

# Problem Set 4

John Hancock

June 28, 2017

## Contents

<b>1</b>	<b>References and License</b>	<b>2</b>
<b>2</b>	<b>Time to failure</b>	<b>3</b>
2.1	$P(X \geq x)$ . . . . .	3
2.2	CDF of Minimum of two exponential random variables . . . . .	5
2.3	Three lightbulbs . . . . .	8
<b>3</b>	<b>Aching Joints</b>	<b>9</b>
3.1	Value of $c$ . . . . .	9
3.2	Marginal cumulative distribution, and probability density functions . . . . .	10
3.3	$E(X)$ , and $\text{Var}(X)$ . . . . .	14
3.4	Variance and Covariance of $X$ and $Y$ . . . .	16

<b>4</b>	<b>Elections</b>	<b>19</b>
4.1	Apply Central Limit Theorem . . . . .	19
4.2	Apply central limit theorem again . . . . .	22
<b>5</b>	<b>Rounding Error</b>	<b>24</b>
<b>6</b>	<b>Independence</b>	<b>26</b>
<b>7</b>	<b>Correlation</b>	<b>27</b>
7.1	Compute Covariance . . . . .	27
7.2	Calculate Correlation . . . . .	29
7.3	Independence, correlation . . . . .	30
<b>8</b>	<b>Joint Uniform Distribution</b>	<b>31</b>
8.1	Joint distribution functions . . . . .	31
8.2	Probability for $A$ . . . . .	32
8.3	Probability for $A$ and $B$ . . . . .	32
8.4	Inequality probability . . . . .	33
8.5	Probability $A$ within range of $B$ . . . . .	35

## 1 References and License

We are answering questions in the material from MIT OpenCourseWare course 18.05, Introduction to Probability and Statistics.

In this document we are answering questions Orloff and Bloom ask in [8].

Please see the references section for detailed citation information.

The material for the course is licensed under the terms at <http://ocw.mit.edu/terms>.

We use documentation in `[logicNot]`, `[proofs]`, `[bars]`, `[packageClash]`, `[curlyBrace]`, `[cases]` to write the L<sup>A</sup>T<sub>E</sub>X source code for this document.

## 2 Time to failure

The first group of problems Orloff and Bloom have for us involve some random variables that follow an exponential distribution.

The exponential distribution they give us to work with has probability density function (pdf):

$$f(x) = \lambda e^{-\lambda x}, x \geq 0. \quad (1)$$

### 2.1 $P(X \geq x)$

We know how to calculate  $P(X < x)$  as a definite integral [2], therefore we will find  $P(X < x)$ , and our final result will be to find  $P(X \geq x) = 1 - P(X < x)$ .

In order to calculate this probability, we will do a change of variable similar to the technique Orloff and Bloom show in section 3.4 of [7].

We change the variable in the pdf  $f(x)$  to  $u$ ; therefore we rewrite the pdf as  $f(u)$ :

$$f(u) = \lambda e^{-\lambda u}. \quad (2)$$

We use this definition, the fact that  $f$  is defined for  $x \geq 0$ , and the definition of probability of continuous random variables [2] to write this equation:

$$P(X < x) = \int_0^x \lambda e^{-\lambda u} du. \quad (3)$$

We substitute the integral on the right hand side of the previous equation with its antiderivative to get:

$$P(X < x) = -e^{-\lambda u} \Big|_{u=0}^x. \quad (4)$$

We evaluate the the antiderivative at the limits of integration:

$$P(X < x) = -e^{-\lambda x} - -e^{-\lambda 0}. \quad (5)$$

Now we simplify the previous equation:

$$P(X < x) = -e^{-\lambda x} + 1. \quad (6)$$

Now, we apply the identity:

$$P(X \geq x) = 1 - P(X < x). \quad (7)$$

Therefore

$$P(X \geq x) = 1 - (-e^{-\lambda x} + 1). \quad (8)$$

The previous equation simplifies to:

$$P(X \geq x) = e^{-\lambda x}. \quad (9)$$

## 2.2 CDF of Minimum of two exponential random variables

In this section Orloff and Bloom ask us to find the cumulative distribution function (CDF) of two independent random variables  $X_1$ , and  $X_2$  that both follow an exponential distribution, and that both have mean  $\frac{1}{\lambda}$ .

In [6] Orloff and Bloom state that the mean of a random variable that has probability mass function (pmf)  $\lambda e^{-\lambda x}$  is  $\frac{1}{\lambda}$ .

Therefore  $X_1$ , and  $X_2$  both have pmf's  $\lambda e^{-\lambda x}$ .

For this problem, Orloff and Bloom let  $T = \min(X_1, X_2)$ .

They ask us for the cdf of  $T$ .

The cdf of  $T$  is a function  $F(t) = P(T < t)$ .

In the previous section, we found that for a random variable  $X$  that has pdf  $\lambda e^{-\lambda x}$ ,

$$P(X \geq x) = e^{-\lambda x}. \quad (10)$$

We use the definition of  $T$  to write the equation:

$$P(T \geq t) = P(\min(X_1, X_2) \geq t). \quad (11)$$

$T = \min(X_1, X_2)$ , so  $T \geq t$  if, and only if,  $X_1 \geq t$ , and  $X_2 \geq t$ .

**Lemma 1.** *If two events  $A$  and  $B$  have a biconditional relation, then*

$$P(A) = P(B). \quad (12)$$

*Proof.* The proof is by contradiction. Assume events  $A$  and  $B$  are biconditionally related, but  $P(A) \neq P(B)$ . Then there would be unequal chances of events  $A$  and  $B$  occurring, which means that one event would occur while the other does not. But  $A$  and  $B$  are biconditionally related, so event  $A$  occurs when, and only when, event  $B$  occurs. This is a contradiction, so  $P(A) = P(B)$ .  $\square$

Therefore

$$P(T \geq t) = P(X_1 \geq t, X_2 \geq t). \quad (13)$$

$X_1$  and  $X_2$  are independent events. In [7] Orloff and Bloom state that random variables  $X_1$  and  $Y_1$  are independent if and only if:

$$P(X_1, X_2) = F_{X_1}(x_1) F_{X_2}(x_2). \quad (14)$$

$F(X_1, X_2)$  is the cdf of  $X_1$ , and  $X_2$ .  $F_{X_1}(x_1)$ , and  $F_{X_2}(x_2)$  are the marginal cumulative distribution functions of  $X_1$ , and  $X_2$ .

We know that  $X_1$ , and  $Y_1$  are exponentially distributed random variables with mean  $\frac{1}{\lambda}$ . The answer we find in the previous section implies that the cdf of  $X_1$  is  $F_{X_1}(x_1) = e^{-\lambda x_1}$ , and the cdf of  $X_2$  is  $F_{X_2}(x_2) = e^{-\lambda x_2}$ .

Since  $X_1$ , and  $X_2$  are independent,

$$F(X_1, X_2) = e^{-\lambda x_1} e^{-\lambda x_2}. \quad (15)$$

We add the exponents of the base  $e$  to simplify the previous equation to:

$$F(X_1, X_2) = e^{-\lambda(x_1+x_2)}. \quad (16)$$

We are finding the cdf of  $T$ .

Consider  $F(X_1 < t, X_2 < t)$ . We use the previous equation to write:

$$F(X_1 < t, X_2 < t) = e^{-\lambda 2t}. \quad (17)$$

Since  $P(T < t) = P(X_1 < t, X_2 < t)$ :

$$F(T < t, X_2 < t) = e^{-\lambda 2t}. \quad (18)$$

### 2.3 Three lightbulbs

In this section we answer a question about three lightbulbs,  $B_1$ ,  $B_2$ , and  $B_3$ , where each lightbulb's lifetime is an exponential random variable with mean values  $\frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{5}$ , respectively. The unit for each mean value is years.

Furthermore, Orloff and Bloom state that the lifetimes of the lightbulbs are independent.

Let  $X_1$ ,  $X_2$ , and  $X_3$  be the random variables equal to the lifetimes of  $B_1$ ,  $B_2$ , and  $B_3$ , respectively.

Since the mean value of  $X_1$  is  $\frac{1}{2}$ , and  $X_1$  follows an exponential distribution,  $X_1 \sim 2e^{-2t}$ .

Similarly,  $X_2 \sim 3e^{-3t}$ , and  $X_3 \sim 5e^{-5t}$

We apply logic similar to what we use in the previous question to state that the cdf of a random variable  $T = \min(X_1, X_2, X_3)$  is

$$F(T < t) = e^{-1(2+3+5)t} = e^{-10t}. \quad (19)$$

The expected value of a random variable with cdf  $e^{-10t}$  is  $\frac{1}{10}$  year.



### 3 Aching Joints

We deal with the joint distribution of two continuous random variables  $X$ , and  $Y$ . The probability density function for  $X$ , and  $Y$  is  $f(x) = c(x^2 + xy)$ . Furthermore,  $f$  is defined on  $[0, 1] \times [0, 1]$ .

#### 3.1 Value of $c$

The first thing Orloff and Bloom ask us for is the value of  $c$  in  $f$  as defined above.

In order for  $f(x, y)$  to be a probability distribution function (PDF),

$$c \int_0^1 \int_0^1 x^2 + xy \, dx \, dy = 1. \quad (20)$$

This is true, if and only if:

$$c \int_0^1 \frac{x^3}{3} + \frac{x^2 y}{2} dy \Big|_{x=0}^1 = 1. \quad (21)$$

We evaluate the anti-derivative over the interval indicated in the equation above to obtain:

$$c \int_0^1 \frac{1}{3} + \frac{y}{2} dy = 1. \quad (22)$$

We now replace the integral of the function of  $y$  above with its anti-derivative:

$$c \frac{y}{3} + \frac{y^2}{4} \Big|_{y=0}^1 = 1. \quad (23)$$

And, we now evaluate the anti-derivative over the interval indicated in the equation above:

$$c \frac{1}{3} + \frac{1}{4} = 1. \quad (24)$$

Now, we simplify the previous equation:

$$c \frac{7}{12} = 1. \quad (25)$$

And, we solve for  $c$ :

$$c = \frac{12}{7}. \quad (26)$$

### 3.2 Marginal cumulative distribution, and probability density functions

Orloff and Bloom are asking us to find four functions

- the marginal CDF  $F_Y(y)$ ,
- the marginal CDF  $F_X(x)$ .
- the marginal PDF  $f_Y(y)$ , and

- the marginal PDF  $f_X(x)$ .

The definition of marginal CDF dictates that we evaluate the CDF at the upper limit of the variable that we are not finding the marginal CDF for.

Therefore,

$$F_Y(y) = F(1, y). \quad (27)$$

In order to find the marginal CDF's, we need to find the anti-derivative of the PDF:

$$F(x, y) = \int \int \frac{12}{7} xy + x^2 dx dy. \quad (28)$$

First we find the anti-derivative with respect to  $x$ :

$$F(x, y) = \int \frac{12}{7} \frac{1}{2} x^2 y + x^2 y dy. \quad (29)$$

Next, we find the anti-derivative with respect to  $y$ :

$$F(x, y) = \frac{12}{7} \frac{x^2 y^2}{2} + \frac{x^2 y}{2}. \quad (30)$$

Now that we have the CDF, we can partially evaluate it to obtain the marginal CDF's.

$$F_Y(y) = F(1, y). \quad (31)$$

We replace the right hand side of the equation above with the CDF we found, and replace  $x$  with the value 1 to find the CDF for  $y$ .

$$F_Y(y) = \frac{12}{7} \frac{y^2}{2} + \frac{2y}{2}. \quad (32)$$

$$F_X(x) = F(x, 1). \quad (33)$$

We replace the right hand side of the equation above with the CDF we found, and replace  $y$  with the value 1 to find the CDF for  $X$ .

$$F_Y(y) = \frac{12}{7} \left( \frac{x^2}{2} + \frac{x^2}{2} \right). \quad (34)$$

The expression above simplifies to:

$$F_Y(y) = \frac{12x^2}{7}. \quad (35)$$

Now we move on to deriving the marginal PDF's. We use the definition of marginal PDF from [7].

First we tackle the marginal PDF  $f_X(x)$ .

$$f_X(x) = \int_0^1 \frac{12}{7} (x^2 + xy) dy \quad (36)$$

We replace the function with its anti-derivative:

$$f_X(x) = \frac{12}{7} \left( x^2 y + \frac{xy^2}{2} + \right) \Big|_{y=0}^1. \quad (37)$$

Now we evaluate the anti-derivative for the interval  $[0, 1]$ .

$$f_X(x) = \frac{12}{7} \left( x^2 + \frac{x^2}{2} + \right). \quad (38)$$

This is the marginal PMF  $f_X(x)$ .

We do a similar integration to derive the marginal PDF  $f_Y(y)$ .

$$f_Y(y) = \frac{12}{7} \left( \int_0^1 x^2 + xy \, dx \right). \quad (39)$$

We replace the integral in the equation above with its anti-derivative.

$$f_Y(y) = \frac{12}{7} \left( \frac{x^3}{3} + \frac{1}{2} x^2 y \Big|_0^1 \right). \quad (40)$$

We evaluate the anti-derivative over the interval  $[0, 1]$ .

$$f_Y(y) = \frac{12}{7} \left( \frac{1}{3} + \frac{1}{2} y \right). \quad (41)$$

This is the marginal PDF for  $f_Y(y)$ .

### 3.3 $E(X)$ , and $\text{Var}(X)$

In order to calculate  $E(X)$ , we integrate the product of the PDF  $f(x, y)$ , and  $x$ .

$$E(X) = \int_0^1 \int_0^1 x \frac{12}{7} (x + xy) dx. \quad (42)$$

We distribute  $x$  in the expression above.

$$E(X) = \int_0^1 \int_0^1 \frac{12}{7} (x^2 + x^2 y) dx dy. \quad (43)$$

First, we integrate with respect to  $x$ .

$$E(X) = \int_0^1 \frac{12}{7} \left( \frac{x^3}{3} + \frac{x^3 y}{3} \Big|_{x=0}^1 \right) dy. \quad (44)$$

Next we integrate with respect to  $y$ :

$$E(X) = \frac{12}{7} \left( \frac{x^3 y}{3} + \frac{x^3 y^2}{6} \Big|_{x=0}^1 \Big|_{y=0}^1 \right). \quad (45)$$

We evaluate the expression at the limits of integration specified to compute:

$$E(X) = \frac{12}{7} \left( \frac{1}{3} + \frac{1}{6} \right) = \frac{6}{7}. \quad (46)$$

In order to compute the variance of  $X$ , we use the identity

$$\text{Var} (X) = E \left( X^2 \right) + E (X)^2 \quad (47)$$

We integrate the product of  $X^2$ , and the PDF Orloff and Bloom give us in order to calculate  $E \left( X^2 \right)$ .

$$E \left( X^2 \right) = \int_0^1 \int_0^1 x^2 \frac{12}{7} (x + xy) \, dx \, dy. \quad (48)$$

We distribute the  $x^2$  term in the expression above to aid in determining the anti-derivative of the integrals.

$$E \left( X^2 \right) = \int_0^1 \int_0^1 \frac{12}{7} (x^3 + x^2 y) \, dx \, dy. \quad (49)$$

First we replace the inner integral with its anti-derivative with respect to  $x$ :

$$E \left( X^2 \right) = \int_0^1 \frac{12}{7} \left( \frac{x^4}{4} + \frac{x^3 y}{3} \right) \Big|_{x=0}^1 \, dy. \quad (50)$$

Next, we replace the remaining integral with its anti-derivative:

$$E \left( X^2 \right) = \frac{12}{7} \left( \frac{x^4 y}{4} + \frac{x^3 y^2}{6} \right) \Big|_{x=0}^1 \Big|_{y=0}^1. \quad (51)$$

Now we evaluate the anti-derivatives to complete the integration:

$$E(X^2) = \frac{12}{7} \left( \frac{1}{4} + \frac{1}{6} \right). \quad (52)$$

We simplify the expression above to:

$$E(X^2) = \frac{5}{7}. \quad (53)$$

Now that we know  $E(X^2)$ , we can calculate  $\text{Var}(X)$ .

$$\text{Var}(X) = E(X^2) - E(X)^2. \quad (54)$$

We substitute the values for  $E(X^2)$ , and  $E(X)^2$  in the equation above to obtain:

$$\text{Var}(X) = \frac{5}{7} - \left( \frac{6}{7} \right)^2 \quad (55)$$

### 3.4 Variance and Covariance of $X$ and $Y$

In order to compute the variance and covariance of  $X$ , and  $Y$ , we will use the property of covariance:

$$\text{Cov} = E(XY) - \mu_x \mu_y \quad (56)$$

However, in order to use the property, we must calculate  $\mu_y$ , and  $E(XY)$ . Note that we computed  $\mu_x$ , in the previous section.



We use the definition of expected value from [5] to compute  $\mu_y$ .

$$\mu_y = \int_0^1 \int_0^1 y \frac{12}{7} (x^2 + xy) dx dy. \quad (57)$$

We distribute  $y$  in order to determine what the anti-derivative is for the integral above:

$$\mu_y = \int_0^1 \int_0^1 y \frac{12}{7} (x^2 y + xy^2) dx dy. \quad (58)$$

We replace the integrals above with their anti-derivatives:

$$\mu_y = \frac{12}{7} \left( \frac{x^3 y^2}{6} + \frac{x^2 y^3}{6} \Big|_{x=0}^1 \Big|_{y=0}^1 \right). \quad (59)$$

And we evaluate the anti-derivative at the limits above to obtain:

$$\mu_y = \frac{12}{7} \left( \frac{1}{6} + \frac{1}{6} \right). \quad (60)$$

The equation above simplifies to:

$$\mu_y = \frac{4}{42}. \quad (61)$$

We continue to collect components of the sum we will compute in order to calculate  $\text{Cov}(X, Y)$ .

We resort to the definition of expected value in order to compute  $E(X, Y)$

$$E(X, Y) = \int_0^1 \int_0^1 \frac{12}{7} xy (x^2 + xy) dx dy. \quad (62)$$

We distribute  $xy$  in order to find the anti-derivatives of the integrals above:

$$E(X, Y) = \int_0^1 \int_0^1 \frac{12}{7} (x^3 y + x^2 y^2) dx dy. \quad (63)$$

Now we replace the integrals above with their anti-derivatives:

$$E(X, Y) = \frac{12}{7} \left( \frac{x^4 y^2}{8} + \frac{x^3 y^3}{9} \Big|_{x=0}^1 \Big|_{y=0}^1 \right). \quad (64)$$

We now evaluate the expression above at the limits of integration to find:

$$E(X, Y) = \frac{12}{7} \left( \frac{1}{8} + \frac{1}{9} \right). \quad (65)$$

This equation above simplifies to:

$$E(X, Y) = \frac{17}{42} \quad (66)$$

We now have all the components we need to compute  $\text{Cov}(X, Y)$ .

$$\text{Cov} = E(XY) - \mu_x \mu_y \quad (67)$$

We substitute the values we computed previously into the equation above:

$$\text{Cov} = \frac{2}{63} - \frac{5}{7} \frac{5}{7} \quad (68)$$

The above simplifies to:

$$\text{Cov} = \frac{211}{567} \quad (69)$$

## 4 Elections

In this section we deal with an election scenario, and a poll of 400 people. In the population: 0.5 of the people support Erika, 0.2 support Ruthi, and the remaining 0.3 support Peter, John or Jerry.

### 4.1 Apply Central Limit Theorem

Orloff and Bloom ask us for the probability that if we poll 400 people, at least 52.5% prefer Erika. Furthermore, they ask us to use the central limit theorem to estimate the probability.

We treat the response of the  $i^{th}$  person polled  $X_i$  as a Bernoulli(0.5) random variable.

We are looking for the probability that the sum:

$$\sum_{i=1}^n x_i = (0.525)400 = 210. \quad (70)$$

We will refer to this sum as  $S$ . Then:

$$P(S > 210) = P\left(\frac{S - \mu}{\sigma} > \frac{210 - \mu}{\sigma}\right). \quad (71)$$

Note: if  $S$  has mean  $\mu$ , then  $S - \mu$  has mean 0. We cite properties of expected value in [4] for the explanation that  $S - \mu$  has mean 0.

Also note: if  $S$  has standard deviation  $\sigma$  then  $\frac{S}{\sigma}$  has standard deviation 1. We use the property of variance [2]:

$$\text{Var}(aX + b) = a^2 \text{Var}(X). \quad (72)$$

Here we let  $a = \frac{1}{\sigma}$ . Then

$$\text{Var}\left(\frac{S}{\sigma}\right) = \frac{1}{\sigma^2} \text{Var}(S). \quad (73)$$

We beg the reader to bear in mind that the standard deviation of  $S$  is  $\sigma$ .

By definition, variance is the square of standard deviation, so the expressions on both sides of the equation above are equal to 1.

In order to procede, we must compute the mean, and standard deviation of  $S$ .

We define  $S$  as the sum of 400 Bernoulli(0.5) random variables,  $X_i$ . We know from [4] that the expected value of  $X_i$  is 0.5. [4] also tells us that the expected value of the sum of the  $X_i$  is 200.

In [9], Orloff and Bloom show the variance of a random variable  $X \sim \text{Binomial}(n, p)$  is  $np(1 - p)$ .

Therefore the variance of  $S$  is

$$400 \times 0.5 \times 0.5 = 100. \quad (74)$$

Since standard deviation is the square root of variance, the standard deviation  $\sigma_S$  of  $S$  is 10.

Now that we have computed the mean and standard deviation of  $S$ , we can substitute them into 79:

$$P(S > 210) = P\left(\frac{S - 200}{10} > \frac{210 - 200}{10}\right). \quad (75)$$

The equation above simplifies to:

$$P(S > 210) = P\left(\frac{S - 200}{10} > 1\right). \quad (76)$$

We may apply the central limit theorem to write:

$$P\left(\frac{S - 200}{10} > 1\right) \approx P(Z > 1). \quad (77)$$

We use the rule of thumb [1] and the expression above to ascertain

$$P\left(\frac{S - 200}{10} > 1\right) \approx 0.68 \quad (78)$$

Recalling 79 we see it is the case that:

$$P(S > 210) \approx 0.68. \quad (79)$$

## 4.2 Apply central limit theorem again

Now, we turn our attention to the question of what would be the probability that at least 0.25 of 400 people polled support Peter, Jon, or Jerry.

We define a new sum of random variables

$$S = \sum_{n=1}^{400} X_i \quad (80)$$

Where each of the  $X_i$  is a Bernoulli random variable that takes the value of 1 with a probability of  $\frac{1}{3}$ , and 0 otherwise.

We define the  $X_i$  in this way because 0.3 of the population support Peter, Jon, or Jerry.

Moreover [9] explains that the mean (expected) value of  $S$  is  $\mu_S = 400 \times 0.3 = 120$ , and that the variance of  $S$  is  $400 \times 0.3 \times 0.7 = 84$ . Hence the standard deviation  $\sigma_S$  of  $S$  is  $\sqrt{84} \approx 9.165$ .

In this case we are looking for

$$P(S < 100) \tag{81}$$

We use the same reasoning as in the previous section to conclude:

$$P(S < 100) = P\left(\frac{S - 120}{9.165} < \frac{100 - 120}{9.165}\right). \tag{82}$$

And we are in a position to use the central limit theorem to obtain:

$$P\left(\frac{S - 120}{9.165} < \frac{100 - 120}{9.165}\right) \approx P(Z < -2.182) \tag{83}$$

The rule of thumb [1] tells us that there is a 95% probability that  $Z$  will be within two standard deviations of the mean. Since  $Z$  has standard deviation 1, there is a 5% chance  $Z$  will have a value larger than 2, or smaller than -2. The

normal distribution is symmetric with respect to the  $x$  axis, so the chance  $Z$  is less than 2 is 0.025.

We can get a more precise answer using R. In R, `qnorm(-2.182) = 0.01455477`.

## 5 Rounding Error

In this section we address the problem Orloff and Bloom pose regarding rounding error, where we round 1,000 random amounts of money to the nearest nickel.

We model the rounding error as a random variable  $X_i, i \in 1, 2, \dots, 1,000$  from a uniform distribution from the set of values:  $\{-2, -1, 0, 1, 2\}$ .

We define  $S$  to be the sum of the  $X_i$ . In order to apply the central limit theorem we must find the mean  $\mu_S$ , and standard deviation  $\sigma_S$  of  $S$ .

Each of the  $X_i$  is uniformly distributed from a set of 5 values; therefore the expected value of  $X_i$  is:

$$0.2(-2) + 0.2(-1) + 0.2(0) + 0.2(1) + 0.2(2) = 0 \quad (84)$$

Therefore the expected value of  $S$  is also 0.

We use the formula for variance of a uniformly distributed random variable from [9].

In this case,

$$Var(X_i) = \frac{24}{12} = 2. \quad (85)$$



Since  $S$  is the sum of the  $X_i$ ,  $\text{Var}(S) = 2,000$ .

Hence,  $\sigma_S = \sqrt{2,000}$ .

Orloff and Bloom ask us to use the central limit theorem to estimate

$$P(S > 100). \quad (86)$$

We standardize the event above to an event of equal probability:

$$P(S > 100) = P\left(\frac{S - \mu_S}{\sigma_S} > \frac{100 - \mu_S}{\sigma_S}\right). \quad (87)$$

We determined  $\mu_S$  and  $\sigma_S$ , so we substitute those values in the expression above to get:

$$P(S > 100) = P\left(\frac{S}{\sqrt{2000}} > \frac{100}{\sqrt{2000}}\right). \quad (88)$$

We simplify the right hand side of the inequality above to:

$$P(S > 100) = P\left(\frac{S}{\sqrt{2000}} > 2.236\right). \quad (89)$$

We are now in a position to apply the central limit theorem [1].  $S$  is the sum of independent identically distributed random variables, and we normalize  $S$  in such a way that it has mean 0 and standard deviation 1. Hence, we apply the central limit theorem, and approximate:

$$P(S > 100) \approx P(Z > 2.236). \quad (90)$$

Where  $Z$  follows the normal distribution  $N(0, 1)$ .

We use the `pnorm` function of the R programming language to find  $P(Z > 2.236) \approx .0254$ .

The R code we write to obtain the probability above is:

```
(1 - pnorm(2.236)) * 2
```

To summarize, there is approximately a 0.025 probability that the rounding error will be greater than or equal to 100 cents.

## 6 Independence

Orloff and Bloom give us a joint distribution of two random variables  $X$ , and  $Y$ , and ask us to determine whether or not they are independent.

In [7] Orloff and Bloom state that two random variables in a joint distribution are independent if the probability of every possible combination of the random variables is the product of the respective marginal probabilities of the combination.

Therefore we will extend the table Orloff and Bloom give for this problem to include the marginal probabilities:

$X/Y$	1	2	3	$f_X$
1	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{3}$
$f_Y$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

We see in the table above that the marginal probabilities are all  $\frac{1}{3}$ . Therefore, the product of any pair of marginal probabilities is  $\frac{1}{9}$ . However, there are entries in the table above that are not  $\frac{1}{9}$ . Therefore  $X$ , and  $Y$  are not independent.

## 7 Correlation

Orloff and Bloom give the following distribution for this problem:

$X/Y$	-1	1	$f_X$
-1	$c$	$\frac{1}{2} - c$	$\frac{1}{2}$
1	$\frac{1}{2} - c$	$c$	$\frac{1}{2}$
$f_Y$	$\frac{1}{2}$	$\frac{1}{2}$	1

### 7.1 Compute Covariance

We use the property of covariance from [3]:

$$\text{Cov}(XY) = E(XY) - \mu_X \mu_Y. \quad (91)$$

Evidently, it is necessary to derive  $E(XY)$ ,  $E(X)$ , and  $E(Y)$ , in order to determine the covariance of  $X$ , and  $Y$ .

$$E(XY) = \sum_{i=1}^2 \sum_{j=1}^2 x_i y_j p(x_i, y_j). \quad (92)$$

We use the distribution in the table above to compute this expected value.

$$E(XY) = (-1)(-1)c + (-1)(1)\left(\frac{1}{2} - c\right) + (-1)(1)\left(\frac{1}{2} - c\right) + (1)(1)c. \quad (93)$$

We can simplify the right hand side to:

$$E(XY) = 4c - 1. \quad (94)$$

We need the expected value of  $X$ , and  $Y$  as well. The expected value  $\mu_X$  of  $X$  is:

$$E(X) = \sum_{i=1}^2 \sum_{j=1}^2 x_i p(x_i, y_j). \quad (95)$$

This sum is:

$$E(X) = (-1)c + (1)\frac{1}{2} - c + (-1)\frac{1}{2} - c + (1)c. \quad (96)$$

The right hand side of the equation above simplifies as follows:

$$E(X) = 0. \quad (97)$$

The the values for  $Y$  in this problem are such that we calculate  $\mu_Y$  to have the same value as we find for  $\mu_X$ . Hence,  $E(Y) = 0$ .

Therefore  $\text{Cov}(X, Y) = 4c - 1$ .

## 7.2 Calculate Correlation

In order to calculate the correlation of  $X$ , and  $Y$ , we must find the standard deviation of  $X$ , and  $Y$ .

We use the property of variance from [9]:

$$\text{Var}(X) = E(X^2) - E(X)^2. \quad (98)$$

We know from our calculations in the previous section that  $E(X)$ , and  $E(Y)$  are both equal to 0.

Therefore we only need to calculate  $E(X^2)$ , and  $E(Y^2)$ .

$$E(X^2) = \sum_{i=1}^2 \sum_{j=1}^2 x_i^2 p(x_i, y_j). \quad (99)$$

This sum is:

$$E(X) = (-1)^2 c + (1)^2 \frac{1}{2} - c + (-1)^2 \frac{1}{2} - c + (1)^2 c. \quad (100)$$

The right hand side of the equation above simplifies to:

$$E(X) = 1. \quad (101)$$

Similar calculations would show that  $E(Y) = 1$ .

Therefore  $\text{Cor}(X, Y) = 4c - 1$ .

### 7.3 Independence, correlation

In this section we answer the questions Orloff and Bloom ask about which values of  $c$  in the distribution they give us for the problem make  $X$ , and  $Y$  independent, and which values make  $X$ , and  $Y$  100% correlated.

In order for  $X$ , and  $Y$  to be independent, each probability in the table above must be the product of the marginal probabilities for the row and column that the probability is in.

All of the marginal probabilities are  $\frac{1}{2}$ , so in order for  $X$ , and  $Y$  to be independent  $c$  and  $\frac{1}{2} - c$  must both equal  $\frac{1}{4}$ . If  $c$  equals  $\frac{1}{4}$ , then  $\frac{1}{2} - c = \frac{1}{4}$ . Hence  $X$ , and  $Y$  are independent when  $c = \frac{1}{4}$ .

In order for  $X$ , and  $Y$  to be 100% correlated,  $\text{Cor}(X, Y) =$

1. Above, we determine:

$$\text{Cor}(X, Y) = 4c - 1 \quad (102)$$

Therefore,  $X$ , and  $Y$  are 100% correlated when  $c = \frac{1}{2}$ .

## 8 Joint Uniform Distribution

In this section, we are dealing with joint, independent, uniformly distributed random variables  $A$  and  $B$ .  $A$  and  $B$  are both uniformly distributed on  $[0, 60]$ .

### 8.1 Joint distribution functions

$A$ , and  $B$  are independently, uniformly distributed random variables. There is a one in 60 chance that  $A$  takes a certain value, and likewise for  $B$ . Therefore, the probability that  $A$ , and  $B$  take a specific pair of values is  $\frac{1}{3,600}$ . Therefore the joint pdf for  $A$ , and  $B$  is:

$$f(a, b) = \frac{1}{3600}. \quad (103)$$

Orloff and Bloom give us that  $A$ , and  $B$  are defined on  $[0, 60]$ . We assume this is a closed subset of the real numbers. Therefore we treat  $A$ , and  $B$  as continuous random variables.

The joint cdf is:

$$F(a, b) = \int_0^a \int_0^b \frac{1}{3600} dx dy. \quad (104)$$

We replace the integrals above with their anti-derivatives:

$$F(a, b) = \frac{ab}{3600} \Big|_0^a \Big|_0^b. \quad (105)$$

## 8.2 Probability for A

Orloff and Bloom ask us to calculate the probability that  $A$  is less than 1,800.

$$P(A < 30) = \int_0^{30} \frac{1}{60} dx. \quad (106)$$

This integral evaluates to  $\frac{1}{2}$ .

## 8.3 Probability for A and B

We find the probability that  $0 \leq A \leq 15$ , and  $30 \leq B \leq 45$ .  $A$  and  $B$  are independent, therefore

$$P(0 \leq A \leq 15 \text{ and } 30 \leq B \leq 45) = P(0 \leq A \leq 15) P(30 \leq B \leq 45). \quad (107)$$

Since  $A$ , and  $B$  are uniformly distributed on  $[0, 60]$ ,  $P(0 \leq A \leq 15) = \frac{15}{60}$ , and  $P(30 \leq B \leq 45) = \frac{15}{60}$ .



Therefore the

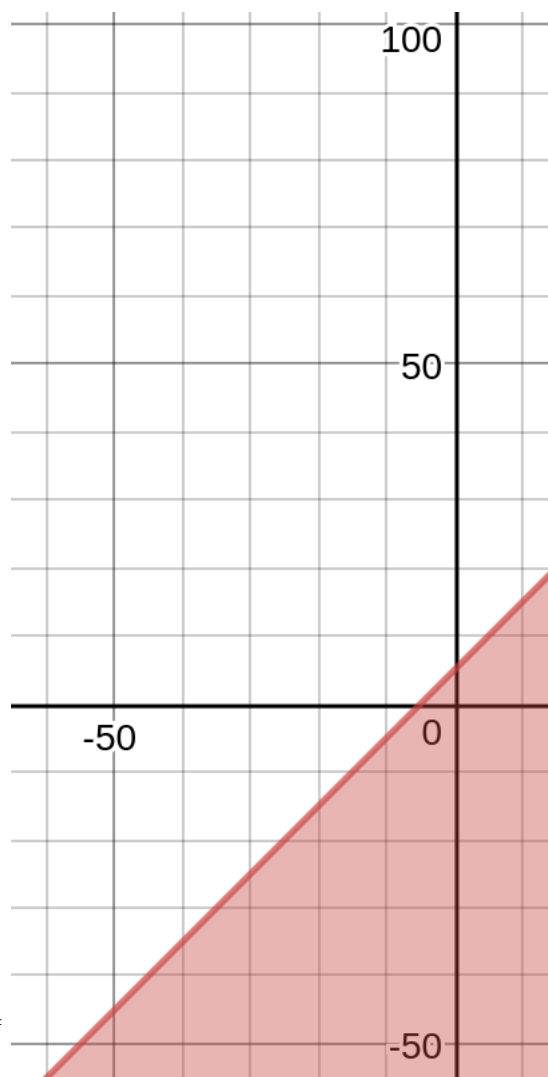
$$P(0 \leq A \leq 15 \text{ and } 30 \leq B \leq 45) = \left(\frac{1}{4}\right) \left(\frac{1}{4}\right). \quad (108)$$

We multiply the fractions on the right hand side of the equation above to get:

$$P(0 \leq A \leq 15 \text{ and } 30 \leq B \leq 45) = \frac{1}{8}. \quad (109)$$

#### **8.4 Inequality probability**

In this section, we will calculate the probability that  $A \leq B + 5$ .



34

Here is a plot of the inequality:

Within that region, we want to know what fraction the shaded portion is of the region.

It will be easier to subtract the area of the unshaded portion from the area of the region to find the area of the shaded portion, since the unshaded portion is a triangle.

The unshaded portion is a right triangle, with one side of length 55, and the other side also of length 55. Therefore unshaded region has area  $\frac{55^2}{2}$ . Hence the shaded region must have area  $3,600 - \frac{55^2}{2} = 2,087.5$ . The probability that  $A \leq B + 5$  is therefore  $\frac{2,087.5}{3,600} = 0.580$ .

## 8.5 Probability $A$ within range of $B$

In this section we calculate the probability that  $B - 15 \leq A \leq B + 15$ .



36

Here is a plot of the inequalities:

We need to know the area of the darkly shaded region within the square where  $A$  and  $B$  are in  $[0, 60] \times [0, 60]$ . We use an approach similar to what we do in the previous section, subtracting the area of triangles that are in the complement of the region whose area we wish to know.

## References

- [1] Jeremy Orloff and Jonathan Bloom. *Central Limit Theorem and the Law of Large Numbers Class 6, 18.05 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading6b.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading6b.pdf) (Spring 2014).
- [2] Jeremy Orloff and Jonathan Bloom. *Continuous Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading5b.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5b.pdf) (Spring 2014).
- [3] Jeremy Orloff and Jonathan Bloom. *Covariance and Correlation Class 7, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading7b.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading7b.pdf) (Spring 2014).

- mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\_05S14\_Reading7b.pdf (Spring 2014).
- [4] Jeremy Orloff and Jonathan Bloom. *Discrete Random Variables: Expected Value Class 4, 18.05, Spring 2014* Jeremy Orloff and Jonathan Bloom. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading4b.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading4b.pdf) (Spring 2014).
- [5] Jeremy Orloff and Jonathan Bloom. *Expectation, Variance and Standard Deviation for Continuous Random Variables Class 6, 18.05, Spring 2014* Jeremy Orloff and Jonathan Bloom. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading6a.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading6a.pdf) (Spring 2014).
- [6] Jeremy Orloff and Jonathan Bloom. *Gallery of Continuous Random Variables Class 5, 18.05, Spring 2014* Jeremy Orloff and Jonathan Bloom. Available at <https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics->

- spring-2014/readings/MIT18\_05S14\_Reading5c.pdf (Spring 2014).
- [7] Jeremy Orloff and Jonathan Bloom. *Joint Distributions, Independence Class 7, 18.05 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading7a.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading7a.pdf) (Spring 2014).
- [8] Jeremy Orloff and Jonathan Bloom. *Joint Distributions, Independence Covariance and Correlation 18.05 Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18\\_05S14\\_class7slides.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/class-slides/MIT18_05S14_class7slides.pdf) (Spring 2014).
- [9] Jeremy Orloff and Jonathan Bloom. *Variance of Discrete Random Variables Class 5, 18.05, Spring 2014 Jeremy Orloff and Jonathan Bloom*. Available at [https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18\\_05S14\\_Reading5a.pdf](https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading5a.pdf) (Spring 2014).