Multi-prover games and their parallel repetition

By

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DECLARATION

This work was carried out at AIMS Rwanda in partial fulfilment of the requirements for a Master of Science Degree.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at AIMS Rwanda or any other University.

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This is optional and should be at most half a page. Thanks Ma, Thanks Pa. One paragraph in normal language is the most respectful.

Do not use too much bold, any figures, or sign at the bottom.

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This is optional.

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A short, abstracted description of your essay goes here. It should be about 100 words long. But write it last.

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The writing style used in an abstract is like the style used in the rest of your essay: concise, clear and direct. In the rest of the essay, however, you will introduce and use technical terms. In the abstract you should avoid them in order to make the result comprehensible to all.

You may like to repeat the abstract in your mother tongue.

Contents

1. Introduction

2. On the Hales-Jewett theorem

In this part, some notions about Hales-Jewett theorem are presented. Firstly, we will start by some basic notions on arithmetic progression, which will be important for understanding the next point. After, we will introduce some elementary notions about Van der Waerden's theorem and Szemerédi's theorem. We will highlight that Van der Waerden's theorem is a particular case of Szemerédi's theorem. Ultimately, we will present the two forms of Hales-Jewett theorem and link these one to the two first theorems.

2.1 Arithmetic progression

[Jan: This section is OK, but since we don't need infinite arithmetic progressions, I would merge those definitions into one, where you just deal with the finite case.] !

2.1.1 Definition. Let be a sequence of numbers: $a_1, a_2, \cdots, a_n, \cdots$.

This sequence of numbers form an **arithmetic progression** if every term of this sequence is obtained by adding a constant to the previous term.

The arithmetic progression is also known as an arithmetic sequence. The constant is also the difference between consecutive terms.

If a_1 and a_n represent the first and the $n-th$ term of a sequence, and d the constant, then the general term a_n of this sequence is expressed as:

$$a_n = a_1 + (n-1)d.$$

Knowing a_m and the constant d, then a_n can be expressed as:

$$a_n = a_m + (n-m)d.$$

2.1.2 Arithmetic progression of length k. Let a and d be two fixed numbers.

An arithmetic progression of length k is an arithmetic progression of k numbers of the form $a+nd$. a is the first term of the arithmetic progression, d is the difference between two consecutive terms and $n = 0, 1, \cdots, k-1$, that is k consecutive values of n.

We denote by AP(k) or AP-k, the arithmetic progression of length k.

2.2 Van der Waerden's theorem

Before stating the Van der Waerden's theorem, let us introduce and define some concepts and notation.

A *partition* of a set A is a collection of nonempty and mutually disjoint subsets A_i of A, such that $A = \cup A_i$ and $A_i \cap A_j = \emptyset, \quad i \neq j$. Thus, a partition is also a sequence $A_1, A_2, \cdots, A_n$ of mutually nonempty and disjoint subsets of set A (Dransfield et al., 2004). A_i are known as *blocks*. [Jan: This is a common definition, no need for a reference.]

We denote by $\mathbb{Z}^+$, the set of positive integers. Let $m \in Z^+$, we designate by $[m]$ the set $\{1, 2, \cdots, m\}$. [Jan: Use ldots instead of cdots].

Let X be a set and r be a positive integer. We want to colour elements of set X with some colours. If C represents the set of colours, then $|C| = r$ is the number of colours.

2.2.1 Definition. An $r - colouring$ of X is a mapping $c : X \longrightarrow [r]$.

[Jan: For words like r-coloring, write "\$r\$-coloring" instead of "\$r-coloring\$".]

If $|X| = n$, then the number of possibilities of colouring the n elements is n^r.

[Jan: Better: "the number of r-colorings of X is"]

Let Y be a subset of X. Y is $monochromatic$ when the restriction $c|Y$ is constant, that is if $c(y)$ is the same for every $y \in Y$.

[Jan: For restriction better use \restriction and write Y in lower script, like this: $c\restriction_Y$.]

According to Polymath (2009) , the Van der Waerden's theorem is stated as follows:

2.2.2 Theorem (Van der Waerden)**.** *For every pair $(k, r) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $N \in \mathbb{Z}^+$ such that for every $r - colouring$ of [N] there is a monochromatic AP-k.*

[Jan: I think it is more consistent with other theorems to state this as: "there exists N_0 such that for every $N \geq N_0$"]

We know that a $r-$colouring is a function called c in definition (2.2.1). So, in other words there exists $a,\ d\ \in \mathbb{N}$ with $d \neq 0$ such that: $c(a) = c(a + d) = c(a + 2d) = \cdots = c(a + (k - 1)d)$.

This Van der Waerden's theorem can also be formulated using partition (Dransfield et al., 2004) as:

2.2.3 Theorem (Van der Waerden)**.** *For every $k, r \in \mathbb{Z}^+$, there exists $N \in \mathbb{Z}^+$ such that for every partition $A_1, \cdots, A_r$ of $[N]$, there is i, $1 \leq i \leq r$, such that block A_i contains an arithmetic progression of least k.*

[Jan: Note that for this version a block in a partition can be empty.]

The existence of the number N for which any $r-$colouring of the integer $\{1, \cdots, N\}$ is certain to have a monochromatic subset of cardinality k of which elements form an arithmetic progression was demonstrated constructively in 1927 by Bartel Leendert van der Waerden Van der Waerden (1927).

Graham and Rothschild (1974) gave a proof of this theorem. The book entitled "*Purely Combinatorial Proofs of Van Der Waerden-Type Theorem*" written by Gasarch et al. (2010) condenses the proof of Van Der Waerden theorem.

[Jan: G. and R. gave *another* proof of this theorem.]

In this theorem, the difficult problem is to find the number N. The least such number is called *Van der Waerden number* denoted as $W(k, l)$. The general expression of $W(k, l)$ is not known, but for some k and l there are exact values or there are some lower and upper bounds (Dransfield et al., 2004).

[Jan: Be consistent: before you used k and r, now it is k and l.]

$W(1, r)$, $W(k, 1)$ and $W(2, r)$ are known as *trivial* Van der Waerden numbers. So,

$W(1, r) = 1$: this is an $AP - 1$. $W(k, 1) = k$: this is an $AP - k$. $W(2, r) = r + 1$: this is an $AP - 2$. [Jan: Either leave the formulas without explanation or explain better.]

For instance, let us find the Van der Waerden number $W(2, 3)$, that is a $2-$colouring of the set $[W(2, 3)]$ such that there is a monochromatic arithmetic progression of length 3. [Jan: Shouldn't it be $W(3, 2)$?]

The value of $W(2, 3)$ is greater than 8 because for any $2-$colouring of $[n]$, $\quad n \in \{3, 4, 5, 6, 7, 8\}$, we can find a $2-$colouring which does not contain a monochromatic arithmetic progression of length 3. [Jan: Don't use \quad like this, normal space is OK. Maybe give an example for $n = 8$?]

So, when $W(2, 3) = 9$ we always find a monochromatic arithmetic progression of length 3 for any $2-$colouring of $[9]$. The table (2.1) shows one of the possibilities of colouring $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If the ninth number is blue, then 3, 6, 9 form an arithmetic progression. If the ninth number is red, then 1, 5, 9 form an arithmetic progression. Therefore, by adding a ninth number and colouring it using any of the two colors, we always create an form an arithmetic progression of length 3.

1	2	3	4	5	6	7	8	9
R	B	B	R	R	B	B	R	

Table 2.1: A $2-$colouring of $\{1, 2, \cdots, 9\}$

The table (2.2) presents the 7 exact non-trivial Van der numbers (when $k \geq 3$) (Dransfield et al., 2004).

$k \setminus r$	2	3	4
3	9	27	76
4	35	293	
5	178		
6	1132		

Table 2.2: The 7 exact non-trivial values of Van der Waerden numbers.

As related previously, searching for the exact value of $W(k, r)$ remains a difficult problem to find solution as the values of k and r increase. However, for some k and r there is an approximation of the lower or upper bound of $W(k,r)$ (Stevens and Shantaram, 1978; Herwig et al., 2007; Beeler and O'neil, 1979; Dransfield et al., 2004; Brown et al., 2008; Rabung and Lotts, 2012; Kouril and Paul, 2008). The table (2.3) summarizes these known lower bounds and includes the seven Van der Waerden numbers known exactly.

k \ r	**2**	**3**	**4**	**5**	**6**
3	9	27	76	>170	>223
4	35	293	>1,048	>2,254	>9,778
5	178	>2,173	>17,705	>98,740	>98,748
6	1,132	>11,191	>91,331	>540,025	>816,981
7	>3,703	>48,811	>420,217	>1,381,687	>7,465,909
8	>11,495	>238,400	>2,388,317	>10,743,258	>57,445,718
9	>41,265	>932,745	>10,898,729	>79,706,009	>458,062,329
10	>103,474	>4,173,724	>76,049,218	>542,694,970	>2,615,305,384
11	>193,941	>18,603,731	>305,513,57	>2,967,283,511	>3,004,668,671

Table 2.3: Some lower bounds and exact values of Van der Waerden numbers $W(k,r)$.

The estimation of lower and upper bounds is still an open problem. There exist some expressions that bound Van der Waerden numbers. Researchers are still looking for closer bound or exact general expression of these numbers. Erdos and Rado (1952), cited by Dransfield et al. (2004) established an inequality for the lower bound for $W(k,r)$.

$$(2(k-1)2^{k-1})^{\frac{1}{2}} < W(k,r). \tag{2.2.1}$$

Berlekamp (1968) found a better bound when $k-1=$ prime number and for $r=2$ (colors). But these bounds still require improvement. [Jan: Do not use symbols inside sentences, write "when $k-1$ is prime".]

$$(k-1)2^{k-1} < W(k,2). \tag{2.2.2}$$

For $p = k-1$, the expression (2.2.2) becomes:

$$p2^p < W(p+1,r). \tag{2.2.3}$$

[Jan: Is r above equal to two?]

So, $W(6,2) > 5 \times 2^5 = 160$, $W(8,2) > 7 \times 2^7 = 896$ and $W(12,2) > 11 \times 2^1 1 = 22528$. (Dransfield et al., 2004) improve this lower bound by using propositional satisfiability solvers for some small van der Waerden numbers for instance $W(2,8) > 1322$. Rabung and Lotts (2012) performs more. Thus, as related in table (2.3), most of the lower bounds used came from Rabung and Lotts (2012). [Jan: W(2, 8) or W(8, 2)?]

The best known upper bound of $W(k,r)$ is the expression (2.3.6) which came from the work of Gowers (2001) on a new proof of Szemerédi's theorem. Section (2.3) will talk about this theorem. Szemerédi's theorem is the extension of Van der Waerden's theorem, that is Van der Waerden's theorem is a particular case of Szemerédi's theorem:

$$W(k,r) \leq 2^{2^{r^{2^{2^{k+9}}}}} \tag{2.2.4}$$

2.3 Szemerédi's theorem

Szemerédi's theorem is merely an extension of Van der Waerden's theorem in terms of *density version*. Below, we show this implication.

Let us consider A a nonempty subset of the set $[N]$. The density of A inside $[N]$ is a positive real number $\delta = \frac{|A|}{N}$. It is clear that $0 < \delta \leq 1$.

2.3.1 Theorem (Szemerédi's theorem). *(Polymath, 2009) For every $k \in \mathbb{Z}^+$ and every $0 < \delta \leq 1$ there exists an integer $N(k, \delta) \geq 1$ such that every subset $A \subseteq [N]$ of size $|A| \geq \delta N$ contains an arithmetic progression of length k.*

[Jan: Don't cite Polymath above. Cite nothing or Szemeredi.]

As conjecture, Szemerédi's theorem was formulated by Erdös and Turán (1936). There are several proofs of this theorem. The cases $k = 1$ and $k = 2$ are trivial. Roth (1953, 1970) proved the case $k = 3$. The case $k = 4$ was proved by Szemerédi (1969) and he gave the general case (Szemerédi, 1975).

Some of proofs necessitated the use of other theories external to combinatoric. Thus, the ergodic theory (*theory related to dynamical system with invariant measures and chaos theory*) has been used to prove this theorem by Furstenberg (1977); Furstenberg et al. (1982). Gowers (1998, 2001) used Fourier analysis and the inverse theory of additive combinatorics. Gowers (2007) used a hypergraph regularity lemma to prove this theorem. A quantitative ergodic theory proof, version of Furstenberg et al. (1982) has been presented by Tao (2006) which does not involve some concepts used in the previous proofs: the axiom of choice, the use of infinite sets or measures, the use of the Fourier transform or inverse theorems from additive combinatorics.

2.3.2 Szemerédi's theorem implies Van der Waerden's theorem..

Proof. Let us assume that all conditions of Szemerédi's theorem (2.3.1) are verified. From Van der Waerden's theorem, let us show that $\forall k, r \in \mathbb{Z}^+, \quad \exists\ N(k, r) \in \mathbb{Z}^+$ such that by $r-$colouring the set $\{1, 2, \cdots, N\}$ (the number $N(k, r)$ depending on k and r), we obtain at least one monochromatic arithmetic progression of length k. Let us notice that we have shown (2.2.2) and (2.2.3) that $r-$colouring a set is to partition it to r blocks. [Jan: Better first sentence: Assume Szemeredi's theorem is true.] [Jan: s/From VDW theorem/To obtain VDW theorem]

Let be a partition of $\{1, \cdots, N\}$ to r blocks, that is $\{1, \cdots, N\} = A_1 \cup A_2 \cup \cdots \cup A_r$, with $A_i \cap A_j \neq 0$. This implies that $A_i \neq \emptyset$. The color of the block A_i is indicated by the number i. There are two blocks with the same colour. [Jan: s/Let be a partition/Partition $[N]$ into r blocks]

Let A_{max} be the set having the largest number of elements. By partitioning $\{1, \cdots, N\}$ to r equal parts, we have: $A_{max} = A_i = \frac{N}{r}$, for $1 \leq i \leq r$. [Jan: I don't get this part.]

If $|A_i| < \frac{N}{r}$, for $1 \leq i \leq r$, then $|A_1| + |A_2| + \cdots + |A_r| < \frac{N}{r} + \cdots + \frac{N}{r} = \frac{rN}{r} = N$, that is $\sum_{i=1}^{r} |A_i| < N$, therefore A_i for $1 \leq i \leq r$ does not form in this case a partition. [Jan: ... which is a contradiction.]

If $|A_i| \leq \frac{N}{r}$, for $1 \leq i \leq r-1$, then $\sum_{i=1}^{r-1} |A_i| \leq \dfrac{(r-1)N}{r}$. There exists a positive integer a such that $\sum_{i=1}^{r-1} |A_i| = \dfrac{(r-1)N}{r} - a$. [Jan: How do you know that a is integer? What if A_{max} is not A_r?]

Thus,

$$
\begin{aligned}
|A_1| + |A_2| + \cdots + |A_r| = N &\iff \frac{(r-1)N}{r} - a + |A_r| = N \\
&\iff |A_r| = \frac{N}{r} + a
\end{aligned}
$$

Therefore, $|A_r| \geq \frac{N}{r}$. In this case, as $A_{max} = A_r$, then $A_{max} \geq \frac{N}{r}$.

We know that A_{max} is the largest block. Let us assume that $A_{max} = A_r$. It is clear that $|A_{max}| \geq |A_i|$, for $1 \leq i \leq r-1$.

So,

$$
\begin{aligned}
|A_1| + |A_2| + \cdots + |A_r| = N &\iff |A_{max}| + |A_{max}| + \cdots + |A_{max}| \geq N \\
&\iff r|A_{max}| \geq N \\
&\iff |A_{max}| \geq \frac{N}{r}.
\end{aligned}
$$

$|A_{max}| \geq \frac{N}{r} \iff |A_{max}| \geq \frac{1}{r}N \iff |A_{max}| \geq \delta N$, with $\delta = \frac{1}{r}$, implies according to Szemerédi's theorem (2.3.1) A_{max} contains an arithmetic progression of length k. [Jan: If N is bigger than what?]

Therefore, A_{max} is monochromatic arithmetic progression of length k. [Jan: A_{max} is not a progression.] □

2.3.3 Quantitative bounds of Szemerédi's theorem. In the previous section (2.3.2) we have shown that Van der Waerden's theorem is a particular case of Szemerédi's theorem. This implies that the Szemerédi's number $N(k, \delta)$ is equal to the Van der Waerden's number $W(k, r)$ when $\delta = \frac{1}{r}$. [Jan: No, you only know that $W(k, r) \leq N(k, 1/r)$.] There is still no a general exact expression of $W(k, r)$, but there are exact values of $W(k, r)$ (7 exact values are known) for some smaller k and r, as far as for the remain cases there are some approximations of the lower and upper bounds of it.

Likewise, for Szemerédi's theorem, the general exact value of $N(k, \delta)$ is not known. The search for this number is an open problem. However, there are some quantitative approximations of the lower and upper bounds of the Szemerédi's number.

Before giving quantitative bounds of Szemerédi's theorem existing in the literature, let us formulate differently the Szemerédi's number. Knowing the number $N(k, \delta)$, all subset A of $[N]$ such that $|A| \geq \delta N$ contains an arithmetic progression of length k. Otherwise, we can define the Szemerédi's number as the largest subset of $[N]$ without containing an arithmetic progression of length k. Let us denote by $r_k(N)$ the size of this largest subset. [Jan: This paragraph is not clear, rewrite it.]

Lower bound Behrend (1946) proved that for $k = 3$, $\epsilon > 0$, $C > 0$ an unspecified constant and $\log = \log_2$:

$$r_3(N) \geq \frac{CN}{2^{2\sqrt{2}(1+\epsilon)\sqrt{\log N}}} \tag{2.3.1}$$

[Jan: Don't write $\log = \log_2$, just say that all your logarithms are binary. Is this for every $\epsilon > 0$?]

Elkin (2010) improved the result of Behrend (2.3.1) by a factor $\Theta(\sqrt{\log N})$ and showed that:

$$r_3(N) \geq \frac{CN(\log N)^{1/4}}{2^{2\sqrt{2}\sqrt{\log N}}} \tag{2.3.2}$$

[Jan: $\sqrt{\log N}$ or $(\log N)^{1/4}$?]

[Jan: Expressions below are very complicated, can you give some explanations or approximations for those numbers?] For $k \geq 1 + 2^{n-1}$, $n = [\log k]$, $\epsilon > 0$, Rankin (1961), cited by O'Bryant (2011) proved that if N is sufficiently large then:

$$r_k(N) \geq \frac{CN}{2^{n2^{(n-1)/2}(1+\epsilon)\sqrt[n]{\log N}}} \tag{2.3.3}$$

Basing on (2.3.1), (2.3.2) and (2.3.3), O'Bryant (2011) constructed the following expressions:

$$r_3(N) \geq N\left(\frac{\sqrt{360}}{e\pi^{3/2}} - \epsilon\right)\frac{\sqrt[4]{2\log N}}{4^{\sqrt{2\log N}}} \tag{2.3.4}$$

$$r_k(N) \geq NC_k 2^{-n2^{(n-1)/2}\sqrt[n]{\log N} + \frac{1}{2n}\log\log N} \tag{2.3.5}$$

where $C_k > 0$ is an unspecified constant. The expression (2.3.5) is presently the best known lower bounds for all k.

Upper bound Gowers (2001) worked on a new proof of Szemerédi's theorem and presented in this work that the upper bound of $r_k(N)$ is:

$$r_k(N) \leq N(\log\log N)^{-2^{-2^{k+9}}} \tag{2.3.6}$$

where $\delta = (\log\log N)^{-2^{-2^{k+9}}}$. [Jan: What is δ for?]

Bloom (2016) improved the upper bound for $r_3(N)$:

$$r_3(N) \leq C\frac{(\log\log N)^4}{\log N}N. \tag{2.3.7}$$

For $k = 4$, Green and Tao (2006) improved the result (2.3.6) of Gowers (2001) as follows:

$$r_4(N) \leq CNe^{-c\sqrt{\log\log N}} \tag{2.3.8}$$

for some absolute constant $c > 0$.

[Jan: It is usual to write e in normal font, not mathrm.]

Therefore, quantitative bounds of $r_k(N)$ are:

$$NC_k 2^{-n2^{(n-1)/2}\sqrt[n]{\log N}+\frac{1}{2n}\log\log N} \leq r_k(N) \leq N\left(\log\log N\right)^{-2^{-2^{k+9}}} \tag{2.3.9}$$

Quantitative bounds for $k=3$ and $k=4$ have been enhanced. Thus, for $k=4$ we have the equation (2.3.8). By combining (2.3.4) and (2.3.7), we have the quantitative bounds of $r_3(N)$, expressed in (2.3.10)

$$N\left(\frac{\sqrt{360}}{\mathrm{e}\pi^{3/2}}-\epsilon\right)\frac{\sqrt[4]{2\log N}}{4^{\sqrt{2\log N}}} \leq r_3(N) \leq C\frac{(\log\log N)^4}{\log N}N \tag{2.3.10}$$

[Jan: Again, if you insert those bounds, please comment on them. What is the nature of expressions on the left and right? How big is the gap?]

2.4 Hales-Jewett theorem

Before stating the Hales-Jewett theorem, let us introduce and define notions about combinatorial lines. Combinatorial line is for Hales-Jewett theorem what arithmetic progression is for Van der Waerden's theorem, that is Hales-Jewett theorem is based on structures called combinatorial lines.

Let k and n be two positive integers. We know that $[k]^n = \underbrace{[k]\times[k]\times\cdots\times[k]}_{n \text{ set-factors of } [k]} = \{(x_1, x_2, \cdots, x_n) :$ $x_i \in [k]\}$. The set $[k]^n$ contains k^n elements.

For instance, $k=3$ and $n=6$, an element of the set $[3]^6$ is : 121132.

Let us consider the set $([k]\times\{x\})^n$. Similarly, the set $([k]\times\{x\})^n$ contains $(k+1)^n$ elements.

Elements of $([k]\times\{x\})^n$ are called *coordinates*. [Jan: No, coordinate is something else.] x is called *wildcard*.

Given $k, n \in \mathbb{N}$, we call $x-string$ (or $n-$dimensional *variable word* on k letters), a finite word $a_1a_2\cdots a_n$ of the symboles $a_i \in [k]\cup\{x\}$, where at least one symbol a_i is x. $w(x)$ denotes an $x-string$. Let V denote the set of all strings: $V=\{w(x)\}$. The cardinality of V is: $V=(k+1)^n - k^n$.

For any integer $i\in[k]$ and $x-string$ $w(x)$, the string obtained from $w(x)$ by replacing each x by i is denoted by $w(x;i)$. A *combinatorial line* is a set of k strings $\{w(x;i) : \ i\in[k]\}$ where $w(x)$ is an $x-string$ (Beck, 2008). That is a combinatorial line is a set of k finite words obtained by replacing x in the word $w(x;i)$ by $i\in\{1,2,\cdots k\}$.

For instance, for $k=3$ and $n=8$, a combinatorial line is :

$\{w(x) = 1xx2x23x : \ x\in[3]\} = \{11121231, 12222232, 13323233\}$. [Jan: This is not consistent with your previous notation.]

Sets which do not contain any combinatorial lines are called a *line-free*.

2.4.1 Theorem (Hales-Jewett theorem)**.** *For every pair of positive integers k and r there exists a positive number $HJ(k,r)$ such that for every $n \geq HJ(k,r)$ and every $r-$colouring of the set $[k]^n$ there is a monochromatic combinatorial line.*

There are several proofs of Hales-Jewett theorem. The original proof has been given by Hales and Jewett (1987). Shelah (1988) proved a primitive recursive bound using simple induction. [Jan: What does primitive recursive mean? When you are writing something, you should be able to explain at least general idea.] Nilli (1990) presented a compact form of Shelah's Proof of the Hales-Jewett Theorem. Matet (2007) presented a variant of Shelah's proof of the Hales–Jewett theorem by replacing Shelah's pigeonhole lemma by an appeal to Ramsey's theorem.

The Hales-Jewett theorem has also a density version. By considering a nonempty subset A of the set $[k]^n$, the density of A inside $[k]^n$ is a positive real number $\delta = \frac{|A|}{k^n}$. Values of δ are bounded by 0 and 1, that is $0 < \delta \leq 1$.

Let denote by $DHJ(k,\delta)$ the density Hales-Jewett number. The density version of Hales-Jewett theorem is announced as follows:

2.4.2 Theorem (Density version of Hales-Jewett theorem)**.** *For any $k \in \mathbb{Z}^+$ and any real number $0 < \delta \leq 1$, there exists a positive integer $DHJ(k,\delta)$ such that if $n \geq DHJ(k,\delta)$ and A is any subset of $[k]^n$ with $|A| \geq \delta k^n$, then A contains a combinatorial line.*

The proof of the density version of Hales-Jewett theorem has been demonstrated by Furstenberg and Katznelson (1991) using ergodic methods. Polymath (2009) gave an elementary non-ergodic proof of the density version of Hales-Jewett theorem by using the equal-slices measure. A simplified version of Polymath (2009) has been given by Dodos et al. (2013) using a purely combinatorial proof of the density Hales–Jewett Theorem.

To show that this density version of Hales-Jewett implies the Hales-Jewett, we need only to set as in (2.3.2), $\delta = \frac{1}{r}$. By $r-$colouring the set $[k]^n$, that is by partitioning to r classes, if A_{max} is the set containing the maximum number then $|A_{max}| \geq \frac{k^n}{r} = \delta k^n$. Hence, according to (2.4.2), A_{max} contains a combinatorial line.

2.4.3 Hales-Jewett theorem implies Van der Waerden's theorem.. To show that the Hales-Jewett theorem implies Van der Waerden's theorem, we need only to show that combinatorial lines corresponds to the arithmetic progression. [Jan: "combinatorial lines correspond to arithmetic progressions".]

In (2.3.2) we have shown that Szemeredi's theorem implies Van der Waerden's theorem. To show that the density version of Hales-Jewett theorem implies Van der Waerden's theorem, we need to show that the density version of Hales-Jewett theorem implies Szemerédi's theorem. Hence, by transitivity, the density version of Hales-Jewett theorem implies Van der Waerden's theorem.

Thereupon, whatever the kind of Hales-Jewett theorem used to establish the implication, we need only to show that the combinatorial line involves the arithmetic progression. [Jan: I'm not sure what you are trying to say here.]

Let us assume that the Hales-Jewett theorem is verified and show that the combinatorial line of k elements contained to the subset A corresponds to the arithmetic progression of length k.

We have defined $[k]$ as the set $\{1, 2, \cdots, k\}$. Instead to start by 1, let us start by 0. In this part, $[k]$ expresses the set $\{0, 1, \cdots, k-1\}$. It is obvious that $[k] = \mathbb{Z}/k\mathbb{Z}$.

Let n be the positive number of the Hales-Jewett theorem, then the set $[k]^n = (\mathbb{Z}/k\mathbb{Z})^n = \{(x_0, x_1, \cdots, x_{n-1}) : \ x_i \in [k]\}$ has k^n elements. Similarly, $[k^n] = \{0, 1, \cdots, k^n - 1\}$ has also k^n elements. The set $[k^n]$ contains natural number (in base 10). While, elements of the set $[k]^n$ are the digits in base$-k$ number system of the numbers $\{0, 1, \cdots, k^n - 1\}$.

Let us consider the bijection $f : [k]^n \longrightarrow [k^n]$ defines as follows:

$$f(y_0, y_1, \cdots, y_{n-1}) = y_0 + y_1 k + y_2 k^2 + \cdots + y_{n-1} k^{n-1}.$$

Let $w(x) \in ([k] \cup \{x\})^n \setminus [k]^n$ be an $x-tring$. The combinatorial line generates by $w(x)$ is a set of k elements.

The difference between two consecutive elements $w(x; i_1)$ and $w(x; i_2)$ of this combinatorial line is a constant. Let us call this constant $l = (l_0, l_1, \cdots, l_{n-1}) = w(x; i_1) - w(x; i_2)$ with $i_1 > i_2$. [Jan: If they are consecutive, why not call them $w(x; c)$ and $w(x; c+1)$?]

For $j \in \{0, 1, \cdots, n-1\}$, l_j has two values:

$l_j = \begin{cases} 1 & \text{if } l_j = x \\ 0 & \text{otherwise} \end{cases}$ [Jan: $l_j = 1$ or $l_j = x$?]

Let $w(x; 0) = (y_{0,0}, y_{1,0}, \cdots, y_{n-1,0})$ be the first element of the combinatorial line generated by $w(x)$. Then, for $0 \leq i \leq n-1$ an element $w(x; i)$ of the combinatorial line can be expressed as:

$$w(x; i) = w(x; o) + il.$$

Let call by a the image of $w(x; 0)$ by f, that is $a = f(w(x; 0))$ and by d the image of l by f, that is $d = f(l)$. We denote by D the set $\{j : \ l_j = x\}$. d can be expressed as:

$$d = f(l) = l_0 + l_1 k + \cdots + l_{n-1} k^{n-1} = \sum_{j=0}^{n-1} l_j k^j = \sum_{j \in D} k^j.$$

Thus, $f(w(x; i)) = a + id$, a and d fixed, $0 \leq i \leq k-1$, the set $\{a + id : \ i \in [k]\}$ forms an arithmetic progression of length k. So, for any combinatorial line of k elements corresponds an arithmetic progression of length k.

We just need to take $N(k, \delta) = k^n$ to establish that the Hales-Jewett theorem implies the Szemerédi's theorem. [Jan: What is n here?] As this latter implies the Van der Waerden's theorem. Similarly, we need to take $N(k, r) = k^n$ to show that the Hales-Jewett theorem implies the Van der Waerden's theorem. [Jan: There is one small additional complication for the density version.]

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