MIT Introduction to Statistics 18.05 Reading 3 - Questions

John Hancock

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
Please see the references section for detailed citation information.
The material for the course is licensed under the terms at `http://ocw.mit.edu/terms`.
We are answering the questions in [3].
We use documentation in [4], [5], [6], and [7] for properly writing the LaTeXsource code for this document.

2 Problem 1

You roll two dice. Consider the following events.
$A =$ 'first die is 3'
$B =$ 'sum is 7'
$C =$ 'sum is greater than or equal to 7'

2.1 Compute $P(B)$; Dice Sum to 7

We are rolling two dice so the sample space, Ω, is $\{(x, y) \mid x, y \in \{1, 2, 3, 4, 5, 6\}\}$
Then B is $\{(x, y) \in \Omega \mid x + y = 7\}$.
Therefore by inspection $B = \{(1, 6), (6, 1), (5, 2), (2, 5), (3, 4), (4, 3)\}$
There are 36 sequences of integers (x, y) for $(x, y) \in \{1, 2, 3, 4, 5, 6\}$. There are 6 elements in B, so $P(B) = \frac{6}{36} \approx 0.1667$.

2.2 Compute $P(B \mid A)$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} \tag{1}$$

We defined elements of B in the previous section, and listed them out.
We define A:

$$A = \{(x, y) \mid x = 3, y \in \{1, 2, 3, 4, 5, 6\}\} \tag{2}$$

A has six elements. Since all events are equally likely, $P(A) = \frac{6}{36} = \frac{1}{6}$.
B has one element where the first element of the sequence is 3, $(3, 4)$.
Therefore $(A \cap B) = (3, 4)$. Since all outcomes are equally likely, $P(A \cap B) = \frac{1}{36}$.
Now, we have all the information we need to calculate $P(B \mid A)$.
We Continue from 2:

$$\frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} \approx 0.167 \tag{3}$$

2.3 Compute $P(B \mid C)$

We define B in the previous section. C is:

$$C = \{(x, y) \mid x + y \geq 7, x, y \in \{1, 2, 3, 4, 5, 6\}\} \tag{4}$$

We count the number of elements of C to find a value for $|C|$.

$$\begin{gathered} C = \{(1, 6), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), (4, 5), (4, 6)\} \\ \cup\{(5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} \end{gathered} \tag{5}$$

We count elements of C listed above to find that $|C| = 21$. However, it behooves us to note that

$$|C| = \sum_{i=1}^{6} i = \frac{6 \times 7}{2} = \frac{42}{2} = 21. \tag{6}$$

Since all outcomes in our current sample space are equally likely, $P(C) = \frac{21}{36}$. Therefore:

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{6}{21} \approx 0.286 \tag{7}$$

2.4 A and B Are Independent

We see in 2.1 $P(A) = \frac{6}{36} \approx 0.167$
We see in 1 that $P(B) = \frac{6}{36} \approx 0.167$
A and B are independent if, and only if,
$P(A \cap B) = P(A) \cdot P(B)$
We see in 2 that $P(A \cap B) = \frac{1}{36}$
We need to check that

$$\frac{1}{36} = \left(\frac{6}{36}\right)\left(\frac{6}{36}\right) \tag{8}$$

Now,

$$\left(\frac{6}{36}\right)\left(\frac{6}{36}\right) = \left(\frac{6}{6^2}\right)\left(\frac{6}{6^2}\right) = \frac{6^2}{6^4} = 6^{2-4} = 6^{-2} = \frac{1}{6^2} = \frac{1}{36} \tag{9}$$

Therefore A and B are independent. ■

2.5 B and C are not independent

We count 21 elements in C in 2.1. Hence $|\Omega| = 36$.
Therefore $P(C) = \frac{21}{36} \approx 0.583$.
In 1 we calculate $P(B) = \frac{1}{6}$
Thus, $P(B) \cdot P(C) = \frac{21}{216}$
However, $B \cap C$ is the set

$$B \cap C = \{(3,4), (3,5), (3,6)\} \tag{10}$$

And, $P(B \cap C) = \frac{3}{36}$.
Since

$$P(B \cap C) \neq P(B) \cdot P(C) \tag{11}$$

B and C are not independent.

3 Problem 2

3.1 $P\left(S_2 \mid S_1^{\complement}\right)$

We calculate this in [1].

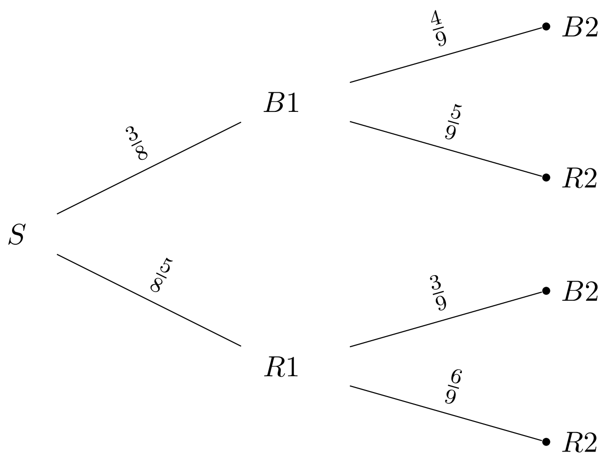
4 Problem 3

4.1 Probability Second Ball Red

We use a probability tree to calculate this probability.
We define four events:

- $R1$ is the event that we draw a red ball for the first drawing, and place two red balls into the urn.
- $R2$ is the event that we draw a red ball for the second drawing.
- $B1$ is the event that we draw a blue ball for the first drawing.
- $B2$ is the event that draw a blue ball for the second drawing.

In addition, we define our starting condition S to be that there are 5 red balls, and 3 blue balls in the urn.



In this problem, Orloff and Bloom are asking us to calculate

$$P\left(R1 \cap R2\right) \cup P\left(B1 \cap R2\right) \tag{12}$$

We calculate this probability as the sum of the products of the probabilities on the edges of the tree above, where the edges are parts of paths that end in $R2$.

This sum of products is

$$P(R1)\,P(R2)+P(B1)\,P(R2) = \left(\frac{5}{8}\right)\left(\frac{6}{9}\right)+\left(\frac{3}{8}\right)\left(\frac{5}{9}\right) = \frac{45}{72} = \frac{5}{8} = 0.625 \tag{13}$$

Therefore the probability that the second ball is red is 0.625.

4.2 Probability First Ball Blue Given Second Ball Red

We calculate the probability that the first ball is blue, given that the second ball is red.
$R1$, $R2$, $B1$, and $B2$ have the same definitions that we give in the previous section.
We use Bayes' Theorem [2] to calculate $P(B1 \mid R2)$.

$$P(B1 \mid R2) = \frac{P(R2 \mid B1)\,P(B1)}{P(R2)} \tag{14}$$

The tree in 4.1 shows $P(R2 \mid B1) = \frac{5}{9}$.
We are given $P(B1) = \frac{3}{8}$, and $P(R2) = \frac{5}{8}$.
We Substitute these values in the right hand side of the equation in 14 to get:

$$P(B1 \mid R2) = \frac{\left(\frac{5}{9}\right)\left(\frac{3}{8}\right)}{\frac{5}{8}} = \left(\frac{5}{9}\right)\left(\frac{3}{8}\right)\left(\frac{8}{5}\right) = \frac{3}{9} \approx 0.333 \tag{15}$$

References

[1] John Hancock. *MIT Introduction to Statistics 18.05 Reading 3 - Think Questions*. Available at https://github.com/jhancock1975/mit-intro-to-stats-18.05/blob/reading-3/reading-3/reading-3.pdf(2017/2/11).

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[3] Jeremy Orloff and Jonathan Bloom. *Reading Questions 3*. Available at https://ocw.mit.edu/courses/mathematics/ 18-05-introduction-to-probability-and-statistics-spring-2014/readings/reading-questions-3/(Spring 2014).

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[5] Stack Exchange Users Will Robertson and Steven. *How can I split an equation over two lines (Answer)*. Available at http://tex.stackexchange.com/questions/3782/how-can-i-split-an-equation-over-two-lines(Spring 2014).

[6] ShareLaTex. *Bold, Italics and Underlining*. Available at https://www.sharelatex.com/learn/Bold,_italics_and_underlining(2017).

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