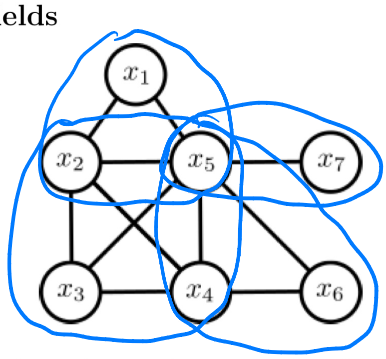
1 Pen and Paper

1.1 Markov Random Fields



a) For the undirected graph $\mathcal{G}$ above, draw a circle around each maximal clique. What is the Markov Random Field $p(x_1, \ldots, x_7)$ of the graph?

$$P(x_1, \cdots, x_7) = \frac{1}{Z} \psi_1(x_1, x_2, x_5)\, \psi_2(x_2, x_3, x_4, x_5)\, \psi_3(x_4, x_5, x_6)\, \psi_4(x_5, x_7)$$

b) Gobal Markov Property: Given $\mathcal{S} = \{x_2, x_5, x_6\}$, define disjoint sets $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \perp\!\!\!\perp \mathcal{B} \mid \mathcal{S}$.
How many of such pairs of sets can you find?
Further, what is the Markov Blanket of x_4?

Instructions on how to count pairs of $\mathcal{A}$ and $\mathcal{B}$:
$\mathcal{A}$ and $\mathcal{B}$ cannot be empty.
Swapping $\mathcal{A}$ and $\mathcal{B}$ does not count as a new pair.
$\mathcal{A} \cup \mathcal{B} \cup \mathcal{S}$ must contain all variables in the graph.

$A = (x_1) \quad B = (x_3, x_4)$

$A = (x_7) \quad B = (x_1)$

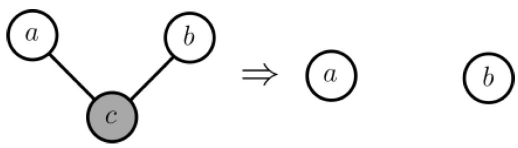
$A = (x_7) \quad B = (x_3, x_4)$

The Markov Blanket of x_4 $\quad ne(x_4) = \{x_2, x_3, x_5, x_6\}$

c) Local Markov Property: Which of the following statements are correct, which are wrong and why?

- $p(x_4 \mid x_1, x_2, x_3, x_5, x_6, x_7) = p(x_4 \mid x_2, x_3, x_5, x_6)$ ✓
- $x_1 \perp\!\!\!\perp x_3 \mid x_5$ ✗
- $x_2 \perp\!\!\!\perp x_6 \mid \{x_4, x_5\}$ ✓
- Marginalizing over x_5 makes x_1 and x_7 dependent. ✓

d) Prove the conditional independence property $a \perp\!\!\!\perp b \mid c$



by showing that $p(a, b \mid c) = p(a \mid c)p(b \mid c)$.

$$P(a,b|c) = \frac{P(a,b,c)}{P(c)} = \frac{\frac{1}{Z}\phi_1(a,c)\,\phi_2(b,c)}{\frac{1}{Z}\sum_{bc}\phi_1(a,c)\,\phi_2(b,c)}$$

$$P(a|c)P(b|c) = \frac{P(a,c)}{P(c)} \cdot \frac{P(b,c)}{P(c)}$$

$$= \frac{\sum_b P(a,b,c) \sum_a P(a,b,c)}{(\sum_{a,b} P(a,b,c))^2}$$

$$= \frac{\left[\sum_b \frac{1}{Z}\phi_1(a,c)\phi_2(b,c)\right]\left[\sum_a \frac{1}{Z}\phi_1(a,c)\phi_2(b,c)\right]}{\left(\sum_{a,b}\frac{1}{Z}\phi_1(a,c)\phi_2(b,c)\right)^2}$$

$$= \frac{\phi_1(a,c)\left[\sum_b \phi_2(b,c)\right]\left[\sum_a \phi_1(a,c)\right]\phi_2(b,c)}{\left[\sum_a \phi_1(a,c)\right]^2\left[\sum_b \phi_2(b,c)\right]^2}$$

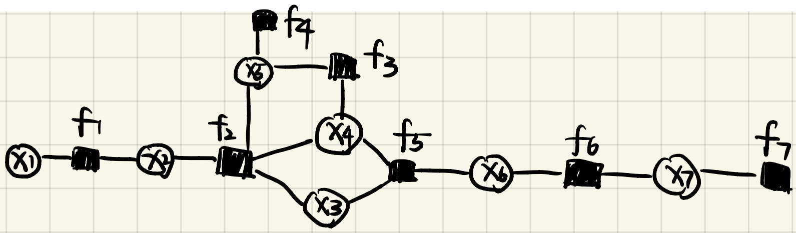
$$= \frac{\frac{1}{Z}\phi_1(a,c)\phi_2(b,c)}{\frac{1}{Z}\sum_a \phi_1(a,c)\sum_b \phi_2(b,c)}$$

$$= \frac{p(a,b,c)}{p(c)} = p(a,b|c)$$

1.2 Factor Graphs

a) Draw the factor graph for function

$$f(x_1,x_2,x_3,x_4,x_5,x_6) = f_1(x_1,x_2)f_2(x_2,x_3,x_4,x_5)f_3(x_4,x_5)f_4(x_5)f_5(x_3,x_4,x_6)f_6(x_6,x_7)f_7(x_7).$$



b) Given $\mathcal{X} = \{x_1,\dots,x_6\}$ with $x_i \in \{0,1\}\ \forall i$, a Markov Random Field is given by the distribution

$$p(x_1,\dots,x_6) = \frac{1}{Z}\phi_1(x_1)\phi_2(x_1,x_2,x_3,x_4)\phi_3(x_3,x_5,x_6).$$

Define the potentials as

$$\begin{aligned} \phi_1(x_1) &= [x_1 = 1] \\ \phi_2(x_1,x_2,x_3,x_4) &= [x_1 = x_2]\cdot[x_3 = x_4] \\ \phi_3(x_3,x_5,x_6) &= 2x_3 + x_5 + x_6 \end{aligned}$$

Calculate the value of the partition function Z. Is $p(x_1,\dots,x_6)$ a Gibbs distribution? Explain your answer.

$$Z = \sum_{x_1,\dots,x_6} \phi_1(x_1)\phi_2(x_1,x_2,x_3,x_4)\phi_3(x_3,x_5,x_6)$$

$x_1 x_2 x_3 x_4 x_5 x_6$	$\phi_1(x_1)\phi_2(x_1,x_2,x_3,x_4)\phi_3(x_3,x_5,x_6)$
1 1 1 1 1 1	4
1 1 0 0 1 1	2
1 1 1 1 1 0	3
1 1 1 1 0 1	3
1 1 0 0 1 0	1
1 1 0 0 0 1	1
1 1 0 0 0 0	0
1 1 1 1 0 0	2

$\therefore Z = 16$

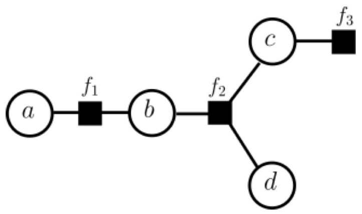
It is a Gibbs distribution unless all potentials are strictly positive.
Hence, $p(x_1,\dots,x_6)$ is not a Gibbs distribution.

1.3 Belief Propagation

a) In one or two sentences, what is the nugget or key observation behind Belief Propagation?

Belief Propagation assumes a single-connected graph $G=(V,E)$, which means it has $|V|-1=O(|V|)$ many edges (in contrast to $|V|(|V|-1)/2=O(V^2)$ of a fully connected graph). That simplifies the computation of any marginal distribution significantly, since most potential of the MRF only depend on a few variables.

b) **Marginal Inference.** Consider the Markov Random Field defined on binary variables $a, b, c, d \in \{0, 1\}$ with the following factor graph:



The potentials are given by $f_1(a, b) = [a = b]$, $f_2(b, c, d) = 0.5b + 0.3c + 0.2d$ and $f_3(c) = [c = 1]$. Compute all marginal distributions using the *Sum-Product Algorithm*. Here, please do not use the log representation for the messages μ.

Initialization: $\mu_{a\to f_1}(a)=1$, $\mu_{d\to f_2}(d)=1$, $\mu_{f_3\to c}(c)=f_3(c)$

Variable-to-Factor: $\mu_{b\to f_2}(b)=\mu_{f_1\to b}(b)=1$

$\mu_{c\to f_2}(c)=\mu_{f_3\to c}(c)=f_3(c)$

$\mu_{b\to f_1}(b)=\mu_{f_2\to b}(b)=b+0.8$

$\mu_{c\to f_3}(c)=\mu_{f_2\to c}(c)=1.2c+1.4$

Factor-to-Variable: $\mu_{f_1\to a}(a)=\sum_b f_1(a,b)\,\mu_{b\to f_1}(b)$

$\mu_{f_1\to b}(b)=\sum_a f_1(a,b)\,\mu_{a\to f_1}(a)=1$

$\mu_{f_2\to b}(b)=\sum_{c,d} f_2(b,c,d)\,\mu_{c\to f_2}(c)\,\mu_{d\to f_2}(d)=b+0.8$

$\mu_{f_2\to d}(d)=\sum_{b,c} f_2(b,c,d)\,\mu_{c\to f_2}(c)\,\mu_{b\to f_2}(b)=0.4d+1.1$

$\mu_{f_2\to c}(c)=\sum_{b,d} f_2(b,c,d)\,\mu_{b\to f_2}(b)\,\mu_{d\to f_2}(d)=1.2c+1.4$

We infer the marginal distributions using $p(x)\propto\prod_{f\in ne(x)}\mu_{f\to x}(x)$

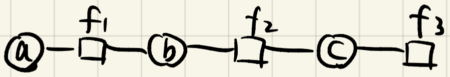
c) Maximum-A-Posteriori. Consider a distribution defined over binary variables $a, b, c \in \{0, 1\}$:

$$p(a, b, c) = \frac{1}{Z} f_1(a, b) f_2(b, c) f_3(c)$$

The factors are given as

$$f_1(a, b) = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}, \qquad f_2(b, c) = \begin{pmatrix} 0.2 & 0.4 \\ 0.1 & 0.3 \end{pmatrix}, \qquad f_3(c) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

so e.g. $f_2(b = 0, c = 1) = 0.4$. What is the most likely joint configuration $a^*, b^*, c^* = \text{argmax}_{a,b,c}\, p(a, b, c)$? Again, please use *Max-Product Algorithm* and do not use the log representation.



Initialization: $\mu_{a \to f_1}(a) = 1$, $\mu_{f_3 \to c}(c) = f_3(c)$

left to right: $\mu_{b \to f_2}(b) = \mu_{f_1 \to b}(b) = \max_a f_1(a, b) \cdot 1 = \begin{cases} 0.5, & b = 0 \\ 0.3, & b = 1 \end{cases}$

$\mu_{c \to f_3}(c) = \mu_{f_2 \to c}(c) = \max_b f_2(b, c) \cdot \mu_{b \to f_2}(b) = \begin{cases} 0.1, & c = 0 \\ 0.2, & c = 1 \end{cases}$

right to left: $\mu_{c \to f_2}(c) = \mu_{f_3 \to c}(c) = f_3(c)$

$\mu_{b \to f_1}(b) = \mu_{f_2 \to b}(b) = \max_c f_2(b, c) \cdot \mu_{c \to f_2}(c) = \begin{cases} 0.2, & b = 0 \\ 0.15, & b = 1 \end{cases}$.

$\mu_{f_1 \to a}(a) = \max_b f_1(a, b) \cdot \mu_{b \to f_1}(b) = \begin{cases} 0.1, & a = 0 \\ 0.045 & a = 1 \end{cases}$

$a^* = \text{argmax}\, \mu_{f_1 \to a}(a) = 0$

$b^* = \text{argmax}\, \mu_{f_1 \to b}(b) \mu_{f_2 \to b}(b) = 0$

$c^* = \text{argmax}\, \mu_{f_2 \to c}(c) \mu_{f_3 \to c}(c) = 1$

d) Consider a pairwise Markov Random Field defined on a ring (a chain that is connected at both ends) with 100 binary variables:

$$p(x) = \phi_0(x_1, x_{100}) \prod_{i=1}^{99} \phi_i(x_i, x_{i+1})$$

Is it possible to compute $\text{argmax}_{x_1, \ldots, x_{100}}\, p(x)$ efficiently?

Hint: Think about what happens if you condition on one of the nodes.

Assume we know the most likely state of one of the nodes, say x_1, the remaining graph is singly connected and we can compute the message passing efficiently. As x_1 is a binary variable, there is only two state of it.

For $x_1 = 0$. $x_2^* x_3^* \cdots x_{100}^* | x_1 = 0 = \text{argmax}_{x_2 \cdots x_{100}}\, p(x_2 \cdots x_{100} | x_1 = 0)$

For $x_1 = 1$. $x_2^* x_3^* \cdots x_{100}^* | x_1 = 1 = \text{argmax}_{x_2 \cdots x_{100}}\, p(x_2 \cdots x_{100} | x_1 = 1)$

Then we can plug these states to $p(x)$ and pick the state with higher potential.

1.4 Graphical Models for Multi-View Reconstruction

a) What are the advantages of using Probabilistic Graphical Models for Computer Vision tasks like Multi-View Reconstruction? And what are the challenges or drawbacks of those models?

Advantages: ① Include prior knowledge into the estimation, constraint only need to be non-negative
② Probabilistic formulation gives a certainty estimate
③ Non linear constraints can be incorporated.

Drawbacks: ① Accurate inference is only given for non-loopy graphs.
② Inference in multiply connected graphs or high order potentials can get more complex and intractable.

b) In the lecture, we discussed a probabilistic dense multi-view reconstruction method using an MRF on a voxel grid. Which assumption(s) is this model based on? And in which cases are those violated?

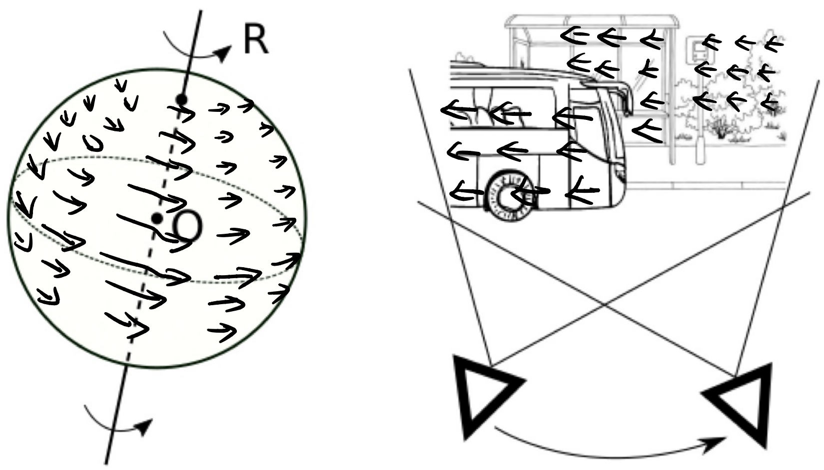
Assumption: Color constancy between pixels showing the same 3D point.

violated cased:

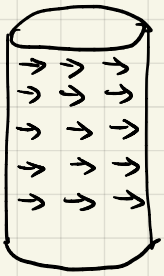
① Non-Lambertian surfaces
② the scene is not static
③ the noise in the measurements is not Guassian or independent

1.5 Graphical Models for Optical Flow

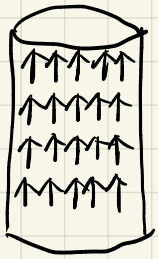
a) Flow fields. Draw a sketch of the optical flow fields for the following two scenarios: The first is a sphere rotating around the shown axis. Please assume the sphere to be textured. The second shows a static bus stop scene with the camera moving from left to right. Indicate with arrows the magnitude of the optical flow for different points in the scene.



b) In the lecture the Barber Pole was discussed as an example for the aperture problem. Answer the question from the slides concerning the Barber Pole: What is the motion field? What is the optical flow field?



motion field

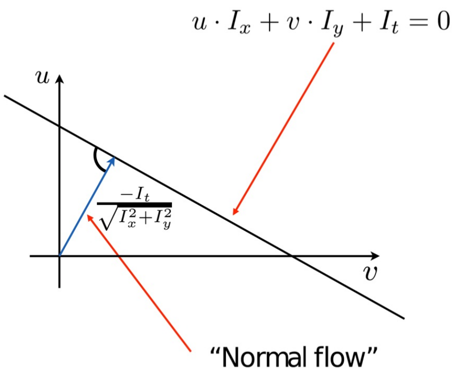


optical flow field

c) Aperture problem. The aperture problem arises from the brightness consistency contraint / optical flow constraint equation

$$u_{x,y} \cdot I_x(x,y) + v_{x,y} \cdot I_y(x,y) + I_t(x,y) = 0$$

This constraint can be graphically represented as follows:



Understand the relationship and explain the aperture problem using the figure above. Additionally, derive the term for the normal flow.

$u \cdot I_x + v \cdot I_y + I_t = 0$ has two unknown variables, u and v.

$$\vec{n} = \binom{I_x}{I_y} / \sqrt{I_x^2 + I_y^2}$$

$$d = (u, v)^T \cdot \vec{n} = \frac{u I_x + v I_y}{\sqrt{I_x^2 + I_y^2}} = \frac{-I_t}{\sqrt{I_x^2 + I_y^2}}$$

add this constraint about d.