The Lake Physical model : primitive equations for velocity, temperature and concentrations

October 8, 2013

Abstract

The equations of the hydrostatic lake model are presented.

In the equations, x, y are the horizontal variables and the vertical variable z is defined such that $z = 0$ at the surface when the fluid is at rest and z is directed upwards.

The graph $h(x, y, t)$ always denotes the free surface perturbation amplitude from rest (i.e. the long waves such as seiche but the wind waves are discarded in the hydrostatic model).

Quantity $\zeta(x, y)$ denotes the layer depth at rest

$H^{(*)}(x, y, t)$ denotes the physical true layer depth with waves

$$H^{(*)}(x, y, t) = \zeta(x, y) + h(x, y, t).$$

We also introduced for the immersed boundary method, quantity $\zeta_1(x, y, t)$ which is not the layer depth at rest but a fictitious moving boundary such that $\zeta_1(x, y, t) > \zeta(x, y)$

A new vertical coordinate is defined such that $\sigma = 1$ at the surface and $\sigma = 0$ at $z = -\zeta_1(x, y, t)$

$$\sigma = \frac{z - h(x, y, t)}{H} + 1, \qquad H(x, y, t) = \zeta_1(x, y, t) + h(x, y, t) \tag{1}$$

For the standard method

$$\zeta_1(x, y, t) = \zeta(x, y), \qquad H(x, y, t) = H^{(*)}(x, y, t)$$

For the immersed boundary method

$$\zeta_1(x,y,t) \neq \zeta(x,y), \qquad H(x,y,t) \neq H^{(*)}(x,y,t)$$

1 The immersed boundary method in the Primitive equations model

In the immersed boundary method, Variable σ is always equal to $\sigma = 1$ at the surface, $\sigma = 0$ at the fictitious bottom. This fictitious moving boundary $\zeta_1(x,y,t)$ is defined such that the true boundary $z = -\zeta(x,y)$ corresponds to a fixed function $\sigma = g(x,y)$ in the sigma-coordinate. This condition imposes a relation between $\zeta_1(x,y,t)$ and $\zeta(x,y)$:

$$g(x,y) = \frac{-\zeta(x,y) - h(x,y,t)}{\zeta_1(x,y,t) + h(x,y,t)} + 1 \tag{2}$$

thus

$$\zeta_1(x,y,t) = \frac{g(x,y)}{1-g(x,y)} h(x,y,t) + \frac{\zeta(x,y)}{1-g(x,y)} \tag{3}$$

$$H(x,y,t) = \zeta_1(x,y,t) + h(x,y,t) = \frac{h(x,y)}{1-g(x,y)} + \frac{\zeta(x,y)}{1-g(x,y)} \tag{4}$$

$$\frac{H^{(*)}(x,y,t)}{H(x,y,t)} = 1 - g(x,y)$$

If one disregards the influence of $h(x,y,t)$, the choice of $\zeta_1(x,y,t)$ is given by $\zeta_1(x,y,t) = \frac{\zeta(x,y)}{1-g(x,y)}$. CARE when one chooses $\zeta(x,y)$ and $\zeta_1(x,y,t)$, $g(x,y)$ is defined.

If $0 \leq g(x,y,t) < 1$ then $\zeta_1(x,y,t) > \zeta(x,y)$ is always satisfied (if one disregards the term in $h(x,y,t)$).

One uses H in the simulations and imposes the bottom boundary condition at $\sigma = g(x,y)$.

1.1 The immersed boundary method with a linear $g(x,y)$: easy but discontinuous derivative

Let us define a critical depth ζ_{cr} such that

$$g(x,y) = 0, \qquad \text{if the depth } \zeta(x,y) > \zeta_{cr}$$

$$g(x,y) = (1 - \frac{\zeta(x,y)}{\zeta_{cr}}) \qquad \text{when } \zeta(x,y) < \zeta_{cr}$$

Since, by definition,

$$H(x,y,t) = \zeta_1(x,y,t) + h(x,y,t) = \frac{h(x,y,t) + \zeta(x,y)}{1 - g(x,y)},$$

when one disregards the first term in $h(x,y,t)$, H is almost constant at the locations of immersed boundaries $\zeta(x,y) < \zeta_{cr}$

$$H(x,y,t) \sim \zeta_c$$

1.2 The immersed boundary method with a quadratic $g(x,y)$: probably better not constant but continuous derivative

Let us define a critical depth ζ_{cr} such that

$$g(x,y) = 0, \qquad \text{if the depth } \zeta(x,y) > \zeta_{cr}$$

$$g(x,y) = C(1 - \frac{\zeta(x,y)}{\zeta_{cr}})^2 \qquad \text{when } \zeta(x,y) < \zeta_{cr}$$

with C a coefficient close to one. Since, by definition,

$$H(x,y,t) = \zeta_1(x,y,t) + h(x,y,t) = \frac{h(x,y,t) + \zeta(x,y)}{1 - g(x,y)},$$

when one disregards the first term in $h(x,y,t)$, H is given at the locations of immersed boundaries $\zeta(x,y) < \zeta_{cr}$

$$H(x,y,t) \sim \frac{\zeta(x,y)}{1 - C(1 - \frac{\zeta(x,y)}{\zeta_{cr}})^2}$$

If one chooses C so that

$$\frac{\zeta_{min}}{1 - C(1 - \frac{\zeta_{min}}{\zeta_{cr}})^2} = \zeta_{cr}$$

then, at the border where $\zeta(x,y) = \zeta_{min}$, $H(x,y,t) \sim \zeta_{cr}$.
In addition at the boundary where $\zeta(x,y) = \zeta_{cr}$, $H(x,y,t)$ almost equals $H(x,y,t) = \zeta_{cr}$. As a consequence, it must be constant as much as possible (TO CHECK NUMERICALLY)

2 Primitive equations model

The primitive equations are

$$\frac{\partial h}{\partial t} + u(x,y,h(x,y,t),t)\frac{\partial h}{\partial x} + v(x,y,h(x,y,t),t)\frac{\partial h}{\partial y} = w(x,y,h(x,y,t),t)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$\frac{\partial P}{\partial z} = -\rho g \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial z}\left(K_v\frac{\partial u}{\partial z}\right) \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial z}\left(K_v\frac{\partial u}{\partial z}\right) \tag{8}$$

where K_v is a **Vertical diffusion for velocity** (see below) .

Boundary conditions

$$p((x,y,h(x,y,t)) = P_{atm}$$

$$\frac{\partial u}{\partial z}(x,y,h(x,y,t)) = \frac{\tau_s}{\rho_{wat}K_v(z=h(x,y,t))}\cos(\theta_W)$$

$$\frac{\partial v}{\partial z}(x,y,h(x,y,t)) = \frac{\tau_s}{\rho_{wat}K_v(z=h(x,y,t))}\sin(\theta_W)$$

with

$$\tau_s = C_D\rho_{air}W^2 \equiv \rho_{Wat}(u_*)2$$

and θ_W the angle of the wind.

2.1 Vertical diffusion

$$K_v = K_{vmin} + K_{turb}$$

K_{vmin} is a minimal diffusion (molecular or given by the turbulent background) which may or may not depend on the given surface velocity.

K_{turb} is a turbulent viscosity to be defined later

Case (a) Constant Vertical diffusion for velocity
For the constant diffusivity case,

$$K_{vmin} \neq 0$$

$$K_{turb} = 0$$

Case (b) Mixing length Vertical diffusion for velocity
For the case with diffusivity which depends on time through shear (USED IN PRIORITY IN THE CODE)

$$K_{vmin} \neq 0$$

K_{turb} is given by a mixing length approximation (see appendix) : the vertical eddy viscosity becomes

$$K_{turb} = \frac{L^2}{(1+\alpha Ri)^\beta}\sqrt{[|\frac{\partial u}{\partial z}|^2 + |\frac{\partial v}{\partial z}|^2]} \tag{9}$$

with

$$L = \kappa(z - z_{bot})(1 - \frac{z - z_{bot}}{z_{top} - z_{bot}}) \tag{10}$$

and the local Richardson number

$$Ri = \frac{N^2}{M^2} \tag{11}$$

with

$$M^2 \equiv |\frac{\partial u}{\partial z}|^2 + |\frac{\partial v}{\partial z}|^2$$

$$N^2 \equiv -g\frac{1}{\rho_0}\frac{\partial \rho}{\partial z}$$

If $-\frac{\partial \rho}{\partial z} < 0$ local Richardson number negative, the factor with Ri It is not considered. It is considered only in the stable case.

Case (c) Parabolic diffusivity Vertical diffusion for velocity (Tsanis case)
In that case,

$$K_{vmin} = 0$$

$$K_{turb} = \lambda u_* H(z'/H + z_{bh})(1 + z_{sh} - z'/H)$$

CARE !! In the code, TSANIS does not work for $u_* = 0$

Case (d) $k - \epsilon$ Model TO BE DONE

$$K_{vmin} \neq 0$$

K_{turb} is given by

$$K_{turb} = c_\mu \frac{k^2}{\epsilon}$$

where c_μ is a function of

$$\frac{k^2}{\epsilon} M^2, \qquad \frac{k^2}{\epsilon} N^2,$$

2.2 Coordinates system and derivatives

$$\sigma = \frac{z + \zeta_1(x,y,t)}{H} = \frac{z - h(x,y,t)}{H} + 1, \qquad H(x,y,t) = \zeta_1(x,y) + h(x,y,t) \quad (12)$$

Variable σ is always equal to $\sigma = 1$ at the surface and $\sigma = g(x,y)$ at the bottom.

$$t \to t_\sigma = t, \quad x \to x_\sigma = x, \quad y \to y_\sigma = y, \quad z \to \sigma$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t_\sigma} + \frac{\partial \sigma}{\partial t}\frac{\partial}{\partial \sigma}$$
$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x_\sigma} + \frac{\partial \sigma}{\partial x}\frac{\partial}{\partial \sigma}$$

$$\frac{\partial}{\partial z} \rightarrow \frac{1}{H}\frac{\partial}{\partial \sigma}$$

with

$$\frac{\partial \sigma}{\partial t} = \frac{-1}{H}[(\sigma-1)\frac{\partial H}{\partial t} + \frac{\partial h}{\partial t}]$$
$$\frac{\partial \sigma}{\partial x} = \frac{-1}{H}[(\sigma-1)\frac{\partial H}{\partial x} + \frac{\partial h}{\partial x}]$$

2.3 Advective Derivative

For any variable q, the advective derivative reads in this σ system of coordinates

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z} = \frac{\partial q}{\partial t_\sigma} + u\frac{\partial q}{\partial x_\sigma} + v\frac{\partial q}{\partial y_\sigma} + \frac{\omega}{H}\frac{\partial q}{\partial \sigma} \tag{13}$$

where the "vertical" velocity ω is related to the true physical vertical velocity w by

$$\omega(x_\sigma, y_\sigma, \sigma, t_\sigma) = w - u((\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}) - v((\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}) - ((\sigma-1)\frac{\partial H}{\partial t_\sigma} + \frac{\partial h}{\partial t_\sigma}) \tag{14}$$

Note that, contrary to some standard approach, ω is a "vertical" velocity not a velocity divided by a length. This relation can be written as well as $\omega = H\frac{d\sigma}{dt}$.

2.4 Continuity equation

In the σ coordinates, the continuity equation (see annex A) reads

$$\frac{\partial H}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}(Hu) + \frac{\partial}{\partial y_\sigma}(Hv) + \frac{\partial \omega}{\partial \sigma} = 0 \tag{15}$$

By using the depth-integrated continuity equation and the boundary condition on the vertical velocity (see annex A), the evolution of the free surface is obtained

$$\frac{\partial h}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{g(x,y)}^{1} u d\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{g(x,y)}^{1} v d\sigma\right) = 0 \tag{16}$$

When immersed boundaries are not used, we set $g(x, y) = 0$.

By integrating the continuity equation from level $g(x, y)$ up to level σ, we obtain ω the "vertical" velocity in σ-coordinates :

$$\omega = (1-\sigma)\frac{\partial H}{\partial t_\sigma} - \frac{\partial h}{\partial t_\sigma} - \int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{17}$$

In the code, this is obtained as follows. First, starting from the bottom $\sigma = g(x, y)$, we compute the integral

$$\int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{18}$$

One may compute $\frac{\partial h}{\partial t}$ by changing the sign of the last integral with $\sigma = 1$.

$$\frac{\partial h}{\partial t_\sigma} = -\int_{g(x,y)}^{1} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{19}$$

Finally, one obtains

$$\omega = \left((1-\sigma)\left(\frac{1}{1-g(x,y)} \right) - 1 \right) \frac{\partial h}{\partial t_\sigma} - \int_{0}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{20}$$

2.5 Advective Derivative : new form

Note that the advective derivative for any variable q multiplied by H reads as

$$H\frac{\partial q}{\partial t_\sigma} + Hu\frac{\partial q}{\partial x_\sigma} + Hv\frac{\partial q}{\partial y_\sigma} + \omega\frac{\partial q}{\partial \sigma} \tag{21}$$

This can be rewritten, thanks to the continuity equation (15), in a conservative way for Hq

$$\frac{\partial Hq}{\partial t_\sigma} + \frac{\partial uHq}{\partial x_\sigma} + \frac{\partial vHq}{\partial y_\sigma} + \frac{\partial \omega q}{\partial \sigma} \tag{22}$$

2.6 Momentum equation in z when density (buoyancy) is a priori not uniform

The vertical momentum equation

$$\frac{\partial P}{\partial z} = -\rho g$$

reads in sigma coordinates as

$$\frac{\partial P}{\partial \sigma} = -\rho g H \tag{23}$$

Its integration leads to

$$P(x_\sigma, y_\sigma, \sigma) = P(x_\sigma, y_\sigma, \sigma = 1) + \int_\sigma^1 \rho g H d\sigma. \tag{24}$$

with

$$P(x_\sigma, y_\sigma, \sigma = 1) = P_{atm}$$

In the momentum equations (7)-(8), one needs the terms $\frac{-1}{\rho_0}\frac{\partial P}{\partial x}$ and $\frac{-1}{\rho_0}\frac{\partial P}{\partial y}$ for z fixed in the (x, y, z) coordinates. Let us now compute these terms in sigma coordinates

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_\sigma} - \frac{1}{\rho_0}\frac{\partial \sigma}{\partial x}\frac{\partial P}{\partial \sigma} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_\sigma} + g\frac{\rho}{\rho_0}\frac{\partial \sigma}{\partial x}H$$

Using equation (24) the first term reads

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x_\sigma} = \frac{\partial}{\partial x_\sigma}(H\int_\sigma^1 [-g - g\frac{\rho-\rho_0}{\rho_0}]d\sigma) = g\frac{\partial H}{\partial x_\sigma}(\sigma - 1) + \frac{\partial}{\partial x_\sigma}(H\int_\sigma^1 b d\sigma)$$

where the buoyancy term b stands for

$$b \equiv -g\frac{\rho-\rho_0}{\rho_0}$$

The second term reads

$$g\frac{\rho}{\rho_0}\frac{\partial \sigma}{\partial x}H = [(\sigma-1)\frac{\partial H}{\partial x} + \frac{\partial h}{\partial x}](b - g)$$

Finally one gets the complete term used in the code

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x} = -g\frac{\partial h}{\partial x_\sigma} + [(\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}]b + \frac{\partial}{\partial x_\sigma}(H\int_\sigma^1 b d\sigma)$$

similarly

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial y} = -g\frac{\partial h}{\partial y_\sigma} + [(\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}]b + \frac{\partial}{\partial y_\sigma}(H\int_\sigma^1 bd\sigma)$$

in other codes
In MARS 3D the equivalent form is used

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x} = -g\frac{\partial h}{\partial x_\sigma} + [\sigma\frac{\partial H}{\partial x_\sigma} - \frac{\partial \zeta}{\partial x_\sigma}]b + \frac{\partial}{\partial x_\sigma}(H\int_\sigma^1 bd\sigma)$$

In Burchard's formulation, it is written as

$$-g\frac{\rho(\sigma=1)}{\rho_0}\frac{\partial h}{\partial x_\sigma} - \frac{g}{\rho_0}\int_\sigma^1 (H\frac{\partial \rho}{\partial x_\sigma} - [(\sigma-1)\frac{\partial H}{\partial x} + \frac{\partial h}{\partial x}]\frac{\partial \rho}{\partial \sigma})d\sigma$$

this can be checked by using an integration by parts.
In Burchard's formulation this term is also written by

$$-\frac{g}{\rho_0}\rho(h)\frac{\partial h}{\partial x} + \int_z^h \frac{\partial b}{\partial x}dz'$$

HOWEVER it seems different from the POM term since the derivative $\frac{\partial h}{\partial y_\sigma}$ is absent in POM !

2.6.1 When density is uniform

In that case, $b = 0$ and

$$P(x_\sigma, y_\sigma, \sigma) = P(x_\sigma, y_\sigma, \sigma = 1) + (1-\sigma)\rho g H \tag{25}$$

$$\frac{-1}{\rho_0}\frac{\partial P}{\partial x} = -g\frac{\partial h}{\partial x_\sigma}$$
$$\frac{-1}{\rho_0}\frac{\partial P}{\partial y} = -g\frac{\partial h}{\partial y_\sigma}$$

In the above momentum equations (26)-((27), one uses the primitive equation with constant density which produces a term in

$$\frac{-1}{\rho_0}\frac{\partial P}{\partial x} = -g\frac{\partial h}{\partial x_\sigma}$$

2.7 Momentum equation

In σ coordinates, the momentum equations (7)-(8) becomes (see POMS, ROMS)
Here written for uniform density

$$\frac{\partial u}{\partial t_\sigma} + u\frac{\partial u}{\partial x_\sigma} + v\frac{\partial u}{\partial y_\sigma} + \frac{\omega}{H}\frac{\partial u}{\partial \sigma} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{1}{H}\frac{\partial}{\partial \sigma}\left(\frac{K_v}{H}\frac{\partial u}{\partial \sigma}\right) + F_x \tag{26}$$

$$\frac{\partial v}{\partial t_\sigma} + u\frac{\partial v}{\partial x_\sigma} + v\frac{\partial v}{\partial y_\sigma} + \frac{\omega}{H}\frac{\partial v}{\partial \sigma} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{1}{H}\frac{\partial}{\partial \sigma}\left(\frac{K_v}{H}\frac{\partial v}{\partial \sigma}\right) + F_y \tag{27}$$

with

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x} = -g\frac{\partial h}{\partial x_\sigma} + [(\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}]b + \frac{\partial}{\partial x_\sigma}(H\int_\sigma^1 bd\sigma)$$

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial y} = -g\frac{\partial h}{\partial y_\sigma} + [(\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}]b + \frac{\partial}{\partial y_\sigma}(H\int_\sigma^1 bd\sigma)$$

where the buoyancy term b stands for

$$b \equiv -g\frac{\rho-\rho_0}{\rho_0}$$

with F_x and F_y are ad-hoc horizontal diffusions terms for velocity

$$F_x = +\frac{1}{H}\frac{\partial}{\partial x_\sigma}\left(H\nu\frac{\partial u}{\partial x_\sigma}\right) + \frac{1}{H}\frac{\partial}{\partial y_\sigma}\left(H\nu\frac{\partial u}{\partial y_\sigma}\right)$$

$$F_y = \frac{1}{H}\frac{\partial}{\partial x_\sigma}\left(H\nu\frac{\partial v}{\partial x_\sigma}\right) + \frac{1}{H}\frac{\partial}{\partial y_\sigma}\left(H\nu\frac{\partial v}{\partial y_\sigma}\right)$$

Using the advective derivatives form and multiplying by H, one gets
Using the continuity equation, one can write this equation in a conservative way

$$\frac{\partial(Hu)}{\partial t_\sigma} = -\frac{\partial J_x^u}{\partial x_\sigma} - \frac{\partial J_y^u}{\partial y_\sigma} - \frac{\partial J_\sigma^u}{\partial \sigma} - \frac{H}{\rho_0}\frac{\partial P}{\partial y} \tag{28}$$

$$\frac{\partial(Hv)}{\partial t_\sigma} = -\frac{\partial J_x^v}{\partial x_\sigma} - \frac{\partial J_y^v}{\partial y_\sigma} - \frac{\partial J_\sigma^v}{\partial \sigma} - \frac{H}{\rho_0}\frac{\partial P}{\partial y} \tag{29}$$

where the fluxes read as :

$$J_x^u = Hu^2 - H\nu\frac{\partial u}{\partial x_\sigma}, \qquad J_x^v = Huv - H\nu\frac{\partial v}{\partial x_\sigma}, \tag{30}$$

$$J_y^u = Huv - H\nu\frac{\partial u}{\partial y_\sigma}, \qquad J_y^v = Hv^2 - H\nu\frac{\partial v}{\partial y_\sigma}, \tag{31}$$

$$J^u_\sigma = \omega u - \frac{K_v}{H}\frac{\partial u}{\partial \sigma}, \qquad J^v_\sigma = \omega v - \frac{K_v}{H}\frac{\partial v}{\partial \sigma} \tag{32}$$

Horizontal diffusion for velocity we assume a constant for horizontal diffusion called ν. It is different from the vertical diffusion K_v.

The order of magnitude is (**this formula is not introduced in the code and should be introduced by changing each time its value**)

$$\nu = Coeff(\Delta x)^{4/3}$$

with

$$Coeff = 510^{-5} m^{2/3}/s$$

(sse Okubo (Cushman p416))
If the horizontal scale is $\Delta x = 10m$ one gets $\nu = 10^{-3} m^2/s$

Vertical diffusion for velocity in σ coordinates

It is given by

$$K_v = K_{vmin} + K_{turb}$$

where K_{vmin} is a minimal diffusion (molecular or given by the turbulent background) which may or may not depend on the given surface velocity u^*.

Case (a) Constant Vertical diffusion for velocity
For the constant diffusivity case,

$$K_{vmin} \neq 0$$

$$K_{turb} = 0$$

Case (b) Mixing length Vertical diffusion for velocity
For the case with diffusivity which depends on time through shear (USED IN PRIORITY IN THE CODE)

$$K_{vmin} \neq 0$$

$$\frac{K_{turb}}{H} = \frac{l^2}{(1+\alpha Ri)^\beta}\sqrt{[|\frac{\partial u}{\partial \sigma}|^2 + |\frac{\partial v}{\partial \sigma}|^2]} \tag{33}$$

where the local Richardson number

$$Ri = H \frac{-g \frac{1}{\rho} \frac{\partial \rho}{\partial \sigma}}{[|\frac{\partial u}{\partial \sigma}|^2 + |\frac{\partial v}{\partial \sigma}|^2]} \tag{34}$$

If $-\frac{\partial \rho}{\partial \sigma} < 0$ local Richardson number negative, the factor with Ri It is not considered. It is considered only in the stable case.

$$l \equiv (L/H) \tag{35}$$

- *In the case of no immersed Boundary*

$$l = \kappa \sigma (1 - \sigma) \tag{36}$$

where H is the true thickness.

- *In the case of immersed Boundary technique*, one must go back to the original case

$$l = \kappa \frac{\sigma - \sigma_{bot}}{1 - \sigma_{bot}} (1 - \sigma) \tag{37}$$

H is not the true thickness since the immersed boundaries are used.

Case (c) Parabolic diffusivity Vertical diffusion for velocity (Tsanis case)
In that case,

$$K_{vmin} = 0$$

$$K_{turb} = \lambda u_* H (z'/H + z_{bh})(1 + z_{sh} - z'/H)$$

CARE !! TSANIS does not work for $u_* = 0$

In the code, it is the value of

$$\frac{K_v}{H} = \lambda u_* (\sigma + z_{bh})(1 + z_{sh} - \sigma)$$

which is introduced (it does not depend on H for the parabolic case (tsanis))

Case (d) $k - \epsilon$ Model
TO BE DONE

2.8 Boundary conditions for velocities

The conditions on boundaries are imposed in subroutine Boundaries or in subroutine Tendencies. The side boundaries of the lake are generally defined in a step like form. When integrated in a box of size $L_x \times L_y$ here taken as an example but the boundary conditions are:

- *On side walls: Impermeability* (in subroutine Boundaries) i.e. normal velocity to the walls is zero. For a square domain this means

$$u(x = 0, y, \sigma) = u(x = L_x, y, \sigma) = 0$$

$$v(x, y = 0, \sigma) = v(x, y = L_y, \sigma) = 0$$

- *On side walls: Free slip (no tangential stress)* (in subroutine Tendencies). For a square domain this means

$$\frac{\partial u}{\partial y_\sigma}(x, y = 0, \sigma) = \frac{\partial u}{\partial y_\sigma}(x, y = L_y, \sigma) = 0$$

$$\frac{\partial v}{\partial x_\sigma}(x = 0, y, \sigma) = \frac{\partial v}{\partial x_\sigma}(x = L_x, y, \sigma) = 0$$

- *On bottom walls: No-slip* (in subroutine Boundaries)

$$u(x_\sigma, y_\sigma, \sigma = 0) = v(x_\sigma, y_\sigma, \sigma = 0) = 0$$

  BUT Turbulent boundary layer....???

- *On the top free surface Momentum transfer* (in subroutine Tendencies) momentum is transferred from the wind in the x and y direction by

$$\frac{\partial u}{\partial \sigma}(x_\sigma, y_\sigma, \sigma = 1) = H \frac{\tau_s}{\rho_{wat} K_v(\sigma = 1)} \cos(\theta_W)$$

$$\frac{\partial v}{\partial \sigma}(x_\sigma, y_\sigma, \sigma = 1) = H \frac{\tau_s}{\rho_{wat} K_v(\sigma = 1)} \sin(\theta_W)$$

  with

$$\tau_s = C_D \rho_{air} W^2.$$

  and θ_W the angle of the wind

We don't need to impose

$$\frac{\partial h}{\partial x_\sigma}(x=0,y) = \frac{\partial h}{\partial x_\sigma}(x=L_x,y) = 0$$
$$\frac{\partial h}{\partial y_\sigma}(x,y=0) = \frac{\partial h}{\partial y_\sigma}(x,y=L_y) = 0$$

because of the conditions on u or v. (normal velocity are zero). However TO BE CHANGED when normal velocity are not zero : case of an inflow or outflow.

3 Temperature and boundary conditions for temperature

On the contrary to these active scalars, the evolution of temperature will be written in the non conservative way (57). This means that the evolution of a temperature T reads :

$$\frac{\partial T}{\partial t_\sigma}+u\frac{\partial T}{\partial x_\sigma}+v\frac{\partial T}{\partial y_\sigma}+\frac{\omega}{H}\frac{\partial T}{\partial \sigma}=-\frac{1}{H}\frac{\partial (HJ_x)}{\partial x_\sigma}-\frac{1}{H}\frac{\partial (HJ_y)}{\partial y_\sigma}-\frac{1}{H}\frac{\partial J_\sigma}{\partial \sigma}+\frac{1}{H}\frac{\partial}{\partial \sigma}(I_{in}) \tag{38}$$

where the source term in I_{in} is due to the absorption of light by water(see below). This term can be seen as a source term given by the equation

$$\frac{1}{H}\frac{\partial I_{in}}{\partial \sigma}=(a+.....)I_{in}.$$

and depends on the values of the temperature, biological tracers at that particular σ.
One can have temperature field without radiation (cf convection at surface without loss by radiation)

The fluxes for temperature are given by :

$$J_x=-\nu_T\frac{\partial T}{\partial x_\sigma} \tag{39}$$

$$J_y=-\nu_T\frac{\partial T}{\partial y_\sigma} \tag{40}$$

$$J_\sigma=-\frac{K_T}{H}\frac{\partial T}{\partial \sigma} \tag{41}$$

Horizontal diffusion for T
we assume a constant horizontal diffusion called ν_T which is set equal to

$$\nu_T=Pr_{turb}\ \nu$$

where Pr_{turb} is the turbulent Prandtl number.

Vertical diffusion for T
we assume a vertical diffusion called K_T which is set equal to

$$K_T = K_{Tmin} + Pr_{turb} \;\; K_{turb}$$

The condition on temperature are imposed in two subroutines

- *On side walls: no flux* (in subroutine tendencies)
- *On bottom walls: adiabatic condition* (in subroutine Boundaries) (see conductivity of the soil)

$$\frac{\partial T}{\partial \sigma}(x_\sigma, y_\sigma, \sigma = 0, t) = 0$$

On the top free surface condition for temperature $T(x_\sigma, y_\sigma, \sigma = 1, t)$ (in subroutine tendencies) :
We must consider several mechanisms which are playing a role in the exchange of energy. We impose the condition :

$$-\rho C_w K_T \frac{\partial T}{\partial z} + 0.55(1 - A)\Phi_{sol} + \Phi_{atm} = \Phi_{convec} + \Phi_{lat} + \Phi_{wat} \qquad (42)$$

- Φ_{sol} is the short wave radiation (Watt.m^{-2}) from the sun

  $$\Phi_{sol} = a\Phi_0(\sin\theta(t))^b(1 - 0.65C^2) \qquad (43)$$

  where $\Phi_0 = 1390$ Watt.m^{-2} (***or $\Phi_0 = 1367$or $\Phi_0 = 1096$??**) is the solar constant, θ is the angle (angle sun with the horizon at time t) , C is the cloud cover (0 for clear skies, 1 for completely cloudy cover). Coefficients a and b are ponderation coefficients which are functions of the skies neatness ($a = 0.81$ $b = 1.15$ for clean sky, $a = 0.77$ $b = 1.22$ for average clean skies, $a = 0.71$ $b = 1.25$ for industrial region).

  From the total flux Φ_{sol}, a part corresponding to $A\Phi_{sol}$ where $A = 0.06$ is the Albedo which depends on the latitude (Klein), is back-scattered to the sky. A part corresponding to $(1 - A)\Phi_{sol}$ are penetrating the water. 55 percent of this last part ($0.55(1 - A)\Phi_{sol}$) penetrates the first few centimeters of the water column (radiation of wavelength larger than $0.7\mu m$) and the remaining 45 percent denoted by the acronym PAR

  $$\Phi_{bulk}(\sigma = 1) = 0.45(1 - A)\Phi_{sol}$$

penetrates into the bulk (DYRESM page 11 or Jellison and Mellack 93).

For completeness 0.218 Joule corresponds to μ mol of photons in the PAR (Reynolds p95).

The term $\Phi_{bulk}(\sigma = 1)$ penetrates in the bulk as a condition on

$$I_{in}(\sigma = 1) = \frac{\Phi_{bulk}(\sigma = 1)}{\rho C_w}.$$

In the above equation, density is assumed constant. For the Buoyancy force, density will depend on temperature (see below).

- Φ_{wat} is the long wave radiation flux (Watt.m^{-2}). It expresses the thermal loss by irradiation from the water surface

$$\Phi_{wat} = \epsilon_{water} \sigma_{Stef} (T_s + 273.15)^4 \tag{44}$$

where the water surface temperature $T_s \equiv T(\sigma = 1)$ is in celsius. Coefficient $\sigma_{Stef} = 5.6710^{-8}$ is the Stefan-Boltzman constant and the water is assumed to be a grey body of emissivity $\epsilon_{water} = 0.97$.

- Φ_{atm} is the long wave radiation (Watt.m^{-2}). It expresses the thermal irradiation emitted from sky and clouds

$$\Phi_{atm} = \epsilon_{atm} \sigma_{Stef} (T_{atm} + 273.15)^4 (1 + 0.17C^2), \tag{45}$$

where the temperature T_∞ in celsius is the air temperature away from the water surface. The atmosphere is assumed to be a grey body of emissivity.

$$\epsilon_{atm} = 0.93710^{-5} (T_{atm} + 273.15)^2 \tag{46}$$

- Φ_{convec} is the sensible heat flux. This flux is assumed positive when directed towards the atmosphere.

$$\Phi_{convec} = \phi_{th} (T_s - T_{atm}) \tag{47}$$

where the heat transfer coefficient ϕ_{th} describes the heat conduction and convection and satisfies a phenomenological model

$$\phi_{th} = \rho_{air} C_{air} (a_1 + b_1 U_2) \tag{48}$$

with the specific air heat capacity at constant pressure $C_{air} = 1002 J kg^{-1} C^{-1}$, air density $\rho_{air} = ** \, kgm^{-3}$. $(a_1 + b_1 U_2)$ is a function of the air velocity at 2 meters from the surface. The coefficient $a_1 = ***$, $b_1 = **$. $0.0017 < a_1 < 0.0035$ (cf salencon).

- Φ_{lat} is the latent heat flux. It expresses the cooling effect due to evaporation. This flux is assumed positive when directed towards the atmosphere.

$$\Phi_{lat} = \phi_{lat}(e(T_s) - e_\infty) \tag{49}$$

where $e(T)$ is the air specific humidity at saturation at temperature T ($kg.kg^{-1}$). Assuming local thermodynamic equilibrium, $e(T)$ directly depends on the surface temperature through the saturated air pressure (in hP)

$$e(T) = \exp[2.3026(\frac{7.5T}{T+237.3} + 0.7858)] \tag{50}$$

This is Magnus-Tetens law. Quantity e_∞ denotes the specific humidity of the air above

The heat transfer coefficient ϕ_{lat} describes the heat conduction and convection and satisfies a phenomenological model

$$\phi_{lat} = \frac{0.622}{P} C_e \rho_{air} U_{10} L(T_s) \tag{51}$$

where P is the air pressure in hP; $C_e = 1.310^{-3}$. Finally $L(T)$ in $J.kg^{-1}$ denotes latent heat of evaporation (or condensation) of water at temperature T To calculate the latent heat in water in the temperature range from -40 to 40 Celsius the following empirical cubic function can be used a formula (Salecon, Bonnet)

$$L(T_s) = (2500.9 - 2.365(T - 273, 15))10^3 \tag{52}$$

or another formula

$$L(T) = (-0.0000614342T^3 + 0.00158927T^2 - 2.36418T + 2500.79)10^3 \tag{53}$$

which is a Cubic fit to Table 2.1,p.16, Textbook: R.R.Rogers and M.K. Yau, A Short Course in Cloud Physics, 3e,(1989), Pergamon press.

If one excepts the term $\Phi_{bulk}(\sigma = 1)0.45(1-A)\Phi_{sol}$ which takes into account the short wave radiation from the sun penetrating in the bulk, the exchange of energy takes place at the surface in the first centimers depth. By imposing the conservation of energy flux at the interface, one obtains a condition for $T(x_\sigma, y_\sigma, \sigma = 1, t)$:

$$-\Psi_{TurbConduct}\frac{\partial T}{\partial z} + 0.55(1-A)\Phi_{sol} + \Phi_{atm} = \Phi_{convec} + \Phi_{lat} + \Phi_{wat} \tag{54}$$

$\Psi_{TurbConduct}$ is a phenomenological thermal conductivity. It is different from the true thermal conductivity $\Psi_{Conduct}$ of water if the turbulence diffusivity K_T is not equal to the true water thermal diffusivity κ at the surface.

$$\Psi_{TurbConduct} = \rho C_w K_T, \quad \Psi_{Conduct} = \rho C_w \kappa$$

with C_w is the water specific heat.

To sum up, the condition for $T(x_\sigma, y_\sigma, \sigma = 1, t)$ reads :

$$-\rho C_w K_T \frac{\partial T}{\partial z} + 0.55(1 - A)\Phi_{sol} + \Phi_{atm} = \Phi_{convec} + \Phi_{lat} + \Phi_{wat} \tag{55}$$

or

$$K_T \frac{\partial T}{\partial z} = \frac{1}{\rho C_w}[0.55(1 - A)\Phi_{sol} + \Phi_{atm} - \Phi_{convec} - \Phi_{lat} - \Phi_{wat}] \tag{56}$$

3.1 Temperature and Buoyancy force

For the Buoyancy force, it will depend on temperature(see below). the dependence of density with temperature is given by

$$\rho(T) = 999.84289 + 10^{-3}(65.4891T - 8.56272T^2 + 0.059385T^3)$$

where ρ is in kg/m^3 and T in degrees Celsius. This equation is valid from 1 to 20 degrees Celsius (cf article Mixing Mechanisms in lakes).

3.2 Temperature and convective adjustement

At each time interval (the value of this time interval has to be set and it is not truly "given" by physical consideration), the stratification is modified to avoid too great overturning. This is performed on scalars and temperature fields. Nothing is done for the velocity which is a bit strange. Convective adjustement acts as a strong diffusion for unstable stratification. It is a way to introduce acceleration along z in a hydrostatic model.

From the top towards the bottom, one looks at each layer i : if the layer below $i - 1$ possesses a minor density ($\rho(i) > \rho(i - 1)$- the density is averaged i.e. the density becomes $\rho^{New}(i) = \rho^{New}(i - 1) = 0.5(\rho(i) + \rho(i - 1))$. Thereafter the newt layer $i - 1$ is examined.

4 Evolution equation for tracer c. Tracers : things which swim and breath or pure sediment

Contrary to some standard approach, we introduce $c(x_\sigma, y_\sigma, \sigma, t)$ as a true density for an active tracer. The evolution of $c(x_\sigma, y_\sigma, \sigma, t)$ is governed by

$$\frac{\partial c}{\partial t_\sigma} + u\frac{\partial c}{\partial x_\sigma} + v\frac{\partial c}{\partial y_\sigma} + \frac{\omega}{H}\frac{\partial c}{\partial \sigma} = -\frac{1}{H}\frac{\partial (HJ_x)}{\partial x_\sigma} - \frac{1}{H}\frac{\partial (HJ_y)}{\partial y_\sigma} - \frac{1}{H}\frac{\partial J_\sigma}{\partial \sigma} + G(c) \quad (57)$$

where $G(c)$ are the sources of the biological tracer and the fluxes due to diffusion or sedimentation are given by :

$$J_x = -\nu_c \frac{\partial c}{\partial x_\sigma} \quad (58)$$

$$J_y = -\nu_c \frac{\partial c}{\partial y_\sigma} \quad (59)$$

$$J_\sigma = w_s c - \frac{K_{vc}}{H}\frac{\partial c}{\partial \sigma} \quad (60)$$

w_s is the "vertical" velocity of sedimentation when < 0, of buoyancy when > 0.

Using the continuity equation, one can write this equation in a conservative way

$$\frac{\partial (Hc)}{\partial t_\sigma} = -\frac{\partial J_x^*}{\partial x_\sigma} - \frac{\partial J_y^*}{\partial y_\sigma} - \frac{\partial J_\sigma^*}{\partial \sigma} + HG(c) \quad (61)$$

where the fluxes read as :

$$J_x^* = uHc + HJ_x = uHc - H\nu_c \frac{\partial c}{\partial x_\sigma} \quad (62)$$

$$J_y^* = vHc + HJ_y = vHc - H\nu_c \frac{\partial c}{\partial y_\sigma} \quad (63)$$

$$J_\sigma^* = \omega c + J_\sigma = (\omega + w_s)c - \frac{K_{vc}}{H}\frac{\partial c}{\partial \sigma} \quad (64)$$

Horizontal diffusion for c
We assume a constant horizontal diffusion called ν_c

$$\nu_c = Sc_{turb}\ \nu$$

where Sc_{turb} is the turbulent Schmidt number.

Vertical diffusion for c

we assume a vertical diffusion called K_{vc} which is set equal to

$$K_{vc} = K_{cmin} + Sc_{turb} \;\; K_{turb}$$

K_{cmin} **may depend on the nature of the material**
CHANGE ****

$$K_{vc} = K_v + sdiff$$

*****

4.1 Flux corrected method for Hc

To avoid negative values of $\psi \equiv Hc$, we use flux corrected method (REF ??). This is why these active scalars are written in conservative equations (61).

A given grid point (i, j, k) possesses a flux
on the east $F^x_{i,j,k}$ and on the west $F^x_{i-1,j,k}$
on the south $F^y_{i,j,k}$ and on the north $F^y_{i,j-1,k}$
on the bottom $F^z_{i,j,k}$ and on the top $F^z_{i,j,k-1}$

At each position (i, j, k), one computes the maximal outgoing flux in each direction

$$(Max(F^x_{i,j,k}, 0) - Min(F^x_{i-1,j,k}, 0))$$
$$(Max(F^y_{i,j,k}, 0) - Min(F^y_{i,j-1,k}, 0))$$
$$(Max(F^z_{i,j,k}, 0) - Min(F^z_{i,j,k-1}, 0))$$

4.1.1 Implicit method in z for Hc

If an implicit method in z is used, one makes a sum of the contributions along x and y but not in z **See JOST**. This defines the factor

$$\gamma_{(i,j,k)} \equiv \frac{(Max(F^x_{i,j,k}, 0) - Min(F^x_{i-1,j,k}, 0))}{\Delta x} + \frac{(Max(F^y_{i,j,k}, 0) - Min(F^y_{i,j-1,k}, 0))}{\Delta y}$$

From a Eulerian prediction one gets

$$\psi_{i,j,k}(t + \Delta t) = \psi_{i,j,k}(t) - \Delta t(\frac{[F^x_{i,j,k} - F^x_{i-1,j,k}]}{\Delta x} + \frac{[F^y_{i,j,k} - F^y_{i,j-1,k}]}{\Delta y})$$

since the grid is regular Δx and Δy are constant.

To avoid $\psi_{i,j,k}(t + \Delta t)$ to become negative, one computes a factor

$$\gamma^{(Vero)}_{(i,j,k)} \equiv Min(\frac{\psi_{(i,j,k)}}{\gamma_{(i,j,k)}\Delta t}, 1)$$

and correct the outgoing fluxes (no incompatibility between neighbour cells) only such that

$$F^{(xNew)}_{i,j,k} = \gamma^{(Vero)}_{i,j,k} F^x_{i,j,k} \qquad \text{if} \qquad F^x_{i,j,k} > 0$$

$$F^{x(New)}_{i,j,k} = \gamma^{(Vero)}_{i+1,j,k} F^x_{i,j,k} \qquad \text{if} \qquad F^x_{i,j,k} < 0$$

$$F^{(yNew)}_{i,j,k} = \gamma^{(Vero)}_{i,j,k} F^y_{i,j,k} \qquad \text{if} \qquad F^y_{i,j,k} > 0$$

$$F^{y(New)}_{i,j,k} = \gamma^{(Vero)}_{i,j+1,k} F^y_{i,j,k} \qquad \text{if} \qquad F^y_{i,j,k} < 0$$

4.1.2 Explicit method in z for Hc

If an explicit method in z is used, one makes a sum of the three contributions and then defines a factor $\gamma_{(i,j,k)}$

$$\frac{Max(F^x_{i,j,k},0) - Min(F^x_{i-1,j,k},0)}{\Delta x} + \frac{Max(F^y_{i,j,k},0) - Min(F^y_{i,j-1,k},0)}{\Delta y} + \frac{Max(F^z_{i,j,k},0) - Min(F^z_{i,j,k-1},0)}{\Delta \sigma_k}$$

here the grid spacing σ_k is

5 Boundary conditions for tracers

To solve the equation for tracers, we impose boundary conditions at the water column boundaries. These conditions are all imposed in subroutine Boundaries or Tendencies.

In subroutine Tendencies one imposes

- The *zero-flux condition at the fluid surface*, (i.e. at $\sigma = 1$). To impose no mass flux at the surface, we set the flux J_σ equal to zero:

  $$J_\sigma(\sigma = 1) = 0. \qquad (65)$$

  For the nutrient we may assume a direct imput due to snow or rain. For the *flux condition at the fluid surface*, (i.e. at $\sigma = 1$), we thus impose a uniform mass flux of nutrient at the surface,

  $$J_\sigma(\sigma = 1) = J_{top}(t). \qquad (66)$$

- The *zero-flux condition at the side walls* : we set the appropriate normal flux J equal to zero. Since boundaries are in a step like form, this means according to the side of the step

$$J_x = 0, \text{ or } J_y = 0, \tag{67}$$

TO BE CHANGED when there is a flux of nutrient from the sides.

In subroutine Boundaries one imposes *the bottom walls conditions.* we have different boundary conditions

We define E_b the erosion (or resuspension) rate expressed as in (Chao, 2008):

$$E_b = \begin{cases} 0 & |\tau_b| < \tau_{ce} \\ \beta_c M \left(\frac{|\tau_b|}{\tau_{ce}} - 1\right) & |\tau_b| \geq \tau_{ce} \end{cases} \tag{68}$$

where τ_b is the bottom shear stress, τ_{ce} is the critical shear stress for erosion, β_c is the percentage of c in the eroded sediment and M is the erodibility coefficient related to the sediment properties.

We define D_b the deposition rate expressed as in (Chao, 2008) (only meaningful when $w_s < 0$)

$$D_b = \begin{cases} 0 & |\tau_b| > \tau_{cd} \\ w_s \quad c(0,t) \left(1 - \frac{|\tau_b|}{\tau_{cd}}\right) & |\tau_b| \leq \tau_{cd} \end{cases} \tag{69}$$

where τ_b is the bottom shear stress,

$$\tau_b = \frac{\rho_{wat} K_v(\sigma = 0)}{H} \sqrt{[|\frac{\partial u}{\partial \sigma}(x_\sigma, y_\sigma, \sigma = 0)|^2 + |\frac{\partial v}{\partial \sigma}(x_\sigma, y_\sigma, \sigma = 0)|^2]}$$

The quantity τ_{cd} is the critical shear stress. The above relation ([**?**]) is related to Parker's resuspension model ($u^* = V_s$ where V_s is the stokes velocity).

- For the sinking phytoplankton, detritus and sediment i.e. $w_s < 0$, we set:

$$J_\sigma(\sigma = 0, t) \quad = \quad D_b + E_b, \tag{70}$$

- For the buoyant phytoplankton , i.e. $w_s > 0$, we set:

$$J_\sigma(\sigma = 0, t) \quad = \quad E_b. \tag{71}$$

- For the zooplankton ($X = Z$), i.e. $v_Z = 0, <, > 0$ we set again:

$$J_\sigma(\sigma = 0, t) \quad = \quad E_b \tag{72}$$

- For the nutrient , i.e. $\omega_c = 0$, we set:

$$J_\sigma(\sigma = 0, t) = J_{in} + E_b. \tag{73}$$

  where the parameter J_{in} represent a constant flux of nutrient release at the bottom-water interface. This constant is of order of $0.1mgm^{-2}$/day to $5mgm^{-2}$/day (see Scheffer p72). Actually the phosphorous release also depends on the oxygen present at the level of the sediment.

  For the nutrient, Klausmeier and Litchman (2001) used the law

$$\frac{\partial c}{\partial z} = \phi_{permea}(c_{in} - c(0, t)) \tag{74}$$

  where the parameter $\phi_{permea} = 0.01m^{-1}$ represented the permeability of the bottom-water interface. However this is not a flux. We did not used this parametrisation.

Note that J_σ is an inappropriate notation since it is really equal to J_z and w_s is the true settling velocity.

6 The distribution of Light for ecosystems

$$\frac{\partial I}{\partial z} = (k_{bg} + k_c c + ...)\, I, \tag{75}$$

$$\tag{76}$$

$$\frac{\partial ln(I)}{\partial \sigma} = H\,(k_{bg} + k_c c + ...) \tag{77}$$

$$\tag{78}$$

For light, one assumes that the incident flux $I(\sigma = 1, t)$ is varying with time (see day and night forcing). It can be expressed simply by a simple law

$$I(\sigma = 1, t) = I_{top}(1 + \tanh[6\sin(2\pi t - \pi/2)])/2. \tag{79}$$

$$I_{top} = \frac{\Phi_{top}}{\mu M}$$

where Φ_{top} is given and μM is a factor of MicroMole of photons with respect to Joules.
If radiation is computed it is given by Φ_{bulk}

$$I(\sigma = 1) = \frac{\Phi_{bulk}(\sigma = 1, t)}{\mu M}$$

Care I_{in} is used in temperature and I is used in ecosystem. They are not exactly the same quantities: one is the flux divided by ρC_w and the other by μM.

7 Problem with numerics

$$CFL = \frac{K\delta t}{(\delta_{sigma})^2}$$

must be in practice less than $CFL_{max} = 0.1 - 0.2$ so that since $\delta_{sigma} = 1/N_{sigma}$ (N_{sigma} number of levels)

$$\delta t \leq \frac{CFL_{max}(\delta_{sigma})^2}{K} \sim \frac{CFL_{max}}{KN_{sigma}^2}$$

7.1 Dt max with diffusivity using mixing length.

In sigma for mixing length one gets

$$K \sim K_v/(H^2) \sim (\kappa^2/16)\frac{\partial u}{\partial z} \sim 0.01\frac{\partial u}{\partial z}$$

$$\delta t \leq \frac{100 * CFL_{max}}{\frac{\partial u}{\partial z}N_{sigma}^2}$$

7.2 Dt max with diffusivity using constant diffusivity.

In sigma for constant diffusivity one gets

$$K \sim K_v/(H^2)$$

$$\delta t \leq \frac{CFL_{max}(\delta_{sigma})^2}{K} \sim \frac{CFL_{max}H^2}{K_v N_{sigma}^2}$$

7.3 CFL with shallow water.

There is a CFL due to the seiche $\sqrt{gH}$ so that

$$\delta t \leq \frac{CFL'_{max}\delta_x}{\sqrt{gH}}$$

with $CFL'_{max} = 0.5$.

Note that coupling between 2D (barotropic) and 3D (surface waves of velocity $\sqrt{gH}$ which necessitate smaller δt that for the slower baroclinic internal modes) is not necessary in lakes because we have smaller dimension for H this restriction for δt is close to the one impose by CFL constraints due to diffusion

A Continuity equation in sigma-coordinates

The continuity equation reads in physical coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0$$

In sigma coordinates this equation multiplied by H becomes

$$H\frac{\partial u}{\partial x_\sigma} + H\frac{\partial v}{\partial y_\sigma} + H\frac{\partial \sigma}{\partial x}\frac{\partial u}{\partial \sigma} + H\frac{\partial \sigma}{\partial y}\frac{\partial v}{\partial \sigma} + \frac{\partial w}{\partial \sigma} = 0 \tag{80}$$

The "vertical" velocity ω is related to the true physical vertical velocity w by

$$\omega(x_\sigma, y_\sigma, \sigma, t_\sigma) = w - u((\sigma-1)\frac{\partial H}{\partial x} + \frac{\partial h}{\partial x}) - v((\sigma-1)\frac{\partial H}{\partial y} + \frac{\partial h}{\partial y}) - ((\sigma-1)\frac{\partial H}{\partial t} + \frac{\partial h}{\partial t})$$

Since h, H and $H^{(*)}$ does not depend on σ one gets for h or H or $H^{(*)}$

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial t_\sigma}, \quad \frac{\partial h}{\partial x} = \frac{\partial h}{\partial x_\sigma}, \quad \frac{\partial h}{\partial y} = \frac{\partial h}{\partial y_\sigma}$$

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial t_\sigma}, \quad \frac{\partial H}{\partial x} = \frac{\partial H}{\partial x_\sigma}, \quad \frac{\partial H}{\partial y} = \frac{\partial H}{\partial y_\sigma}$$

$$\frac{\partial H^{(*)}}{\partial t} = \frac{\partial H^{(*)}}{\partial t_\sigma}, \quad \frac{\partial H^{(*)}}{\partial x} = \frac{\partial H^{(*)}}{\partial x_\sigma}, \quad \frac{\partial H^{(*)}}{\partial y} = \frac{\partial H^{(*)}}{\partial y_\sigma}$$

$$\frac{H^{(*)}(x,y,t)}{H(x,y,t)} = 1 - g(x,y)$$

One obtains

$$\omega(x_\sigma, y_\sigma, \sigma, t_\sigma) = w - u((\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}) - v((\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}) - ((\sigma-1)\frac{\partial H}{\partial t_\sigma} + \frac{\partial h}{\partial t_\sigma}) \tag{81}$$

This relation can be differentiated with respect to σ

$$\frac{\partial \omega}{\partial \sigma}(x_\sigma, y_\sigma, \sigma, t_\sigma) = \frac{\partial w}{\partial \sigma} - \frac{\partial u}{\partial \sigma}((\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}) - u\frac{\partial H}{\partial x_\sigma} - \frac{\partial v}{\partial \sigma}((\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}) - v\frac{\partial H}{\partial y_\sigma} - \frac{\partial H}{\partial t_\sigma} \tag{82}$$

By summing equations (80) and (82) and taking into account that

$$-H\frac{\partial \sigma}{\partial y} = [(\sigma-1)\frac{\partial H}{\partial y_\sigma} + \frac{\partial h}{\partial y_\sigma}]$$

$$-H\frac{\partial \sigma}{\partial x} = [(\sigma-1)\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x_\sigma}]$$

one gets an equation in the volume

$$\frac{\partial H}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}(Hu) + \frac{\partial}{\partial y_\sigma}(Hv) + \frac{\partial \omega}{\partial \sigma} = 0 \tag{83}$$

Without immersed boundary, this was

$$\frac{\partial h}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}(Hu) + \frac{\partial}{\partial y_\sigma}(Hv) + \frac{\partial \omega}{\partial \sigma} = 0$$

The integration along the σ coordinate from $g(x,y)$ to 1 of the above continuity equation, leads to

$$(1-g(x,y))\frac{\partial H}{\partial t_\sigma} + \int_{g(x,y)}^{1} d\sigma \frac{\partial}{\partial x_\sigma}(Hu) + \int_{g(x,y)}^{1} d\sigma \frac{\partial}{\partial y_\sigma}(Hv) + \omega(1) - \omega(g(x,y)) = 0$$

Using the equality

$$\frac{\partial}{\partial x_\sigma}\int_{g(x,y)}^{1} Hud\sigma + \frac{\partial g(x,t)}{\partial x_\sigma} Hu(x,y,g(x,y),t) = \int_{g(x,y)}^{1} \frac{\partial}{\partial x_\sigma}(Hu)d\sigma$$

one obtains

$$(1-g(x,y))\frac{\partial H}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{g(x,y)}^{1} ud\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{g(x,y)}^{1} ud\sigma\right)$$

$$+\frac{\partial g(x,t)}{\partial x_\sigma} Hu(x,y,g(x,y),t) + \frac{\partial g(x,t)}{\partial y_\sigma} Hv(x,y,g(x,y),t) + \omega(1) - \omega(g(x,y)) = 0$$

The condition of no slip at the bottom $\sigma = g(x,y)$

$$u(x_\sigma, y_\sigma, \sigma = g(x,y), t_\sigma) = v(x_\sigma, y_\sigma, \sigma = g(x,y), t_\sigma) = 0$$

imposes

$$(1-g(x,y))\frac{\partial H}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{g(x,y)}^{1} ud\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{g(x,y)}^{1} ud\sigma\right) + \omega(1) - \omega(g(x,y)) = 0 \tag{84}$$

Let us consider the boundary condition on the vertical velocity, the evolution of the free surface is obtained by the kinematical condition on the free surface

$$\frac{\partial h}{\partial t} + u(x, y, h(x, y, t), t)\frac{\partial h}{\partial x} + v(x, y, h(x, y, t), t)\frac{\partial h}{\partial y} = w(x, y, h(x, y, t), t)$$

This can be written for $\sigma = 1$

$$\frac{\partial h}{\partial t_\sigma} + u(x_\sigma, y_\sigma, \sigma = 1, t)\frac{\partial h}{\partial x_\sigma} + v(x_\sigma, y_\sigma, \sigma = 1, t)\frac{\partial h}{\partial y_\sigma} = w(x_\sigma, y_\sigma, \sigma = 1, t)$$

leading to

$$\omega(x_\sigma, y_\sigma, \sigma = 1, t_\sigma) = 0 \tag{85}$$

In addition impermeability at the true boundary $z = -\zeta(x, y)$ reads

$$u(x, y, -\zeta(x, y), t)\frac{\partial \zeta(x, y)}{\partial x} + v(x, y, -\zeta(x, y), t)\frac{\partial \zeta(x, y)}{\partial y} = w(x, y, -\zeta(x, y), t)$$

or in sigma coordinates written at $\sigma = g(x, y)$

$$u(x_\sigma, y_\sigma, \sigma = g(x, y), t_\sigma)\frac{\partial \zeta(x, y)}{\partial x} + v(x_\sigma, y_\sigma, \sigma = g(x, y), t_\sigma)\frac{\partial \zeta(x, y)}{\partial y} = w(x_\sigma, y_\sigma, \sigma = g(x, y), t_\sigma)$$

Since $u(\sigma = g(x, y), t_\sigma) = v(\sigma = g(x, y), t_\sigma) = 0$, $w(x_\sigma, y_\sigma, \sigma = g(x, y), t_\sigma) = 0$.
This boundary condition reads as well

$$\omega(g(x, y)) = -((g(x, y) - 1)\frac{\partial H}{\partial t_\sigma} + \frac{\partial h}{\partial t_\sigma})$$

$$(1-g(x, y))\frac{\partial H}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{g(x,y)}^{1} u d\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{g(x,y)}^{1} u d\sigma\right) + (g(x, y)-1)\frac{\partial H}{\partial t_\sigma} + \frac{\partial h}{\partial t} = 0$$

Finally equation (84) becomes

$$\frac{\partial h}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{g(x,y)}^{1} u d\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{g(x,y)}^{1} v d\sigma\right) = 0 \tag{86}$$

By integrating the continuity equation (83) from $g(x, y)$ up to level σ, we obtain ω the "vertical" velocity in σ-coordinates :

$$\omega = -((g(x, y) - 1)\frac{\partial H}{\partial t_\sigma} + \frac{\partial h}{\partial t_\sigma}) - \int_{g(x,y)}^{\sigma}\left(\frac{\partial H}{\partial t_\sigma} + \frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma}\right) d\sigma$$

$$\omega = (1-\sigma)\frac{\partial H}{\partial t_\sigma} - \frac{\partial h}{\partial t_\sigma} - \int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{87}$$

In the code this is obtained as follows. First we compute starting from the bottom $\sigma = g(x,y)$ the integral

$$\int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{88}$$

From (84), one may compute $\frac{\partial h}{\partial t_\sigma} = \frac{\partial h}{\partial t}$ by changing the sign of the last integral with $\sigma = 1$.

$$\frac{\partial h}{\partial t_\sigma} = -\int_{g(x,y)}^{1} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{89}$$

Then

$$\omega = (1-\sigma)\frac{\partial H}{\partial t_\sigma} - \frac{\partial h}{\partial t_\sigma} - \int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma$$

From

$$\zeta_1(x,y,t) = \frac{g(x,y)}{1-g(x,y)} h(x,y) + \frac{\zeta(x,y)}{1-g(x,y)}$$

$$\frac{\partial H}{\partial t_\sigma} = \frac{\partial h}{\partial t_\sigma} + \frac{\partial \zeta_1}{\partial t_\sigma} = \left(\frac{1}{1-g(x,y)} \right) \frac{\partial h}{\partial t_\sigma}$$

$$\omega = (1-\sigma)\frac{\partial H}{\partial t_\sigma} - \frac{\partial h}{\partial t_\sigma} - \int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{90}$$

$$\omega = \left((1-\sigma) \left(\frac{1}{1-g(x,y)} \right) - 1 \right) \frac{\partial h}{\partial t_\sigma} - \int_{g(x,y)}^{\sigma} \left(\frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma} \right) d\sigma \tag{91}$$

B Primitive equations Stansby model without immersed boundary

The vertical coordinate is

$$\sigma = \frac{z - h(x,y,t)}{H} \tag{92}$$

with $H(x,y,t) = \zeta(x,y) + h(x,y,t)$ where ζ is the layer depth at rest and h the free surface perturbation amplitude. z is zero at the surface and directed upwards. σ is 0 at the surface and -1 at the bottom.

$$t \to t_\sigma, x \to x_\sigma, \quad y \to y_\sigma, \quad , z \to \sigma$$

$$\frac{\partial h}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}(Hu) + \frac{\partial}{\partial y_\sigma}(Hv) + \frac{\partial \omega}{\partial \sigma} = 0 \tag{93}$$

$$\frac{\partial h}{\partial t_\sigma} + \frac{\partial}{\partial x_\sigma}\left(H\int_{-1}^{0} u d\sigma\right) + \frac{\partial}{\partial y_\sigma}\left(H\int_{-1}^{0} v d\sigma\right) = 0 \tag{94}$$

$$\omega = w - u(\sigma\frac{\partial H}{\partial x_\sigma} + \frac{\partial h}{\partial x}) - v(\sigma\frac{\partial H}{\partial y} + \frac{\partial h}{\partial y}) - (\sigma\frac{\partial H}{\partial t} + \frac{\partial h}{\partial t}) \tag{95}$$

From $\partial h/\partial t_\sigma$ we obtain the "vertical" velocity in σ-coordinates, ω:

$$\omega = -\int_{0}^{\sigma}\left(\frac{\partial h}{\partial t_\sigma} + \frac{\partial uH}{\partial x_\sigma} + \frac{\partial vH}{\partial y_\sigma}\right) d\sigma \tag{96}$$

Note that, contrary to some standard approach, ω is a "vertical" velocity not a velocity divided by a length. Actually the relationship between the "vertical" velocity ω and the true vertical velocity w is as follows

This can be written as well as

$$\omega = H\frac{d\sigma}{dt} \tag{97}$$

C The distribution of vertical viscosity (Pacanowski)

The eddy viscosity has a dimension $[K_v] = [L][V]$ or $[K_v] = [L]^2[V/L]$. In a mixing length approximation, we write the vertical eddy viscosity with $L = \kappa min(z, H - z, 02H)$ where $\kappa = 0.4$ is the Von Karman constant or else (cf Cugier)

$$L = \kappa H\sigma(1-\sigma)$$

We have used a mixing length approximation we write the vertical eddy viscosity as

$$K_v = K_{vmin} + \frac{L^2}{(1+\alpha Ri)^\beta}\sqrt{[|\frac{\partial u}{\partial z}|^2 + |\frac{\partial v}{\partial z}|^2]} \tag{98}$$

or

$$K_v = K_{vmin} + \frac{Hl^2}{(1+\alpha Ri)^\beta}\sqrt{[|\frac{\partial u}{\partial \sigma}|^2 + |\frac{\partial v}{\partial \sigma}|^2]} \tag{99}$$

with

$$l = (L/H) = \kappa\sigma(1-\sigma) \tag{100}$$

local Richardson number

$$Ri = H\frac{-g\frac{1}{\rho}\frac{\partial\rho}{\partial\sigma}}{[|\frac{\partial u}{\partial\sigma}|^2 + |\frac{\partial v}{\partial\sigma}|^2]} \tag{101}$$

If $-\frac{\partial\rho}{\partial\sigma} < 0$ local Richardson number negative, the factor with Ri It is not considered. It is considered only in the stable case.

For scalars we use $K_v + sdiff$

The boundary layer at the bottom for sediment is of length $D/v_{settling}$. It should be greater than the minimum value of δz.

D The distribution of vertical viscosity (Tsanis)

Following (Wu and Tsanis 1995a,b) we write the vertical eddy viscosity as $z' = z + \zeta$

$$K_v = \lambda u_* H(z'/H + z_{bh})(1 + z_{sh} - z'/H) = \lambda u_* H(\sigma + z_{bh})(1 + z_{sh} - \sigma) \tag{102}$$

where z' is a vertical coordinate starting from 0 at the bottom, z_{bh} and z_{sh} are measures of the relative bottom and surface boundary layer thicknesses and $u_* = \sqrt{\tau_s/\rho_{wat}}$.
The vertical velocity profile is written as

$$u(z) = Au_* \log[1 + z'/(Hz_{bh})] + Bu_* \log[1 - z'/(Hz_{sh} + H)] \tag{103}$$

$$u(\sigma) = Au_* \log[1 + \sigma/z_{bh}] + Bu_* \log[1 - \sigma/(z_{sh} + 1)] \tag{104}$$

and from imposing the surface boundary condition

$$\frac{\partial u}{\partial\sigma}(\sigma = 1) = H\tau_s/(\rho_{wat}K_v)$$

the no-slip at the bottom and requesting that the depth integrated velocity is zero, we get the following expressions for A and B:

$$A = u_*q_2/(q_2p_1 + q_1p_2), \quad B = -u_*q_1/(q_2p_1 + q_1p_2) \tag{105}$$

$$p_1 = z_{sh}\lambda, \quad p_2 = (1 + z_{bh})\lambda, \tag{106}$$

$$q_1 = (1 + z_{bh})\log(1 + 1/z_{bh}) - 1, \quad q_2 = z_{sh}\log(1 + 1/z_{sh}) - 1 \tag{107}$$