

Multi-prover games and their parallel repetition

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DECLARATION

This work was carried out at AIMS Rwanda in partial fulfilment of the requirements for a Master of Science Degree.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at AIMS Rwanda or any other University.

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ACKNOWLEDGEMENTS

This is optional and should be at most half a page. Thanks Ma, Thanks Pa. One paragraph in normal language is the most respectful.

Do not use too much bold, any figures, or sign at the bottom.

¹⁸ DEDICATION

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Abstract

A short, abstracted description of your essay goes here. It should be about 100 words long. But write it last.

An abstract is not a summary of your essay: it's an abstraction of that. It tells the readers why they should be interested in your essay but summarises all they need to know if they read no further.

The writing style used in an abstract is like the style used in the rest of your essay: concise, clear and direct. In the rest of the essay, however, you will introduce and use technical terms. In the abstract you should avoid them in order to make the result comprehensible to all.

You may like to repeat the abstract in your mother tongue.

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1. Introduction

2. On the Hales-Jewett theorem

In this part, some notions about Hales-Jewett theorem are presented. Firstly, we will start by some basic notions on arithmetic progression, which will be important for understanding the next point. After, we will introduce some elementary notions about Van der Warden's theorem and Szemerédi's theorem. We will highlight that Van der Waerden's theorem is a particular case of Szemerédi's theorem. Ultimately, we will present the two forms of Hales-Jewett theorem and link these one to the two first theorems.

2.1 Arithmetic progression

2.1.1 Definition. Let be a sequence of numbers: $a_1, a_2, \dots, a_n, \dots$.

This sequence of numbers form an **arithmetic progression** if every term of this sequence is obtained by adding a constant to the previous term.

The arithmetic progression is also known as an arithmetic sequence. The constant is also the difference between consecutive terms.

If a_1 and a_n represent the first and the n -th term of a sequence, and d the constant, then the general term a_n of this sequence is expressed as:

$$a_n = a_1 + (n - 1)d.$$

Knowing a_m and the constant d , then a_n can be expressed as:

$$a_n = a_m + (n - m)d.$$

2.1.2 Arithmetic progression of length k. Let a and d be two fixed numbers.

An arithmetic progression of length k is an arithmetic progression of k numbers of the form $a + nd$. a is the first term of the arithmetic progression, d is the difference between two consecutive terms and $n = 0, 1, \dots, k - 1$, that is k consecutive values of n .

We denote by AP(k) or AP- k , the arithmetic progression of length k .

2.2 Van der Waerden's theorem

Before stating the Van der Waerden's theorem, let us introduce and define some concepts and notation.

A *partition* of a set A is a collection of nonempty and mutually disjoint subsets A_i of A , such that $A = \cup A_i$ and $A_i \cap A_j = \emptyset$, $i \neq j$. Thus, a partition is also a sequence A_1, A_2, \dots, A_n

of mutually nonempty and disjoint subsets of set A (Dransfield et al., 2004). A_i are known as blocks.

We denote by \mathbb{Z}^+ , the set of positive integers. Let $m \in \mathbb{Z}^+$, we designate by $[m]$ the set $\{1, 2, \dots, m\}$.

Let X be a set and r be a positive integer. We want to colour elements of set X with some colours. If C represents the set of colours, then $|C| = r$ is the number of colours.

2.2.1 Definition. An r -colouring of X is a mapping $c : X \rightarrow [r]$.

If $|X| = n$, then the number of possibilities of colouring the n elements is n^r .

Let Y be a subset of X . Y is *monochromatic* when the restriction $c|_Y$ is constant, that is if $c(y)$ is the same for every $y \in Y$.

According to Polymath (2009), the Van der Waerden is stated as follows:

2.2.2 Theorem (Van der Waerden). For every pair $(k, r) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $N \in \mathbb{Z}^+$ such that for every r -colouring of $[N]$ there is a monochromatic AP- k .

We know that a r -colouring is a function called c in definition (2.2.1). So, in other words there exists $a, d \in \mathbb{N}$ with $d \neq 0$ such that: $c(a) = c(a + d) = c(a + 2d) = \dots = c(a + (k - 1)d)$.

This Van der Waerden's theorem can also be formulated using partition (Dransfield et al., 2004) as:

2.2.3 Theorem (Van der Waerden). For every $k, r \in \mathbb{Z}^+$, there exists $N \in \mathbb{Z}^+$ such that for every partition A_1, \dots, A_r of $[N]$, there is i , $1 \leq i \leq r$, such that block A_i contains length at one arithmetic progression of least k .

The existence of the number N for which any r -colouring of the integer $\{1, \dots, N\}$ is certain to have a monochromatic subset of cardinality k of which elements form an arithmetic progression was demonstrated constructively in 1927 by Bartel Leendert van der Waerden (Van der Waerden (1927)).

Graham and Rothschild (1974) gave a proof of this theorem. The book entitled "Purely Combinatorial Proofs of Van Der Waerden-Type Theorem" written by Gasarch et al. (2010) condenses the proof of Van Der Waerden theorem.

In this theorem, the difficult problem is to find the number N . The least such number is called *Van der Waerden number* denoted as $W(k, l)$. The general expression of $W(k, l)$ is not known, but for some k and l there are exact values or there are some lower and upper bounds (Dransfield et al., 2004).

$W(1, r)$, $W(k, 1)$ and $W(2, r)$ are known as *trivial* Van der Waerden numbers. So,

$W(1, r) = 1$: this is an AP - 1. $W(k, 1) = k$: this is an AP - k . $W(2, r) = r + 1$: this is an AP - 2.

For instance, let us find the Van der Waerden number $W(2, 3)$, that is a 2-colouring of the set $[W(2, 3)]$ such that there is a monochromatic arithmetic progression of length 3.

The value of $W(2, 3)$ is greater than 8 because for any 2-colouring of $[n]$, $n \in \{3, 4, 5, 6, 7, 8\}$, we can find a 2-colouring which does not contain a monochromatic arithmetic progression of length 3.

So, when $W(2, 3) = 9$ we always find a monochromatic arithmetic progression of length 3 for any 2-colouring of $[9]$. The table (2.1) shows one of the possibilities of colouring $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If the ninth number is blue, then 3, 6, 9 form an arithmetic progression. If the ninth number is red, then 1, 5, 9 form an arithmetic progression. Therefore, by adding a ninth number and colouring it using any of the two colors, we always create an arithmetic progression of length 3.

1	2	3	4	5	6	7	8	9
R	B	B	R	R	B	B	R	

Table 2.1: A 2-colouring of $\{1, 2, \dots, 9\}$

The table (2.2) presents the 7 exact non-trivial Van der numbers (when $k \geq 3$) (Dransfield et al., 2004).

$k \setminus r$	2	3	4
3	9	27	76
4	35	293	
5	178		
6	1132		

Table 2.2: The 7 exact non-trivial values of Van der Waerden numbers.

As related previously, searching the non-trivial value of $W(k, r)$ remains a difficult problem to find solution as long as the values of k and r increase. However, for some k and r there is an approximation of the lower or upper bound of $W(k, r)$ (Stevens and Shantaram, 1978; Herwig et al., 2007; Beeler and O'neil, 1979; Dransfield et al., 2004; Brown et al., 2008; Rabung and Lotts, 2012; Kouril and Paul, 2008). The table (2.3) summaries these known lower bounds and includes the seven Van der Waerden numbers known exactly.

The estimation of lower and upper bounds is still an open problem. There exists some expression that bounded Van der Waerden numbers. Researchers are still looking for closer bound or exact general expression of these numbers. Erdos and Rado (1952), cited by Dransfield et al. (2004) established an inequality for the lower bound for $W(k, r)$.

$$(2(k-1)2^{k-1})^{\frac{1}{2}} < W(k, r). \quad (2.2.1)$$

Berlekamp (1968) found a better bound when $k-1$ = prime number and for $r = 2$ (colors). But

$k \setminus r$	2	3	4	5	6
3	9	27	76	>170	>223
4	35	293	>1,048	>2,254	>9,778
5	178	>2,173	>17,705	>98,740	>98,748
6	1,132	>11,191	>91,331	>540,025	>816,981
7	>3,703	>48,811	>420,217	>1,381,687	>7,465,909
8	>11,495	>238,400	>2,388,317	>10,743,258	>57,445,718
9	>41,265	>932,745	>10,898,729	>79,706,009	>458,062,329
10	>103,474	>4,173,724	>76,049,218	>542,694,970	>2,615,305,384
11	>193,941	>18,603,731	>305,513,57	>2,967,283,511	>3,004,668,671

Table 2.3: Some lower bounds and exact values of Van der Waerden numbers $W(k, r)$.

these bounds still require improvement.

$$(k-1)2^{k-1} < W(k, 2). \quad (2.2.2)$$

For $p = k - 1$, the expression (2.2.2) becomes:

$$p2^p < W(p+1, 2). \quad (2.2.3)$$

So, $W(6, 2) > 5 \times 2^5 = 160$, $W(8, 2) > 7 \times 2^7 = 896$ and $W(12, 2) > 11 \times 2^{11} = 22528$. (Dransfield et al., 2004) improve this lower bound by using propositional satisfiability solvers for some small van der Waerden numbers for instance $W(2, 8) > 1322$. Rabung and Lotts (2012) performs more. Thus, as related in table (2.3), most of the lower bounds used came from Rabung and Lotts (2012).

The best known upper bound of $W(k, r)$ is the expression (2.3.6) which came from the work of Gowers (2001) on a new proof of Szemerédi's theorem. The section (2.3) will talk about this theorem. Szemerédi's theorem is the extension of Van der Waerden's theorem, that is Van der Waerden's theorem is a particular case of Szemerédi's theorem.

$$W(k, r) \leq 2^{2^{r \cdot 2^{k+9}}} \quad (2.2.4)$$

2.3 Szemerédi's theorem

Szemerédi's theorem is merely an extension of Van der Waerden's theorem in terms of *density version*. Below, we show this implication.

Let us consider A a nonempty subset of the set $[N]$. The density of A inside $[N]$ is a positive real number $\delta = \frac{|A|}{N}$. It is clear that $0 < \delta \leq 1$.

2.3.1 Theorem (Szemerédi's theorem). (Polymath, 2009) For every $k \in \mathbb{Z}^+$ and every $0 < \delta \leq 1$ there exists an integer $N(k, \delta) \geq 1$ such that every subset $A \subseteq [N]$ of size $|A| \geq \delta N$ contains an arithmetic progression of length k .

As conjecture, Szemerédi's theorem was formulated by Erdős and Turán (1936). There are several proofs of this theorem. The cases $k = 1$ and $k = 2$ are trivial. Roth (1953, 1970) proved the case $k = 3$. The case $k = 4$ was proved by Szemerédi (1969) and he gave the general case (Szemerédi, 1975).

Some of proofs necessitated the use of other theories external to combinatoric. Thus, the ergodic theory (*theory related to dynamical system with invariant measures and chaos theory*) has been used to prove this theorem by Furstenberg (1977); Furstenberg et al. (1982). Gowers (1998, 2001) used Fourier analysis and the inverse theory of additive combinatorics. Gowers (2007) used a hypergraph regularity lemma to prove this theorem. A quantitative ergodic theory proof, version of Furstenberg et al. (1982) has been presented by Tao (2006) which does not involve some concepts used in the previous proofs: the axiom of choice, the use of infinite sets or measures, the use of the Fourier transform or inverse theorems from additive combinatorics.

2.3.2 Szemerédi's theorem implies Van der Waerden's theorem..

Proof. Let us assume that all conditions of Szemerédi's theorem (2.3.1) are verified. From Van der Waerden's theorem, let us show that $\forall k, r \in \mathbb{Z}^+, \exists N(k, r) \in \mathbb{Z}^+$ such that by r -colouring the set $\{1, 2, \dots, N\}$ (the number $N(k, r)$ depending on k and r), we obtain at least one monochromatic arithmetic progression of length k . Let us notice that we have shown (2.2.2) and (2.2.3) that r -colouring a set is to partition it to r blocks.

Let be a partition of $\{1, \dots, N\}$ to r blocks, that is $\{1, \dots, N\} = A_1 \cup A_2 \cup \dots \cup A_r$, with $A_i \cap A_j \neq \emptyset$. This implies that $A_i \neq \emptyset$. The color of the block A_i is indicated by the number i . There are two blocks with the same colour.

Let A_{max} be the set having the largest number of elements. By partitioning $\{1, \dots, N\}$ to r equal parts, we have: $A_{max} = A_i = \frac{N}{r}$, for $1 \leq i \leq r$.

If $|A_i| < \frac{N}{r}$, for $1 \leq i \leq r$, then $|A_1| + |A_2| + \dots + |A_r| < \frac{N}{r} + \dots + \frac{N}{r} = \frac{rN}{r} = N$, that is $\sum_{i=1}^r |A_i| < N$, therefore A_i for $1 \leq i \leq r$ does not form in this case a partition.

If $|A_i| \leq \frac{N}{r}$, for $1 \leq i \leq r-1$, then $\sum_{i=1}^{r-1} |A_i| \leq \frac{(r-1)N}{r}$. There exists a positive integer a such

that $\sum_{i=1}^{r-1} |A_i| = \frac{(r-1)N}{r} - a$.

Thus,

$$\begin{aligned} |A_1| + |A_2| + \dots + |A_r| = N &\iff \frac{(r-1)N}{r} - a + |A_r| = N \\ &\iff |A_r| = \frac{N}{r} + a \end{aligned}$$

Therefore, $|A_r| \geq \frac{N}{r}$. In this case, as $A_{max} = A_r$, then $A_{max} \geq \frac{N}{r}$.

We know that A_{max} is the largest block. Let us assume that $A_{max} = A_r$. It is clear that $|A_{max}| \geq |A_i|$, for $1 \leq i \leq r-1$.

So,

$$\begin{aligned} |A_1| + |A_2| + \cdots + |A_r| = N &\iff |A_{max}| + |A_{max}| + \cdots + |A_{max}| \geq N \\ &\iff r|A_{max}| \geq N \\ &\iff |A_{max}| \geq \frac{N}{r}. \end{aligned}$$

176 $|A_{max}| \geq \frac{N}{r} \iff |A_{max}| \geq \frac{1}{r}N \iff |A_{max}| \geq \delta N$, with $\delta = \frac{1}{r}$, implies according to Szemerédi's
177 theorem (2.3.1) A_{max} contains an arithmetic progression of length k .

178 Therefore, A_{max} is monochromatic arithmetic progression of length k . □

179 **2.3.3 Quantitative bounds of Szemerédi's theorem.** In the previous section (2.3.2) we have
180 shown that Van der Waerden's theorem is a particular case of Szemerédi's theorem. This implies
181 that the Szemerédi's number $N(k, \delta)$ is equal to the Van der Waerden's number $W(k, r)$ when
182 $\delta = \frac{1}{r}$. There is still no a general exact expression of $W(k, r)$, but there are exact values of
183 $W(k, r)$ (7 exact values are known) for some smaller k and r , as far as for the remain cases there
184 are some approximations of the lower and upper bounds of it.

185 Likewise, for Szemerédi's theorem, the general exact value of $N(k, \delta)$ is not known. The seek
186 of this number is an open problem. However, there are some quantitative approximations of the
187 lower and upper bounds of the Szemerédi's number.

188 Before giving quantitative bounds of Szemerédi's theorem existing in the literature review, let us
189 formulate differently the Szemerédi's number. Knowing the number $N(k, \delta)$, all subset A of $[N]$
190 such that $|A| \geq \delta N$ contains an arithmetic progression of length k . Otherwise, we can define the
191 Szemerédi's number as the largest subset of $[N]$ without containing an arithmetic progression of
192 length k . Let us denote by $r_k(N)$ the size of this largest subset.

193 **Lower bound** Behrend (1946) proved that for $k = 3$, $\epsilon > 0$, $C > 0$ an unspecified constant and
194 $\log = \log_2$:

$$r_3(N) \geq \frac{CN}{2^{2\sqrt{2}(1+\epsilon)\sqrt{\log N}}} \quad (2.3.1)$$

195 Elkin (2010) improved the result of Behrend (2.3.1) by a factor $\Theta(\sqrt{\log N})$ and showed
196 that:

$$r_3(N) \geq \frac{CN(\log N)^{1/4}}{2^{2\sqrt{2}\sqrt{\log N}}} \quad (2.3.2)$$

197 For $k \geq 1 + 2^{n-1}$, $n = \lceil \log k \rceil$, $\epsilon > 0$, Rankin (1961), cited by O'Bryant (2011) proved that
198 if N is sufficiently large then:

$$r_k(N) \geq \frac{CN}{2^{n2^{(n-1)/2}(1+\epsilon)\sqrt[n]{\log N}}} \quad (2.3.3)$$

Basing on (2.3.1), (2.3.2) and (2.3.3), O'Bryant (2011) constructed the following expressions:

$$r_3(N) \geq N \left(\frac{\sqrt{360}}{e\pi^{3/2}} - \epsilon \right) \frac{\sqrt[4]{2 \log N}}{4\sqrt{2 \log N}} \quad (2.3.4)$$

$$r_k(N) \geq NC_k 2^{-n2^{(n-1)/2} \sqrt[n]{\log N} + \frac{1}{2n} \log \log N} \quad (2.3.5)$$

where $C_k > 0$ is an unspecified constant. The expression (2.3.5) is presently the best known lower bounds for all k .

Upper bound Gowers (2001) worked on a new proof of Szemerédi's theorem and presented in this work that the upper bound of $r_k(N)$ is:

$$r_k(N) \leq N (\log \log N)^{-2^{-2^{k+9}}} \quad (2.3.6)$$

where $\delta = (\log \log N)^{-2^{-2^{k+9}}}$.

Bloom (2016) improved the upper bound for $r_3(N)$:

$$r_3(N) \leq C \frac{(\log \log N)^4}{\log N} N. \quad (2.3.7)$$

For $k = 4$, Green and Tao (2006) improved the result (2.3.6) of Gowers (2001) as follows:

$$r_4(N) \leq CN e^{-c\sqrt{\log \log N}} \quad (2.3.8)$$

for some absolute constant $c > 0$.

Therefore, quantitative bounds of $r_k(N)$ are:

$$NC_k 2^{-n2^{(n-1)/2} \sqrt[n]{\log N} + \frac{1}{2n} \log \log N} \leq r_k(N) \leq N (\log \log N)^{-2^{-2^{k+9}}} \quad (2.3.9)$$

Quantitative bounds for $k = 3$ and $k = 4$ have been enhanced. Thus, for $k = 4$ we have the equation (2.3.8). By combining (2.3.4) and (2.3.7), we have the quantitative bounds of $r_3(N)$, expressed in (2.3.10)

$$N \left(\frac{\sqrt{360}}{e\pi^{3/2}} - \epsilon \right) \frac{\sqrt[4]{2 \log N}}{4\sqrt{2 \log N}} \leq r_3(N) \leq C \frac{(\log \log N)^4}{\log N} N \quad (2.3.10)$$

2.4 Hales-Jewett theorem

Before stating the Hales-Jewett theorem, let us introduce and define notions about combinatorial lines. Combinatorial line is for Hales-Jewett theorem what arithmetic progression is for Van der Waerden's theorem, that is Hales-Jewett theorem is based on structures called combinatorial lines.

Let k and n be two positive integers. We know that $[k]^n = \underbrace{[k] \times [k] \times \cdots \times [k]}_{n \text{ set-factors of } [k]} = \{(x_1, x_2, \dots, x_n) : x_i \in [k]\}$. The set $[k]^n$ contains k^n elements.

For instance, $k = 3$ and $n = 6$, an element of the set $[3]^6$ is : 121132.

Let us consider the set $([k] \times \{x\})^n$. Similarly, the set $([k] \times \{x\})^n$ contains $(k + 1)^n$ elements.

Elements of $([k] \times \{x\})^n$ are called *coordinates*. x is called *wildcard*.

Given $k, n \in \mathbb{N}$, we call x -string (or n -dimensional *variable word* on k letters), a finite word $a_1 a_2 \cdots a_n$ of the symbols $a_i \in [k] \cup \{x\}$, where at least one symbol a_i is x . $w(x)$ denotes an x -string. Let V denote the set of all string: $V = \{w(x)\}$. The cardinal of V is: $V = (k + 1)^n - k^n$.

For any integer $i \in [k]$ and x -string $w(x)$, the string obtained from $w(x)$ by replacing each x by i is denoted by $w(x; i)$. A *combinatorial line* is a set of k strings $\{w(x; i) : i \in [k]\}$ where $w(x)$ is an x -string (Beck, 2008). That is a combinatorial line is a set of k finite words obtained by replacing x in the word $w(x; i)$ by $i \in \{1, 2, \dots, k\}$.

For instance, for $k = 3$ and $n = 8$, a combinatorial line is :

$\{w(x) = 1xx2x23x : x \in [3]\} = \{11121231, 12222232, 13323233\}$.

Sets which do not contain any combinatorial lines is called a *line-free*.

2.4.1 Theorem (Hales-Jewett theorem). *For every pair of positive integers k and r there exists a positive number $HJ(k, r)$ such that for every $n \geq HJ(k, r)$ and every r -colouring of the set $[k]^n$ there is a monochromatic combinatorial line.*

There are several proofs of Hales-Jewett theorem. The original proof has been given by Hales and Jewett (1987). Shelah (1988) proved through primitive recursive bound using simple induction. Nilli (1990) presented a compact form of Shelah's Proof of the Hales-Jewett Theorem. Matet (2007) presented a variant of Shelah's proof of the Hales-Jewett theorem by replacing Shelah's pigeonhole lemma by an appeal to Ramsey's theorem.

The Hales-Jewett's theorem has also a density version. By considering a nonempty subset A of the set $[k]^n$, the density of A inside $[k]^n$ is a positive real number $\delta = \frac{|A|}{k^n}$. Values of δ are bounded by 0 and 1, that is $0 < \delta \leq 1$.

Let denote by $DHJ(k, \delta)$ the density Hales-Jewett number. The density version of Hales-Jewett theorem is announced as follows:

2.4.2 Theorem (Density version of Hales-Jewett theorem). *For any $k \in \mathbb{Z}^+$ and any real number*

245 $0 < \delta \leq 1$, there exists a positive integer $DHJ(k, \delta)$ such that if $n \geq DHJ(k, \delta)$ and A is any
 246 subset of $[k]^n$ with $|A| \geq \delta k^n$, then A contains a combinatorial line.

247 The proof of the density version of Hales-Jewett theorem has been demonstrated by Furstenberg
 248 and Katznelson (1991) using ergodic methods. Polymath (2009) gave an elementary non-ergodic
 249 proof of the density version of Hales-Jewett theorem by using the equal-slices measure. A simpli-
 250 fied version of Polymath (2009) has been given by Dodos et al. (2013) using a purely combinatorial
 251 proof of the density Hales-Jewett Theorem.

252 To show that this density version of Hales-Jewett implies the Hales-Jewett, we need only to set
 253 as in (2.3.2), $\delta = \frac{1}{r}$. By r -colouring the set $[k]^n$, that is by partitioning to r classes, if a_{max} is
 254 the set containing the maximum number then $|A_{max}| \geq \frac{k^n}{r} = \delta k^n$. Hence, according to (2.4.2),
 255 A_{max} contains a combinatorial line.

256 **2.4.3 Hales-Jewett theorem implies Van der Waerden's theorem..** To show that the Hales-
 257 Jewett theorem implies Van der Waerden's theorem, we need only to show that combinatorial
 258 lines corresponds to the arithmetic progression.

259 In (2.3.2) we have shown that Szemerédi's theorem implies Van der Waerden's theorem. To show
 260 that the density version of Hales-Jewett theorem implies Van der Waerden's theorem, we need
 261 to show that the density version of Hales-Jewett theorem implies Szemerédi's theorem. Hence,
 262 by transitivity, the density version of Hales-Jewett theorem implies Van der Waerden's theorem.

263 Thereupon, whatever the kind of Hales-Jewett theorem used to establish the implication, we need
 264 only to show that the combinatorial line involves the arithmetic progression.

265 Let us assume that the Hales-Jewett theorem is verified and show that the combinatorial line of
 266 k elements contained to the subset A corresponds to the arithmetic progression of length k .

267 We have defined $[k]$ as the set $\{1, 2, \dots, k\}$. Instead to start by 1, let us start by 0. In this part,
 268 $[k]$ expresses the set $\{0, 1, \dots, k-1\}$. It is obvious that $[k] = \mathbb{Z}/k\mathbb{Z}$.

269 Let n be the positive number of the Hales-Jewett theorem, then the set $[k]^n = (\mathbb{Z}/k\mathbb{Z})^n =$
 270 $\{(x_0, x_1, \dots, x_{n-1}) : x_i \in [k]\}$ has k^n elements. Similarly, $[k^n] = \{0, 1, \dots, k^n - 1\}$ has also
 271 k^n elements. The set $[k^n]$ contains natural number (in base 10). While, elements of the set $[k]^n$
 272 are the digits in base- k number system of the numbers $\{0, 1, \dots, k^n - 1\}$.

273 Let us consider the bijection $f : [k]^n \longrightarrow [k^n]$ defines as follows:

$$f(y_0, y_1, \dots, y_{n-1}) = y_0 + y_1 k + y_2 k^2 + \dots + y_{n-1} k^{n-1}.$$

274 Let $w(x) \in ([k] \cup \{x\})^n \setminus [k]^n$ be an x -tring. The combinatorial line generates by $w(x)$ is a
 275 set of k elements.

276 The difference between two consecutive elements $w(x; i_1)$ and $w(x; i_2)$ of this combinatorial line
 277 is a constant. Let us call this constant $l = (l_0, l_1, \dots, l_{n-1}) = w(x; i_1) - w(x; i_2)$ with $i_1 > i_2$.

278 For $j \in \{0, 1, \dots, n-1\}$, l_j has two values:

$$l_j = \begin{cases} 1 & \text{if } l_j = x \\ 0 & \text{otherwise} \end{cases}$$

Let $w(x; 0) = (y_{0,0}, y_{1,0}, \dots, y_{n-1,0})$ be the first element of the combinatorial line generated by $w(x)$. Then, for $0 \leq i \leq n-1$ an element $w(x; i)$ of the combinatorial line can be expressed as:

$$w(x; i) = w(x; 0) + il.$$

Let call by a the image of $w(x; 0)$ by f , that is $a = f(w(x; 0))$ and by d the image of l by f , that is $d = f(l)$. We denote by D the set $\{j : l_j = x\}$. d can be expressed as:

$$d = f(l) = l_0 + l_1k + \dots + l_{n-1}k^{n-1} = \sum_{j=0}^{n-1} l_j k^j = \sum_{j \in D} k^j.$$

Thus, $f(w(x; i)) = a + id$, a and d fixed, $0 \leq i \leq k-1$, the set $\{a + id : i \in [k]\}$ forms an arithmetic progression of length k . So, for any combinatorial line of k elements corresponds an arithmetic progression of length k .

We just need to take $N(k, \delta) = k^n$ to establish that the Hales-Jewett theorem implies the Szemerédi's theorem. As this latter implies the Van der Waerden's theorem. Similarly, we need to take $N(k, r) = k^n$ to show that the Hales-Jewett theorem implies the Van der Waerden's theorem.

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