Differential Equations, Part 3

Jay Havaldar

1 Higher Order Linear Equations

We move onto higher order linear differential equations. As before, we need some suitable existence and uniqueness theorem which works for the general case.

1.1 Existence and Uniqueness

In general, a linear differential equation is of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y = g(t)$$

Where a_n are functions and $y^{(n)}$ represents the nth derivative of y.

Theorem

> Suppose we have a linear differential equation with coefficients $a_i(t)$ and right hand side equal to $g(t)$ defined in an interval I. Then, given n initial conditions $y^{(n)}(t_0) = c_n$, there exists a unique solution to the differential equation in the interval.

Note that we already know how to solve this equation in many cases by borrowing techniques from the first two chapters:

- If the coefficients are constant and the right hand side is zero, we use the same methods as in Chapter 2.
- If the coefficients are constant and the right hand side is of a particular form, we can use the method of undetermined coefficients.
- If we know one solution, we can reduce our order n equation into an order $n-1$ equation using the method of reduction of order.

Furthermore, we still know the form of a general solution of a non-homogeneous linear equation as a sum of a complementary and a particular solution; and the Wronskian can still be used to check for linear independence of solutions. The following theorem tells us that the solution space for a homogeneous equation is exactly dimension n.

Theorem

> Suppose we have an equation of the form:
>
> $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y = 0$$
>
> Where a_i are continuous on an interval. Then, if we find $y_1, y_2, ...y_n$ linearly independent solutions to the equation, then every solution can be expressed as a linear combination of $y_1, ..., y_n$.

We call such a solution set a **fundamental set of solutions**.

1.2 Variation of Parameters

Finally, we arrive at the most important tool we have for solving a non-homogeneous nth order linear differential equation.

Suppose we have an equation of the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = f(t)$$

In other words, the first coefficient is 1.

Taking the associated homogeneous equation, let $y_1, ..., y_n$ be its fundamental solutions. Then we try to find a solution to the non-homogeneous equation of the form:

$$y = u_1 y_1 + u_2 y_2 + ... + u_n y_n$$

Where u_i are functions.

To solve uniquely for each u_i, we need n equations. To find them, we first take the derivative of y:

$$y' = \sum_{i=1}^{n} u_i' y_i + u_i y_i'$$

To make things easier for us, we set the condition $\sum_{i=1}^{n} u_i' y_i = 0$. Taking the second derivative, we get:

$$y' = \sum_{i=1}^{n-1} u_i' y_i + u_i y_i'$$

And similarly, we set the condition $\sum_{i=1}^{n} u_i' y_i' = 0$. Repeating this process until we hit the $n-1$th derivative, we get $n-1$ conditions. The final condition comes from substituting in our trial solution into the differential equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(t)$$

Now substituting in $y^{(n)} = \sum_{i=1}^{n} u_i y_i^{(n)} + u_i' y_i^{(n-1)}$, we get:

$$\sum_{i=1}^{n} u_i y_i^{(n)} + u_i' y_i^{(n-1)} + a_{n-1} \sum u_i y_i^{(n-1)} + ... + a_0 \sum u_i y_i = f(t)$$

But note that for each y_i, since it is a solution to the associated homogeneous equation, we have:

$$y_i^{(n)} + a_{n-1} y_i^{(n-1)} + ... + a_0 y_i = 0$$

So most of these terms drop out! The only terms that remain give us:

$$\sum_{i=1}^{n} u_i' y_i^{(n-1)} = f(t)$$

To summarize, our conditions look like:

$$\begin{aligned} u_1'y_1 + ... + u_1'y_n &= 0 \\ u_1'y_1' + ... + u_1'y_n' &= 0 \\ &\vdots \\ u_1'y_1^{(n-1)} + ... + u_1'y_n^{(n-1)} &= f(t) \end{aligned}$$

And to put this in matrix form:

$$\begin{bmatrix} f_1 & \dots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{n-1} & \dots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}' = \begin{bmatrix} 0 \\ \vdots \\ f(t) \end{bmatrix}$$

So, as long as $W \neq 0$, we can solve for u_i' and thus integrate to find u_i.

Summary of Method Suppose we have an equation of the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = f(t)$$

Where we have a set of fundamental solutions y_i to the associated homogeneous equation. Then, we can write a solution as:

$$y = u_1 y_1 + ... + u_n y_n$$

Where u_i are found by solving the equation:

$$\begin{bmatrix} f_1 & \dots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{n-1} & \dots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}' = \begin{bmatrix} 0 \\ \vdots \\ f(t) \end{bmatrix}$$