Chap 4 notes: Probability

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Outline

- In oder to study a population we
    1. obtain a sample
    2. analyze the sample data
    3. make inferences about the population

Our inferences may be correct or incorrect, i.e. there is some "uncertainty" associated with any inference.

Why?

- Probability is a measure of the likelihood of an event.

  Example:

  Toss a fair coin once

$$P(H) = \frac{1}{2} \qquad P(T) = \frac{1}{2} \tag{1}$$

  These are "theoritical" probabilities.

  Example:

  But if I toss the coin 10 times I might not get exactly 5 H and 5 T.

  Why?

Probability - Relative Frequency Definitiion

Definition:

Suppose an experiment consists of n trials, and k of these trials result in event E. Then

$$\hat{P}(H) = \frac{K}{n} = \frac{\textit{no. of successful repetitions}}{\textit{no of repetitions}} \tag{2}$$

- Note:

  This is called the empirical probability of an event or the relative frequency of the event, or the experimental probability of the event.

Example:

I toss a fair coin 100 times and observe 46 heads.
Then

$$\hat{P}(H) = \frac{k}{n} = \frac{46}{100} = 0.46 \tag{3}$$

and

$$\hat{P}(T) = \frac{54}{100} = 0.54 \tag{4}$$

. . . See Table 4.2 Text

- Note:

1. n must be *large*, the larger the better.
2. Usually, as n increase

$$\hat{P}(E) \rightarrow P(E)$$

empirical probability approaches theoretical probability

3. Two researchers, conducting similar experiments, may obtain different values for

$$\hat{P}(E)$$

4. The relative frequency definition of probability may be impractical - may not be possible to conduct n = 1000 or more trials.
5. Unless otherwise stated, when we say *"Probability"* we will mean *"Theoretical probability"*.

Probability Models

Definition:

A trial is an action that results in one of several possible outcomes.

Definition:

An experiment is a single trial or a series of trials.

Definition:

The sample space of an experiment is the set of all possible outcomes.

Definition:

An event is a set of outcomes with something in common.

Example:

Roll fair 6 - sided die once
Sample Space = 1, 2, 3, 4, 5, 6
Let E = roll a number less than 3
Then E = 1,2
and

$$P(E) = \frac{2}{6}$$

Probability - Equally Likely Outcomes

Definition:

Suppose an experiment can result in one of m equally likely outcomes. Suppose that r of these outcomes result in event A occurring. Then the theoretical probability of event A is

$$P(A) = \frac{r}{m} = \frac{\textit{no. of outcomes in event A}}{\textit{total no. of possible outcomes}} \tag{5}$$

- Note:

For each outcome in Sample Space

$$P(\textit{outcome}) = \frac{1}{\textit{total no. of possible outcomes}}$$

Example:

Randomly select one card from an ordinary deck of 52 playing cards

$$P(King) = \frac{4}{52}$$

$$P(Heart) = \frac{13}{52}$$

$$P(Face\ card) = \frac{12}{52}$$

Example:

Example 4.6 Text [1]

[1] Probability Distribution for Single Toss of a Fair Die

Discrete Probability Distributions

Discrete random variable can have only certain values, like 0, 1, 2, 3, etc. , which are usually obtained by counting.

A *discrete probability distribution* is a list (or description) of the values the random variable can have, along with the associated probabilities.

Example:

Suppose the probability that a baby will be male is 0.50 (so the probability that the baby will be female is also 0.50) . A couple plans to have two children. Let X represents the number of male children.

Construct the probability distribution for X.

We can do this using a <u>probability tree</u>.

First Child	Second Child	Outcome	Probability
M(=0.50)	M(=0.50)	MM	0.25
	F(=0.50)	MF	0.25
F(=0.50)	M(=0.50)	FM	0.25
	F(=0.5)	FF	0.25

X = no. of Males	0	1	2
probability	0.25	(0.25 + 0.25) = 0.50	0.25

Rules:

1. The probability of an event E is always between 0 and 1, inclusive:

$$0 \leq P(E) \leq 1$$

$$P(E) = 0 \rightarrow \text{event E can not occur}$$

$$P(E) = 0 \rightarrow \text{event E is guaranteed to occur}$$

2. The probability of event A is equal to the sum of the probabilities of the outcomes in the event A.

$$P(A) = \sum_{\text{all outcomes in A}} P(outcome)$$

Example:

Roll a fair die once
SS = {1, 2, 3, 4, 5, 6}

Outcome	*1*	*2*	*3*	*4*	*5*	*6*
P(outcome)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let event B= roll even = {2, 4, 6}

Then
$P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$

Complementary Event

Definition:

Suppose A is an event. The complement of event A, denoted as "not A", is the event "A does not occur".

Rule of Complementary Events:

$$P(\text{not A}) = 1 - P(A)$$

Example:

Randomly select one card

E = select Ace

Then

$$\begin{aligned} \textit{not } E &= \textit{do not select Ace} \\ &= \textit{select "non - Ace"} \end{aligned}$$

So,

$$\begin{aligned} P(\textit{not } E) &= 1 - P(E) \\ &= 1 - \frac{4}{52} \\ &= \frac{52}{52} - \frac{4}{52} \\ &= \frac{48}{52} \end{aligned}$$

Probability and Compound Events

Definition:

An event comprised of two (or more) events is a compound event.

Example:

Roll a fair 6 - sided die once

$$\begin{aligned} E_1 &= \textit{roll even number} \\ &= \{2, 4, 6\} \end{aligned}$$

$$\begin{aligned} E_2 &= \textit{roll number beginning with "t"} \\ &= \{2, 3\} \end{aligned}$$

- The compound event

$$E_1 \cup E_2 = E_1 or\ E_2 (\text{or both})$$

occurs

$$\begin{aligned} if &\quad E_1 \text{ happens or} \\ if &\quad E_2 \text{ happens or} \\ if &\quad \text{both events happen} \end{aligned}$$

So,

$$\begin{aligned} E_1 \cup E_2 &= \text{roll an even number or a number} \\ &\quad \text{beginning with "t" or both} \\ &= \{2, 3, 4, 6\} \end{aligned}$$

Thus,

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1 or\ E_2\ or\ both) \\ &= \frac{4}{6} \end{aligned}$$

The addition law (the "OR" Law)

$$P(E_1 \text{OR } E_2) = P(E_1) + P(E_2) + P(E_1 \text{and } E_2)$$

Example:

Roll fair die once

$$\begin{aligned} E_1 &= \text{roll even} \\ E_2 &= \text{roll number starting with "t"} \\ E_1 \cap E_2 &= \text{roll even starting with "t"} \\ E_1 \, and \; E_2 &= \{2\} \end{aligned}$$

$$\begin{aligned} P(E_1 \, or \; E - 2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} \\ &= \frac{4}{6} \end{aligned}$$

Mutually Exclusive Events

Definition:

Event E_1 and event E_2 are mutually exclusive if the occurrence of one event precludes the occurrence of the other event.

- Note:

  Two events are mutually exclusive if they can not occur simultaneously.
  In symbols

$$P(E_1 and E_2) = 0$$

Example:

Roll a fair die once

$$\begin{aligned} E_1 &= \text{Roll even} \\ E_2 &= \text{Roll a 5} \end{aligned}$$

These events are mutually exclusive.

Special Addition Law

IF events E_1 and event E_2 are mutually exclusive. Then

$$P(E_1 or E_2) = P(E_1) + P(E_2)$$

Example: (Continued)

$$\begin{aligned} P(E_1 \text{or} E_2) &= P(E_1) + P(E_2) \\ &= \frac{3}{6} + \frac{1}{6} \\ &= \frac{4}{6} \end{aligned}$$

- Note:

  IF *E_1, E_2 are mutually exclusive.*

  Then

$$P(E_1|E_2) = 0$$

  and

$$P(E_2|E_1) = 0$$

  and

$$P(E_1 \cap E_2) = 0$$

- The compound event

$$E_1 \cap E_2 = E_1 \text{ AND } E_2$$

occurs if and only if both event E_1 and event E_2 occurs.

So,

$$\begin{aligned} E_1 \cap E_2 &= \text{roll a number which is even} \\ &\quad \text{and begins with "t"} \\ &= \{2\} \end{aligned}$$

Thus,

$$P(E_1 \cap E_2) = \frac{1}{6}$$

Probability Trees

Example:

Select two marbles without replacement from box containing 3 Red and 2 White.

First Marble	Second Marble	Probability
R (.6)	R (.5)	P(R and R)=(.6*.5)=.3
	W (.5)	P(R and W)=(.6*.5)=.3
W(.4)	R(.75)	P(W and R)=(.4*.75)=.3
	W(.25)	P(W and W)=(.4*.25)=.1

The Multiplication Law (The "AND" Law)

Conditional probability

$$P(A|B)$$

P(A | B) : Probability (denoted by P) of B given that (denoted by |) A has occurred.

Example:

randomly select two marbles without replacement, from a bag containing 14 White and 6 Red marbles. Find the probability that

a. *the second marble is Red given the first marble is White.*

$$P(R_2|W_1) = \frac{6}{19}$$

b. *Find*

$$P(R_2|R_1) = \frac{5}{19}$$

General Multiplication Law (The "AND" Law)

$$P(A \text{ and } B) = P(A).P(B|A)$$

Example: (Continued)

c. Find the probability that both marbles are Red.

$$\begin{aligned} P(\text{both R}) &= P(R_1 \text{ and } R_2) \\ &= P(R_1).P(R_2|R_1) \\ &= \frac{6}{20}.\frac{5}{19} \\ &= 0.0789 \\ &\simeq 0.08 \end{aligned}$$

Sometimes event B does not depend on whether A has occurred.

Example: (Continued)

d. Suppose the two marbles are selected with replacement.
   Find the probability that both marbles are Red.

$$\begin{aligned} P(\text{both R}) &= P(R_1 \text{and} R_2) \\ &= P(R_1).P(R_2) \\ &= \frac{6}{20}.\frac{6}{20} \\ &= 0.900 \end{aligned}$$

- Note:

  In this example, the second selection (B) does not depend on the first selection.

This example illustrates the:

Special Multiplication Law

IF event A and event B are statistically independent, then

$$P(A(and)B) = P(A).P(B)$$

- Two events are independent if the occurrence of one event does not affect the probability of the other event,
  i.e.

$$P(E_1) = P(E_1|E_2)$$

and

$$P(E_2) = P(E_2|E_1)$$

IF two events are not independent, the events are said to be *dependent*.

- For some situations, it may be "easy" to determine
  if two events are independent or dependent,
  e.g., for "selection with replacement" experiments,
  the selections are independent,
  but for "selection without replacement" experiments,
  the selections are dependent.

  For othe situations, it may be more difficult to determine
  if two events are independent.

Example:

Roll a fair 6 - sided die once

$$\begin{aligned} A &= \textit{roll odd} \\ B &= \textit{roll number greater than 3} \end{aligned}$$

Are event A and event B independent?

(Continued)

First,

$$P(A) = \frac{3}{6} = 0.6$$

$$P(B) = \frac{3}{6} = 0.5$$

Next,

$$\begin{aligned} P(A|B) &= \frac{P(A \; and \; B)}{P(B)} \\ &= \frac{\frac{1}{6}}{\frac{3}{6}} \\ &= \frac{1}{3} \end{aligned}$$

(Continued)

and

$$P(B|A) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

Does P(A) = P(A | B) ?

Does P(B) = P(B | A) ?

No !

Thus, these two events are *dependent*.

Simulation

- Simulation methods (or Monte Carlo simulation) are techniques which use a computer to perform a large number of times.

- Five Step Simulation Method

1. Choose a Probability Model
2. Define one Simulation
3. Define event of interest
4. Repeat Simulation n times
5. Compute experimental probability

Example:

Example 4.13

Random Number Table B.2

Example

Example 4.14

Random Number Table B.1

Example

Example 4.17

Random Number Table B.3

Simulating Random Sampling via a Box Model

- *Box Modeling* is a method of simulating data by randomly drawing a number from a box repeatedly, either with or without replacement.The box may contain any number of real numbers, some (or all) of which may appear in the box more than once.

Example:

To model a toss of a fair six - sided die

- Note:

  A box models may contain objects other than numbers.

Example:

Select two marbles from box containing 5 Red and 3 Blue.

Random Sampling - With or Without Replacement

- When sampling *with replacement* entails replacing the selected ball before drawing the next selection.
- When sampling *without replacement*, the selected ball is *not* replaced before the next selection.

▶ Note:

When sampling without replacement, the selections are not independent.

Multiplication Rule For Multistage Experiments

- Suppose an experiment is conducted in k stages. Stage 1 can be performed in any of N_1 ways; stage 2 can be performed in any of N_2 ways; etc. Then there are

$$N_1 * N_2 * N_3 \cdots * N_k$$

ways of performing the experiment.

Example:

A student owns 5 shirts, 4 pants, and 2 pair of shoes. How many distinct outfits can he wear?

- [Answer]

$$5 * 4 * 2$$

Example:

Example 4.28 Text