Chap. 5 Notes: Expected Value and Simulation

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Outline

Expected Values:

Definition:

The **experimental expected value** of a random variable X is the average value of X in a very long sequence of replications of an experiment.

Example:

Toss a fair six-sided die.
X = number of dots on upward side.
What is expected value of X?

contd. from the previous example

Five-step simulation method:

1. **Choose Probability model:**
   Select 1, 2, 3, 4, 5, 6 at random with equal probability $\frac{1}{6}$.
2. **Define one simulation**
   One toss of the die corresponds to the selection of a random number from Table B.2[1].
3. **define event of interest:**
   Observe the random number selected from Table B.2.

[1] refer to the Page: 693 of the Text Book

4. **Repetitions of the simulation:**
   Simulate 10,000 tosses.
5. **Find experimental mean of the event of interest:**
   Compute sample mean $\bar{X}$ for the 10,000 numbers selected from Table B.2.

- Then the experimental expected value is 3.5101.

Definition:

For a discrete random variable X with probability distribution p(X), the **theoretical expected value** of X is

$$E(X) = \sum x.p(x)$$

Note:

This is often called the **theoretical mean** of random variable X and is denoted by μ_X

Example:

Toss a fair six-sided die once,
X = number of dots on the upward side.
What is the the theoretical mean of random variable X?

contd. from the previous example

The probability distribution of X is,

X	1	2	3	4	5	6
p(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

then,

$$\begin{aligned} E(X) &= \mu_X = \sum x.p(x) \\ &= 1.(\frac{1}{6}) + 2.(\frac{1}{6}) + 3.(\frac{1}{6}) + 4.(\frac{1}{6}) + 5.(\frac{1}{6}) + 6.(\frac{1}{6}) \\ &= 3.5 \end{aligned}$$

Note:

The experimental expected value of X, $\hat{E}(X) = 3.5101$ is an estimate of the theoretical expected value of X, E(X) = 3.5.

Law of Large Numbers for Independent Repetitions:

- Consider a random variable X which represents the outcome of an experiment. In a very long sequence of independent repetitions of the experiment, the average $\bar{X}$ of the observed values of X will be very close to the theoretical mean μ_X.

Other Expected Value Formulae:

Example:

Toss a fair coin 10 times,
X = number of heads observed.
What is E(X)?

$$E(X) = 10 * \frac{1}{2} = 5$$

Example:

Examples 5.3 - 5.6 of Text[2].

[2] refer to the Page: 284-285 of the Text Book

The Formula $\mu_X = n.p$

Suppose an event has probability p of occurring on a single trial and n trails are conducted. Let random variable X count the number of times the event of interest occurs in n trials. Then the expected value of X is

$$E(X) = \mu_X = n.p$$

Law of Large Numbers for Population Surveys:

- Suppose $X_1, X_2, \ldots, X_n$ are selected randomly without replacement from a very large population.

  IF the sample size n is large, then the sample mean $\bar{X}$ of the observed values will be close to μ, the mean of the population.

Box Models and Expected Values:

- Suppose random variable X is the <u>**sum**</u> of repeated random draws from a box then,

$$E(X) = \mu_X = \text{box mean} * \text{number of draws}$$

In particular, if we make just one draw, then

$$E(X) = \mu_X = \text{mean}$$

Example: Section. 4.4 of the Text Book[3]

- The formula above works whether we make our draws with replacement or without replacement.

[3] refer to Page: 256 of the Text book

Five-Step simulation to Estimate Mean Values

Example:

Examples From 5.1 lecture notes.

Example:

Example 5.10 of the Text Book[4].

[4] refer to Page: 294 of the Text Book

Standard Deviation of a Random Variable:

Definition:

Suppose random variable X has probability distribution p(X). Then the **(theoretical) variance** of X is,

$$\sigma_X = \sum (X - \mu_X)^2 . p(X)$$

and the **(theoretical) standard deviation** of X is,

$$\sigma_X = \sqrt{\sum (X - \mu_X)^2 . p(X)}$$

Example:

Roll a fair six-sided die once,
X = number of dots on the upward side.

The probability distribution of X is

X	1	2	3	4	5	6
p(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

We found $E(X) = \mu_X$ is[5].

$$\begin{aligned}
\mu_X &= \sum x.p(x) \\
&= 1.(\frac{1}{6}) + 2.(\frac{1}{6}) + 3.(\frac{1}{6}) + 4.(\frac{1}{6}) + 5.(\frac{1}{6}) + 6.(\frac{1}{6}) \\
&= 3.5
\end{aligned}$$

[5] refer to Sect. 5.1

- The Variance of X is,

$$
\begin{aligned}
\sigma_X^2 &= \sum (X - \mu_X)^2 . p(X) \\
&= (1 - 3.5)^2 . (\frac{1}{6}) + (2 - 3.5)^2 . (\frac{1}{6}) + \\
&\qquad \ldots + (6 - 3.5)^2 . (\frac{1}{6}) \\
&= (-2.5)^2 . (\frac{1}{6}) + (-1.5)^2 . (\frac{1}{6}) + \ldots + (2.5)^2 . (\frac{1}{6}) \\
&= \frac{35}{12} \\
&= 2.91666
\end{aligned}
$$

- So the standard deviation of X is,

$$
\begin{aligned}
\sigma_X &= \sqrt{2.916667} \\
&= 1.7078
\end{aligned}
$$

Box Models:

- Consider a box with N numbered balls. The box mean is,

$$\mu_{box} = \frac{\sum X}{N}$$

and the box variance is,

$$\sigma^2_{box} = \frac{\sum (X - \mu_{box})^2}{N}$$

and the box standard deviation is,

$$\sigma_{box} = \sqrt{\frac{\sum (X - \mu_{box})^2}{N}}$$

- Now, suppose X is the sum of n randomly selected balls from the box containing N balls.
  Then,
  $$E(X) = \mu_X = n * \mu_{box}$$
  (with or without replacement)
- **IF** the balls are selected with replacement, then,
  $$\sigma_X = \sqrt{n} * \sigma_{box}$$
- **IF** the balls are selected without replacement, then,
  $$\sigma_X = \sqrt{n} * \sigma_{box} * \sqrt{\frac{N-n}{N-1}}$$

Example:

Example 5.13 of the Text Book[6].

[6] refer to Page: 308 of the Text Book