

Classifying 2-Stratifolds with Finite Fundamental Group



FLORIDA STATE UNIVERSITY
Mathematics

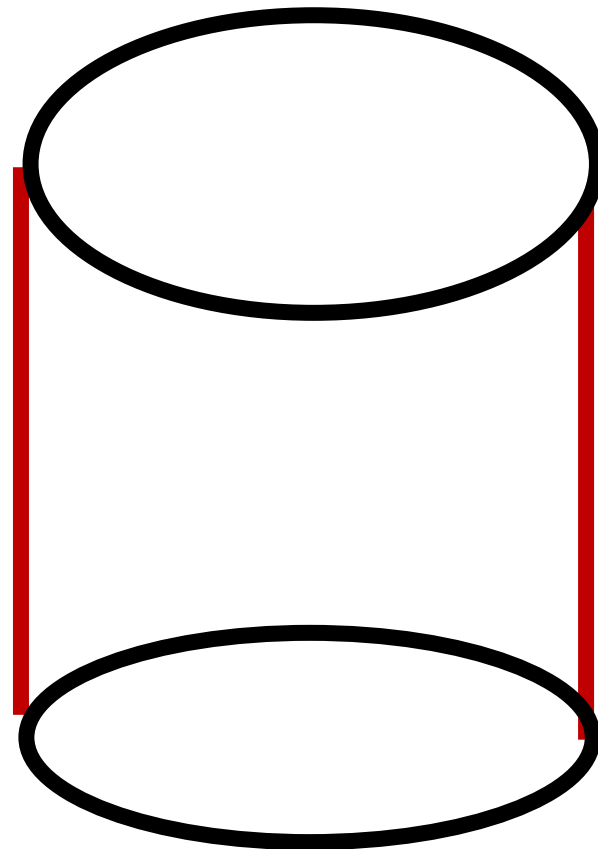
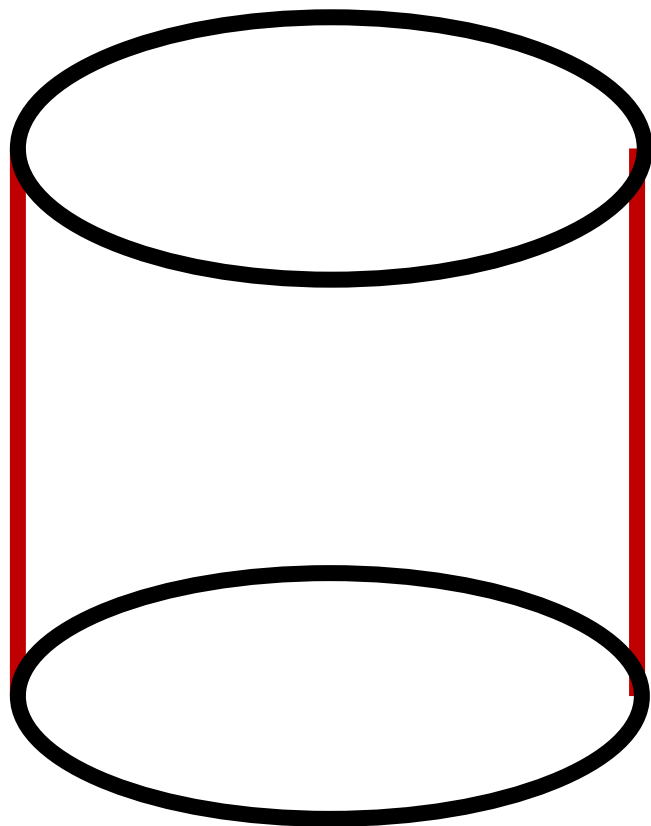
John Bergschneider
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GSCAGT 2019

Outline

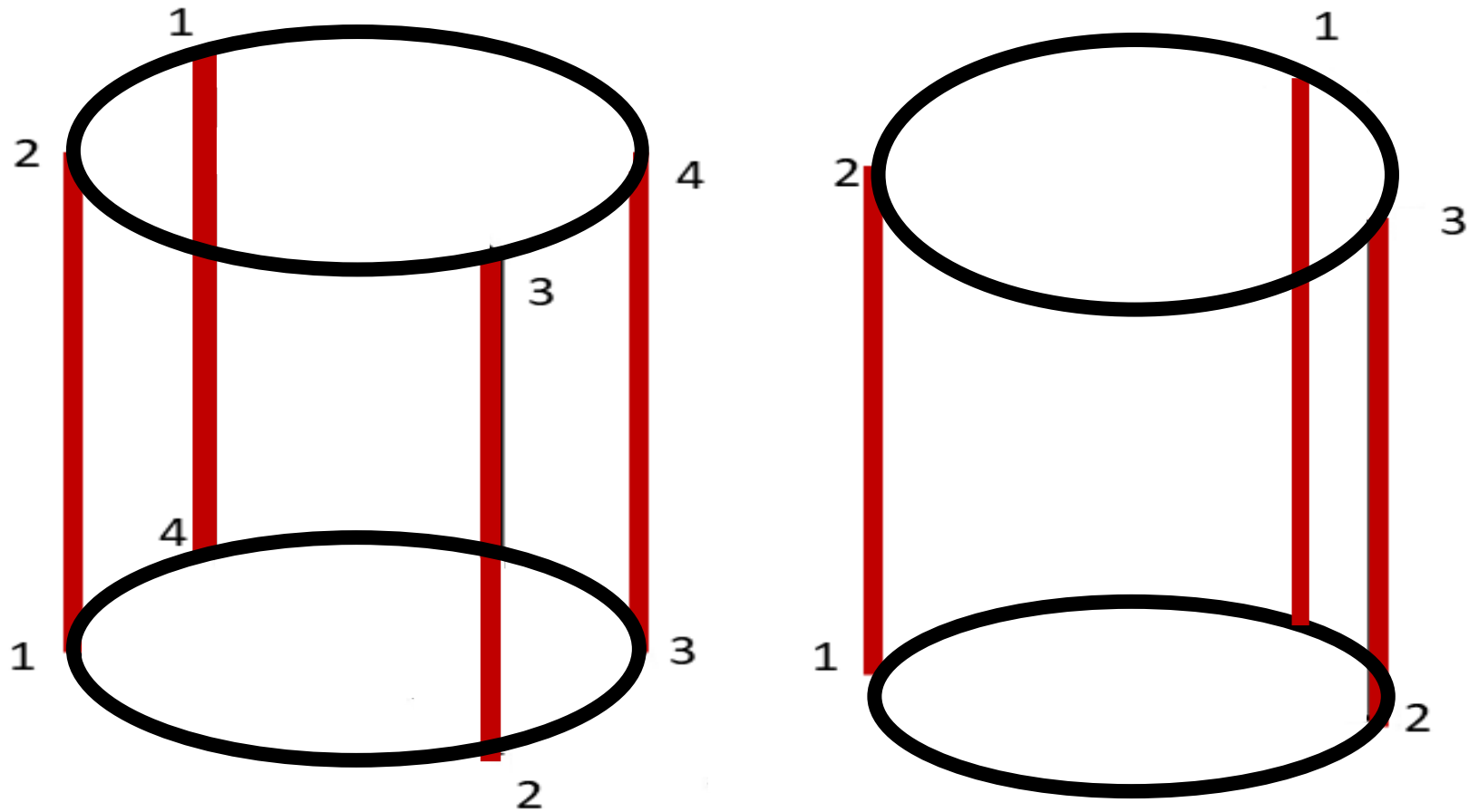
- Part 1 : Definitions and Basic Properties of 2-stratifolds
- Part 2 : Theorems and Classifying 2-stratifolds
- Part 3 : Classification of Trivalent 2-Stratifolds with finite fundamental group

Part 1 : Definitions and Basic Properties

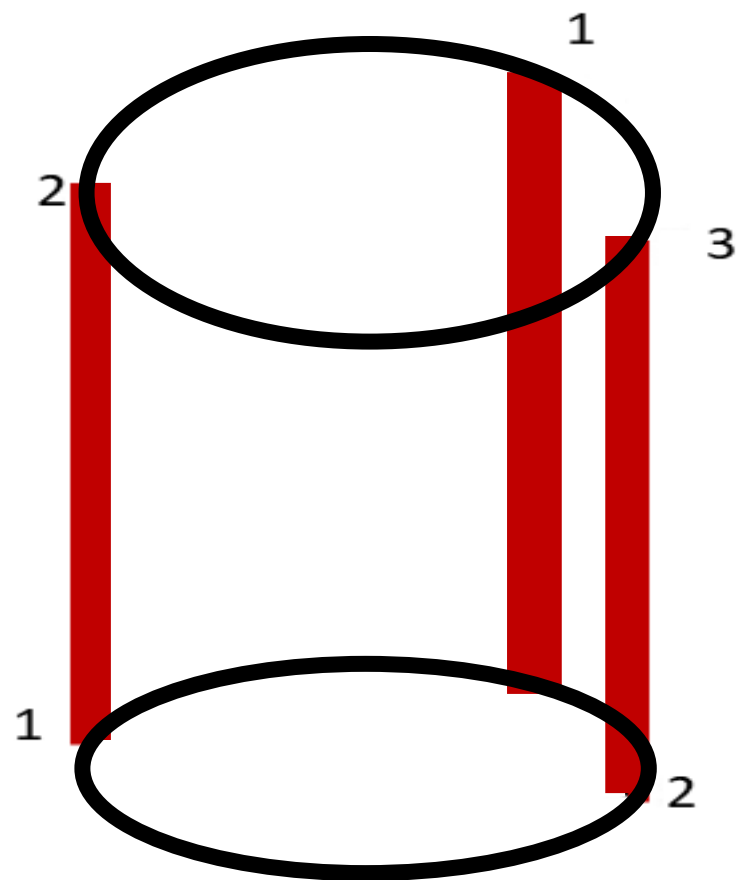
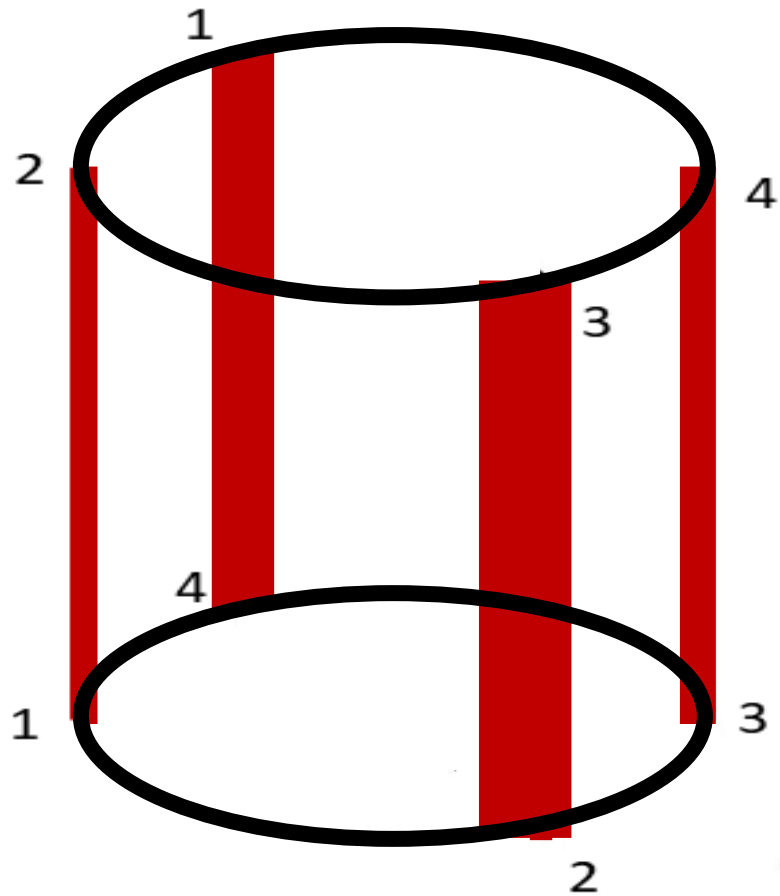
Seifert Fiber Space



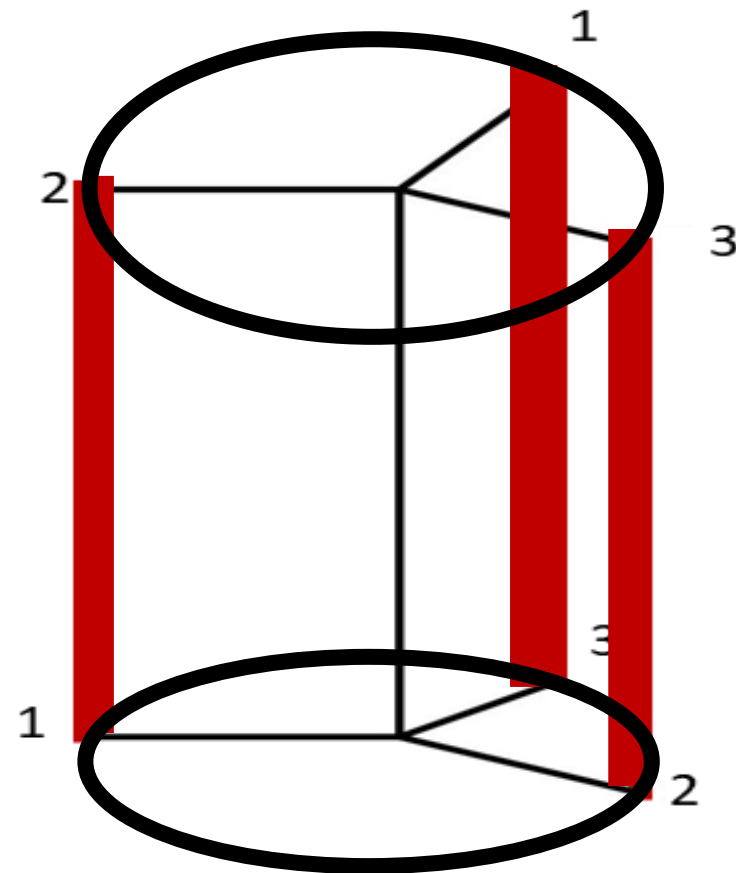
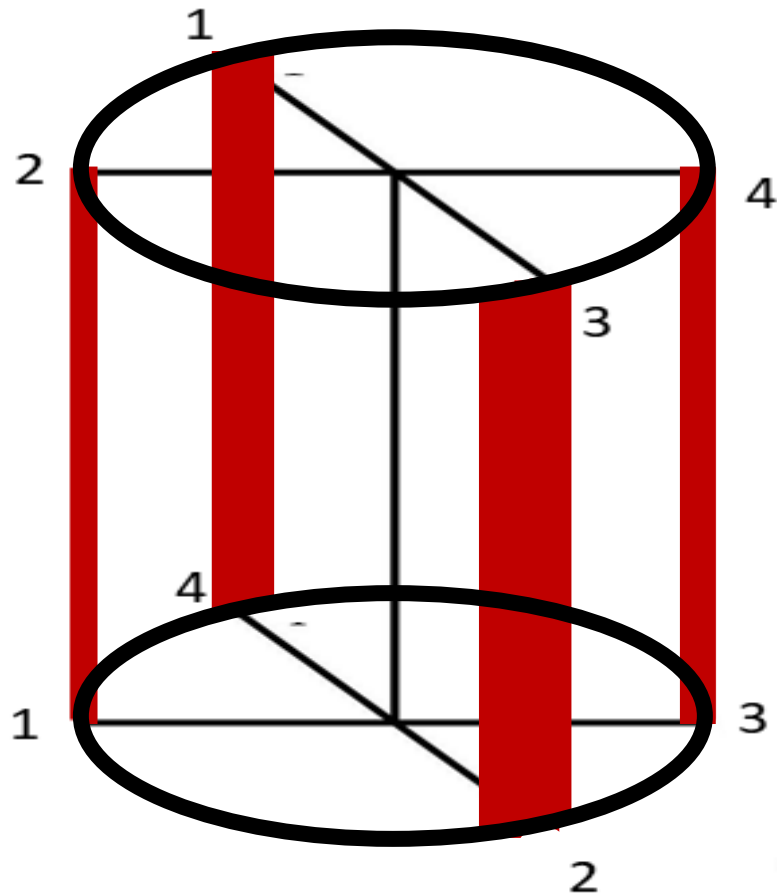
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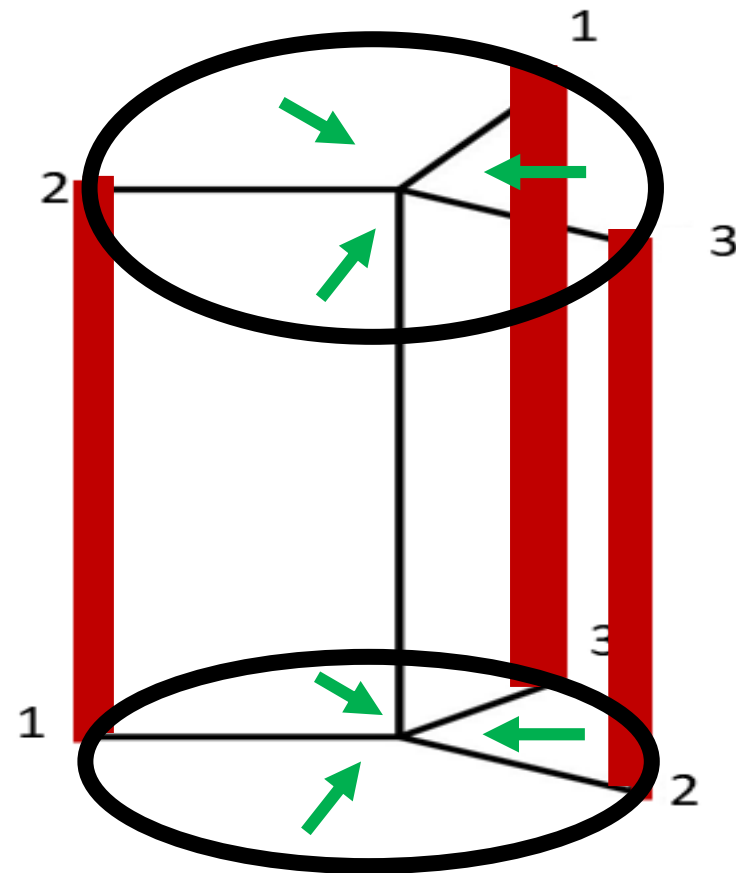
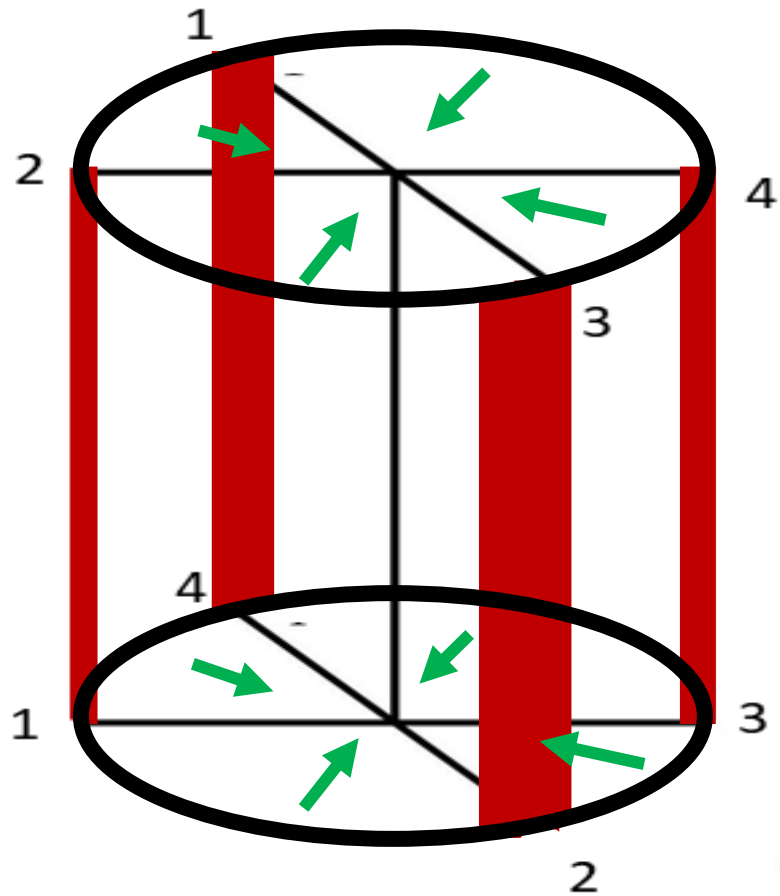
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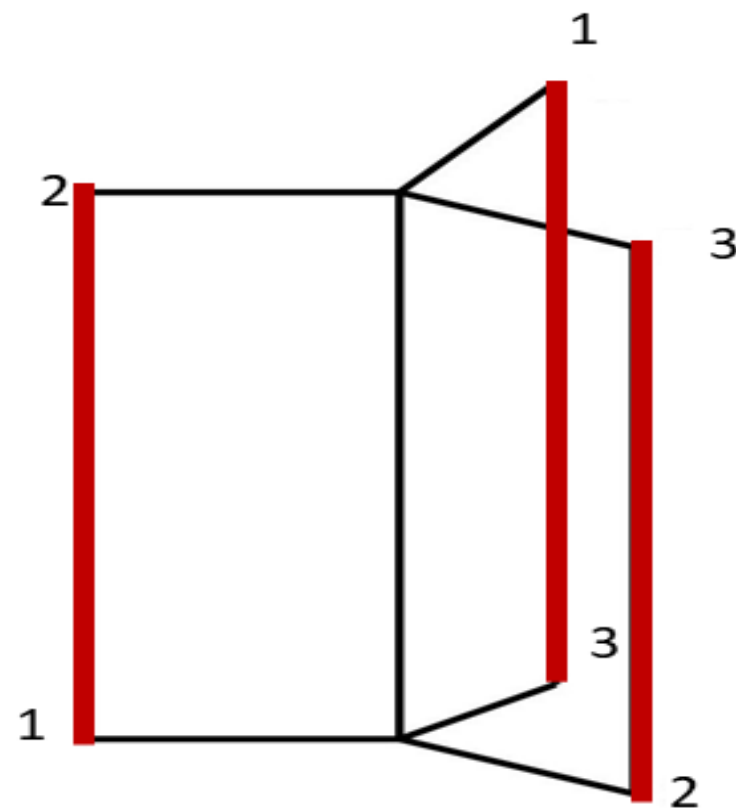
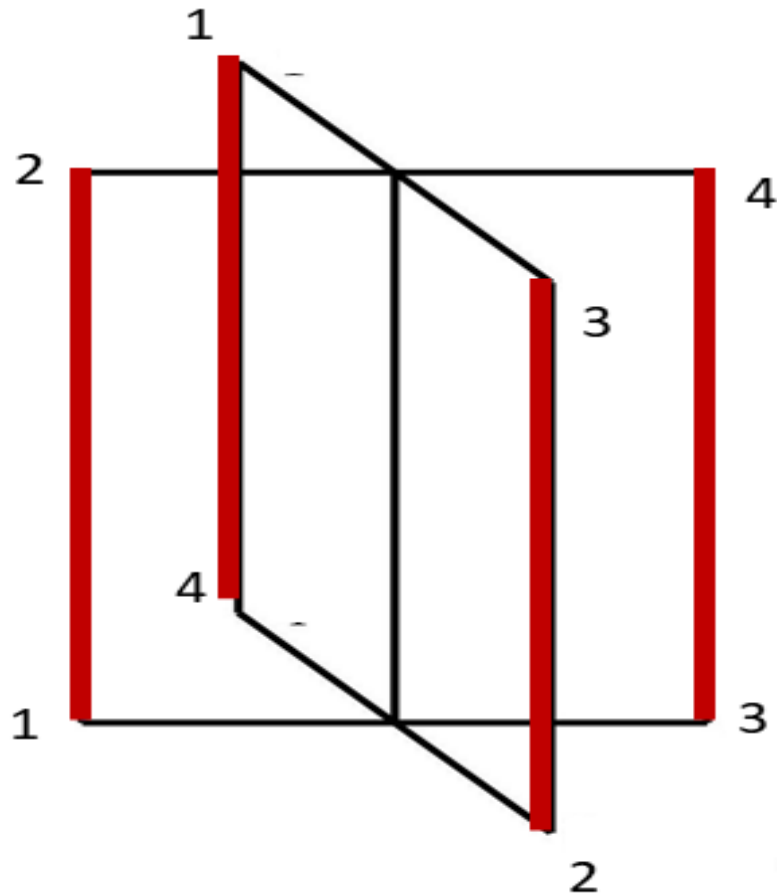
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Seifert Fiber Space



Seifert Fiber Space



2-Stratifold

A closed 2-stratifold X is a 2-complex where

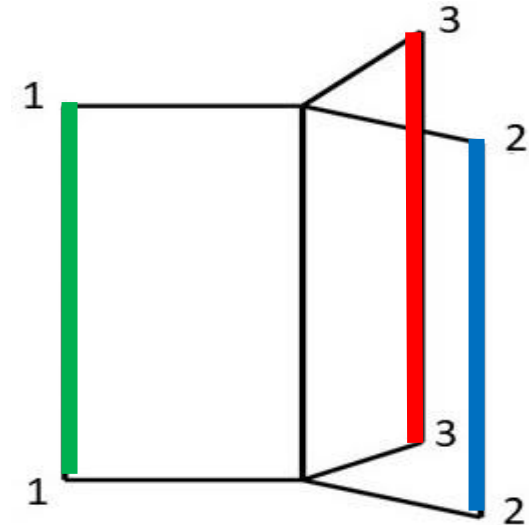
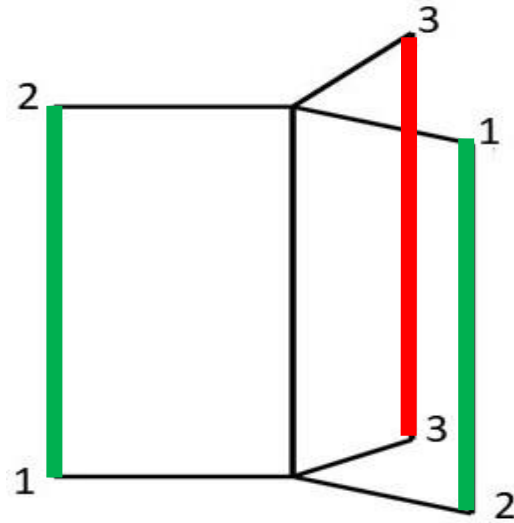
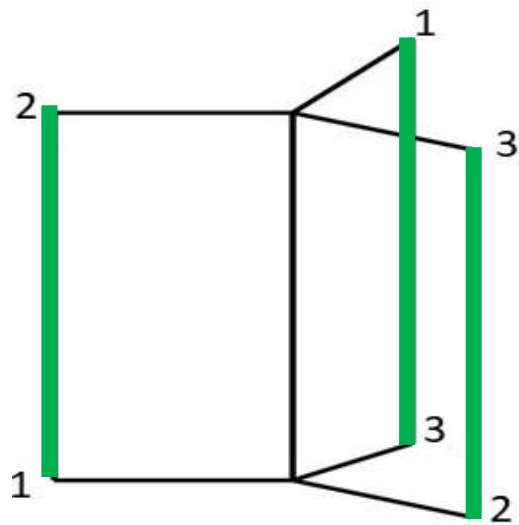
- X contains a collection S of finitely many s.c.c, such that the closure of $X - S$ is a compact surface,
- and a neighborhood of each component in S consists of more than 2 sheets.

Branch Neighborhoods

A simple closed curve in S

- is called a **singular curve**
- and a regular neighborhood of a **singular curve** is called a **branch neighborhood**.

Trivalent Branch Neighborhoods



Gluing Action : (123)

Boundary Components : 1

Boundary Words : $3a$

Gluing Action : $(12)(3)$

Boundary Components : 2

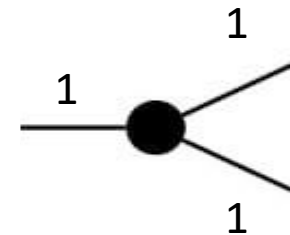
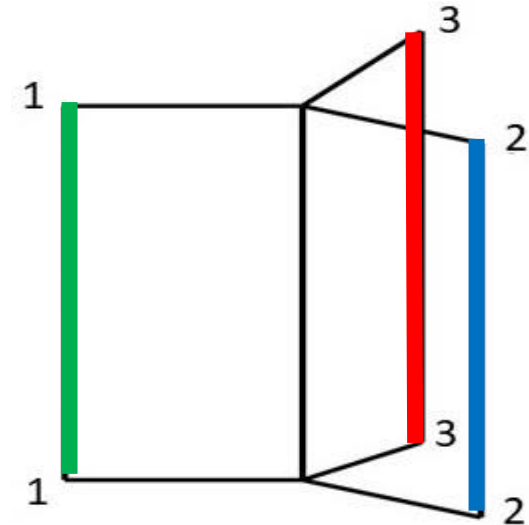
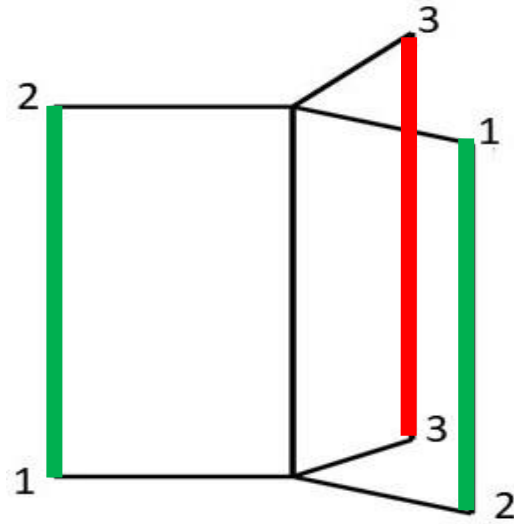
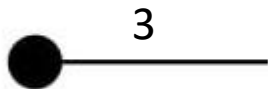
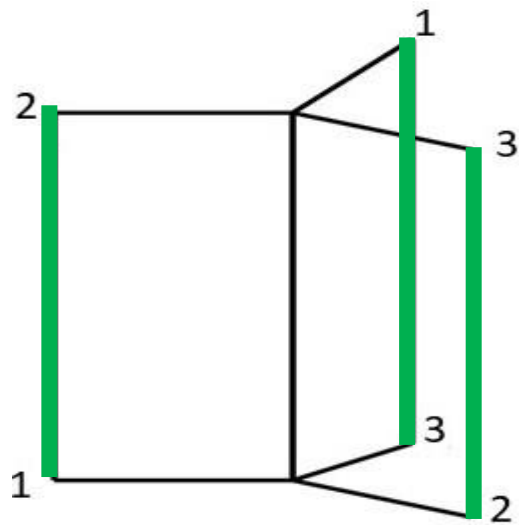
Boundary Words : $2a, a$

Gluing Action : $(1)(2)(3)$

Boundary Components : 3

Boundary Words : a, a, a

Trivalent Branch Neighborhoods



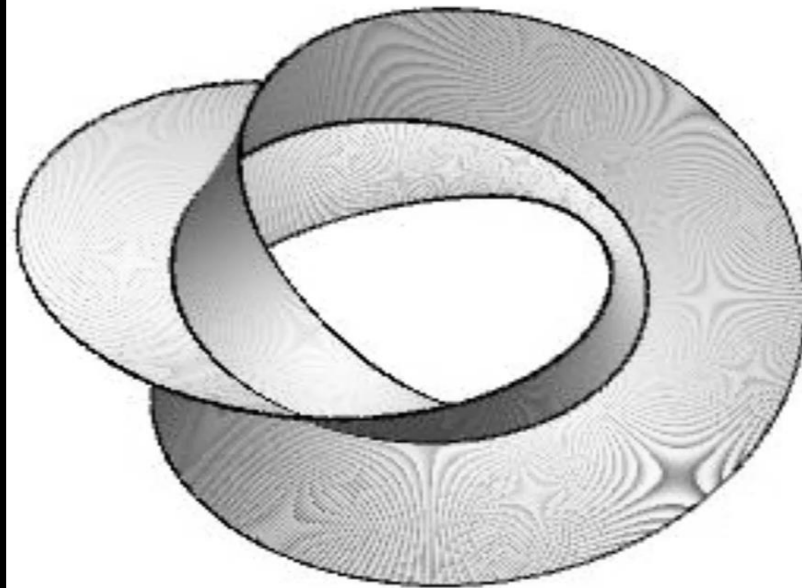
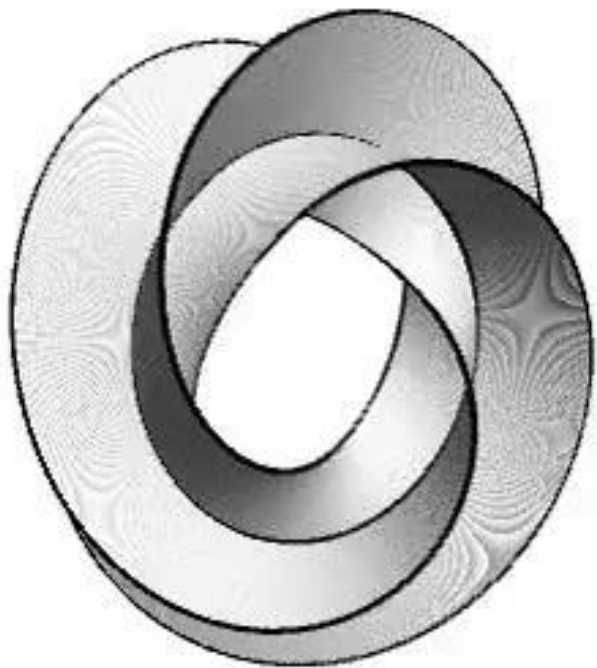
Embedding 2-Stratifolds into Manifolds

- Every 2-stratifold embeds into 4-space.

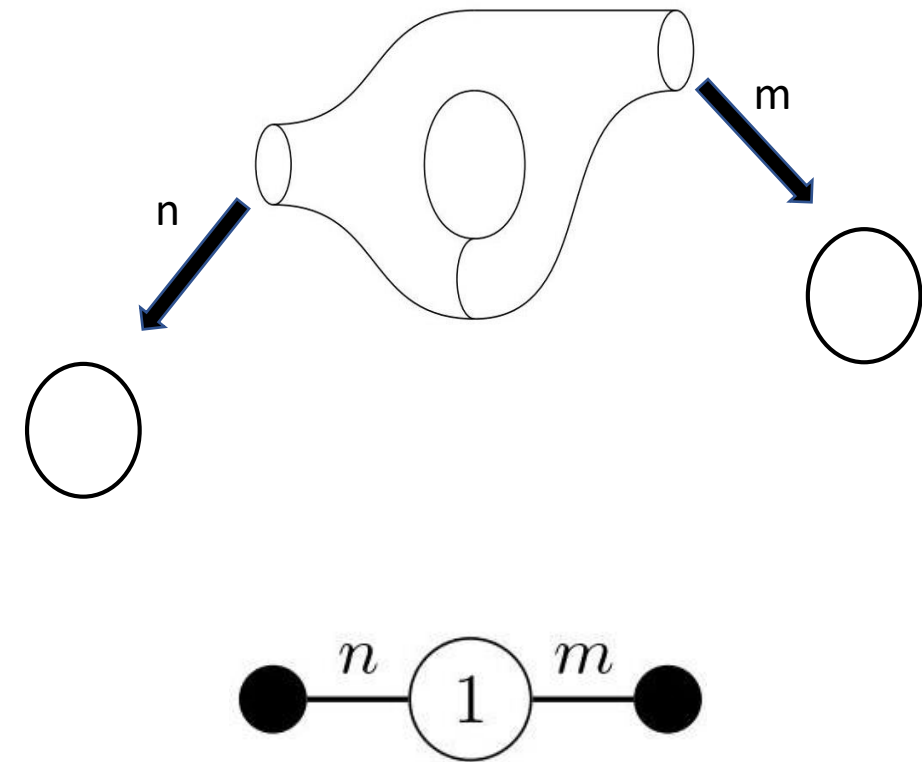
S. Matsuzaki, M. Ozawa

- For a 2-stratifold to be embeddable in a 3-manifold it is necessary that for each branch neighborhood the boundary words are the same.

Embeddings of Branch Neighborhoods



Associated Graph



Generators

Generators from singular curves

$$\underline{C, D, E, F}, \underline{b_1, b_2}$$

Generators from Torus

Relations

$$b_1^n = E, b_2^m = F, [C, D]EF = 1$$

Associated Graph

- 2-Stratifolds are determined almost uniquely by their associated labeled graph

Part 2 : Theorems and Classifying 2-Stratifolds

Classifying 2-stratifolds

Classification of
closed surfaces
groups



Classification of closed
surfaces

Classification of closed
2-stratifold groups



Classification of
closed 2-stratifolds

Some Goals

Main Goal. Given a 2-stratifold group G , enumerate all possible 2-stratifolds whose fundamental group is G .

Secondary Goal. Given a bipartite labeled graph representing a 2-stratifold, determine if the fundamental group is of a given type.

Initial Questions

Question 1. What are the finite 2-stratifolds groups?

Answer 1. Finite Fuchsian Groups.

Question 2. What is the graph type of a 2-stratifold with finite fundamental group?

Answer 2. Tree where almost all white vertices have genus zero and at most one black terminal vertex.

Trivalent Classification

2-stratifolds where at most 3 sheets meet are trivalent.

- Trivalent 2-stratifold with **trivial** or **infinite cyclic fundamental group**.

J.C. Gómez-Larrañaga, F. González-Acuña, and W. Heil

- Trivalent 2-stratifolds with **finite fundamental group**.

B.

Trivalent Algorithm

Given a trivalent 2-stratifold it can be determined if

- It is has **trivial or infinite cyclic fundamental group.**

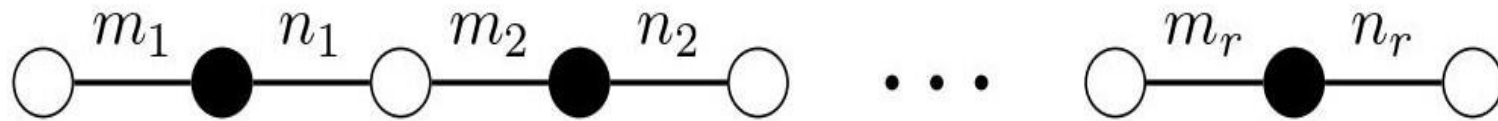
J.C. Gómez-Larrañaga, F. González-Acuña, and W. Heil

- It has **finite fundamental group.**
B.

Part 3 : Classifying Trivalent 2-Stratifolds with Finite Fundamental group



Linear Stratifolds



$$X_1^{n_1} = X_2^{m_2} \dots X_{r-1}^{n_{r-1}} = X_r^{m_r}$$

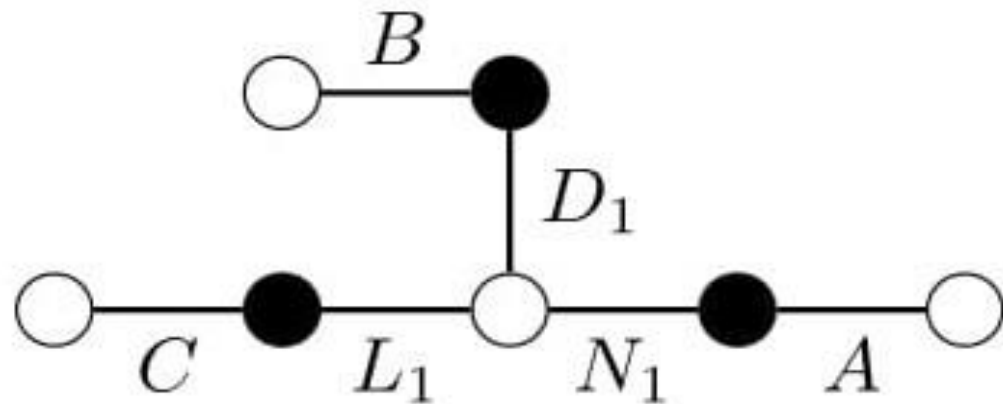
$$X_1^{m_1} = 1 \qquad 1 = X_r^{n_r}$$

Finite π_1



Finite Cyclic

Star Stratifolds



$$\begin{aligned} X_1^A &= 1 & X_1^{N_1} X_2^{D_1} X_3^{L_1} &= 1 \\ X_2^B &= 1 \\ X_3^C &= 1 \end{aligned}$$

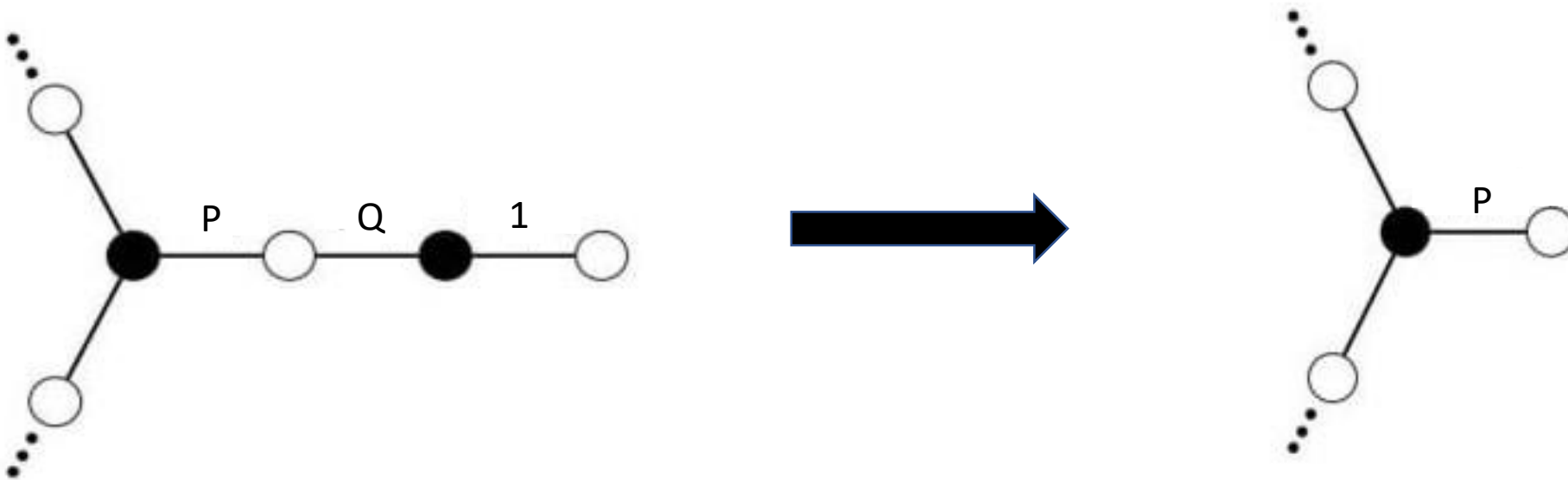
Finite π_1



Dihedral,
Tetrahedral,
Octahedral, or
Dodecahedral

Pruned Trivalent Stratifolds

- No terminal edges have label 1



Some Facts

Lemma. If a trivalent 2-stratifold has finite fundamental group then

- the associated graph is a tree,
- almost all white vertices are genus zero,
- and there is at most one of either a white vertex of degree 3, a black terminal vertex, or a terminal white vertex of genus -1.

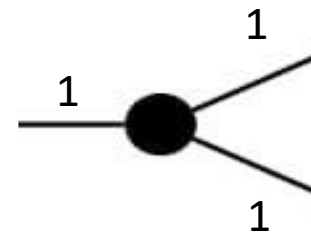
Some Facts

Corollary. The trivalent finite 2-stratifold groups are

- Cyclic of order 2^n
- Cyclic of order $3(2^n)$
- Dihedral of order 2^n

Trivalent Branch Neighborhoods

Trivalent Branch Neighborhoods



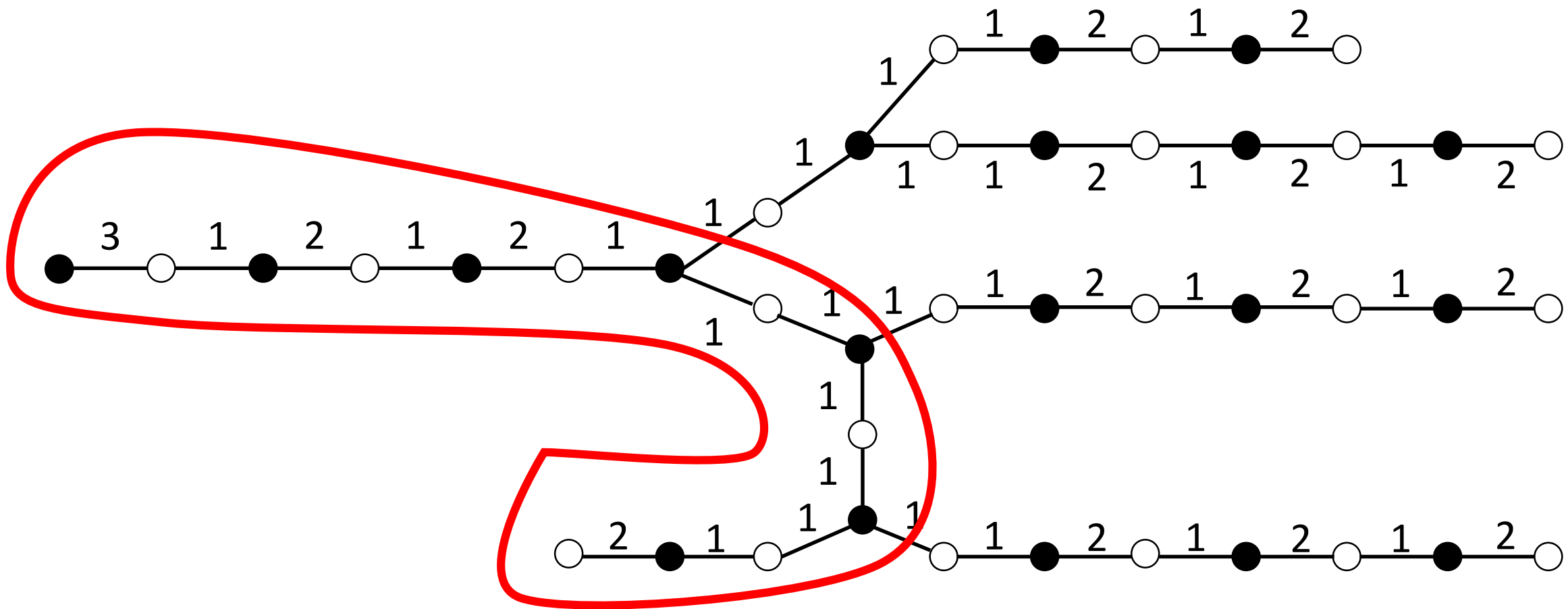
Cyclic Fundamental Group

Let X be a trivalent pruned 2-stratifold with cyclic fundamental group. If the fundamental group of X is cyclic of order $(3)2^n$ then the associated graph has

1. only white vertices of degree < 3 and
2. exactly one terminal black vertex.

Furthermore, all linear subgraphs $L(V,T)$ starting at the black terminal vertex V and ending at a terminal vertex T has the labeling $(12)(11)\dots(11)(12)(11)\dots(12)$ with at least n labels being 2 and there is at least one linear subgraph L having exactly n copies of 2 in their labeling.

Cyclic Fundamental Group



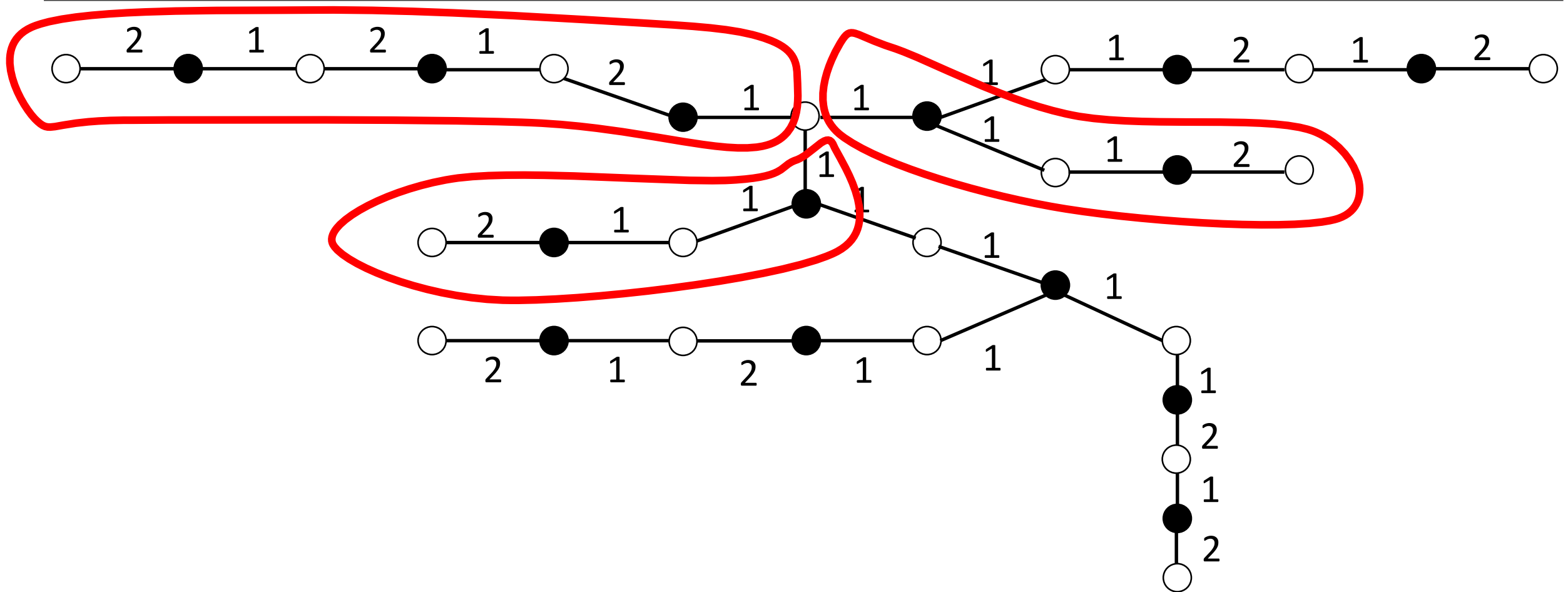
Dihedral Group

Let X be a trivalent pruned 2-stratifold with dihedral fundamental group. If the fundamental group of X is of order 2^n then the associated graph has

1. exactly one white vertex V of degree = 3,
2. all white terminal vertices,
3. and all other white vertices are of degree < 3 .

Furthermore, all linear subgraphs $L(V, T)$ starting at the white vertex V and ending at a terminal vertex T has the labeling $(12)(11)\dots(11)(12)(11)\dots(12)$. Two of the subtrees must contain a linear subgraph H that starts at V and ends at terminal white vertex and contains only a single 2 label.

Dihedral Group



Thank You!



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