MIT Introduction to Statistics 18.05 Problem Set 2

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Contents

1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.
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We are answering the questions that Orloff and Bloom ask in [3].
We use documentation in to write LaTeXsource code for this document.

2 'Boy or girl' paradox

In order to write this solution, we rely on the answer to this problem in [4], and the treatment of the 'Boy or girl,' paradox in [1].
For these questions on the 'Boy or girl paradox we deal with events B, "the child is a boy," and G, "the child is a girl."
We assume B, and G have the same properties as the B and G events Orloff and Bloom analize in example 9 of [6]. These properties are that B, and G are independent, and they have probability $\frac{1}{2}$.
We use these properties to define 4 more events, BB, BG, GB, and GG. These events are: "the younger child is a boy, and the older child is a boy," "the younger child is a boy, and the older child is a girl," "the younger child is a girl, and the older child is a boy," "the younger child is a girl, and the

older child is a girl," respectively. We use the properties of B, and G, of example 9 to compute that the probabilities of BB, BG, GB, GG, $P(BB)$, $P(BG)$, $P(GB)$, $P(GG)$, are all equal to $\frac{1}{4}$.

2.1 Probability of girls

The question Orloff and bloom quote is, "Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?"
We can restate the question above as, "Given event BG or event GG, what is the probability GG?"
We use the definition of conditional probability. We also use the law of total probability to comupte $P(GG \cup BG)$.
Therefore we write the equation:

$$P(GG \mid GG \cup BG) = \frac{P(GG \cap (GG \cup BG))}{P(GG \cup BG)} \tag{1}$$

$$\frac{P(GG \cap (GG \cup BG))}{P(GG \cup BG)} = \frac{P(GG)}{P(GG \cup BG)} \tag{2}$$

$$\frac{P(GG)}{P(GG \cup BG)} = \frac{\frac{1}{4}}{\frac{1}{2}} \tag{3}$$

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \left(\frac{1}{4}\right)\left(\frac{2}{1}\right) = \frac{1}{2} \tag{4}$$

Therefore if Mr. Jones' older child is a girl, there is a probabiility of $\frac{1}{2}$ that the younger child is also a girl.

2.2 Probability of boys

In this section, Orloff and Bloom quote another question for us to answer here.
The question is, "Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?"
We use the definitions from the previous section for events, BB, BG, GB and GG. We use the probabilities we found in the first section for these events as well.
The author of this question is giving us that three possible events have occured: BB, BG, or GB. Furthermore the question asks for the conditional probability of BB.
We use the definition of conditional probability, and the law of total probability to compute:

$$P(BB \mid BB \cup BG \cup GB) = \frac{P(BB \cap (BB \cup BG \cup GB))}{P(BB \cup BG \cup GB)} \tag{5}$$

$$\frac{P(BB \cap (BB \cup BG \cup GB))}{P(BB \cup BG \cup GB)} = \frac{P(BB)}{P(BB \cup BG \cup GB)} \tag{6}$$

$$\frac{P(BB)}{P(BB \cup BG \cup GB)} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \left(\frac{1}{4}\right)\left(\frac{4}{3}\right) = \frac{1}{3} \tag{7}$$

If at least one of Mr. Smith's children is a boy, then there is a probability of $\frac{1}{3}$ that both children are boys.

3 The blue taxi

In order to solve this problem we will write a confusion matrix [2]. This is the term we found for the kind of table Orloff and Bloom write in [5] that organizes false positive rates, false negative rates, etc. Into a table.
We define the following sets:

- $D+$, "The car is blue."
- $D-$, "The car is green."
- $T+$, "The witness reports seing a blue car."
- $T-$, "The witness reports seing a green car."

Orloff and Bloom give us the following sizes of the probabilities:

- $P(D+) = 0.01$
- $P(D-) = 0.99$
- $P(T- \mid D-) = 0.99$
- $P(T- \mid D+) = 0.02$

In order to make our case, we need to know $P(D+ \mid T+)$.
We are not given $P(D+ \mid T+)$. However, we can apply Bayes' theorem to $P(T+ \mid D+)$ in order to compute $P(D+ \mid T+)$.
We are not given $P(T+ \mid D+)$ either, but we can compute this value with the aid of a confusion matrix. Ironically the confusion matrix lends us understanding.
We write several versions of the confusion matrix, first with only symbolic values, fill in what we know. Then we start compute values we do not know in order to complete a version of the confusion matrix with numeric values.

	$D+$	$D-$	Total
$T+$	$P(T+ \mid D+)$	$P(T+ \mid D-)$	$P(T+)$
$T-$	$P(T- \mid D+)$	$P(T- \mid D-)$	$P(T-)$
total	$P(D+)$	$P(D-)$	$P(D+ \cup D-)$

References

[1] 113.161.72.37 et al. *Boy or Girl paradox*. Available at https://en.wikipedia.org/w/index.php?title=Boy_or_Girl_paradox&oldid=766674814 (Spring 2014).

[2] Howard Hamilton. *Confusion Matrix*. Available at http://www2.cs.uregina.ca/~dbd/cs831/notes/confusion_matrix/confusion_matrix.html (2012/6/8).

[3] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 2, Spring 2014*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps2.pdf (Spring 2014).

[4] Jeremy Orloff and Jonathan Bloom. *18.05 Problem Set 2, Spring 2014 Solutions*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/assignments/MIT18_05S14_ps2_solutions.pdf (Spring 2014).

[5] Jeremy Orloff and Jonathan Bloom. *Conditional Probability, Independence and Bayes Theorem Class 3, 18.05, Spring 2014*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading3.pdf (Spring 2014).

[6] Jeremy Orloff and Jonathan Bloom. *Discrete Random Variables Class 4, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading4a.pdf (Spring 2014).