Differential Geometry, Part 2

Part II: Frame Fields

This section is mostly concerned with constructing a "natural" frame for curves in $\mathbb{R}^3$, with the Frenet apparatus, and state some fundamental theorems using those constructions. This approach will be generalized when studying surfaces by looking at curves on the surface.

Definition: A set e_1, e_2, e_3 of mutually orthogonal vectors is a frame if $e_i \cdot e_j = \delta_{ij}$.

Definition: A regular curve $\alpha(t)$ has $\alpha'(t) = 0$. Basically, it has no cusps.

1.1 Curves

Definition: The arclength s of a curve $\alpha(t)$ is computed as $s = \int |\alpha'(t)| dt$. It is not hard to prove by the chain rule that the arclength is the same independent of parametrization.

Theorem There is a reparametrization of any regular curve $\alpha(t)$ with unit speed. In fact, this is the parametrization by arclength. Take a curve $\alpha(t)$ with $\beta(s) = \alpha(t(s))$. We define $t(s)$ to be the inverse function of the arclength function $s(t)$ (which exists by the inverse function theorem since by hypothesis the curve is regular). Then, $\beta'(s) = \alpha'(t(s)) \cdot \frac{dt}{ds} = \frac{ds}{dt}(t(s)) \cdot \frac{dt}{ds}(s) = 1$.

Definition: A reparametrization $\alpha(h)$ is called orientation preserving if $h' \geq 0$ and orientation reversing if $h' \leq 0$.

Definition: A vector field Y on a curve α assigns to each t a tangent vector $Y(t)$ at $\alpha(t)$; for example, velocity is a vector field defined on a curve.

We can differentiate vector fields, and operations on vector fields can give you more vector fields (addition, cross product, scalar multiplication by a real-valued function) or even a scalar field (dot product). We define the derivative of a vector field componentwise. Analogously for the definition for tangent vectors, Y is defined to be **parallel** if its vector components everywhere are the same; it is clear that this is equivalent to the condition that $Y' = 0$.

1.2 Frenet Formulas

Finally we get around to defining a local orthonormal nautral frame defined everywhere on a curve. Take a unit speed curve, then you can define its velocity vector ,T, which is called the unit tangent vector field. T' measures the amount the curve is turning away from its trajectory and is called *curvature*. A short summary of the Frenet apparatus follows:

- $T = \alpha'(s)$ where α is a unit speed curve is called the **tangent** vector field.
- $\kappa(s) = |T'(s)|$ is the **curvature**.
- $N = \frac{T}{\kappa}$ is a unit vector field called the (principal) **normal** vector field. It is perpendicular to T.
- $B = T \times N$ is the **binormal** vector field, completing the Frenet frame.
- $\tau = -|B'(s)|$ is the **torsion**. The reason for the minus sign will be apparent later on.

Theorem The key here is that we can represent the derivatives of T, N, B in terms of this frame. We can write the matrix equation:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Note that the matrix is skew symmetric! The spectral theorem tells us that a real symmetric matrix has only real eigenvalues, and similarly a real skew symmetric matrix has only pure imaginary eigenvalues.

Generally, geometric problems are solved using just the Frenet formulas and looking at derivatives. We can prove that a curve is a plane curve, that it is a circle, or place a lower bound on the curvature of a curve on a sphere. Here are some easy to prove results using the Frenet frame:

- $\kappa = 0$ iff the curve is a straight line.
- $\tau = 0$ iff the curve lies on a plane.
- κ is constant and $\tau = 0$ iff the curve is a circle of radius $\frac{1}{\kappa}$.
- $\frac{\tau}{\kappa}$ constant and nonzero iff the curve is a cylindrical helix.
- κ is constant and $\tau = 0$ iff the curve is a circular helix.

1.3 Covariant Derivatives

Recall that for a function, we were able to define the directional derivative of a real-valued function f with respect to a vector v:

$$v[f] = \frac{d}{dt} f(p + tv)_{t=0}$$

We will do a similar construction with vector fields instead of real-valued functions.

Definition: Let W be a vector field, v is a tangent vector field at p. Then the **covariant derivative** of W with respect to v is the tangent vector:

$$\nabla_v W = \frac{d}{dt} W(p + tv)_{t=0}$$

The definition is almost analogous, except in this case the result is a vector and not a scalar. The covariant derivative of a vector field with respect to a vector is clearly also a tangent vector, since it depends on a point of application p. The covariant derivative measures the initial rate of change of $W(p)$ along the v direction. Let's see an example.

Example Let $W = x^2 U_1 + yz U_3$, where U_i are the Euclidean coordinate functions. Pick a tangent vector $v = (-1, 0, 2)$ and a point $p = (2, 1, 0)$. Then we have the line:

$$p + tv = (2 - t, 1, 2t)$$

This gives rise to a vector field $W(p + tv) = (2 - t)^2 U_1(p + tv) + 2tU_3(p + tv)$. Finally we can compute:

$$\nabla_v W = W(p + tv)'(0) = -4U_1(p) + 2U_3(p)$$

And we can evaluate this vector field by computing the coordinates $U_1(p)$ and $U_2(p)$, which of course are x and y unit vectors everywhere. So indeed, the result lies in the tangent space at p.

Lemma The example above clues us into a simpler formula for the covariant derivative. Suppose $W = \sum w_i U_i$, and we take a tangent vector v at p. Then we extend the notion of the directional derivative to write:

$$\begin{aligned}\nabla_v W &= W'(p+tv)(0)\\ &= \frac{d}{dt}\Big|_{t=0}\sum w_i(p+tv)U_i(p+tv)\\ &= \sum \frac{d}{dt}\Big|_{t=0} w_i(p+tv)U_i(p+tv)\\ &= \sum v[w_i]U_i(p)\end{aligned}$$

To apply $\nabla_v W$ to a vector field, apply v to each of its Euclidean coordinate functions (take the directional derivative of each of its components in the direction of v). We can prove various properties similar to derivative properties using this definition.

We can also define the covariant derivative of one vector field. We define $\nabla_V W$ to be at each point defined as $\nabla_{V(p)} W$. You can think of it as directional derivative of W along $V(p)$, then projecting it onto the tangent space given by V. In short, the covariant derivative measures the initial rate of change of $W(p)$ as p moves in the direction of $V(p)$. It's difficult to formulate a geometric meaning for what this is, since W has many components as does V; it is best to just think of it as some algebraically consistent way of extending the notion of the derivative consistently to vector fields. By consistently, what I mean is that we preserve linearity (with respect to both vector field arguments at a given point), as well as some suitable Leibniz-esque property for products. In the case of vector fields, we handle both scalar and vector (inner) products.

Properties of the Covariant Derivative We have the following properties, analogously with our other conceptions of the derivative.

- $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$
- $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$
- $\nabla_v(fY) = \nabla_v f\ Y(p) + f(p)\nabla_v Y$
- $\nabla_v(Y \cdot Z) = \nabla_v Y \cdot V(p) + Y(p) \cdot \nabla_v Z$

And all these follow fairly straightforwardly from applying the definition. If instead we are taking the covariant derivative along a varying vector field, we have:

- $\nabla_{fV+gW} Y = f\nabla_V + g\nabla_W Y$
- $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$
- $\nabla_V(fY) = \nabla_V\ fY + f\nabla_V Y$
- $\nabla_V(Y \cdot Z) = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$

Frame Fields **Definition:** Vector fields E_1, E_2, E_3 form a frame field on $\mathbb{R}^3$ if everywhere $E_i \cdot E_j = \delta_{ij}$.

With a frame field, we can define coordinate functions, i.e. for a vector field V and a frame field E_i, then we can write $V = \sum f_i E_i$ by orthonormal expansion, and $V \cdot E_i$ are called the coordinate functions. We can define dot products by multiplying componentwise with respect to these coordinate functions.

1.3.1 Connection Forms

We can, similarly to the construction with Frenet fields, express the covariant of coordinate fields in terms of the coordinate fields themselves. For example, take an arbitrary vector tangent v at a point p. Then we can indeed write:

$$\begin{bmatrix} \nabla_v E_1 \\ \nabla_v E_2 \\ \nabla_v E_3 \end{bmatrix} = \omega \cdot \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Where $\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p)$. In fact, each $\omega_{ij}(v)$ is a 1-form, called the **connection form**. Recall that a 1-form is a linear function in its arguments, which is clearly true here by the linear properties of the covariant derivative. Another way to look at it is that w_{ij} are elements of the dual space of the tangent space (hence, the cotangent space, or something). Furthermore, we can prove:

$$\begin{aligned} \nabla_v (E_i \cdot E_j) &= \nabla_v E_i \cdot E_j + \nabla_v E_j \cdot E_i \\ &= \omega_{ij}(v) + \omega_{ji}(v) \end{aligned}$$

This means that ω is a skew-symmetric matrix with 0 along the diagonals. By now, you should be reminded of the Frenet formulas, which had the same exact setup. The physical intuition is that $\omega_{ij}(v)$ is the initial rate at which E_i rotates towards E_j as p moves in the v direction.

We can now get to the connection equations of the frame field:

$$\nabla_V (E_i) = \sum_j \omega_{ij}(V) E_j$$

This bears a striking resemblance to the Frenet equations:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Note that, the only key difference is that $\omega_{13}(V)$ is zero, since we define the derivative of the tangent vector such that it is entirely in the direction of N (in fact, that is how we pick N), so it makes sense that the rate at which T rotates towards B is always zero. The rates of change here are computed along T alone, but the coefficients can also apply to arbitrary vector fields. This is why the connection forms are 1-forms, as they are functions of vectors and not real-valued functions. In this sense, the connection forms allow us to encapsulate information about all the covariant derivatives at the same time.

We can find an explicit formula for the connection forms of an arbitrary frame field E_i. First, we express each E_i in terms of U_i, the natural frame field, to obtain a matrix:

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = A \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

The matrix A is called the **attitude matrix** of the frame field. This matrix is orthogonal (its rows are orthogonal unit vectors). We define its differential dA to be the differential of each of the 1-forms in its entries, i.e. $dA_{ij} = d(a_{ij})$.

Theorem We can write the connection forms simply as:

$$\omega = dA \cdot A^T$$

Proof The proof is a little bit tedious. I'm warning you! First, recall that a_{ij} is the jth coordinate of E_i with respect to the natural coordinates U_i. We will then say that $a_{ij} = E_i^{(j)}$. Then, we write out an arbitrary w_{ij}:

$$\begin{aligned} \omega_{ij}(v) = (dA \cdot A^T)_{ij} &= \sum_k d(a_{ik})(v) \cdot A_{jk} \\ &= \sum_k v[a_{ik}] \cdot A_{jk} \\ &= \sum_k v[E_i^{(k)}] \cdot E_j^{(k)} \\ &= \nabla_v E_i \cdot E_j \end{aligned}$$

And we are done. This formulation will make some later proofs a lot easier.

1.4 The Structural Equations

Definition: The dual 1-forms of a frame field E_i are the 1-forms θ_i defined such that $\theta_i(v) = v \cdot E_i(p)$ for each tangent vector v.

For example, in the case of the natural frame field, $\theta_i = dx_i$ since $dx_i(v) = v_i$. Using dual 1-forms, we can write any vector field V in the tangent space in a different basis as: $V = \sum \theta_i(V)E_i$. In fact, θ_i is the dual basis of E_i.

Lemma We let θ_i be the dual forms of a frame field E_i. Then any 1-form ϕ has a unique representation $\phi = \sum \phi(E_i)\theta_i$; in this sense the dual forms form a basis for the 1-forms. This is the generalization of the earlier statement that any 1-form on $\mathbb{R}^3$ can be written in the form $f dx + g dy + h dz$.

Note that $\theta_i(U_j) = E_i \cdot U_j = A_{ij}$. So therefore, by the preceding lemma, we can concisely write the dual forms in the matrix equation:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = A \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

Now we are faced with the question of finding exterior derivatives of our new objects, the connection forms and the dual forms. These are represented by Cartan's structural equations.

The Structural Equations The first structural equations are:

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$$

Compare to the connection equations:

$$\nabla_V(E_i) = \sum_j \omega_{ij}(V)E_j$$

And the second equations:

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$$

The first equations mirror closely the connection equations, since θ_i are the dual forms of E_i. To prove these two, first denote ξ to be the column vector consisting of the natural coordinates x_i of Euclidean space (in other words, 0-forms). Then we can write:

$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ and $d\xi = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$

So the formula for the dual forms can be written succinctly:

$$\theta = Ad\xi$$

Though matrix multiplication works as normal, recall that 1-forms have a separate multiplication operation.

Proof of First Structural Equation We have $d(d\xi) = 0$ by definition of the exterior derivative. Recall that A is orthogonal so:

$$d\theta = d(A\ d\xi) = dA \cdot d\xi + 0 = dA \cdot A^T \cdot A \cdot d\xi = \omega\theta$$

Which puts the first structural equations all in one matrix. ω and θ have 1-forms as their entries.

Proof of Second Structural Equation For functions f and g, we have:

$$d(g\ df) = -df \wedge dg$$

This is from the modified product rule for exterior derivatives. We apply this formula to ω:

$$\begin{aligned} d(\omega) &= d(dA \cdot A^T) = -dA \cdot d(A^T) \\ &= -dA \cdot A^T \cdot A \cdot (dA)^T \\ &= -\omega \cdot \omega^T \\ &= \omega^2 \end{aligned}$$

The last equality follows from the skew-symmetry of ω.

1.5 Summary

The first major idea here is that we can generalize our work with the triple of vector fields that is the Frenet frame to study arbitrary frames, and represent the covariant derivatives in terms of the frame itself with Cartan's equations. We can also exent these formulas to a "dual" notion by considering dual frames, which are not triples of vector fields but triples of cotangents, which are functionals on tangent vectors.