# Analysis, Part III: Uniform Functions on Metric Spaces

*Jay Havaldar*

We know that the following holds for sequences in a metric space, if $f$ is continuous:

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

The weakest form of convergence is **pointwise convergence**.

**Definition:** $f_n \to f$ pointwise if for every $x$ and for every $\epsilon > 0$, there exists $N$ so that if $n \geq N$ then $d(f_n(x), f(x)) < \epsilon$.

But this definition does not allow us to swap the order of limits.

A stronger condition, however, does allow this:

**Definition:** $f_n \to f$ pointwise if for every $\epsilon > 0$, there exists $N$ so that for all $x$, if $n \geq N$ then $d(f_n(x), f(x)) < \epsilon$.

The key distinction here is that the value of $N$ works for all $x$. As alluded to earlier, we have the following proposition.

**Proposition**

A sequence of continuous functions converges uniformly to a continuous function.

**Corollary**

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f(x)$$

We also have this nice theorem:

**Proposition**

Let $f_n$ be a sequence of continuous functions which converge uniformly to $f$. Then for any sequence $x_n$ which converges to $x$, $f_n(x_n)$ converges to $f(x)$.

Similarly, uniform limits preserve boundedness in a nice way.

**Proposition**

A sequence of bounded functions converges uniformly to a bounded function.

## 0.1 Metrics of Uniform Convergence

**Definition:** Let $X, Y$ be metric spaces. Let $B(X \to Y)$ denote the space of bounded functions from $X \to Y$. We let:

$$d_\infty(f, g) = \sup_{x\in X} d(f(x), g(x))$$

This is called the $L^\infty$ metric or the ``sup norm'' metric. Since $f, g$ are assumed bounded in this space, $d(f,g)$ is bounded.

This space with this norm has the nice property that convergence in this space directly correlates with uniform convergence in the regular sense.

**Proposition**

> Let $f_n$ be a sequence of functions in $B(X \to Y)$; then $f_n$ converges to $f \in B(X \to Y)$ if and only if $f_n$ converges to $f$ uniformly.

In particular, we define the continuous and bounded functions.

**Definition:** $C(X \to Y)$ is the space of bounded and continuous functions from $X$ to $Y$. This a closed subspace of $B(X \to Y)$

And finally, we have Cauchy sequences:

**Proposition**

> If $Y$ is a complete metric space, then the space $C(X \to Y)$ is a complete subspace of $B(X \to Y)$.

## 0.2 The Weierstrass M-Test

**Definition:** A series of functions $f_n$ converges pointwise to $f(x)$ if the partial sums converge pointwise to $f(x)$

In a similar way we have uniform convergence:

**Definition:** A series of functions $f_n$ converges uniformly to $f(x)$ if the partial sums converge uniformly to $f(x)$

We can find an easy condition for uniform convergence using the following norm:

**Definition:** If $f : X \to \mathbb{R}$ is a bounded function, define the sup norm to be the number:

$$|f|_\infty = \sup_{x \in X} f(x)$$

And finally we have the immensely useful Weierstrass $M$-Test:

**Theorem**

> Let $f_n$ be a series of bounded real-valued functions such that $\sum_{n=1}^{\infty} |f_n|_\infty$ is convergent.
>
> Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuous function $f$ on $X$.

## 0.3 Uniform Convergence & Integration/Differentation

**Theorem**

> Let $[a, b]$ be an interval and $f_n$ Riemann integrable functions. Suppose $f_n$ converges uniformly on $[a, b]$ to a function $f$. Then $f$ is also Riemann integrable and:
>
> $$\lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} \lim_{n\to\infty} f_n = \int_{[a,b]} f$$

An analogy of this theorem exists for series.

**Corollary**

> Let $[a, b]$ be a an interval. Let $f_n$ be a sequence of Riemmann integrable functions on $[a, b]$ such that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Then we can say:
>
> $$\sum_{n=1}^{\infty} \int_{[a,b]} f_n = \int_{[a,b]} \sum_{n=1}^{\infty} f_n$$

### 0.3.1 Derivatives and Uniform Convergence

We ask if the same is true for derivatives. If $f_n$ converges uniformly to $f$, then if we require $f_n$ to be differentiable is $f$ differentiable? And do $f_n'$ converge to $f'$? In general, the answer is no.

As a counterexample consider:

$$f_n(x) = n^{1/2} \cos nx$$

This sequences converges uniformly to $0$ if we look at the sup norm. However, its derivative at the origin is never $0$. Thus, $f_n'$ need not converge to $f'$.

Similarly, consider the sequence:

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

We can check that this sequence converges uniformly to $|x|$, which is not differentiable at the origin. This, however, is the theorem we're looking for -- it gives us more or less the converse, so long as we have pointwise convergence at least at one point.

**Theorem**

> Let $[a, b]$ be an interval and $f_n$ be differentiable functions with continuous derivative on the interval. Suppose the derivatives $f_n'$ converge uniformly to a function $g : [a, b] \to \mathbb{R}$.

Suppose also that there exists a point $x_0$ so that the limit $\lim_{n\to\infty} f_n(x_0)$ exists (i.e. we have pointwise convergence at some point). Then the functions $f_n$ converge uniformly to a differentiable function $f$ such that the derivative $f' = g$.

**Corollary**

Let $[a,b]$ be an interval and $f_n$ be differentiable functions with continuous derivative on the interval. Suppose the series $\sum_{n=1}^{\infty} |f_n'|$ converges absolutely.

Suppose also that there exists a point $x_0$ so that $\sum_{n=1}^{\infty} f_n(x_0)$ converges. Then the series $\sum_{n\to\infty} f_n$ converges uniformly on $[a,b]$ to a differentiable function $f$ such that:

$$\frac{d}{dx}\sum_{n\to\infty} f_n(x) = \sum_{n\to\infty} \frac{d}{dx} f_n(x)$$

**Example** Let $f : \mathbb{R} \to \mathbb{R}$ be the function:

$$f(x) = \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$$

This series is uniformly convergent (Weierstrass M-Test) and continuous, but nowhere differentiable.