Vector Spaces and Subspaces

Vector space :-

The space $\mathbb{R}^n$ consists of all column vectors 'v' with 'n' components.

→ should obey 8 conditions.

Similarly, M ⇒ The vector space of all real 2×2 matrices.

F ⇒ The vector space of all real functions f(x)

Z ⇒ The vector space with only a zero vector.

→ No space can do without zero vector. Each space has it's own zero vector, the zero matrix, the zero function..

Subspace :-

A subspace of a vector space is a set of vectors (including 0) that satisfies two requirements.

If v and w are vectors in the subspace and c is any scalar, then

i) $v + w$ is in the subspace

ii) cv is in the subspace.

→ Every subspace contains the zero vector. From rule ii) if $c = 0$, then $cv = 0$ should be in subspace.

→ For $\mathbb{R}^3$, possible subspaces are

i) L → Any line through (0,0,0)

ii) P → Any plane through (0,0,0)

iii) $\mathbb{R}^3$ → The whole space

iv) Z → The single vector (0,0,0)

→ A subspace containing v and w must contain all linear combinations $cv + dw$.

Column space:-

It consists of all linear combinations of the columns.

The system $Ax=b$ is solvable if and only if b is in the column space of A.

$$\underset{m\times n}{\overset{A}{\begin{bmatrix} & \\ & \end{bmatrix}}}\ \underset{n\times 1}{\overset{x}{\begin{bmatrix} \\ \end{bmatrix}}} = \underset{m\times 1}{\begin{bmatrix} \\ \end{bmatrix}}$$

Ex: $C(A)$ of $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is the whole space $\mathbb{R}^2$

Is it a subspace? Yes

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$$v_1=\begin{bmatrix}a\\c\end{bmatrix} \quad w=\begin{bmatrix}b\\d\end{bmatrix}$$

$$v+w=\begin{bmatrix}a+b\\c+d\end{bmatrix}$$

$$q(v+w)=\begin{bmatrix}qa+qb\\qa+qd\end{bmatrix}$$

Null Space :-

The nullspace of A consists of all solutions to $Ax=0$.

Is is a subspace? Yes.

Suppose x and y are in $N(A)$ $\Rightarrow$ $Ax=0$ and $Ay=0$

then from rules of matrix multiplication, $A(x+y)=0+0$

also $A(cx)=c0$. Therefore $x+y$ and cx are also in the $N(A)$.

Ex:- $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$

$$Ax=0 \Rightarrow \begin{aligned} x_1+2x_2&=0\\ 3x_1+6x_2&=0\end{aligned} \quad\rightarrow\quad \begin{aligned}x_1+2x_2&=0\\0&=0\end{aligned}$$

$$x_1=-2x_2 \Rightarrow$$

special solution. $\begin{bmatrix}-2\\1\end{bmatrix}$

$N(A)$ contains all multiples of $s=\begin{bmatrix}-2\\1\end{bmatrix}$

Ex:- $x + 2y + 3z = 0$ $\Rightarrow$ $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

for $y = 1$ & $z = 0$ $\Rightarrow x = -2$

for $z = 1$ & $y = 0$ $\Rightarrow x = -3$

$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ two special solutions

Nullspace is span of s_1, s_2

5. Multiplica...

9. Linear Independence, Basis, and Dimension:-

→ Suppose A is m by n with $m < n$. Then there are non-zero solutions to $Ax = 0$ (more unknowns than equations)

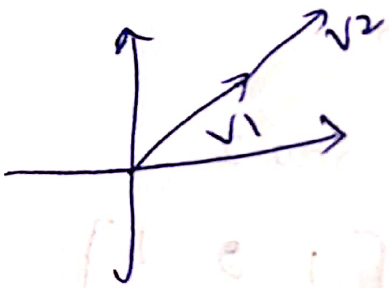
$\overset{A}{[\quad]}_{m\times n} \overset{x}{[\;]} = [\;]$

Reason:- There will be free variables !!

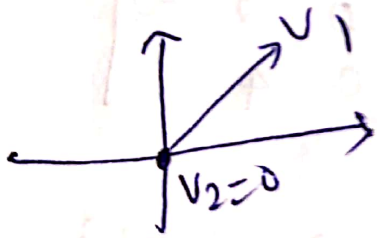
Independence:- vectors $x_1, x_2, \ldots, x_n$ are independent if no combination gives zero vector (except the zero combination → all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \neq 0$$

Ex:-



$2V_1 - V_2 = 0$ dependent



$0V_1 + cV_2 = 0$ independent

Repeat:- $V_1, \ldots V_n$ are ~~inde~~ columns of A

They are independent if nullspace of A is {zero vector}. They are dependent if $Ac = 0$ for some non-zero c.

Rank:- Independent → rank = n $N(A) = \{0\}$
Dependent → rank < n   → free variables.

Span:- Vectors $v_1, \dots, v_l$ span a space means the space consists of all combinations of those vectors.

Basis for a space is a sequence of vectors $v_1, \dots, v_d$ with 2 properties
1) they are independent
2) they span the space.

(If given basis, then everything about the space will be understood)

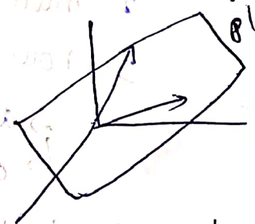
Example:-
Space is $\mathbb{R}^3$
i) one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

The N(Identity matrix) is $\begin{bmatrix}0\\0\\0\end{bmatrix}$

ii) Another basis is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

→ 'n' vectors give basis if the $n \times n$ matrix with those column's is invertible.

iii) $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$



* Any invertible matrix, it's columns are basis for $\mathbb{R}^3$ [millions of basis vectors].

* ~~Every~~ Given a space, every basis for the space has the same number of vectors.
dimension of the space

Example:-

$$A = \begin{bmatrix}1 & 2 & 3 & 1\\1 & 1 & 2 & 1\\1 & 2 & 3 & 1\end{bmatrix}$$

They are not independent

$$N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$$

2 = rank (A) = # pivot columns = dimension of Columnspace(A)

another basis for $C(A) = \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}7\\5\\7\end{bmatrix}$

$\dim[C(A)] = \text{rank}$

$N(A)$ can be : $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

$\dim[N(A)] = \#$ free variables $= n - r$

10. Four Fundamental Subspaces (for Matrix A)

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$ is the standard basis of $\mathbb{R}^3$

Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix} \begin{bmatrix}2\\2\\5\end{bmatrix} \begin{bmatrix}3\\3\\8\end{bmatrix}$ is said independent but not

The rows 1 & 2 are dependent & inverse won't exists.

4 Subspaces :
1. Column space $C(A)$
2. null space $N(A)$
3. row space $R(A) = C(A^T)$
   - → all combinations of rows of A
   - ↳ all comb's of columns of A^T
4. Null space of $A^T = N(A^T)$
   (left nullspace of A)

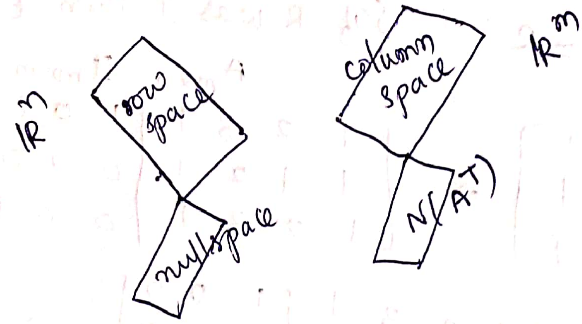
If A is $m \times n$

$C(A) \rightarrow \mathbb{R}^m$

$N(A) \rightarrow \mathbb{R}^n$

$R(A) = C(A^T) \rightarrow \mathbb{R}^n$

$N(A^T) \rightarrow \mathbb{R}^m$



basis ?

Dimension ?

1) $C(A) \Rightarrow$ Basis is # pivot columns

dimension is 'r'

2) $R(A) \Rightarrow$ dimension is 'r'

Basis

3) $N(A) \Rightarrow$ Basis is special solutions

dimension $= n - r$

$n - r =$ free variables

4) $N(A^T) \Rightarrow$ dimension $= m - r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$C(A) \neq C(R)$

Basis for row space is first r rows of R

From $R \rightarrow$ we can get any row vector of A

Ex:- $R_1 + R_2 = A_2$

$R_1 + 2R_2 = A_3$

4^{th} space : $N(A^T)$

$$A^T y = 0$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \, \end{bmatrix} = \begin{bmatrix} \, \end{bmatrix}_0$$

$$y^T (A^T)^T = 0^T$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 & \; \end{bmatrix}$$

rrEf $\begin{bmatrix} A_{m \times n} & I_{m \times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m \times n} & E_{m \times m} \end{bmatrix}$

Method 2

$$EA = R \qquad \{\text{If } R \text{ was } I \text{ then } E = A^{-1}$$

$$A_{m\times n} \quad I_{m\times m}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{cccc|cc} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_2} \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\quad}_{R} \qquad \underbrace{\quad}_{E}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ +1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & \cancel{1} & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N(A^T) \quad \because \quad -1\begin{bmatrix} 1 & 2 & 3 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}$$

$$+ -1\begin{bmatrix} 1 & 2 & 3 & 1 \end{bmatrix} = 0$$