Differential Equations, Part 2

Jay Havaldar

1 Part II: Second Order Linear Equations

We move onto second order linear differential equations.

Homogeneous Equations with Constant Coefficients We have a differential equation of the form:

$$ay'' + by' + cy = 0$$

We could then we could use a trial solution $y = e^{rt}$, which yields the associated auxiliary equation (called the characteristic equation):

$$ar^2 + br + c = 0$$

Since $e^{rt} \neq 0$. Solving this yields two solutions.

First note that for a homogeneous linear equation, linear combinations of solutions yield solutions; this creates a vector space structure for the solutions as a subspace of smooth functions. In general, for a linear constant coefficient differential equation of order n, the subspace of solutions is dimension n. So we only need to construct n linearly independent solutions, which we do as follows:

- For a real root, we use e^{rt}
- For a real root repeated with multiplicity m, we add $e^{rt}, te^{rt}, ..., t^{m-1}e^{rt}$. You can check that these are all solutions to the differential equation and are linearly independent away from $t = 0$.
- For complex roots $a \pm bi$ we can write $e^{at} \cos bt, e^{a} t \sin bt$.
- For repeated complex roots $a \pm bi$ we can write $e^{at} \cos bt, e^{a} t \sin bt, te^{at} \cos bt, te^{a} t \sin bt$, etcetera.

Reduction of Order So how did we solve linear constant coefficient homogeneous differential equations as above in the case of a repeated root? The answer lies in the method known as reduction of order.

Suppose we have a linear differential equation of the form:

$$y'' + p(t)y'' + q(t)y = r(t)$$

With p, q, r continuous and r possibly zero. And suppose that we know some solution y_1 (with continuous derivative) which is never zero in an interval. Then, we can try a solution of the form $y_2 = vy_1$ for some twice differentiable function v. Substitution and some algebra yields:

$$v'' + (\frac{2y_1'}{y_1} + p)v' = \frac{r}{y_1}$$

Since this is a linear first order differential equation and y_1', p are assumed to be continuous, we know there is a unique solution in our interval.

Note that though we are essentially dealing with a first order differential equation for v', when we integrate v' we obtain an integration constant. We can WLOG write $v = \tilde{v} + c$, and thus write:

$$y_2 = \tilde{v}y_1 + cy_1$$

Existence and Uniqueness We need a suitable theorem guaranteeing uniqueness. The theorem goes just as we expected.