Estimation of Signal Parameters via Rotational Invariance Techniques[1]– *ESPRIT*

A. PAULRAJ, R. ROY AND T. KAILATH

Information Systems Laboratory
Stanford University
Stanford, CA 94305

Abstract—**A new subspace approach (*ESPRIT*) to signal parameter estimation is described. The technique is discussed in the context of direction-of-arrival estimation, though it can be applied to a wide variety of problems. *ESPRIT* exploits an underlying rotational invariance among signal subspaces induced by an array of sensors with a translational invariance structure (*e.g.*, pairwise matched and co-directional antenna element doublets). The new approach has several remarkable advantages over earlier techniques such as MUSIC and also provides asymptotically unbiased and efficient estimates. Results of computer simulations carried out to evaluate the new algorithm are also presented.**

I. INTRODUCTION

High resolution direction-of-arrival (DOA) estimation is important in many sensor systems such as radar, sonar, electronic surveillance measures (ESM), *etc.* Over the years, several methods have been proposed such as the so-called maximum likelihood (ML) method due to Capon (1969) [1], the maximum entropy (ME) method due to Burg (1967) [2], and others. These methods were recently overshadowed by the signal subspace method (also called MUSIC or the eigenstructure method) due to Schmidt (1979, 1981) [3,4] and Bienvenu (1979) [5]. Several variations of these basic approaches have also been studied by several researchers (*cf.* Haykin (1984) [6] for a discussion). Among all the methods proposed to date, MUSIC is known to yield the best results in most practically interesting situations (*cf.* [7]), results which are known to be asymptotically[2] unbiased and efficient.

MUSIC derives its properties from exploitation of the underlying model of finite (low) rank signals (*e.g.*, spatially coherent wavefronts) in additive noise, a situation typical in many sensor array environments. The MUSIC algorithm first determines the so-called *signal subspace* from the array measurements. Introducing the concept of an array manifold as the set of all possible array responses as functions of the parameter(s) to be estimated, intersections between this manifold and the estimated signal subspace are then sought. This search is typically carried out by computing a weighted norm (Hermitian form) using the direction vectors for each angle of interest and a kernel obtained from the so-called noise eigenvectors of the data covariance matrix. Essentially the same computation also underlies the earlier methods (ML, ME, *etc.*) with the only difference being in the choice of norms (kernels). Note that this search procedure requires a complete knowledge of the array geometry and sensor element characteristics (the array response or direction vectors) and is also the reason for the heavy computational burden of such algorithms.

In this paper, a new approach (***ESPRIT***) to the signal parameter estimation problem is presented (*cf.* [8]). It is similar to MUSIC in that it exploits the underlying signal and noise models and generates estimates that are asymptotically exact and efficient. In addition, it has many important advantages over MUSIC.

- The algorithm does not require knowledge of the array geometry and element characteristics (*i.e.*, directional pattern or gain/phase response).
- It is computationally much less complex because it does not need the search procedure inherent in other algorithms.
- It does not require a calibration of the array (a difficult if not sometimes impossible task), therefore completely eliminating the need for the associated storage of the array manifold which can become very large for multidimensional problems.
- The method *simultaneously* estimates the number of sources and DOA's.

However, restrictions to planar wavefronts and pairwise matched co-directional doublets are present, and in this sense, ***ESPRIT*** is not completely general.

[1]This work was supported in part by Joint Services Program at Stanford University under Contract DAAG29-81-K-0057. and by the Office of Naval Research under Contract N00014-85-K-0550.

[2]Herein *asymptotically* refers to the amount of data or the signal-to-noise ratio (SNR) approaching infinity.

II. Problem Formulation

The basic problem under consideration is that of estimation of parameters of finite dimensional signal processes given measurements from an array of sensors. This general problem appears in many different fields including radio astronomy, geophysics, sonor signal processing, electronic surveillance, structural (vibration) analysis, temporal frequency estimation, *etc.* In order to simplify the description of the basic ideas behind ***ESPRIT***, the ensuing discussion is couched in terms of the problem of multiple source direction-of-arrival (DOA) estimation from data collected by an array of sensors. Though easily generalized to higher dimensional parameter spaces, the discussion and results presented deal only with single dimensional parameter spaces, *i.e.,* azimuth only direction finding (DF) of far-field point sources. Furthermore, narrowband signals of known center frequency will be assumed. A DOA/DF problem is classified as *narrowband* if the signal bandwidth is small compared to the inverse of the transit time of a wavefront across the array. The generality of the fundamental concepts on which ***ESPRIT*** is based makes the extension to signals containing multiple frequencies straightforward as discussed later. Note that wideband signals can also be handled by decomposing them into narrowband signal sets using comb filters.

Consider a planar array of arbitrary geometry composed of m matched sensor doublets whose elements are translationally separated by a known constant displacement vector as shown in Figure 1. The element characteristics such as element gain and phase pattern, polarization sensitivity, *etc.*, may be arbitrary for each doublet as long as the elements are *pairwise identical.* Assume there are $d < m$ narrowband stationary zero-mean sources centered at frequency ω_0, and located sufficiently far from the array such that in homogenous isotropic transmission media, the wavefronts impinging on the array are planar. Additive noise is present at all the $2m$ sensors and is assumed to be a stationary zero-mean random process that is uncorrelated from sensor to sensor.

In order to exploit the translational invariance property of the sensor array, it is convenient to describe the array as being comprised of two subarrays, X and Y, identical in every respect although physically displaced (not rotated) from each other by a known displacement vector. The signals received at the i^{th} doublet can then be expressed as:

$$\begin{aligned} x_i(t) &= \sum_{k=1}^{d} s_k(t) a_i(\theta_k) + n_{x_i}(t) \\ y_i(t) &= \sum_{k=1}^{d} s_k(t) e^{j\omega_0 \Delta \sin\theta_k / c} a_i(\theta_k) + n_{y_i}(t) \end{aligned} \tag{1}$$

where $s_k(\cdot)$ is the k^{th} signal (wavefront) as received at sensor 1 (the reference sensor) of the X subarray, θ_k is

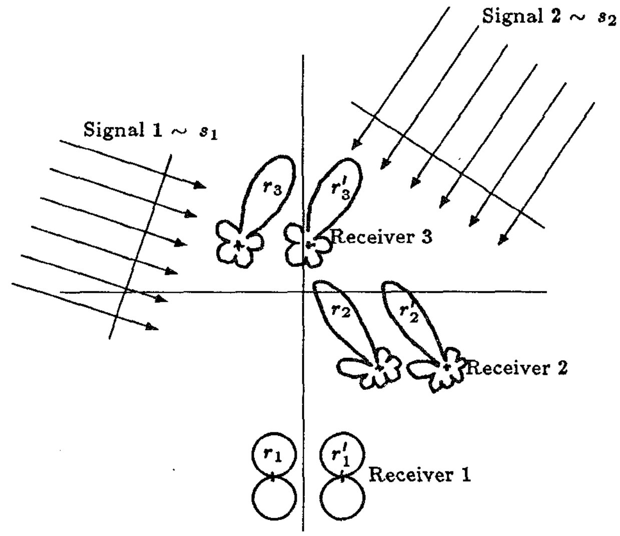

Figure 1: Multiple Source DOA Estimation using *ESPRIT*

the direction of arrival of the k^{th} source relative to the direction of the translational displacement vector, $a_i(\theta_k)$ is the response of the i^{th} sensor of either subarray relative to its response at sensor 1 of the same subarray when a single wavefront impinges at an angle θ_k, Δ is the magnitude of the displacement vector between the two arrays, c is the speed of propagation in the transmission medium, $n_{x_i}(\cdot)$ and $n_{y_i}(\cdot)$ are the additive noises at the elements in the i^{th} doublet for subarrays X and Y respectively.

Combining the outputs of each of the sensors in the two subarrays, the received data vectors can be written as follows:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}\mathbf{s}(t) + \mathbf{n}_x(t) \;, \\ \mathbf{y}(t) &= \mathbf{A}\Phi\mathbf{s}(t) + \mathbf{n}_y(t) \;; \end{aligned} \tag{2}$$

where:

$$\begin{aligned} \mathbf{x}^T(t) &= [x_1(t), \ldots, x_m(t)], \\ \mathbf{n}_x^T(t) &= [n_{x_1}(t), \ldots, n_{x_m}(t)], \end{aligned} \tag{3}$$

and $\mathbf{y}(t)$ and $\mathbf{n}_y(t)$ are similarly defined. The vector $\mathbf{s}(t)$ is a $d \times 1$ vector of impinging signals (wavefronts) as observed at the reference sensor of subarray X. The matrix Φ is a diagonal $d \times d$ matrix of the phase delays between the doublet sensors for the d wavefronts, and can be written as:

$$\Phi = diag\left[e^{j\phi_1}, \ldots, e^{j\phi_d}\right]; \quad \phi_k = \omega_0 \Delta \sin\theta_k / c \,. \tag{4}$$

Note that Φ is a unitary matrix (operator) that relates the measurements from subarray X to those from subarray Y. In the complex field, Φ is a simple scaling operator. However, it is isomorphic to the real two-dimensional rotation operator and is herein referred to as a *rotation* operator. The $m \times d$ matrix $\mathbf{A}$ is the *direction matrix* whose columns $\{\mathbf{a}(\theta_k),\ k = 1, \ldots, d\}$ are the signal *direction vectors* for the d wavefronts.

$$\mathbf{a}^T(\theta_k) = [a_1(\theta_k), \ldots, a_m(\theta_k)]. \quad (5)$$

The auto-covariance of the data received by subarray X is given by:

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}^*(t)] = \mathbf{ASA}^* + \sigma^2\mathbf{I}\,, \quad (6)$$

where $\mathbf{S}$ is the $d \times d$ covariance matrix of the signals $\mathbf{s}(t)$, *i.e.*,

$$\mathbf{S} = E[\mathbf{s}(t)\mathbf{s}^*(t)]\,, \quad (7)$$

and σ^2 is the covariance of the additive uncorrelated white noise that is present at all sensors. Note that $(\cdot)^*$ is used herein to denote the Hermitean conjugate, or complex conjugate transpose operation. Similarly, the cross-covariance between measurements from subarrays X and Y is given by:

$$\mathbf{R}_{xy} = E[\mathbf{x}(t)\mathbf{y}^*(t)] = \mathbf{AS\Phi^*A}^*\,. \quad (8)$$

This completes the definition of the signal and noise model, and the problem can now be stated as follows: *Given measurements* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$, *and making no assumptions about the array geometry, element characteristics, DOA's, noise powers, or the signal (wavefront) correlation, estimate the signal DOA's.*

III. Invariant Subspace Approach

The basic idea behind the new technique is to exploit the rotational invariance of the underlying signal subspaces induced by the translational invariance of the sensor array. The following theorem provides the foundation for the results presented herein.

Theorem: *Define* Γ *as the generalized eigenvalue matrix associated with the matrix pencil* $\{(\mathbf{R}_{xx} - \lambda_{\min}\mathbf{I}), \mathbf{R}_{xy}\}$ *where* $\lambda_{\min}$ *is the minimum (repeated) eigenvalue of* $\mathbf{R}_{xx}$. *Then, if* $\mathbf{S}$ *is nonsingular, the matrices* Φ *and* Γ *are related by*

$$\Gamma = \begin{bmatrix} \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (9)$$

to within a permutation of the elements of Φ.

Proof: First it is shown that $\mathbf{ASA}^*$ is rank d and $\mathbf{R}_{xx}$ has a multiplicity $(m-d)$ of eigenvalues all equal to σ^2. From linear algebra,

$$\rho(\mathbf{ASA}^*) = \min(\rho(\mathbf{A}), \rho(\mathbf{S})) \quad (10)$$

where $\rho(\cdot)$ denotes the rank of the matrix argument. Assuming that the array geometry is such that there are no ambiguities (at least over the angular interval where signals are expected), the columns of the $m \times d$ matrix $\mathbf{A}$ are linearly independent and hence $\rho(\mathbf{A}) = d$. Also, since $\mathbf{S}$ is a $d \times d$ matrix and is nonsingular, $\rho(\mathbf{S}) = d$. Therefore, $\rho(\mathbf{ASA}^*) = d$, and consequently $\mathbf{ASA}^*$ will have $m - d$ zero eigenvalues. Equivalently $\mathbf{ASA}^* + \sigma^2\mathbf{I}$ will have $m - d$ minimum eigenvalues all equal to σ^2. If $\{\lambda_1 > \lambda_2 > \ldots > \lambda_m\}$ are the ordered eigenvalues of $\mathbf{R}_{xx}$, then

$$\lambda_{d+1} = \ldots = \lambda_m = \sigma^2\,. \quad (11)$$

Hence,

$$\mathbf{R}_{xx} - \lambda_{\min}\mathbf{I} = \mathbf{R}_{xx} - \sigma^2\mathbf{I} = \mathbf{ASA}^*\,. \quad (12)$$

Now consider the matrix pencil

$$\mathbf{C}_{xx} - \gamma\mathbf{R}_{xy} = \mathbf{ASA}^* - \gamma\mathbf{AS\Phi^*A}^* = \mathbf{AS}(\mathbf{I} - \gamma\Phi^*)\mathbf{A}^*\,; \quad (13)$$

where $\mathbf{C}_{xx} \doteq \mathbf{R}_{xx} - \lambda_{\min}\mathbf{I}$. By inspection, the column space of both $\mathbf{ASA}^*$ and $\mathbf{AS\Phi^*A}^*$ are identical. Therefore, $\rho(\mathbf{ASA}^* - \gamma\mathbf{AS\Phi^*A}^*)$ will in general be equal to d. However, if

$$\gamma = e^{j\omega_0\Delta\sin\theta_i/c}, \quad (14)$$

the i^{th} row of $(\mathbf{I} - e^{j\omega_0\Delta\sin\theta_i/c}\Phi^*)$ will become zero. Thus,

$$\rho(\mathbf{I} - e^{j\omega_0\Delta\sin\theta_i/c}\Phi^*) = d - 1\,. \quad (15)$$

Consequently, the pencil $(\mathbf{C}_{xx} - \gamma\mathbf{R}_{xy})$ will also decrease in rank to $d-1$ whenever γ assumes values given by (14). However, by definition these are exactly the *generalized eigenvalues* (GE's) of the matrix pair $\{\mathbf{C}_{xx}, \mathbf{R}_{xy}\}$. Also, since both matrices in the pair span the same subspace, the GE's corresponding to the common null space of the two matrices will be zero, *i.e.*, d GE's lie on the unit circle and are equal to the diagonal elements of the rotation matrix Φ, and the remaining $m-d$ GE's are at the origin. This completes the proof of the theorem.

Once Φ is known, the DOA's can be calculated from:

$$\theta_k = \arcsin\{c\phi_k/\omega_0\Delta\}\,. \quad (16)$$

Due to errors in estimating $\mathbf{R}_{xx}$ and $\mathbf{R}_{xy}$ from finite data as well as errors introduced during the subsequent finite precision computations, the relations in (9) and (11) will not be exactly satisfied. At this point, a procedure is proposed which is not globally optimal, but utilizes some well established, stepwise-optimal techniques to deal with such issues.

IV. Subspace Rotation Algorithm

The key steps of the covariance matrix formulation of *ESPRIT* are:

1. Find the auto- and cross-covariance matrix estimates $\hat{\mathbf{R}}_{xx}$ and $\hat{\mathbf{R}}_{xy}$ from the data.

2. Compute the eigen-decomposition of $\hat{\mathbf{R}}_{xx}$ and $\hat{\mathbf{R}}_{xy}$ and then estimate the number of sources $\hat{d}$ and the noise variance $\hat{\sigma}^2$.

3. Compute rank $\hat{d}$ approximations to $\mathbf{ASA^*}$ and $\mathbf{AS\Phi^*A^*}$ given $\hat{\sigma}^2$.

4. The d GE's of the estimates of $(\mathbf{ASA^*}, \mathbf{AS\Phi^*A^*})$ that lie close to the unit circle determine the subspace rotation operator Φ and hence, the DOA's.

Details of the covariance algorithm are now discussed, and after some remarks on the implementation, a discussion of the use of generalized singular value decompositions of data matrices to solve for the signal parameters is given as well.

Covariance Estimation

In order to estimate the required covariances, observations $\mathbf{x}(t_j)$ and $\mathbf{y}(t_j)$ at time instants t_j are required; the subarrays must be sampled simultaneously. Maximum likelihood estimates of the auto- and cross-covariance matrices are then given by

$$\begin{aligned} \hat{\mathbf{R}}_{xx} &= \tfrac{1}{N}\textstyle\sum_{j=1}^{N}\mathbf{x}(t_j)\mathbf{x}^*(t_j) \\ \hat{\mathbf{R}}_{xy} &= \tfrac{1}{N}\textstyle\sum_{j=1}^{N}\mathbf{x}(t_j)\mathbf{y}^*(t_j)\,. \end{aligned} \tag{17}$$

The number of *snapshots*, N, needed for an adequate estimate of the covariance matrices depends upon the signal-to-noise ratio at the array input and the desired accuracy of the DOA estimates. In the absense of noise, $N > d$ is required in order to completely *span* the signal subspaces. In the presence of noise, it has been shown by Brennan and Reed [9] that N must be at least m^2.

Estimating d and σ^2

Due to errors in $\hat{\mathbf{R}}_{xx}$, its eigenvalues will be perturbed from their true values and the true multiplicity of the minimal eigenvalue may not be evident. A popular approach for determining the underlying eigenvalue multiplicity is an information theoretic method based on the minimum description length (MDL) criterion proposed by Wax and Kailath (1985) [10]. The estimate of the number of sources d is given by the value of k for which the following MDL function is minimized:

$$\mathrm{MDL}(k) = -\log\left\{\frac{\prod_{i=k+1}^{m}\hat{\lambda}_i^{\frac{1}{m-k}}}{\frac{1}{m-k}\sum_{i=k+1}^{m}\hat{\lambda}_i}\right\}^{(m-k)N} + \frac{k}{2}(2m-k)\log N\,; \tag{18}$$

where $\hat{\lambda}_i$ are the eigenvalues of $\mathbf{R}_{xx}$. The MDL criterion is known to yield asymptotically consistent estimates.

Having obtained an estimate of d, the maximum likelihood estimate of σ^2 conditioned on $\hat{d}$ is given by the average of the smallest $m - \hat{d}$ eigenvalues *i.e.,*

$$\hat{\sigma}^2 = \frac{1}{m-d}\sum_{i=d+1}^{m}\hat{\lambda}_i\,. \tag{19}$$

Estimating $\mathbf{ASA^*}$ *and* $\mathbf{AS\Phi^*A^*}$

Using the results from the previous step, and making no assumptions about the array geometry, the maximum likelihood estimate $\hat{\mathbf{C}}_{xx}$ of $\mathbf{ASA^*}$, conditioned on $\hat{d}$ and $\hat{\sigma}^2$, is the maximum Frobenius norm (F-norm) rank d approximation of $\hat{\mathbf{R}}_{xx} - \hat{\sigma}^2\mathbf{I}$, [11];

$$\hat{\mathbf{C}}_{xx} = \sum_{i=1}^{d}(\hat{\lambda}_i - \hat{\sigma}^2)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_i^*\,; \tag{20}$$

where; $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ are the eigenvectors corresponding to the ordered eigenvalues of $\hat{\mathbf{R}}_{xx}$.

Similarly, given $\hat{\mathbf{R}}_{xy}$ and $\hat{d}$, the maximum likelihood estimate $\mathbf{AS\Phi^*A^*}$ is the maximum F-norm rank d approximation of $\hat{\mathbf{R}}_{xy}$

$$\mathbf{AS\Phi^*A^*} = \sum_{i=1}^{d}\hat{\lambda}_i^{xy}\hat{\mathbf{e}}_i^{xy}\hat{\mathbf{e}}_i^{xy^*}\,; \tag{21}$$

where, $\{\hat{\lambda}_1^{xy} > \hat{\lambda}_2^{xy} > \ldots > \hat{\lambda}_m^{xy}\}$ and $\{\hat{\mathbf{e}}_1^{xy}, \hat{\mathbf{e}}_2^{xy}, \ldots, \hat{\mathbf{e}}_m^{xy}\}$ are the eigenvalues and the eigenvectors of $\hat{\mathbf{R}}_{xy}$.

The information in $\hat{\mathbf{R}}_{xx}$ and $\hat{\mathbf{R}}_{xy}$ can be jointly exploited to improve the estimates of the underlying subspace and therefore of the estimates of $\mathbf{ASA^*}$ and $\mathbf{AS\Phi^*A^*}$. In situations where the array geometry (*i.e.,* the manifold on which the columns of $\mathbf{A}$ lie) is known, these estimates can be further improved, but this is not pursued here since no knowledge of the array geometry is assumed.

Estimating Directions of Arrival

The estimates of the DOA's now follow by computing the the m GE's of the matrix pair $\mathbf{ASA^*}$ and $\mathbf{AS\Phi^*A^*}$. This is a singular generalized eigen-problem and needs more care than the regular case to obtain stable estimates of the GE's (*cf.* Demmel (1985) [12] for a discussion on preferred methods). Note that since the subspaces spanned by the two matrix estimates cannot be expected to be identical, the $m-d$ *noise* GE's will not be zero. Furthermore, the *signal* GE's will not lie exactly on the unit circle. In practice, d GE's will lie close to the unit circle and the remaining $m-d$ GE's well inside and close to the origin. The d values near the unit circle are the desired estimates of Φ_{kk}, and (16) can be used to obtain estimates of the source directions.

V. Some Remarks

Estimation of the Number of Signals

In the algorithm detailed above, an estimate of the number of sources $\hat{d}$ is obtained as one of the first steps in the algorithm. This estimate is then used in subsequent steps as the rank of the approximations to covariance matrices. This approach has the disadvantage that an error (particularly underestimation) in determining d may result in severe biases in the final DOA estimates. Therefore, if an estimator for σ^2 can be found which is independent of d (*e.g.,* $\hat{\sigma}^2 = \hat{\lambda}_{\min}$), estimation of d and the DOA's can be performed simultaneously. Simulation results have shown that the estimates of Φ have low sensitivity to errors in estimating σ^2. This implies that the rank d estimates of $\mathbf{ASA}^*$ and $\mathbf{AS\Phi^*A}^*$ can be dispensed with and the GEV's computed directly from the matrix pair $\{\hat{\mathbf{R}}_{xx} - \hat{\sigma}^2\mathbf{I}, \hat{\mathbf{R}}_{xy}\}$. This ability to simultaneously estimate d and the parameters of interest is another advantage of ***ESPRIT*** over MUSIC.

Extensions to Multiple Dimensions

The discussion hitherto has considered only single dimensional parameter estimation. To extend ***ESPRIT*** to multidimensional parameter vectors, measurements must be made by arrays manifesting the the shift invariant structure in the appropriate dimension. For example, co-directional sensor doublets are used to estimate DOA's in a plane (*e.g.,* azimuth) containing the doublet axes. Elevation angle is unobservable with such an array as a direct consequence of the rotational symmetry about the reference direction defined by the doublet axes (*cf. cones* of ambiguity). To obtain elevation estimates, another pair of subarrays sensitive to elevation angle is necessary. Geometrically, this provides an independent set of *cones*, and the intersections of the two sets of cones yield the desired estimates. Note that the parameter estimates (*e.g.,* azimuth and elevation) can be calculated independently. This results in the computational load in ***ESPRIT*** growing *linearly* with the dimension of the signal parameter vector, whereas in MUSIC it increases *exponentially.*

If the signals impinging on the array are not monochromatic, but are composed of sums of cisoids of fixed frequencies, ***ESPRIT*** can also estimate the frequencies. This requires temporal (doublet) samples which can be obtained for example by adding a uniform tapped delay line ($p+1$ taps) behind each sensor. The frequencies estimates are obtained (independent of the DOA estimates) from the $mp \times mp$ auto- and cross-covariance matrices of two (temporally) displaced data sets.

Array Ambiguities

Array ambiguities are discussed below in the context of DOA estimation, but can be extended to other problems as well. Ambiguities in ***ESPRIT*** arise from two sources. First, ***ESPRIT*** inherits the ambiguity structure of a single doublet, independent of the global geometry of the array. Any distribution of co-directional doublets contains a *symmetry* axis, the doublet axis. Even though the individual sensor elements may have directivity patterns which are functions of the angle in the *other* dimension (*e.g.,* elevation), for a given elevation angle the directional response of each element in any doublet is the same, and the phase difference observed between the elements of any doublet depends only on the *azimuthal* DOA. The MUSIC algorithm, on the other hand, can (generally) determine azimuth and elevation without ambiguity given this geometry since knowledge of the directional sensitivites of the individual sensor elements is assumed.

Other doublet related ambiguities can also arise if the sensor spacing within the doublets is larger than $\lambda/2$. In this case, ambiguities are generated at angles $\arcsin\{\lambda(\Phi_{ii} \pm 2n\pi)/2\pi\Delta\}, n = 0, 1, \ldots$, a manifestation of undersampling and the aliasing phenomenon.

ESPRIT is also heir to the subarray ambiguities usually classified in terms of first-order, second-order, and higher order ambiguities of the array manifold (*cf.* Schmidt [4] for a detailed discussion). These ambiguities manifest themselves in the same manner as in MUSIC where they bring about a collapse of the signal subspace dimensionality.

VI. Algorithm Extensions

There are parameters other than DOA's and temporal frequencies that are often of interest in array processing problems. Extensions of ***ESPRIT*** to provide such estimates are described below. ***ESPRIT*** can also be easily extended to solve the signal copy problem, a problem which is of particular interest in communications applications.

Array Response Vector Estimation

Let $\mathbf{e}_i$ be the generalized eigenvector (GEV) corresponding to the generalized eigenvalue (GE) γ_i. By definition, $\mathbf{e}_i$ satisfies the relation

$$\mathbf{AS}(\mathbf{I} - \gamma_i\Phi^*)\mathbf{A}^*\mathbf{e}_i = 0\,. \tag{22}$$

Since the column space of the pencil $\mathbf{AS}(\mathbf{I} - \gamma_i\Phi^*)\mathbf{A}^*$ is same as the subspace spanned by the vectors $\{\mathbf{a}_j, j \neq i\}$, it follows that $\mathbf{e}_i$ is orthogonal to all direction vectors, except $\mathbf{a}_i$. Assuming for now that the sources are uncorrelated, *i.e.,* , $\mathbf{S} = diag[\sigma_1^2, \ldots, \sigma_d^2]$. Multiplying $\mathbf{C}_{xx}$ by $\mathbf{e}_i$ yields the desired result:

$$\begin{aligned} \mathbf{C}_{xx}\mathbf{e}_i &= \mathbf{AS}[0, \ldots, 0, \mathbf{a}_i^*\mathbf{e}_i, 0, \ldots, 0]^T \\ &= \mathbf{a}_i(\sigma_i^2\mathbf{a}_i^*\mathbf{e}_i) \\ &= scalar \times \mathbf{a}_i\,. \end{aligned} \tag{23}$$

The result can be normalized to make the response at sensor 1 equal to unity, yielding:

$$\mathbf{a}_i = \frac{\mathbf{C}_{xx}\mathbf{e}_i}{\mathbf{u}^T\mathbf{C}_{xx}\mathbf{e}_i}\,, \tag{24}$$

where $\mathbf{u} = [1, 0, 0, \ldots, 0]^T$.

Source Power Estimation

Assuming that the estimated array response vectors have been normalized as described above (*i.e.*, unity response at sensor 1), the source powers follow from (23):

$$\sigma_i^2 = \frac{|\mathbf{u}^T \mathbf{C}_{xx} \mathbf{e}_i|^2}{\mathbf{e}_i^* \mathbf{C}_{xx} \mathbf{e}_i}. \tag{25}$$

Note that these estimate are only valid if sensor 1 is omnidirectional, *i.e.*, has the same response to a given source in all directions. If this is not the case, the estimates will be in error.

Signal Copy (SC)

Signal copy refers to the weighted combination of the sensor measurements such that the output contains the desired signal while completely rejecting the other $d-1$ signals. From (22), $\mathbf{e}_i$ is orthogonal to all wavefront direction vectors except the i^{th} wavefront, and is therefore the desired weight vector for signal copy of the i^{th} signal. Note that this is true even for correlated signals. If a unit response to the desired source is required, once again the assumption of a unit response at sensor 1 to this source becomes necessary. The weight vector is now a scaled version of $\mathbf{e}_i$ and using the constraint $\mathbf{a}_i^* \mathbf{w}_i^{SC} = 1$ can be shown to be

$$\mathbf{w}_i^{SC} = \mathbf{e}_i \left\{ \frac{|\mathbf{u}^T \mathbf{C}_{xx} \mathbf{e}_i|}{\mathbf{e}_i^* \mathbf{C}_{xx} \mathbf{e}_i} \right\}. \tag{26}$$

In the presence of correlated signals as often arises in situations where multipath is present, it is useful to combine the information in the various wavefronts (paths). This leads to a maximum likelihood (ML) beamformer [13] which is given by:

$$\mathbf{w}_i^{ML} = \mathbf{R}_{xx}^{-1} \mathbf{C}_{xx} \mathbf{e}_i. \tag{27}$$

In the absence of noise, $\mathbf{R}_{xx} = \mathbf{C}_{xx}$ and $\mathbf{w}_i^{ML} = \mathbf{w}_i^{SC}$. Similarly, optimum weight vectors for other types of beamformers can be determined.

VII. A Generalized SVD Approach

The details of the computations in *ESPRIT* presented so far have been based upon auto- and cross-covariances of the subarray sensor data. Since the a step in the algorithm involves determining the GE's of a singular matrix pair, it is preferable to avoid using covariance matrices, choosing instead to operate directly on the *data*. This approach leads to a generalized singular value decomposition (GSVD) of data matrices.

Let $\mathbf{X}$ and $\mathbf{Y}$ be $m \times N$ data matrices containing N simultaneous snapshots $\mathbf{x}(t)$ and $\mathbf{y}(t)$ respectively;

$$\begin{aligned} \mathbf{X} &= [\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_N)], \\ \mathbf{Y} &= [\mathbf{y}(t_1), \mathbf{y}(t_2), \ldots, \mathbf{y}(t_N)]. \end{aligned} \tag{28}$$

The GSVD of the matrix pair $(\mathbf{X}, \mathbf{Y})$ is given by:

$$\mathbf{X} = \mathbf{U}_X \Sigma_X \mathbf{V}^*; \quad \mathbf{Y} = \mathbf{U}_Y \Sigma_Y \mathbf{V}^*; \tag{29}$$

where $\mathbf{U}_X$ and $\mathbf{U}_Y$ are the $m \times m$ unitary matrices containing the left generalized singular vectors (LGSV's), Σ_X and Σ_Y are $m \times N$ real rectangular matrices that have zero entries everywhere except on the main diagonal (whose pairwise ratios are the generalized singular values), and $\mathbf{V}$ is a nonsingular matrix.

Assuming for a moment that there is no additive noise, both $\mathbf{X}$ and $\mathbf{Y}$ will be rank d. Now consider the pencil

$$\mathbf{X} - \gamma \mathbf{Y} = \mathbf{A}(\mathbf{I} - \gamma \Phi)[\mathbf{s}(t_1), \ldots, \mathbf{s}(t_N)]. \tag{30}$$

Similar to previous discussions, whenever $\gamma = \Phi_{ii}^*$, this pencil will decrease in rank from d to $d-1$. Now consider the same pencil written in terms of its GSVD:

$$\begin{aligned} \mathbf{X} - \gamma \mathbf{Y} &= (\mathbf{U}_X \Sigma_X - \gamma \mathbf{U}_Y \Sigma_Y) \mathbf{V}^*, \\ &= \mathbf{U}_X \Sigma_X (\mathbf{I} - \gamma \Sigma_X^{-1} \mathbf{U}_X^* \mathbf{U}_Y \Sigma_Y) \mathbf{V}^*. \end{aligned} \tag{31}$$

This pencil will loose rank whenever γ^* is an eigenvalue of $(\Sigma_X^{-1} \mathbf{U}_X^* \mathbf{U}_Y \Sigma_Y)$. Therefore, the desired Φ_{ii} are the eigenvalues of the product $\Sigma_X^{-1} \mathbf{U}_X^* \mathbf{U}_Y \Sigma_Y$. However, from the underlying model in (1) and (2), it can be shown that in the absence of noise $\Sigma_X = \Sigma_Y$, in which case Φ_{ii} are also the eigenvalues of $\mathbf{U}_X^* \mathbf{U}_Y$.

In presence of additive white sensor noise, we can show that asymptotically (*i.e.*, for large N) the GSVD of the data matrices converges to the GSVD obtained in the noiseless case except that Σ_X and Σ_Y are augmented by $\sigma^2 \mathbf{I}$. Therefore, the LGSV matrices in the presence of noise are asymptotically equal to $\mathbf{U}_X$ and $\mathbf{U}_Y$ computed in the absence of noise, and the earlier result is still applicable. Estimates for other model parameters as discussed previously can be computed in a similar manner (*cf.* [14]).

VIII. Simulation Results

Computer simulations were carried out to verify the performance of *ESPRIT*. The simulation consisted of an array with 8 doublets. The elements in each of the doublets were spaced a quarter of a wavelength apart. The array geometry was generated by randomly scattering the doublets on a line 10 wavelengths in length such that the doublet axes were all parallel to the line. Three planar and weakly correlated signal wavefronts impinged on the array at angles 20°, 22°, and 60°, with SNRs of 10, 13 and 16 db relative to the additive uncorrelated noise present at the sensors. The covariance estimates were computed from 100 snapshots of data and several simulation runs were made using independent data sets.

The covariance version of *ESPRIT* was implemented with $\hat{\lambda}_{\min}$ being used as an estimate of σ^2. Figure 2 shows a plot of the GE's obtained from 10 runs. The GE's on the unit circle are closely clustered and the two sources 2° apart are easily resolved.

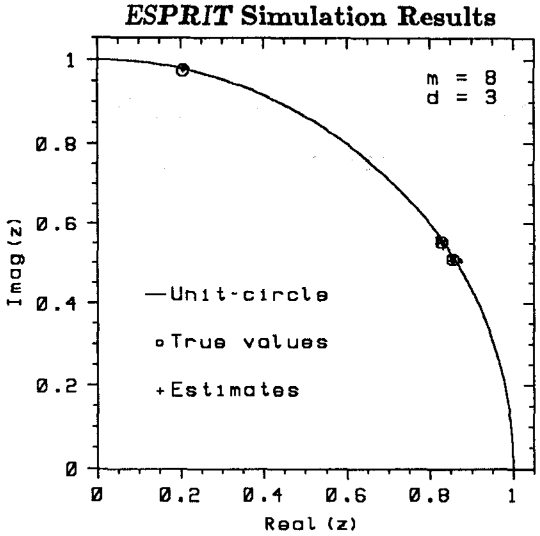

Figure 2: GE Estimates using *ESPRIT*

IX. CONCLUDING REMARKS

In this paper, a new algorithm for signal parameter estimation of finite dimensional low-rank signals received by an array having a known translational invariance has been described. The method shows considerable promise and has several advantages over previous algorithms including speed, storage, and indifference to array calibration. For example, with a 20 element array covering an arc of 2 radians with a one milliradian resolution in both azimuth and elevation, ***ESPRIT*** has a computational advantage on the order of 10^5 over MUSIC. Furthermore, while MUSIC needs about 20 megabytes of storage for the array manifold (using 16 bit words), ***ESPRIT*** requires no storage. Its ability to work without array calibration is also very attractive in applications such as space antennas, sonobuoys, *etc.*where the array geometry may not be known. The new technique has the potential to make high resolution DOA estimation, signal copy, *etc.*feasible in the sense of being simpler and cheaper to implement in many applications.

REFERENCES

[1] J. Capon, High resolution frequency wave number spectrum analysis, *Proc. IEEE*, 57:1408–1418, 1969.

[2] J. P. Burg, Maximum entropy spectral analysis, In *Proceedings of the* 37^{th} *Annual International SEG Meeting*, Oklahoma City, OK., 1967.

[3] R. O. Schmidt, Multiple emitter location and signal parameter estimation, In *Proc. RADC Spectrum Estimation Workshop*, pages 243–258, Griffiths AFB, N.Y., 1979.

[4] R. O. Schmidt, *A Signal Subspace Approach to Multiple Emitter Location and Spectral Estimation*, PhD thesis, Stanford University, Stanford, CA., 1981.

[5] G. Bienvenu and L. Kopp, Principle de la goniometrie passive adaptive, In *Proc. 7'eme Colloque GRESIT*, pages 106/1–106/10, Nice, France, 1979.

[6] S. Haykin, *Array Signal Processing*, Prentice-Hall, Englewood Cliffs, NJ., 1984.

[7] A. J. Barabell, J. Capon, D. F. Delong, J. R. Johnson, and K. Senne, *Performance Comparison of Superresolution Array Processing Algorithms*, Technical Report TST-72, Lincoln Laboratory, M.I.T., 1984.

[8] A. Paulraj, R. Roy, and T. Kailath, Patent Application: *Methods and Means for Signal Reception and Parameter Estimation*, Stanford University, Stanford, Ca., 1985.

[9] I. S. Reed, J. D. Mallet, and L. E. Brennan, Rapid convergence rate in adaptive arrays, *IEEE Trans. on AES*, AES-10:853–864, 1974.

[10] M. Wax and T. Kailath, Detection of signals by information theoretic criteria, *IEEE Trans. on ASSP*, ASSP-33(2):387–392, 1985.

[11] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD., 1984.

[12] J. Demmel and B. Kågström, *Computing Stable Eigendecompositions of Matrix Pencils*, Technical Report 162, Dept. of Comp. Sci., Courant Inst. of Math. Sci., May 1985.

[13] A. Paulraj and T. Kailath, On beamforming in the presence of multipath, In *Proc. IEEE ICASSP*, pages 15.4.1–15.4.4, Tampa, Fla., March 1985.

[14] A. Paulraj, R. Roy, and T. Kailath, Extensions to the subspace invariance approach to signal parameter estimation, In *Proc. Int. Conf. on Acoustics, Speech and Sig. Proc. (submitted)*, Tokyo, Japan, 1986.