Representation Theory, Part I: Basics

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Definition: A **representation** of a finite group G on a finite dimensional vector space V (WLOG the vector space is assumed to be over the complex numbers).

This map gives V the structure of a module over G, because for $g \in G$, we have:

$$g(v+w) = gv + gwg(hv) = (gh)v$$

Sometimes, V is itself called the representation of the group; thus, we identify a representation of a group as a vector space on which G acts linearly.

Definition: A map φ between two representations V, W of G (also called a G-linear map) is a vector space map $\varphi : V \to W$ such that for any $g \in G$ and $v \in V$:

$$g\varphi(v) = \varphi(gv)$$

Definition: A **subrepresentation** of a representation V is a vector subpace of W of V which is invariant under G.

Definition: A representation V is called **irreducible** if there is no proper nonzero invariant subspace W of V.

Given two represetnations, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations. The latter is given by:

$$g(v \otimes w) = gv \otimes gw$$

Similarly, the nth tensor power can be constructed from a representation, and similarly the exterior powers and symmetric powers as subrepresentations.

0.0.1 Duals and Tensor Products of Representations; Representation of $Hom(V, W)$

The dual V^* of a vector space is a representation as well. We wish to respect the natural pairing between V^* and V, given by:

$$\langle v^*, v \rangle = v^*(v)$$

So we need to define the dual representation such that:

$$\langle \rho^*(g)v^*, \rho(g)v \rangle = \langle v^*, v \rangle$$

And this forces us to define the representation as follows. Note that by the definition of the transpose:

$$\rho(g^{-1})^T v^*(gv) = v^*(g^{-1}gv) = v^*(v)$$

So we define:

$$\rho^*(g) = \rho(g^{-1})^T : V^* \to V^*$$

Now that we have defined the dual and the tensor product of representations, we can show that $Hom(V, W)$ is a representation. Note that there is a natural identification:

$$V^* \otimes W \to Hom(V, W)a^* \otimes b \to (v \mapsto a^*(v)b)$$

It is not hard to show that this identification is surjective and injective, and hence an isomorphism of vector spaces. Now, we take an arbitrary element $a^* \otimes b \in V^* \otimes W$. We identify this element naturally with $\varphi \in Hom(V, W)$:

$$\varphi : v \mapsto a^*(v)b$$

Now we consider $g\varphi = g(a^* \otimes b)$. We have:

$$g(a^* \otimes b) = ga^* \otimes gb = (g^{-1})^T a^* \otimes gb$$

Where we have used the definition of the dual representation. By the natural identification again we have:

$$g\varphi : v \mapsto (g^{-1})^T a^*(v)gb = ga^*(g^{-1}v)b$$

But this is simply telling us that:

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

And this gives us the representation of the space $Hom(V, W)$.

Proposition 1

> The vector space of G-linear maps between two representations V, W of G is the subspace of $Hom(V, W)$ which is fixed by G, often denoted $Hom_G(V, W)$

Note that if we have a G-linear map φ, then by definition:

$$g\varphi(v) = \varphi(gv)$$

Note that the representation of $Hom(V, W)$ however is given by:

$$(g\varphi)(v) = g\varphi(g^{-1}v) = \varphi(gg^{-1}v) = \varphi(v)$$

So indeed φ is fixed under the action of G. The converse holds evidently as well; if φ is fixed by G, then it follows that φ is G-linear.

Finally, if X is any finite set and G acts on X, then G naturally is embedded into the permutation group $Aut(X)$ of X. So we can construct a vector space with basis
$e_x \ : \ x \in X$ and the action of G is then given by:

$$g \sum a_x e_x = \sum a_x e_{gx}$$

Definition: The **regular representation** R_G or R corresponds to the action of G on itself. We could alternatively define it as the space of complex-valued functions on G where:

$$(g\alpha)(h) = \alpha(g^{-1}h)$$

To prove that these are equivalent, we identify e_x with the function f_x which takes the value 1 on x and 0 elsewhere. Then we have:

$$(gf_x)(h) = f_x(g^{-1}h)$$

And evidently this function takes value 1 where $g^{-1}h = x$ or equivalently $h = gx$. Thus we can write:

$$(gf_x) = f_{gx}$$

0.0.2 Complete Reducibility; Schur's Lemma

Proposition 2 (Maschke's Theorem)

> If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V, so that $V = W \oplus W'$

We define the complement as follows. Chose an arbitary subspace U which is complementary to W. Then we can write:

$$V \cong W \oplus U$$

So for any $v \in V$, we can identify it with some pair (w, u). Define the natural projection map $\pi_0 : V \mapsto W$ as:

$$\pi_0(w, u) = w$$

This map is G-linear. Then, we define a new map π:

$$\pi(v) = \sum_{g \in G} g\pi_0(g^{-1}v)$$

Since π_0 is G-linear, it follows that this map is G linear. In fact on W, we have:

$$\pi(w)=\sum_{g\in G} g\pi_0(g^{-1}w)=\sum_{g\in G} gg^{-1}\pi_0(w)=|G|w$$

So this map is nothing more than multiplication by $\|G\|$ on W. Therefore, its kernel is a subspace of V which is invariant under G and is complementary to W.

Corollary

> Any representation is a direct sum of irreducible representations.

Now we move on to Schur's Lemma, one of the more useful theorems in basic representation theory.

Proposition 3 (Schur's Lemma)

> If V, W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then: - Either φ is an isomorphism, of $\varphi = 0$. - If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$.

The first claim follows from the fact that if φ is a module homomorphism, then its kernel and image are subspaces of V, W respectively. Furthermore, for $v \in \ker\varphi$:

$$\varphi(gv) = g\varphi(v) = 0$$

So that the kernel is invariant under G. Similarly, for $\varphi(v)$ in the image we have:

$$g\varphi(v) = \varphi(gv)$$

And so $g\varphi(v)$ also lies in the image. Thus, we have shown the kernel and image of φ are subrepresentations of V and W respectively. The only possibilities are that the kernel is trivial and the image is W (yielding an isomorphism), or the kernel is V and the image is trivial (i.e. $\varphi = 0$).

To prove the second claim, φ must have an eigenvalue λ so that $\varphi - \lambda I$ has nonzero kernel. But if the kernel is nonzero, then by the above argument, the kernel is the V. So identically we indeed have:

$$\varphi - \lambda I = 0$$

Proposition 4

> For any representation V of a finite group G, there is a decomposition:
>
> $$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$
>
> Where V_i are distinct irreducible representations. The decomposition is furthermore unique.

This is a straightforward consequence of Schur's Lemma. Occasionally this decomposition is written:

$$V = a_1 V_1 \oplus \cdots \oplus a_k V_k = a_1 V_1 + \cdots + a_k V_k$$

Where the a_i denote multiplicities.

0.0.3 Examples: Abelian Groups; S_3

In general,if V is a repreentation of a finite group G, then each $g \in G$ gives a map $\rho(g) : V \to V$. However, in general, this map is not a G-module homomorphism (G-linear), i.e. in general we do not have:

$$g(h(v)) = h(g(v))$$

Indeed, $\rho(g)$ is G-linear for every ρ iff g is in $Z(G)$. Then g commutes with h and the above holds. In particular if G is abelian, the above holds. But if V is an irreducible representation, by Schur's Lemma each $g \in G$ acts on V by a scalar multiple, so every subspace is invariant. Thus, V is one dimensional.

Therefore, the irreducible representations of an abelian group G correspond to homomorphisms:

$$\rho : G \to \mathbb{C}$$

Next, we look at S_3. There are two one dimensional representations, given by the trivial representation (U) and the alternating representation U' given by:

$$gv = \text{sgn}(g)v$$

Naturally, we ask if there are any others. Since G is a permutation group, it has a natural permutation representation, where it acts on $\mathbb{C}^3$ by permuting the basis vectors. The representation is not irreducible since it has the invariant subspace spanned by $(1, 1, 1)$. The complementary subspace is given by:

$$V = \{(z_1, z_2, z_3) \ : \ z_1 + z_2 + z_3 = 0\}$$

And this is irreducible since it has no invariant subspaces. It is called the standard representation.

In general, we take a representaiton W of S_3 and look at the action of the abelian subgroup $\mathbb{Z}/3$ on W. If τ is a generator of this subgroup (a 3-cycle), then the space W is spanned by eigenvectors for the action of τ. Furthermore, since $\tau^3 = 1$, the eigenvalues are all third roots of unity. We write $\tau(v) = \omega^i v$ where ω^i is one of the roots of unity.

Let σ be a transposition in S_3. Then we have the relation:

$$\sigma\tau\sigma = \tau^2$$

So therefore we can write:

$$\begin{aligned}\tau(\sigma(v)) &= \sigma(\tau^2(v)) \\ &= \sigma(\omega^{2i}v) \\ &= \omega^{2i}\sigma(v)\end{aligned}$$

So if v is an eigenvector for τ with eigenvalue ω^i, then $\sigma(v)$ is an eigenvector for τ with eigenvalue ω^{2i}.

If v is an eigenvector of τ with eigenvalue $\omega^i \neq 1$, then $\sigma(v)$ is an eigenvector with a different eigenvalue and hence independent. Thus, $v, \sigma(v)$ span a two dimensional subspace of W which is invariant under S_3.

On the other hand, if $w^i = 1$, then $\sigma(v)$ may or may not be linearly independent to v. If it is not, then v spans a one-dimensional subrepresentation, isomorphic to the trivial representation if $\sigma(v) = v$ and the alternating representation if $\sigma(v) = -v$. If $\sigma(v)$ and v are linearly independent, then $v + \sigma(v)$ and $v - \sigma(v)$ span one dimensional represnetations of W isomorphic to the trivial and alternating representations, respectively.

This is not the best approach to find the decomposition of any representation of S_3, but it is one way to do it.