MIT Introduction to Statistics 18.05 Problem Set 2

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1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
In this document we are answering questions Orloff and Bloom ask in [1].
Please see the references section for detailed citation information.
The material for the course is licensed under the terms at `http://ocw.mit.edu/terms`.
We use documentation in to write LaTeXsource code for this document.

2 Variance of Bernoulli random variable

The first question Orloff and Bloom ask in the lecture 5 slides is for a proof that if $X \sim \text{Bernoulli}(p)$, then $\text{Var}(X) = p(1-p)$.
Orloff and Bloom prove this in [2].

3 Variance of a binomial random variable

Next, Orloff and Bloom ask for a proof that the variance of a random variable $X \sim \text{binomial}(n, p) = np(1-p)$.
Orloff and Bloom also prove this in [2].

4 Variance of a sum

In this section Orloff and Bloom pose the question:
Suppose $X_1, X_2, \ldots, X_n$ are all independent random variables with $\sigma = 2$. Define a new random variable, $\bar{X}$ that is the average of $X_1, X_2, \ldots, X_n$.
They ask, "What is the standard deviation of $\bar{X}$?"
We know from [2] that, for two independent random variables X, and Y, $\mathrm{Var}\,(X+Y) = \mathrm{Var}\,(X) + \mathrm{Var}\,(Y)$
To extend this property to a sum of more than two independent random variables, we let $Y = Z + W$, where Z, and W are independent random variables.
Then $\mathrm{Var}\,(Z+W) = \mathrm{Var}\,(Z) + \mathrm{Var}\,(Z)$, and $\mathrm{Var}\,(X+Y) = \mathrm{Var}\,(X) + \mathrm{Var}\,(Z) + \mathrm{Var}\,(b)$.
We continue to rewrite the last term in the sum of variances until we have an expression on the right hand side of the sum that is the sum of variances of the independent random variables whose sum we wish to know the variance of.
$\bar{X}$ is the average of the random variables $X_1, X_2, \ldots, X_n$, so:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

We apply the variance function to both sides of the equation above:

$$\mathrm{Var}\,\left(\bar{X}\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \tag{2}$$

In [2] Orloff and Bloom show that for constants a, b:

$$\mathrm{Var}\,(aX+b) = a^2\mathrm{Var}\,(X) \tag{3}$$

Therefore

$$\mathrm{Var}\,\left(\bar{X}\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \tag{4}$$

Recall what we showed regarding extending the property of variance to the sum of multiple independent random variables. Because it is true, we can write

$$\mathrm{Var}\,\left(\bar{X}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\,(X_i) \tag{5}$$

Orloff and Bloom give us that $\sigma(X_i) = 2$, so

$$\mathrm{Var}\left(\bar{X}\right) = \frac{1}{n^2}\sum_{i=1}^{n}(4) \tag{6}$$

We evaluate the sum:

$$\mathrm{Var}\left(\bar{X}\right) = \frac{1}{n^2}4n \tag{7}$$

We simplify the right hand side of the equation above:

$$\mathrm{Var}\left(\bar{X}\right) = \frac{4}{n} \tag{8}$$

Since the standard deviation is defined as the square root of the variance, we apply this definition to arrive at the answer to the question:

$$\sigma\left(\bar{X}\right) = \frac{2}{\sqrt{n}} \tag{9}$$

References

[1] Jeremy Orloff and Jonathan Bloom. *Variance; Continuous Random Variables 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom.* Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class5slides.pdf` (Spring 2014).

[2] Jeremy Orloff and Jonathan Bloom. *Variance of Discrete Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom.* Available at `https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5a.pdf` (Spring 2014).