MIT Introduction to Statistics 18.05 Problem Set 2

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
Please see the references section for detailed citation information.
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We are answering the questions that Orloff and Bloom ask in [3].
We use documentation in to write LaTeXsource code for this document.

2 'Boy or girl' paradox

In order to write this solution, we rely on the answer to this problem in [4], and the treatment of the 'Boy or girl,' paradox in [1].
For these questions on the 'Boy or girl paradox we deal with events B, "the child is a boy," and G, "the child is a girl."
We assume B, and G have the same properties as the B and G events Orloff and Bloom analize in example 9 of [6]. These properties are that B, and G are independent, and they have probability $\frac{1}{2}$.
We use these properties to define 4 more events, BB, BG, GB, and GG. These events are: "the younger child is a boy, and the older child is a boy,"

"the younger child is a boy, and the older child is a girl," "the younger child is a girl, and the older child is a boy," "the younger child is a girl, and the older child is a girl," respectively. We use the properties of B, and G, of example 9 to compute that the probabilities of BB, BG, GB, GG, $P(BB)$, $P(BG)$, $P(GB)$, $P(GG)$, are all equal to $\frac{1}{4}$.

2.1 Probability of girls

The question Orloff and bloom quote is, "Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?"
We can restate the question above as, "Given event BG or event GG, what is the probability GG?"
We use the definition of conditional probability. We also use the law of total probability to comupte $P(GG \cup BG)$.
Therefore we write the equation:

$$P(GG \mid GG \cup BG) = \frac{P(GG \cap (GG \cup BG))}{P(GG \cup BG)} \tag{1}$$

$$\frac{P(GG \cap (GG \cup BG))}{P(GG \cup BG)} = \frac{P(GG)}{P(GG \cup BG)} \tag{2}$$

$$\frac{P(GG)}{P(GG \cup BG)} = \frac{\frac{1}{4}}{\frac{1}{2}} \tag{3}$$

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \left(\frac{1}{4}\right)\left(\frac{2}{1}\right) = \frac{1}{2} \tag{4}$$

Therefore if Mr. Jones' older child is a girl, there is a probabiility of $\frac{1}{2}$ that the younger child is also a girl.

2.2 Probability of boys

In this section, Orloff and Bloom quote another question for us to answer here.
The question is, "Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?"
We use the definitions from the previous section for events, BB, BG, GB and GG. We use the probabilities we found in the first section for these events as well.
The author of this question is giving us that three possible events have occured: BB, BG, or GB. Furthermore the question asks for the conditional probability of BB.
We use the definition of conditional probability, and the law of total probability to compute:

$$P(BB \mid BB \cup BG \cup GB) = \frac{P(BB \cap (BB \cup BG \cup GB))}{P(BB \cup BG \cup GB)} \tag{5}$$

$$\frac{P(BB \cap (BB \cup BG \cup GB))}{P(BB \cup BG \cup GB)} = \frac{P(BB)}{P(BB \cup BG \cup GB)} \tag{6}$$

$$\frac{P(BB)}{P(BB \cup BG \cup GB)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \left(\frac{1}{4}\right)\left(\frac{4}{3}\right) = \frac{1}{3} \tag{7}$$

If at least one of Mr. Smith's children is a boy, then there is a probability of $\frac{1}{3}$ that both children are boys.

3 The blue taxi

In order to solve this problem we will write a confusion matrix [2]. This is the term we found for the kind of table Orloff and Bloom write in [5] that organizes false positive rates, false negative rates, etc. Into a table.
We define the following sets:

- $D+$, "The car is blue."
- $D-$, "The car is green."
- $T+$, "The witness reports seing a blue car."
- $T-$, "The witness reports seing a green car."

Orloff and Bloom give us the following probabilities:

- $P(D+) = 0.01$
- $P(D-) = 0.99$
- $P(T+ \mid D+) = 0.99$
- $P(T+ \mid D-) = 0.02$

In order to make our case, we need to know $P(D+ \mid T+)$. That is the probability that, given a blue car, the witness saw a blue car.
This table summarizes the information we know. Note the small ratio of blue taxis to all taxis: $\frac{1}{100}$.

	Green	Blue
Sees Blue	$P(T+ \mid D-) = 0.02$	$P(T+ \mid D+) = 0.99$
Total	$P(D-) = 0.99$	$P(D+) = 0.01$

We apply Bayes' theorem [5] to $P(T+ \mid D+)$ in order to compute $P(D+ \mid T+)$.

$$P(D+ \mid T+) = \frac{P(T+ \mid D+)\, P(D+)}{P(T+)} \tag{8}$$

We use the Law of total probability to rewrite the denominator of the fraction on the righthand side of 8

$$\frac{P(T+ \mid D+)\, P(D+)}{P(T+)} = \frac{P(T+ \mid D+)\, P(D+)}{P(T+\cap D+) + P(T+\cap D-)} \tag{9}$$

We now use the definition of conditional probabilty to rewrite the probabilities in the denomenator of the equation in the right hand side of the equation above:

$$\frac{P(T+ \mid D+)\, P(D+)}{P(T+)} = \frac{P(T+ \mid D+)\, P(D+)}{P(T+ \mid D+)\, P(D+) + P(T+ \mid D-)\, P(D-)} \tag{10}$$

The terms of the righthand side of the equation above are all in our table, so we now have a way to compute $P(D+ \mid T+)$:

$$\frac{P(T+ \mid D+)\, P(D+)}{P(T+ \mid D+)\, P(D+) + P(T+ \mid D-)\, P(D-)} = \frac{0.99 \times 0.01}{0.99 \times 0.01 + 0.02 \times 0.99} \tag{11}$$

Now we simplify the right hand side of the euqation above to arrive at a value for $P(D+ \mid T+)$.

$$\frac{0.99 \times 0.01}{(0.01 + 0.02) \times 0.99} = \frac{0.99 \times 0.01}{0.03 \times 0.99} = \frac{1}{3} \tag{12}$$

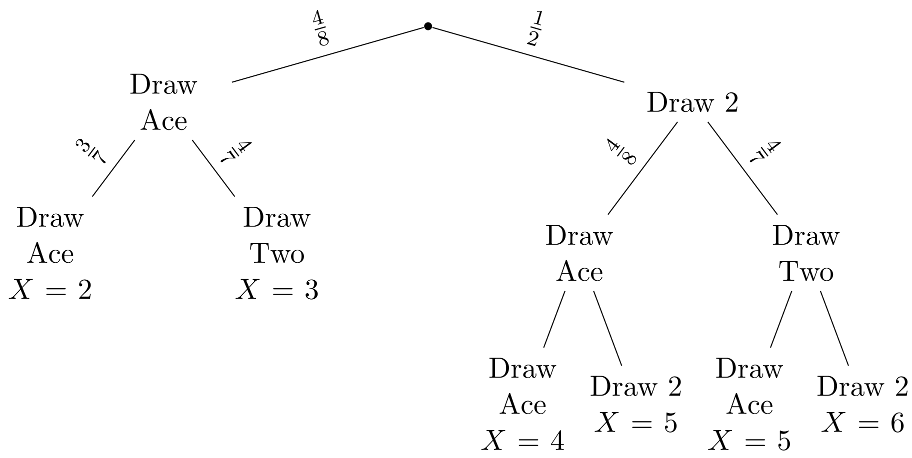
Therefore there is a $\frac{1}{3}$ that given a blue taxi, the witness sees a blue taxi. This is a less than 50% chance that the witness actually saw a blue taxi. Hence we have a reasonable doubt that the witness saw a blue taxi.

4 Trees of cards

In this section we answer Orloff and Bloom's question in [3] about the expected value of a random variable.
The random variable is the value of the sum of cards we draw from a hat.
We refer to a card of rank one as an ace, and a card of rank two as a two.
There are four aces, and four two's in the hat.
Rules govern the way we draw the cards from the hat. The rules are:

- If we draw an ace first, then we draw one more card.
- If we draw a two first, then we draw two more cards.

We assign one to the value of the ace, and two to the value of the two.
We assign the random variable X the value of the sum of the values of the cards we draw.



References

[1] 113.161.72.37 et al. *Boy or Girl paradox*. Available at `https://en.wikipedia.org/w/index.php?title=Boy_or_Girl_paradox&oldid=766674814` (Spring 2014).

[2] Howard Hamilton. *Confusion Matrix*. Available at `http://www2.cs.uregina.ca/~dbd/cs831/notes/confusion_matrix/confusion_matrix.html` (2012/6/8).

[3] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 2, Spring 2014*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps2.pdf` (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 2, Spring 2014 Solutions*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps2_solutions.pdf` (Spring 2014).

[5] Jeremy Orloff and Jonathan Bloom. *Conditional Probability, Independence and Bayes Theorem Class 3, 18.05, Spring 2014*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading3.pdf` (Spring 2014).

[6] Jeremy Orloff and Jonathan Bloom. *Discrete Random Variables Class 4, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading4a.pdf` (Spring 2014).