{36} = \frac{25}{72} \approx 0.347$

We use the addition rule for independent outcomes to compute the probability for the outcome of path 1, or path 2, or path 3.

The sum of the probabilities is approximately $0.069 + 0.347 + 0.174 = 0.59$

4 Birthdays: counting and simulation

Note: we did not find this problem to be trivial, so we performed a Google search on the term, "birthday problem," to get some clues on how to proceed with a solution. We find the web pages [7], and [1] helpful in writing this solution.

4.1 Probability function for Ω

For this problem, the elements ω of Ω are sequences of n birthdays, one for each person in the group.
An example ω when $n = 5$ is $(1, 2, 2, 1, 1))$.
All birthdays are equally likely.
Therefore we can write any sequence of birthdays by sampling the integers $1, 2, 3...365$ with replacement.
Therefore the probability function

$$P(\Omega) = \frac{1}{365^n} \tag{6}$$

4.2 Careful Descriptions of Subsets

In this problem Ω is the set

$$\{\omega \mid \omega = (x_1, x_2, ..., x_n), x_i \in \{1, 2, 3, ..., 365\}\} \tag{7}$$

4.2.1 Someone Shares Our Birthday

Let S be the set of events where someone shares our birthday b, $b \in \{1, 2, ..., 365\}$.
Then we define S as

$$\{\omega \mid \omega = (x_1, x_2, ..., x_n), x_i \in \{1, 2, 3, ..., 365\}, \forall\omega (\exists x_i \mid x_i = b)\} \tag{8}$$

4.2.2 Two People Share a Birthday

Let S be the set of sequences of n birthdays where at least two elements of every sequence are equal.
Then we can express S as

$$\{\omega \mid \omega = (x_1, x_2, ..., x_n), x_i \in \{1, 2, 3, ..., 365\}, \forall\omega (\exists x_i, x_j \mid x_i = x_j)\} \tag{9}$$

Let S be the set of sequences of n birthdays where at least three elements of every sequence are equal.
Then we can express S as

$$\{\omega \mid \omega = (x_1, x_2, ..., x_n), x_i \in \{1, 2, 3, ..., 365\}, \forall \omega (\exists x_i, x_j, x_k \mid x_i = x_j = x_k)\} \tag{10}$$

4.3 Exact Formula For Someone Sharing Our Birthday

We want to compute an exact formula for $P(A)$ where A is the event that some sequence of birthdays ω contains our birthday, and $P(A)$ is the probability that a sequence of birthdays ω contains our birthday b.
There are 365^n sequences of birthdays for groups of people of size n or smaller.
There is *binomn*1 ways to choose an element of *omega* to be the element that is equal to b.
Therefore the exact formula for $P(A)$ is

$$\frac{n}{365^n} \tag{11}$$

References

[1] Better Explained. *Birthday Problem.* Available at https://betterexplained.com/articles/understanding-the-birthday-paradox/ (2007/12/31).

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[3] James Grime and David Spiegelhalter. *Non-transitive Dice.* Available at http://www.youtube.com/watch?v=zWUrwhaqq_c (2010/08/05).

[4] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 1, Spring 2014.* Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps1.pdf (Spring 2014).

[5] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 1, Spring 2014 Solutions.* Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps1_solutions.pdf (Spring 2014).

[6] Jeremy Orloff and Jonathan Bloom. *Probability: Terminology and Examples 18.05 Spring 2014.* Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class2slides.pdf(Spring 2014).

[7] Wikipedia. *Birthday Problem.* Available at https://en.wikipedia.org/wiki/Birthday_problem (2017/01/23).