Problem Set 4

John Hancock

June 21, 2017

Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [7].
Please see the references section for detailed citation information.

We use documentation in [**logicNot**], [**proofs**], [**bars**], [**packageClash**], [**curlyBrace**], [**cases**] to write the LaTeXsource code for this document.

2 Time to failure

The first group of problems Orloff and Bloom have for us involve some random variables that follow an exponential distribution.
The exponential distribution they give us to work with has probability density function (pdf):

$$f(x) = \lambda e^{-\lambda x}, x \geq 0. \tag{1}$$

2.1 $P(X \geq x)$

We know how to calculate $P(X < x)$ as a definite integral [2], therefore we will find $P(X < x)$, and our final result will be to find $P(X \geq x) = 1 - P(X < x)$.
In order to calculate this probability, we will do a change of variable similar to the technique Orloff and Bloom show in section 3.4 of [3].
We change the variable in the pdf $f(x)$ to u; therefore we rewrite the pdf as $f(u)$:

$$f(u) = \lambda e^{-\lambda u}. \tag{2}$$

We use this definition, the fact that f is defined for $x \geq 0$, and the definition of probability of continuous random variables [2] to write this equation:

$$P\left(X < x\right) = \int_0^x \lambda e^{-\lambda u}\, du. \tag{3}$$

We substitute the integral on the right hand side of the previous equation with its antiderivative to get:

$$P\left(X < x\right) = -e^{-\lambda u}\Big|_{u=0}^{x}. \tag{4}$$

We evaluate the the antiderivative at the limits of integration:

$$P\left(X < x\right) = -e^{-\lambda x} - -e^{-\lambda 0}. \tag{5}$$

Now we simplify the previous equation:

$$P\left(X < x\right) = -e^{-\lambda x} + 1. \tag{6}$$

Now, we apply the identity:

$$P\left(X \geq x\right) = 1 - P\left(X <\right). \tag{7}$$

Therefore

$$P\left(X \geq x\right) = 1 - \left(-e^{-\lambda x} + 1\right). \tag{8}$$

The previous equation simplifies to:

$$P\left(X \geq x\right) = e^{-\lambda x}. \tag{9}$$

2.2 CDF of Minimum of two exponential random variables

In this section Orloff and Bloom ask us to find the cumulative distribution function (CDF) of two independent random variables X_1, and X_2 that both follow an exponential distribution, and that both have mean $\frac{1}{\lambda}$.
In [6] Orloff and Bloom state that the mean of a random variable that has probability mass function (pmf) $\lambda e^{-\lambda x}$ is $\frac{1}{\lambda}$.
Therefore X_1, and X_2 both have pmf's $\lambda e^{-\lambda x}$.
For this problem, Orloff and Bloom let $T = \min(X_1, X_2)$.
They ask us for the cdf of T.
The cdf of T is a function $F(t) = P(T < t)$.
In the previous section, we found that for a random varialbe X that has pdf $\lambda e^{-\lambda x}$,

$$P(X \geq x) = e^{-\lambda x}. \tag{10}$$

We use the definition of T to write the equation:

$$P(T \geq t) = P(\min(X_1, X_2) \geq t). \tag{11}$$

$T = \min(X_1, X_2)$, so $T \geq t$ if, and only if, $X_1 \geq t$, and $X_2 \geq t$.

Lemma 1. *If two events A and B have a biconditional relation, then*

$$P(A) = P(B). \tag{12}$$

Proof. The proof is by contradiction. Assume events A and B are biconditionally related, but $P(A) \neq P(B)$. Then there would be unequal chances of events A and B occuring, which means that one event would occur while the other does not. But A and B are biconditionally related, so event A occurs when, and only when, event B occurs. This is a contradiction, so $P(A) = P(B)$. $\square$

Therefore

$$P(T \geq t) = P(X_1 \geq t, X_2 \geq t). \tag{13}$$

X_1 and X_2 are independent events. In [3] Orloff and Bloom state that random variables X_1 and Y_1 are independent if and only if:

$$P(X_1, X_2) = F_{X_1}(x_1)\, F_{X_2}(x_2). \tag{14}$$

$F(X_1, X_2)$ is the cdf of X_1, and X_2. $F_{X_1}(x_1)$, and $F_{X_2}(x_2)$ are the marginal cumulative distribution functions of X_1, and X_2.
We know that X_1, and Y_1 are exponentially distributed random variables with mean $\frac{1}{\lambda}$. The answer we find in the previous section implies that the cdf of X_1 is $F_{X_1}(x_1) = e^{-\lambda x_1}$, and the cdf of X_2 is $F_{X_2}(x_2) = e^{-\lambda x_2}$.

Since X_1, and X_2 are independent,

$$F(X_1, X_2) = e^{-\lambda x_1} e^{-\lambda x_2}. \tag{15}$$

We add the exponents of the base e to simplify the previous equation to:

$$F(X_1, X_2) = e^{-\lambda(x_1 + x_2)}. \tag{16}$$

We are finding the cdf of T.
Consider $F(X_1 < t, X_2 < t)$. We use the previous equation to write:

$$F(X_1 < t, X_2 < t) = e^{-\lambda 2t}. \tag{17}$$

Since $P(T < t) = P(X_1 < t, X_2 < t)$:

$$F(T < t, X_2 < t) = e^{-\lambda 2t}. \tag{18}$$

2.3 Three lightbulbs

In this section we answer a question about three lightbulbs, $B1$, $B2$, and $B3$, where each lighbulb's lifetime is an exponential random variable with mean values $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$, respectively. The unit for each mean value is years.
Furthermore, Orloff and Bloom state that the lifetimes of the lightbulbs are independent.

Let X_1, X_2, and X_3 be the random variables equal to the lifetimes of B_1, B_2, and B_3, respectively.
Since the mean value of X_1 is $\frac{1}{2}$, and X_1 follows an exponential distribution, $X_1 \sim 2e^{-2t}$.
Similarly, $X_2 \sim 3e^{-3t}$, and $X_3 \sim 5e^{-5t}$
We apply logic similar to what we use in the previous question to state that the cdf of a random variable $T = \min\left(X_1, X_2, X_3\right)$ is

$$F\left(T < t\right) = e^{-1(2+3+5)t} = e^{-10t}. \tag{19}$$

The expected value of a random variable with cdf e^{10t} is $\frac{1}{10}$ year.

3 Aching Joints

We deal with the joint distribution of two continuous random variables X, and Y. The probability density function for X, and Y is $f\left(x\right) = c\left(x^2 + xy\right)$. Furthermore, f is defined on $[0, 1] \times [0, 1]$.

3.1 Value of c

The first thing Orloff and Bloom ask us for is the value of c in f as defined above.

In order for $f(x, y)$ to be a probabilty distribution function (PDF),

$$c \int_0^1 \int_0^1 x^2 + xy \, dx \, dy = 1. \tag{20}$$

This is true, if and only if:

$$c \int_0^1 \frac{x^3}{3} + \frac{x^2 y}{2} \, dy \bigg|_{x=0}^{1} = 1. \tag{21}$$

We evaluate the anti-derivative over the interval indicated in the equation above to obtain:

$$c \int_0^1 \frac{1}{3} + \frac{y}{2} \, dy = 1. \tag{22}$$

We now replace the integral of the function of y above with its anti-derivative:

$$c \frac{y}{3} + \frac{y^2}{4} \bigg|_{y=0}^{1} = 1. \tag{23}$$

And, we now evaluate the anti-derivative over the interval indicated in the equation above:

$$c \frac{1}{3} + \frac{1}{4} = 1. \tag{24}$$

Now, we simplify the previos equation:

$$c\frac{7}{12} = 1. \tag{25}$$

And, we solve for c:

$$c = \frac{12}{7}. \tag{26}$$

3.2 Marginal cumulative distribution, and probability density functions

Orloff and Bloom are asking us to find four functions

- the marginal CDF $F_Y(y)$,
- the marginal CDF $F_X(x)$.
- the marginal PDF $f_Y(y)$, and
- the marginal PDF $f_X(x)$.

The definition of marginal CDF dictates that we evaluate the CDF at the upper limit of the variable that we are not finding the marginal CDF for.
Therefore,

$$F_Y(y) = F(1, y). \tag{27}$$

In order to find the marginal CDF's, we need to find the anti-derivative of the PDF:

$$F\left(x,y\right)=\int\int\frac{12}{7}xy+x^{2}\,dx\,dy. \tag{28}$$

First we find the anti-derivative with respect to x:

$$F\left(x,y\right)=\int\frac{12}{7}fracx^{2}y2+fracx^{2}2\,dy. \tag{29}$$

Next, we find the anti-derivative with respect to y:

$$F\left(x,y\right)=\frac{12}{7}\frac{x^{2}y^{2}}{2}+\frac{x^{2}y}{2}. \tag{30}$$

Now that we have the CDF, we can partially evaluate it to obtain the marginal CDF's.

$$F_{Y}\left(y\right)=F\left(1,y\right). \tag{31}$$

We replace the right hand side of the equation above with the CDF we found, and replace x with the value 1 to find the CDF for y.

$$F_{Y}\left(y\right)=\frac{12}{7}\frac{y^{2}}{2}+\frac{2y}{2}. \tag{32}$$

$$F_{X}\left(x\right)=F\left(x,1\right). \tag{33}$$

We replace the right hand side of the equation above with the CDF we found, and replace y with the value 1 to find the CDF for X.

$$F_Y(y) = \frac{12}{7}\left(\frac{x^2}{2} + \frac{x^2}{2}\right). \tag{34}$$

The expression above simplifies to:

$$F_Y(y) = \frac{12x^2}{7}. \tag{35}$$

Now we move on to deriving the marginal PDF's. We use the definition of marginal PDF from [3].
First we tackle the marginal PDF $f_X(x)$.

$$f_X(x) = \int_0^1 \frac{12}{7}\left(x^2 + xy+\right)\, dy \tag{36}$$

We replace the function with its anti-derivative:

$$f_X(x) = \frac{12}{7}\left(x^2 y + \frac{xy^2}{2}+\right)\Bigg|_{y=0}^{1}. \tag{37}$$

Now we evaluate the anti-derivative for the interval $[0, 1]$.

$$f_X(x) = \frac{12}{7}\left(x^2 + \frac{x^2}{2}+\right). \tag{38}$$

This is the marginal PMF $f_X(x)$.
We do a similar integration to derive the marginal PDF $f_Y(y)$.

$$f_Y(y) = \frac{12}{7}\left(\int_0^1 x^2 + xy\,dx\right). \tag{39}$$

We replace the integral in the equation above with its anti-derivative.

$$f_Y(y) = \frac{12}{7}\left(\frac{x^3}{3} + fracx^2y2\bigg|_0^1\right). \tag{40}$$

We evaluate the anti-derivative over the interval $[0, 1]$.

$$f_Y(y) = \frac{12}{7}\left(\frac{1}{3} + fracy2\right). \tag{41}$$

This is the marginal PDF for $f_Y(y)$.

3.3 $E(X)$, and Var(X)

In order to calculate $E(X)$, we integrate the product of the PDF $f(x, y)$, and x.

$$E(X) = int_0^1 int_0^1 x\frac{12}{7}(x + xy)\;dx. \tag{42}$$

We distribute x in the expression above.

$$E(X) = int_0^1 int_0^1 \frac{12}{7}\left(x^2 + x^2y+\right)\;dx\,dy. \tag{43}$$

First, we integrate with respect to x.

$$E\left(X\right) = int_0^1 \frac{12}{7}\left(\frac{x^3}{3} + \frac{x^3 y}{3}\Bigg|_{x=0}^{1}\right)\,dy. \tag{44}$$

Next we integrate with respect to y:

$$E\left(X\right) = \frac{12}{7}\left(\frac{x^3 y}{3} + \frac{x^3 y^2}{6}\Bigg|_{x=0}^{1}\Bigg|_{y=0}^{1}\right). \tag{45}$$

We evaluate the expression at the limits of integtation specified to compute:

$$E\left(X\right) = \frac{12}{7}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{6}{7}. \tag{46}$$

In order to compute the variance of X, we use the identity

$$\mathrm{Var}\left(X\right) = E\left(X^2\right) + E\left(X\right)^2 \tag{47}$$

We integrate the product of X^2, and the PDF Orloff and Bloom give us in order to calculate $E\left(X^2\right)$.

$$E\left(X^2\right) = \int_0^1 \int_0^1 x^2 \frac{12}{7}\left(x + xy\right)\,dx\,dy. \tag{48}$$

We distribute the x^2 term in the expression above to aid in determining the anti-derivative of the integrals.

$$E\left(X^2\right) = \int_0^1 \int_0^1 \frac{12}{7}\left(x^3 + x^2 y\right)\,dx\,dy. \tag{49}$$

First we replace the inner integral with its anti-derivative with respect to x:

$$E\left(X^2\right) = \int_0^1 \frac{12}{7}\left(\frac{x^4}{4} + \frac{x^3 y}{3}\right)\Bigg|_{x=0}^{1} dy. \tag{50}$$

Next, we replace the remaining integral with its anti-derivative:

$$E\left(X^2\right) = \frac{12}{7}\left(\frac{x^4 y}{4} + \frac{x^3 y^2}{6}\right)\Bigg|_{x=0}^{1}\Bigg|_{x=0}^{1}. \tag{51}$$

Now we evalute the anti-derivatives to complete the integration:

$$E\left(X^2\right) = \frac{12}{7}\left(\frac{1}{4} + \frac{1}{6}\right). \tag{52}$$

We simplify the expression above to:

$$E\left(X^2\right) = \frac{5}{7}. \tag{53}$$

Now that we know $E\left(X^2\right)$, we can calculate $\mathrm{Var}(X)$.

$$\mathrm{Var}\left(X\right) = E\left(X^2\right) - E\left(X\right)^2. \tag{54}$$

We substitute the values for $E\left(X^2\right)$, and $E\left(X\right)^2$ in the equation above to obtain:

$$\mathrm{Var}\left(X\right) = \frac{5}{7} - \left(\frac{6}{7}\right)^2 \tag{55}$$

3.4 Variance and Covariance of X and Y

In order to compute the variance and covariance of X, and Y, we will use the property of covariance:

$$\mathrm{Cov} = E\left(XY\right) - \mu_x \mu_y \tag{56}$$

However, in order to use the property, we must calculate μ_y, and $E\left(XY\right)$. Note that we computed mu_x, in the previous section.
We use the definition of expected value from [5] to compute mu_y.

$$\mu_y = int_0^1 int_0^1 y \frac{12}{7}\left(x^2 + xy\, dx\, dy\right). \tag{57}$$

We distribute y in order to determine what the anti-derivative is for the intrgral above:

$$\mu_y = int_0^1 int_0^1 \frac{12}{7}\left(x^2 y + xy^2\, dx\, dy\right). \tag{58}$$

We replace the integrals above with their anti-derivatives:

$$\mu_y = \frac{12}{7}\left(\frac{x^3 y^2}{6} + \frac{x^2 y^3}{6}\Big|_{x=0}^{1}\Big|_{x=0}^{1}\right). \tag{59}$$

And we evaluate the anti-derivative at the limits above to obtain:

$$\mu_y = \frac{12}{7}\left(\frac{1}{6}+\frac{1}{6}\right). \tag{60}$$

The equation above simplifies to:

$$/mu_y = \frac{4}{42}. \tag{61}$$

We continue to collect components of the sum we will compute in order to calculate $\text{Cov}(X, Y)$.
We resort to the definition of expected value in order to compute $E\left(X, Y\right)$

$$E\left(X, Y\right) = \int_0^1 \int_0^1 \frac{12}{7} xy\left(x^2 + xy\, dx\, dy\right). \tag{62}$$

We distribute xy in order to find the anti-derivatives of the integrals above:

$$E\left(X, Y\right) = \int_0^1 \int_0^1 \frac{12}{7}\left(x^3 y + x^2 y^2\, dx\, dy\right). \tag{63}$$

Now we replace the integrals above with their anti-derivatives:

$$E\left(X, Y\right) = \frac{12}{7}\left(\frac{x^4 y^2}{8} + \frac{x^3 y^3}{9}\Bigg|_{x=0}^{1}\Bigg|_{y=0}^{1}\right). \tag{64}$$

We now evaluate the expression above at the limits of integration to find:

$$E\left(X,Y\right)=\frac{12}{7}\left(\frac{1}{8}+\frac{1}{9}\right). \tag{65}$$

This equation above simplifies to:

$$E\left(X,Y\right)=\frac{17}{42} \tag{66}$$

We now have all the components we need to compute $\mathrm{Cov}(X,Y)$.

$$\mathrm{Cov}=E\left(XY\right)-\mu_x\mu_y \tag{67}$$

We substitute the values we computed previously into the equation above:

$$\mathrm{Cov}=\frac{2}{63}-\frac{5}{7}\frac{5}{7} \tag{68}$$

The above simplifies to:

$$\mathrm{Cov}=\frac{211}{567} \tag{69}$$

4 Elections

In this section we deal with an election scenario, and a poll of 400 people. In the population: 0.5 of the people support Erika, 0.2 support Ruthi, and the remaining 0.3 support Peter, John or Jerry.

4.1 Apply Central Limit Theorem

Orloff and Bloom ask us for the probabilty that if we poll 400 people, at least %52.5 prefer Erika. Furthermore, they ask us to use the central limit theorem to estimate the probabilty.
We treat the response of the i^th person polled X_i as a Bernoulli(0.5) random variable.
We are looking for the probability that the sum:

$$\sum_{i=1}^{n} x_i = (0.525)400 = 210. \tag{70}$$

We will refer to this sum as S. Then:

$$P\left(S > 210\right) = P\left(\frac{S-\mu}{\sigma} > \frac{210-\mu}{\sigma}\right). \tag{71}$$

Note: if S has mean μ, then $S - \mu$ has mean 0. We cite properties of expexted value in [4] for the explanation that $S - \mu$ has mean 0.

Also note: if S has standard deviation σ then $\frac{S}{\sigma}$ has standard deviation 1. We use the property of variance [2]:

$$\mathrm{Var}\left(aX+b\right)=a^2\mathrm{Var}\left(X\right). \tag{72}$$

Here we let $a=\frac{1}{\sigma}$. Then

$$\mathrm{Var}\left(\frac{S}{\sigma}\right)=\frac{1}{\sigma^2}\mathrm{Var}\left(S\right). \tag{73}$$

We beg the reader to bear in mind that the standard deviation of S is σ.
By definition, variance is the square of standard deviation, so the expressions on both sides of the equation above are equal to 1.
In order to procede, we must compute the mean, and standard deviation of S.
We define S as the sum of 400 Bernoulli(0.5) random variables, X_i. We know from [4] that the expected value of X_i is 0.5. [4] also tells us that the expected value of the sum of the X_i is 200.
In [8], Orloff and Bloom show the variance of a random variable $X\sim\mathrm{Binomial}\left(n,p\right)$ is $np\left(1-p\right)$.
Therefore the variance of S is

$$400\times0.5\times0.5=100. \tag{74}$$

Since standard deviation is the square root of variance, the standard deviation σ_S of S is 10.

Now that we have computed the mean and standard deviation of S, we can substitute them into 79:

$$P\left(S > 210\right) = P\left(\frac{S-200}{10} > \frac{210-200}{10}\right). \tag{75}$$

The equation above simplifies to:

$$P\left(S > 210\right) = P\left(\frac{S-200}{10} > 1\right). \tag{76}$$

We may apply the central limit theorem to write:

$$P\left(\frac{S-200}{10} > 1\right) \approx P\left(Z > 1\right). \tag{77}$$

We use the rule of rule of thumb [1]and the expression above to ascertain

$$P\left(\frac{S-200}{10} > 1\right) \approx 0.68 \tag{78}$$

Recalling 79 we see it is the case that:

$$P\left(S > 210\right) \approx 0.68. \tag{79}$$

4.2 Apply central limit theorem again

Now, we turn our attention to the question of what would be the probabilty that at least 0.25 of 400 people polled support Peter, Jon, or Jerry.
We define a new sum of random variables

$$S = \sum_{n=1}^{400} X_i \tag{80}$$

Where each of the X_i is a Bernoulli random variable that takes the value of 1 with a probabilty of $\frac{1}{3}$, and 0 otherwise. We define the X_i in this way because 0.3 of the population support Peter, Jon, or Jerry.
Moreover [8] explains that the mean (expected) value of S is $\mu_S = 400 \times 0.3 = 120$, and that the variance of S is $400 \times 0.3 \times 0.7 = 84$. Hence the standard deviation σ_S of S is $sqrt84 \approx 9.165$.
In this case we are looking for

$$P\left(S < 100\right) \tag{81}$$

We use the same reasoning as in the previous section to conclude:

$$P\left(S < 100\right) = P\left(\frac{S - 120}{9.165} < \frac{100 - 120}{9.165}\right). \tag{82}$$

And we are in a position to use the central limit theorem to obtain:

$$P\left(\frac{S-120}{9.165} < \frac{100-120}{9.165}\right) \approx P\left(Z < -2.182\right) \qquad (83)$$

The rule of thumb [1] tells us that there is a 95% probabilty that Z will be within two standard deviations of the mean. Since Z has standard deviation 1, there is a 5% chance Z will have a value larger than 2, or smaller than 2. The normal distribution is symmetric with respect to the x axis, so the chance Z is less than 2 is 0.025.
We can get a more precise answer using R. In R, qnorm$(-2.182) =$ 0.01455477.

5 Rounding Error

In this section we address the problem Orloff and Bloom pose regarding rounding error, where we round 1,000 random amounts of money to the nearest nickel.
We model the rounding error as a random variable $X_i, i \in 1, 2, \ldots, 1,000$ from a uniform distribution from the set of values: $\{-2, -1, 0, 1, 2\}$.
We define S to be the sum of the X_i. In order to apply the central limit theorem we must find the mean μ_S, and standard deviation σ_S of S.

Each of the X_i is uniformly distributed from a set of 5 values; therefore the expected value of X_i is:

$$0.2\,(-2) + 0.2\,(-1) + 0.2\,(0) + 0.2\,(1) + 0.2\,(2) = 0 \tag{84}$$

Therefore the expected value of S is also 0.
We use the formula for variance of a uniformly distributed random variable from [8].
In this case,

$$Var\,(X_i) = \frac{24}{12} = 2. \tag{85}$$

Since S is the sum of the X_i, $\mathrm{Var}(S) = 2,000$.
Hence, $\sigma_S = \sqrt{2,000}$.
Orloff and Bloom ask us to use the central limit theorem to estimate

$$P\,(S > 100)\,. \tag{86}$$

We standardize the event above to an event of equal probabilty:

$$P\,(S > 100) = P\left(\frac{S - \mu_S}{\sigma_S} > \frac{100 - \mu_S}{\sigma_S}\right). \tag{87}$$

We determined μ_S and σ_S, so we substitute those values in the expression above to get:

$$P\,(S > 100) = P\left(\frac{S}{\sqrt{2000}} > \frac{100}{\sqrt{2000}}\right). \tag{88}$$

We simplify the right hand side of the inequality above to:

$$P\left(S > 100\right) = P\left(\frac{S}{\sqrt{2000}} > 2.236\right). \tag{89}$$

We are now in a position to apply the central limit theorem [1]. S is the sum of independent identically distributed random variables, and we normalize S in such a way that it has mean 0 and standard deviation 1. Hence, we apply the central limit theorem, and approximate:

$$P\left(S > 100\right) \approx P\left(Z > 2.236\right). \tag{90}$$

Where Z follows the normal distribution $N\left(0, 1\right)$.
We use the pnorm function of the R programming language to find $P\left(Z > 2.236\right) \approx .0254$.
The R code we write to obtain the probabilty above is:

```
(1- pnorm(2.236))*2
```

To summarize, there is approximately a 0.025 probability that the rounding error will be greater than or equal to 100 cents.

6 Independence

Orloff and Bloom give us a joint distribution of two random variables X, and Y, and ask us to determine whether or

not they are independent.
In [**reading7a**] Orloff and Bloom state that two random variables in a joint distribution are independent if the probability of every possible combination of the random variables is the product of the respective marginal probabilities of the combination.
Therefore we will extend the table Orloff and Bloom give for this problem to include the marginal probabilities:

References

[1] Jeremy Orloff and Jonathan Bloom. *Central Limit Theorem and the Law of Large Numbers Class 6, 18.05 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading6b.pdf` (Spring 2014).

[2] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf` (Spring 2014).

[3] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *Discrete Random Variables: Expected Value Class 4, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading4b.pdf (Spring 2014).

[5] Jeremy Orloff and Jonathan Bloom. *Expectation, Variance and Standard Deviation for Continuous Random Variables Class 6, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading6a.pdf (Spring 2014).

[6] Jeremy Orloff and Jonathan Bloom. *Gallery of Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https:

//ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5c.pdf (Spring 2014).

[7] Jeremy Orloff and Jonathan Bloom. *Joint Distributions, Independence Covariance and Correlation 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class7slides.pdf (Spring 2014).

[8] Jeremy Orloff and Jonathan Bloom. *Variance of Discrete Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5a.pdf (Spring 2014).