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How Random is Random?

Using Markov Chains and Linear Algebra to Model Random Events

Abstract

From simply predicting the weather for the next day to the Google page rank algorithm, Markov Chains equip us with the tools needed to mathematically determine the outcome of seemingly random events. By understanding Markov Chains we can make meaningful predictions for events—including some that could be life saving, as we will explore using the SIR model. This module has been developed to review the Linear Algebra topic of Markov Chains through videos and a practice problem to then use Markov Chains in several real-world applications.

Introduction of Concepts

For the first part of the module I wanted to focus on introducing the question we are attempting to answer and reviewing Markov Chains from elementary linear algebra. I started with a YouTube video "Introducing Markov Chains" posted by HarvardX. This five minute video is not mathematically detailed, but I like how the chosen situation of a worker spinning a spinner to randomly decide where they will visit next demonstrates how random is not always as random as we initially thought. This video also gives a nice visual representation of the stationary distribution, which is really useful in further applications ("Introducing Markov Chains").

The second YouTube video "Vector and Matrix Algebra Tutorial: Markov Chains 1" posted by Complexity Explorer goes into more detail about the math behind Markov Chains. I picked this video because the user clearly defines state vectors, probability vectors, and stochastic matrices ("Vector and Matrix Algebra Tutorial").

After this linear algebra review, I gave the students a practice problem to work on in groups. The purpose of this problem was to practice using Markov Chains on a simple yet relevant example. This question is taken from the YouTube video "The Transition Matrix"[3] by

William Lindsey, which I adapted into a document and separated the different steps into different parts of the problem. The question is:

In a certain city, if today is sunny, tomorrow will be sunny 80% of the time. If today is cloudy, tomorrow will be cloudy 60% of the time. Supposing today is sunny, what is the probability it will be cloudy the day after tomorrow?

1. Find the transition matrix
2. Find the distribution matrix for "today"
3. Find the probability of tomorrow being either sunny or cloudy
4. FInd the probability it will be cloudy the day after tomorrow

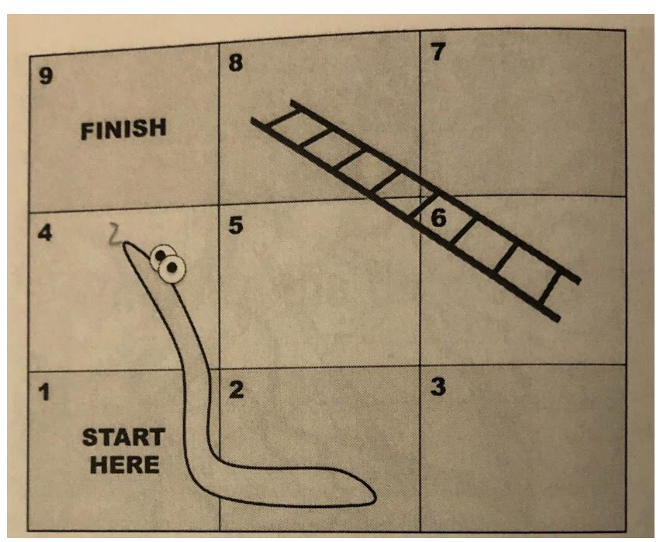
The solutions to this problem can be found at the end of the document ("The Transition Matrix").

Applications

Markov Chains have several important applications. We will look at board games, the Google Page Rank Algorithm, and the SIR model.

Games

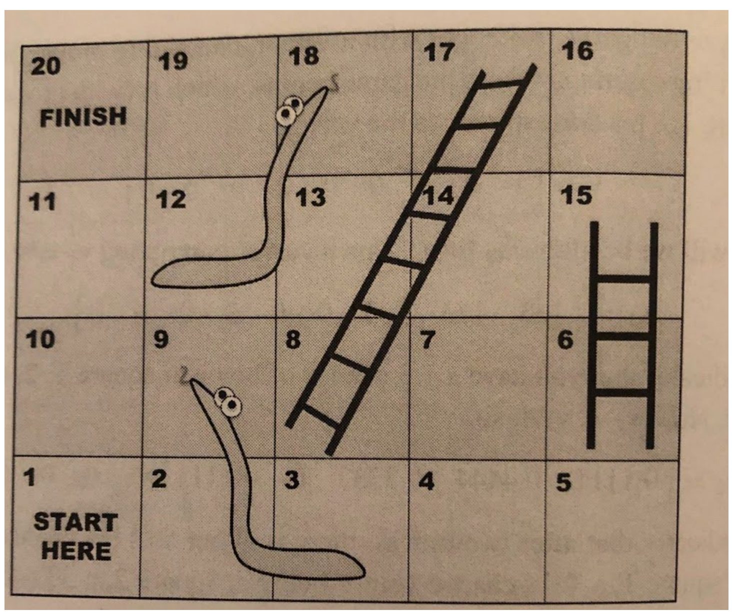
In his text "When Life is Linear", Tim Chartier opens his Chapter 10.1 with the question we've all asked during the fifth hour of our game of Monopoly, "How short a game is possible?" (Chartier, 92). Chartier then uses Markov Chains and the much simpler game of Chutes and Ladders to explore different seemingly random aspects of board games. In Chutes and Ladders, players take turns. Each turn, the player rolls a die. A roll of 1 or 2 means the player does not move, a 3 or 4 moves the player one square, and a 5 or 6 moves the player 2 squares. If the square has the foot of the ladder the player "climbs" the ladder to the higher square, and if the square has the head of the snake the player "slides" back to the lower square. Chartier produces the transition matrix for the following version of the board in his book, and I will explain the first two rows.



From square 1, a player can either stay on square 1, move one space to square 2, or move two spaces to square 3. This corresponds to [⅓ ⅓ ⅓ 0 0 0 0 0 0] as the first row in the transition matrix. From square 2, a player can stay on square 2, move one square to square 3, or move two squares to square 4, which would result in sliding down the snake back to square 2. Therefore, the second row in the transition matrix is [0 ⅔ ⅓ 0 0 0 0 0 0]. The rest of the transition matrix follows similarly.

After constructing the transition matrix, we can calculate both the minimum number of moves needed to win the game and after how many turns it is 50% likely someone has won the game. To calculate the minimum number of moves needed to win the game, we would start with our transition matrix A and the initial vector v0=[1 0 0 0 0 0 0 0 0], since the probability we start on square 1 is 1. Multiplying A by v0 gives us v1, the probabilities of being on the different squares after one move. We would then multiply A by v1 to get the probabilities after the second move, and so on. As soon as there is a value other than 0 in column 9 of vn, we know the minimum number of turns a player can win the game in is n. For this example, the minimum number of turns is 4. To calculate after how many turns it is 50% likely someone has won the game, we would continue multiplying A by vn until the value in column 9 is at least .50. N would be the number of turns needed to reach this. For this example n is 10, meaning that it is 50% likely someone has won after 10 turns (Chartier, 92-96).

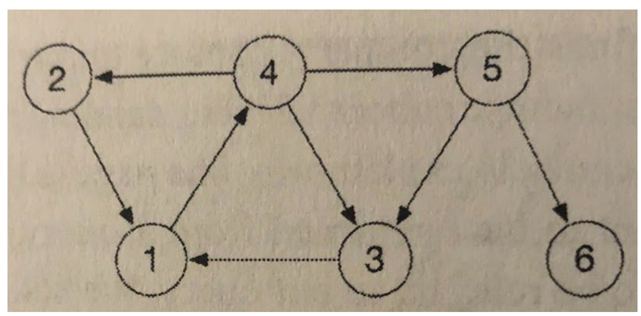
For this application, the activity I designed was to have the students play a game of Chutes and Ladders in small groups using the board below, taken from page 96 of Chartier's text. Students can use a real die or an online dice simulator. While doing this activity, the students should talk about the probability of being on each square (and compute the transition matrix if feeling inspired). It should be recognized that some squares will never or almost never be landed on due to the placement of the snakes and ladders on the board. Note that there is no solution for this activity. Similar methods could be used for games like Monopoly, however the cards like "Move to x Property" or the added elements of jail and that players go around the board several times make these kinds of



games complicated to model. Material that goes into further detail about Monopoly can be found in the references section.

Google Page Rank Algorithm

The goal of the Google Page Rank Algorithm is to predict the probability of a user being at a certain web page in a network of web pages after a certain number of pages. The presentation put together by Jeff Jauregui from the University of Pennsylvania, linked to in further reading, introduces Markov Chains and then presents several proposed ways of ranking web pages before moving on to the method in use, which I will focus on here. Proposed by Brin and Page, the Google Page Rank Algorithm is based on a "random surfer" who will click a link to a new page in the network of web pages 85% of the time and randomly jump to a new page in the network 15% of the time. Any of the links are equally likely to be followed, and if a page has no links there is an equal chance of jumping to any of the pages in the network. A transition matrix can be constructed from a network of webpages, and the transition matrix can be used to find the probability of being at a certain webpage after a certain number of page changes and the steady-state vector for the network. We can look at pages 98-100 of Chartier's text to see a simple example of the Google Page Rank Algorithm. Chartier gives the network as a graph of nodes and edges. We can see that an arrow out of a node is a link from that node to the node the arrow is pointing to. To construct the first row of the transition matrix, we need to calculate the probability of going to each page from page 1.



Here is a table of the calculations:

From Node 1 to Node 1	15% of jumping to page 1 x (⅙)	(15/100)x(⅙) = 1/40
From Node 1 to Node 2	15% of jumping to page 2 x (⅙)	(15/100)x(⅙) = 1/40

From Node 1 to Node 3	15% of jumping to page 3 x (⅙)	(15/100)x(⅙) = 1/40
From Node 1 toNode 4	15% of jumping to page 4 x (⅙) + 85% of clicking to link 4	(15/100)x(⅙) + (85/100)x(1/1) = 1/40 + 34/40 = 35/40
From Node 1 to Node 5	15% of jumping to page 5 x (⅙)	(15/100)x(⅙) = 1/40
From Node 1 to Node 6	15% of jumping to page 6 x (⅙)	(15/100)x(⅙) = 1/40

Therefore, the first row of the transition matrix would be [1/40 1/40 1/40 35/40 1/40 1/40]. Notice that the probabilities in the row add up to 1. The rest of the transition matrix can be produced using similar logic. Assuming we start at web page 1, our initial vector would be v0 = [1 0 0 0 0 0], since the probability we are on page 1 is 1. If we multiply the transition matrix by v0 we get v1, which is the probability of being at each web page after one step. If we continue this process, we can produce the steady-state matrix to find the probabilities of each page being visited in the long run. Why does this work? Perron's Theorem states that every square matrix with positive entries has a unique eigenvector with all positive entries: this eigenvector's corresponding eigenvalue has only one associated eigenvector, and the eigenvalue is the largest of the eigenvalues. Basically, since our square transition matrix has all positive entries we get a unique steady-state vector that represents the probabilities in the long run. One of the main takeaways from this is that the page rank of a web page is based on the page rank of those that link to it. Therefore, in the real world—well, the internet—having a link from a more powerful page will attract more attention to your web page than from a small page (Chartier, 96-101).

SIR Model

The last application of Markov Chains I would like to explore is the SIR Model for diseases, used in Epidemiology. The SIR Model is used to estimate how many individuals in a population are Susceptible to a certain disease, Infected with the disease, or Recovered from the disease at certain time increments. There are several different versions of the SIR Model, but I will be focusing on a simplistic version that uses Markov Chains, which have been the main focus throughout the rest of this module. These more complicated models make use of differential equations and take into account more variable factors, such as a difference in the probability of certain individuals passing on the disease, that individuals have a varying number of contacts, or that "R" could also be people who have passed from the disease. Links to literature that uses the SIR Model with differential equations is linked in the further reading section.

The first step to using the Markov Chains to model the spread of a disease is to set up the transition matrix. A sample transition matrix is:

$$T = \begin{array}{c|ccc} & S & I & R \\ \hline S & .85 & .15 & 0 \\ I & 0 & .12 & .88 \\ R & 0 & 0 & 1 \end{array}$$

This transition matrix tells us that individuals who are susceptible to the disease have an 85% chance of staying in their current state and a 15% chance of becoming infected over the next time interval, individuals who are infected with the disease have a 12% chance of staying infected and an 88% chance of becoming recovered in the next time interval, and individuals who are recovered have a 100% chance of being recovered. It is important to note that Recovered is an absorbing state, meaning that once an individual is recovered they will not go back to either the susceptible or infected state. This is a large assumption on behalf of the SIR Model, and may not always be the case with real diseases. The 0 in the susceptible row shows that it is not possible for a person to go from a susceptible state to a recovered state in one jump, and the 0 in the infected row shows that it is not possible for an individual to go from an infected state back to a susceptible state.

To see the SIR Model in action, let's set an initial vector and calculate the amount of people in each state after the first time interval.

$$v0 = \begin{array}{c} .90 \\ .07 \\ .03 \end{array} \qquad v1 = T*v0 = \begin{array}{c} .7755 \\ .0348 \\ .03 \end{array}$$

After the first time interval, 77.55% of the population is susceptible, 3.48% is infected, and 3% is recovered. With tracking the spread of diseases we are usually interested in many time periods from now. We can calculate the number of people in each state n time intervals from now by calculating $T^n * v0$. This can be used to calculate the steady-state vector, which is how the probabilities settle in the long run and therefore the amount of people in each state in the long run. The SIR model can be useful in determining the rate of growth and the final size of an epidemic. It is with noting that this model does not predict every disease perfectly as diseases all spread and develop differently, but it is a simple model that can explore the underlying patterns and assist in making decisions on how to stop the spread of a disease (Perham & Perham).

Reflection

In terms of content, I believe my module met the requirements listed on the checklist. I believe my strongest areas were introducing the topic with videos and a practice problem, followed by the Chutes and Ladders discussion. I think the math behind some of the more complicated applications is my weakest point, as I don't have too much experience with really complicated formulas and calculations. This could be strengthened by spending more time trying to understand these concepts. As for the class presentation, I similarly believe that the beginning of my presentation was stronger than the end. To improve my presentation, I think it would have been more helpful to put together my own set of slides rather than using premade presentations that went into detail that was over my head. I also would have liked to improve on the Chutes and Ladders group activity to have the students actually compute at least part of the probability matrix. Being the first presentation I think I have more hindsight into things I would have done differently after seeing the rest of the presentations and how they have evolved with each module. I also think that I did not have as much time as I would have liked to fully understand and feel confident in talking about these topics, and I hoped to expand on my understanding and explanation in this paper. I am glad that I got to add information about the SIR model in this paper because after my presentation I felt like I could have done more in terms of finding applications outside of the chapter in Chartier. I think Markov Chains are really interesting even after spending the semester developing this module and paper and would maybe be interested in doing a project applying them to something!

Works Cited

Chartier, Tim. *When Life Is Linear: From Computer Graphics to Bracketology*. The Mathematical Association of America, 2015.

"Introducing Markov Chains." *YouTube*, uploaded by HarvardX, 28 Feb. 2020, https://www.youtube.com/watch?v=JHwyHIz6a8A.

Perham, Bernadette H. and Perham, Arnold E. "Markov Chains." *Topics in Discrete Mathematics: Markov Chain Theory.* https://www.rose-hulman.edu/~bryan/lottamath/transmat.pdf.

"The Transition Matrix." *YouTube*, uploaded by William Lindsey, 13 Apr. 2015, https://www.youtube.com/watch?v=4zg5bNlHZRg.

"Vector and Matrix Algebra Tutorial: Markov Chains 1." *YouTube*, uploaded by Complexity Explorer, 3 Apr. 2019, https://www.youtube.com/watch?v=9CJBrwQ3lbM.

Further Reading

1. https://web.williams.edu/Mathematics/sjmiller/public_html/hudson/Li_Markov%20Chains%20in%20the%20Game%20of%20Monopoly.pdf (Monopoly)
2. http://jnsilva.ludicum.org/HMR13_14/LuckLogicLies.pdf (Ch. 16 - Monopoly)
3. https://www.math.upenn.edu/~kazdan/312F12/JJ/MarkovChains/markov_google.pdf (Google Page Ramk)
4. https://www.maa.org/press/periodicals/loci/joma/the-sir-model-for-spread-of-disease-the-differential-equation-model (SIR Model - Differential Equations)
5. https://www.maplesoft.com/applications/download.aspx?SF=127836/SIRModel.pdf (SIR Model - Differential Equations)

Solutions

1. Transition Matrix:

$$\begin{array}{c|cc} & \text{Sunny} & \text{Cloudy} \\ \hline \text{Sunny} & .8 & .4 \\ \text{Cloudy} & .2 & .6 \end{array}$$

Using the probabilities given in the problem. Set up as current state being the column and next state being the row, each column adds to 1

Distribution Matrix for “today”:

$$\begin{array}{c|c} \text{Sunny} & 1 \\ \text{Cloudy} & 0 \end{array}$$

The problem states it is sunny today, so the probability today is sunny is 1

Find the probability of tomorrow being either sunny or cloudy:

$$\begin{matrix} .8 & .4 \\ .2 & .6 \end{matrix} \times \begin{matrix} 1 \\ 0 \end{matrix} = \begin{matrix} .8 \\ .2 \end{matrix}$$

There is a .8 chance of tomorrow being sunny and a .2 chance of tomorrow being cloudy

Find the probability it will be sunny the day after tomorrow:

$$\begin{matrix} .8 & .4 \\ .2 & .6 \end{matrix} \times \begin{matrix} .8 \\ .2 \end{matrix} = \begin{matrix} .72 \\ .28 \end{matrix}$$

There is a .72 chance the day after tomorrow will be sunny and a .28 chance the day after tomorrow will be sunny