AN INEQUALITY WITH APPLICATIONS TO STATISTICAL ESTIMATION FOR PROBABILISTIC FUNCTIONS OF MARKOV PROCESSES AND TO A MODEL FOR ECOLOGY

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1. Summary. The object of this note is to prove the theorem below and sketch two applications, one to statistical estimation for (probabilistic) functions of Markov processes [1] and one to Blakley's model for ecology [4].

2. Result.

THEOREM. *Let $P(x)=P(\{x_{ij}\})$ be a polynomial with nonnegative coefficients homogeneous of degree d in its variables $\{x_{ij}\}$. Let $x=\{x_{ij}\}$ be any point of the domain D: $x_{ij}\geqq 0$, $\sum_{j=1}^{q_i} x_{ij}=1$, $i=1,\cdots,p$, $j=1,\cdots,q_i$. For $x=\{x_{ij}\}\in D$ let $\mathfrak{J}(x)=\mathfrak{J}\{x_{ij}\}$ denote the point of D whose i, j coordinate is*

$$\mathfrak{J}(x)_{ij}=\left(x_{ij}\left.\frac{\partial P}{\partial x_{ij}}\right|_{(x)}\right)\bigg/\sum_{j=1}^{q_i} x_{ij}\left.\frac{\partial P}{\partial x_{ij}}\right|_{(x)}.$$

Then $P(\mathfrak{J}(x))>P(x)$ unless $\mathfrak{J}(x)=x$.

Notation. μ will denote a doubly indexed array of nonnegative integers: $\mu=\{\mu_{ij}\}$, $j=1,\cdots,q_i$, $i=1,\cdots,p$. x^μ then denotes $\prod_{i=1}^{p}\prod_{j=1}^{q_i}x_{ij}^{\mu_{ij}}$. Similarly, c_μ is an abbreviation for $c_{\{\mu_{ij}\}}$. The polynomial $P(\{x_{ij}\})$ is then written $P(x)=\sum_\mu c_\mu x^\mu$.

In our notation:

$$\mathfrak{J}(x)_{ij}=\left(\sum_\mu c_\mu\mu_{ij}x^\mu\right)\bigg/\sum_{j=1}^{q_i}\sum_\mu c_\mu\mu_{ij}x^\mu. \tag{1}$$

We wish to prove

$$P(x)=\sum_\mu c_\mu x^\mu\leqq\sum_\mu c_\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\mathfrak{J}(x)_{ij}^{\mu_{ij}}. \tag{2}$$

PROOF.

$$P(x)=\sum_\mu\left\{c_\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\mathfrak{J}(x)_{ij}^{\mu_{ij}}\right\}^{1/d+1}\times\left\{c_\mu^{d/d+1}\,x^\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\left(\frac{1}{\mathfrak{J}(x)_{ij}}\right)^{\mu_{ij}/d+1}\right\}.$$

We apply Hölder's inequality [6, p. 21] to obtain

$$P(x)\leqq\left\{\sum_\mu c_\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\mathfrak{J}(x)_{ij}^{\mu_{ij}}\right\}^{1/d+1}\times\left\{\sum_\mu c_\mu x^\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\left(\frac{x_{ij}}{\mathfrak{J}(x)_{ij}}\right)^{\mu_{ij}/d}\right\}^{d/d+1}. \tag{3}$$

(In the last braces we have used $(x^\mu)^{d+1/d}=x^\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}x_{ij}^{\mu_{ij}/d}$.) Since $\sum_{i=1}^{p}\sum_{j=1}^{q_i}\mu_{ij}/d\equiv 1$ by homogeneity of P, we can apply the inequality of geometric and arithmetic means [6, p. 16] to the double products of the second brace of (3) to conclude:

$$\sum_\mu c_\mu x^\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\left(\frac{x_{ij}}{\mathfrak{J}(x)_{ij}}\right)^{\mu_{ij}/d}\leqq\sum_\mu c_\mu x^\mu\sum_{i=1}^{p}\sum_{j=1}^{q_i}\frac{\mu_{ij}}{d}\,\frac{x_{ij}}{\mathfrak{J}(x)_{ij}}. \tag{4}$$

We now substitute the definition (1) of $\mathfrak{J}(x)_{ij}$ in the expression on the right of (4) and interchange the order of summation to obtain:

$$\begin{aligned}\sum_\mu c_\mu x^\mu\sum_{i=1}^{p}\sum_{j=1}^{q_i}\frac{\mu_{ij}}{d}\,\frac{x_{ij}}{\mathfrak{J}(x)_{ij}}&=\frac{1}{d}\sum_\mu c_\mu x^\mu\sum_{i=1}^{p}\sum_{j=1}^{q_i}\mu_{ij}x_{ij}\\&\quad\cdot\left(\sum_{j_0=1}^{q_i}\sum_{\mu'}c_{\mu'}\mu'_{ij_0}x^{\mu'}\right)\bigg/\left(\sum_{\mu'}c_{\mu'}\mu'_{ij}x^{\mu'}\right)\\&=\frac{1}{d}\sum_{i=1}^{p}\sum_{j=1}^{q_i}x_{ij}\left[\left(\sum_\mu\mu_{ij}c_\mu x^\mu\right)\bigg/\left(\sum_{\mu'}\mu'_{ij}c_{\mu'}x^{\mu'}\right)\right]\\&\quad\cdot\sum_{j_0=1}^{q_i}\sum_{\mu'}c_{\mu'}\mu'_{ij_0}x^{\mu'}.\end{aligned} \tag{5}$$

For each $\langle i,j\rangle$ the expression within the brackets is $=1$ and by hypothesis for each i, $\sum_{j=1}^{q_i}x_{ij}=1$. Hence the whole last expression of (5) reduces to $(1/d)\sum_{i=1}^{p}\sum_{j_0=1}^{q_i}\sum_{\mu'}c_{\mu'}\mu'_{ij_0}x^{\mu'}$. But this is just $(1/d)\sum_{ij_0}x_{ij_0}\cdot(\partial P/\partial x_{ij_0})$ so by the Euler theorem for homogeneous functions it is equal to $\Sigma_\mu c_\mu x^\mu$.

Finally, if we use this upper bound $\Sigma_\mu c_\mu x^\mu$ for the expression within the second braces in (3), raise both sides of (3) to the $(d+1)$st power, and divide both sides of the resulting inequality by $(\Sigma_\mu c_\mu x^\mu)^d$ we obtain the desired inequality (2).

That $P(\mathfrak{J}\{x_{ij}\})>P\{x_{ij}\}$ if $\{x_{ij}\}\neq\{x_{ij}\}$ follows from (4) and the strictness of the inequality of geometric and arithmetic means if all summands are not equal.

3. **Application 1.** The first application of this theorem is to statistical estimation for (probabilistically) lumped Markov chains. Let S be the finite state space of a Markov chain. Let f be a function from S to R. Let $y \in R^T$, T an integer, be an observation. In [1] the problem is considered of estimating the transition probabilities a_{ij} for $i, j \in S$, given y.

Let $X = (f^T)^{-1}(y)$. $X \subseteq S^T$. For $x \in X$, $i, j \in S$, let $\nu_{ij}(x)$ be the number of times the pattern $\cdot, \cdot, \cdot, i, j, \cdot, \cdot, \cdot$ occurs in x. The function $P(\{a_{ij}\}) = \sum_{x \in X} \prod_{i,j \in S} a_{ij}^{\nu_{ij}(x)}$ may be interpreted as the "probability of observing y given the transition probabilities $\{a_{ij}\}$." Note that P is a homogeneous polynomial of degree T with nonnegative (integer) coefficients in the variables a_{ij}.

An iterative procedure for estimating the transition probabilities $\{a_{ij}\}$ given y is suggested in [1]. If $\{a_{ij}\}$ is an *a priori* estimate, let $A'_{ij} = (\sum_{x \in X} \nu_{ij}(x) \prod_{k,l \in S} a_{kl}^{\nu_{kl}(x)} / P(\{a_{ij}\})$. A'_{ij} may be interpreted as the "*a posteriori* expected value of the frequency of transition from state i to state j given y and the *a priori* probabilities $\{a_{ij}\}$." Thus $A'_{ij}/\Sigma_j A'_{ij}$ may be thought of as an "*a posteriori* estimate of the transition probabilities given y." Since

$$A'_{ij}/\Sigma_j A'_{ij} = a_{ij}(\partial P/\partial a_{ij})/\Sigma_j a_{ij}(\partial P/\partial a_{ij})$$

by our theorem applied to the transformation $\mathfrak{I}\{a_{ij}\} = \{A'_{ij}/\Sigma A'_{ij}\}$ we conclude that $P(\mathfrak{I}\{a_{ij}\}) \geqq P(\{a_{ij}\})$. In other words the *a posteriori* estimate of transition probabilities increases the likelihood of the given observation y.

Various results on the convergence of hill climbing iteration procedures [2], [3], [5] may be adduced to show that for almost all starts successive iterations will converge to a connected component of the local maximum set of P. If P has only finitely many local maxima then successive iterates converge to a point.

This is the usual case in the more general situation considered in [1] in which the observation y_t at time t is obtained from the Markov state x_t at time t according to $P(y_t = k \mid x_t = j) = b_{jk}$ where b_{jk} is an $s \times r$ stochastic matrix which is also to be estimated. Here the identifiability problem does not arise since, according to a theorem of Ted Petrie [7], "in general" no other (a_{ij}), (b_{jk}) yields the same y probabilities as a given (a^0_{ij}), (b^0_{jk}) (save for the $s!$ relabellings of states).

The second application is to some results of Blakley and Dixon [2], [3], [4]. Let Γ be a symmetric p-linear form on R^n that has nonnegative coefficients with respect to the standard basis for R^n. Let $g(\eta) = \Gamma(\eta, \eta, \cdots, \eta)$ where η is a vector in R^n. Since g is then just a pth degree homogeneous polynomial with nonnegative coefficients of the components of η we may apply the theorem of this note to it. In Blakley's model g is the adaptation (rate of growth) of a population. The transformation in Blakley's model $\sigma(\eta) = \eta_i(\partial g(\eta)/\partial \eta_i)/pg(\eta)$ is the same as the transformation $\mathfrak{I}\{x_{ij}\}$ where $x_{ij} = \eta_j$, $i = 1$, $j = 1, \cdots, n$.

In Blakley's model if η is the distribution of genotypes at time t, then $\sigma(\eta)$ is the distribution at time $t+1$. Thus it follows from the theorem in this note with $i = 1$ that adaptation is nondecreasing with time when evolution of the genotypes at a single locus is considered. Our theorem with $i > 1$ yields the same conclusion under natural hypotheses for evolution of the genotypes at several loci. This nondecreasing of the adaptation with time is clearly a desirable feature of the model.

References

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