Classifying 2-Stratifolds with Finite Fundamental Group

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Advisor: Wolfgang Heil
GSCAGT 2019

Outline

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- Part 1 : Definitions and Basic Properties of 2-stratifolds

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- Part 2 : Theorems and Classifying 2-stratifolds

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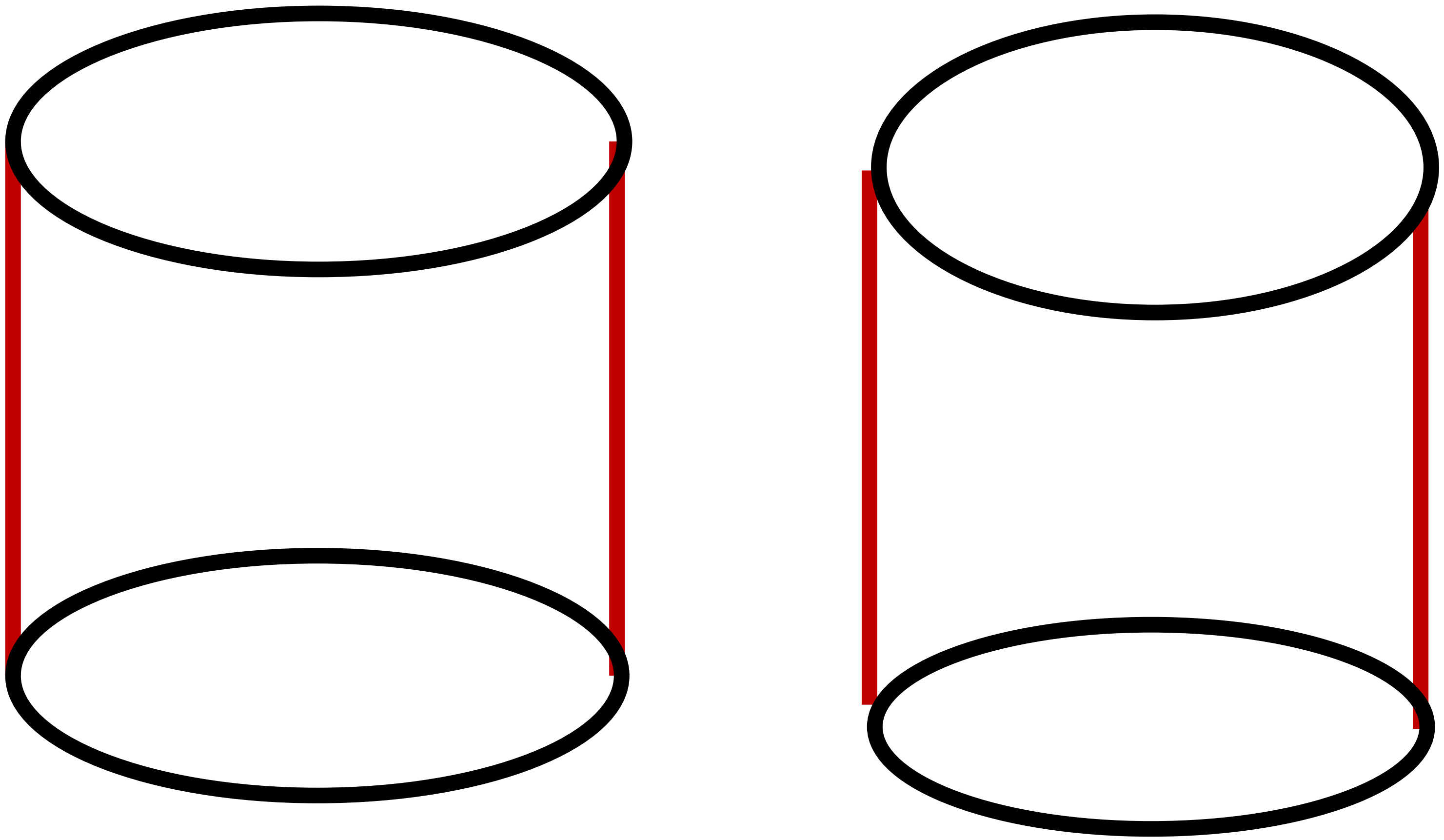
- Part 2 : Theorems and Classifying 2-stratifolds
- Part 3 : Classification of Trivalent 2-Stratifolds with finite fundamental group

Part 1 : Definitions and Basic Properties

Seifert Fiber Space



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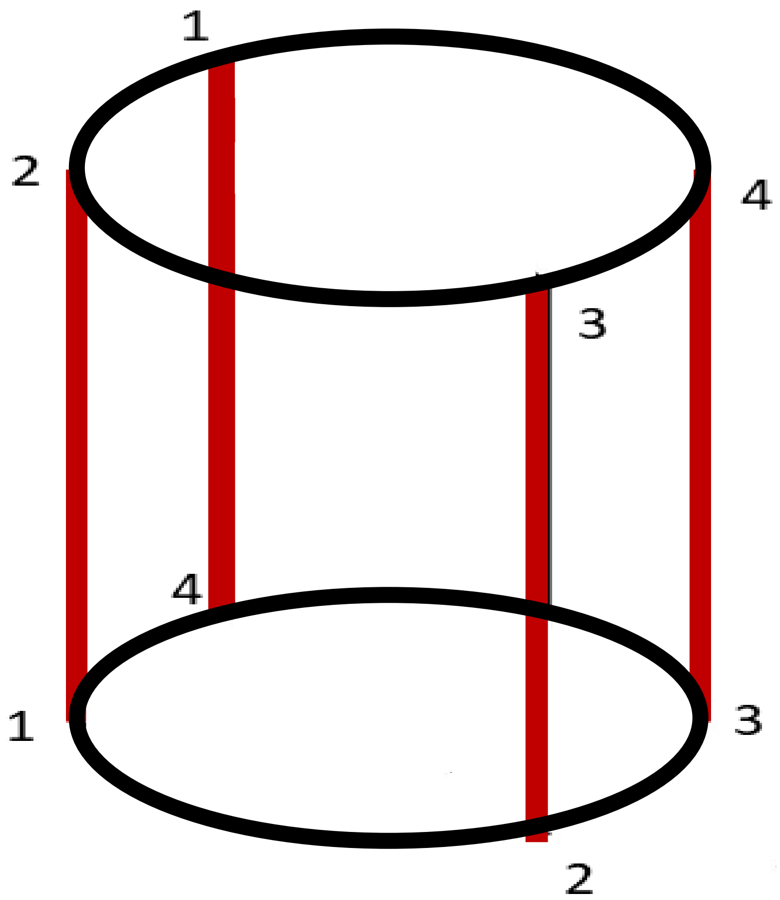

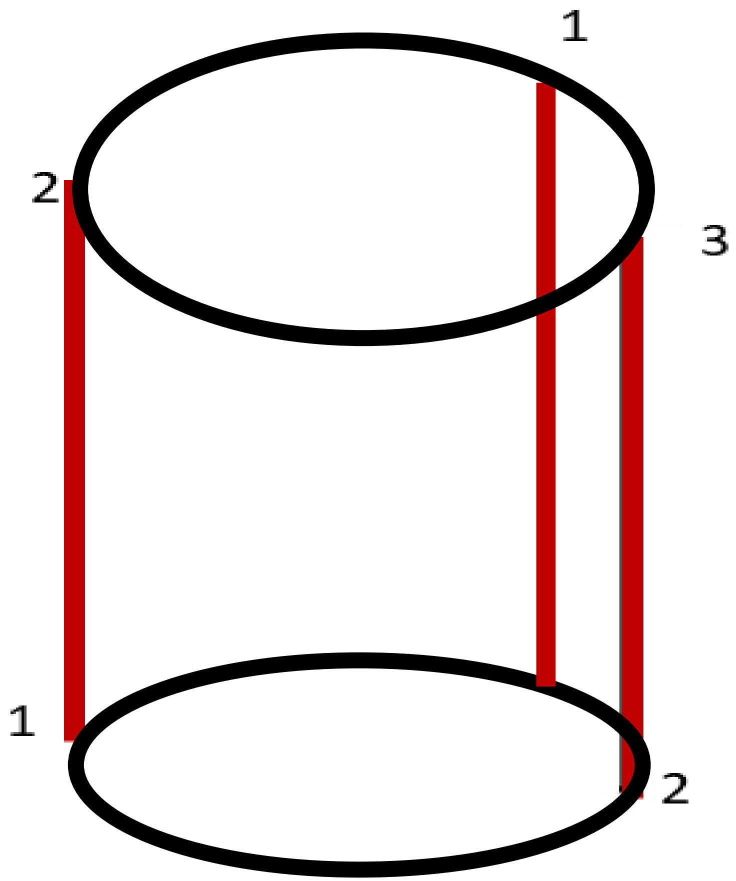

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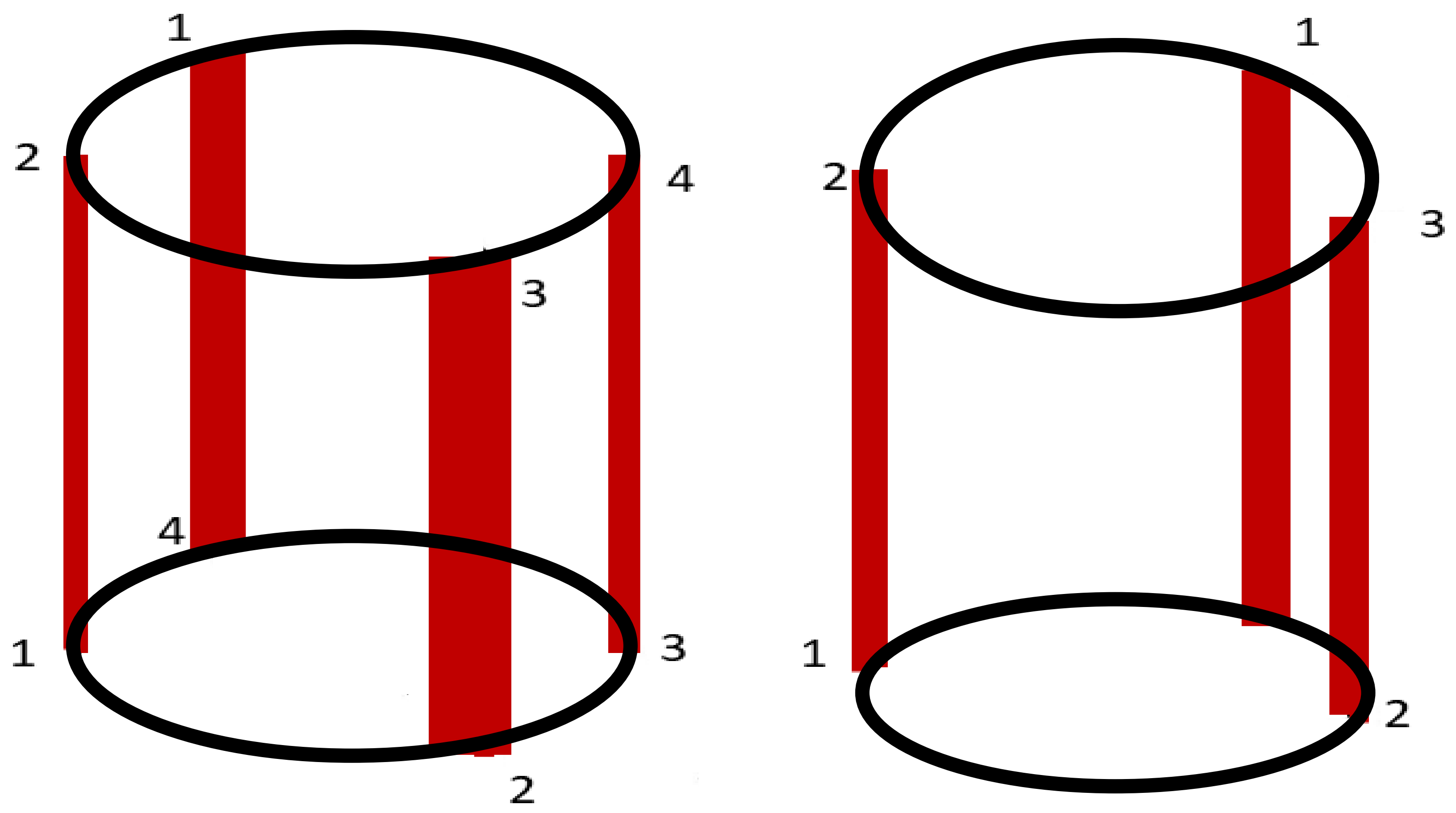

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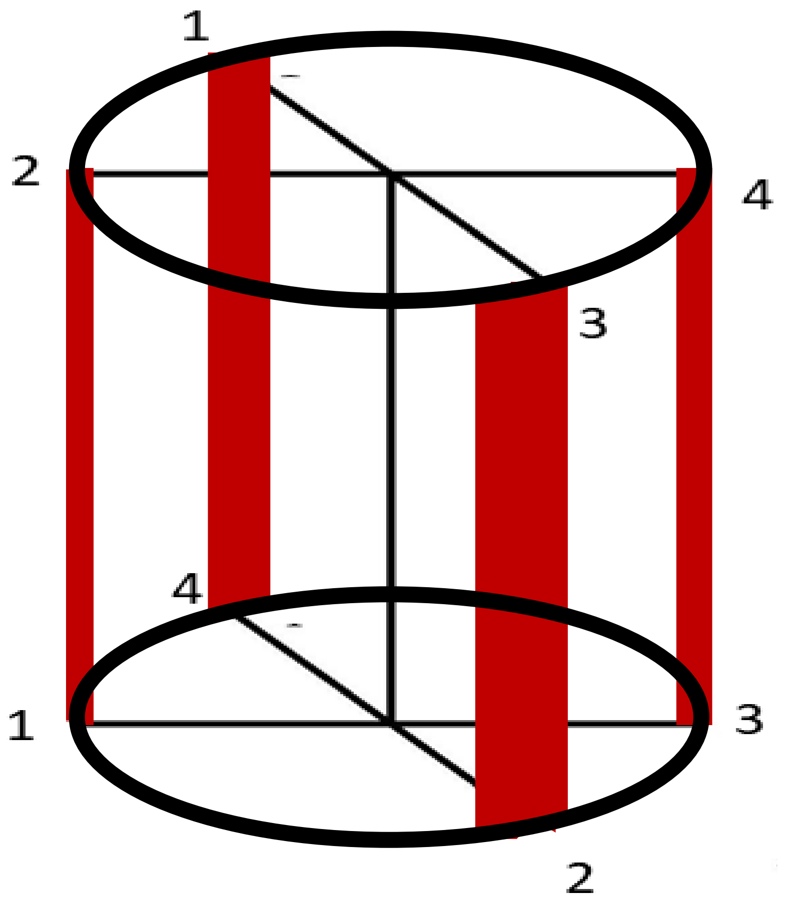

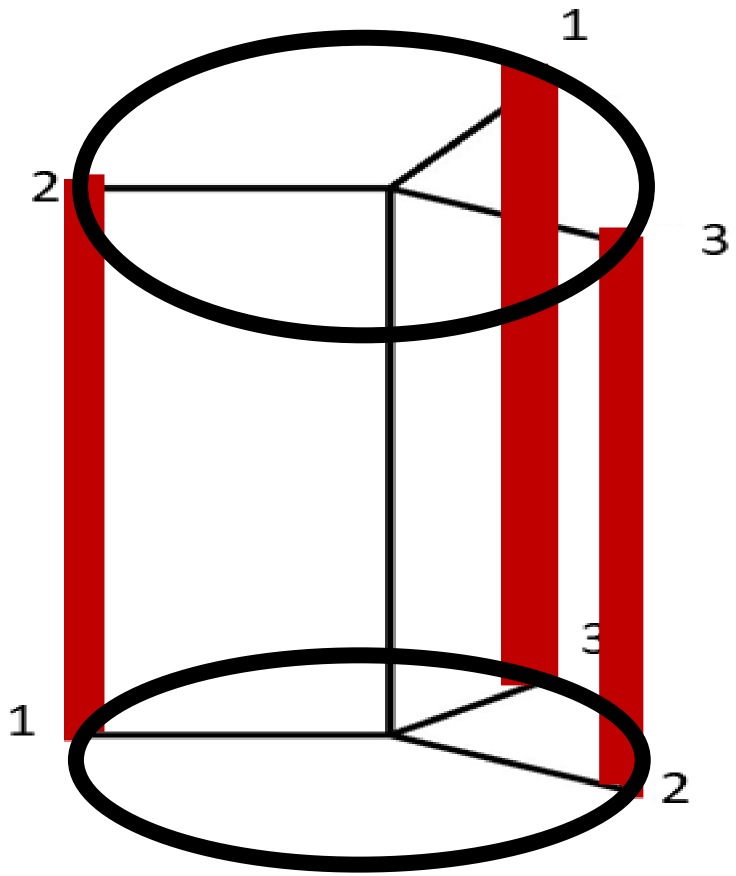

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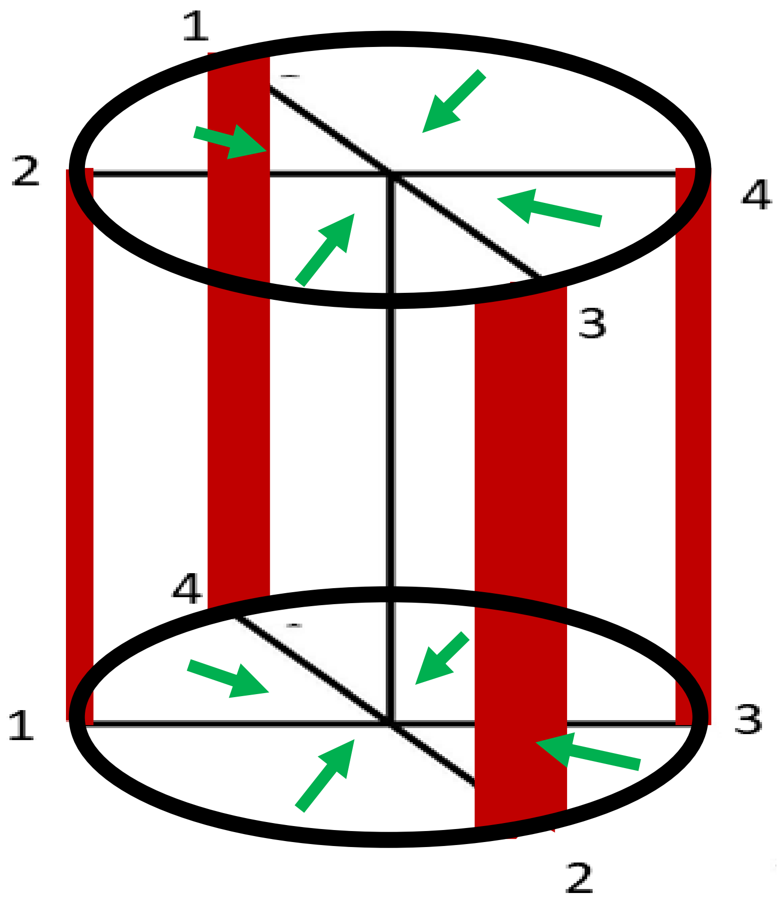

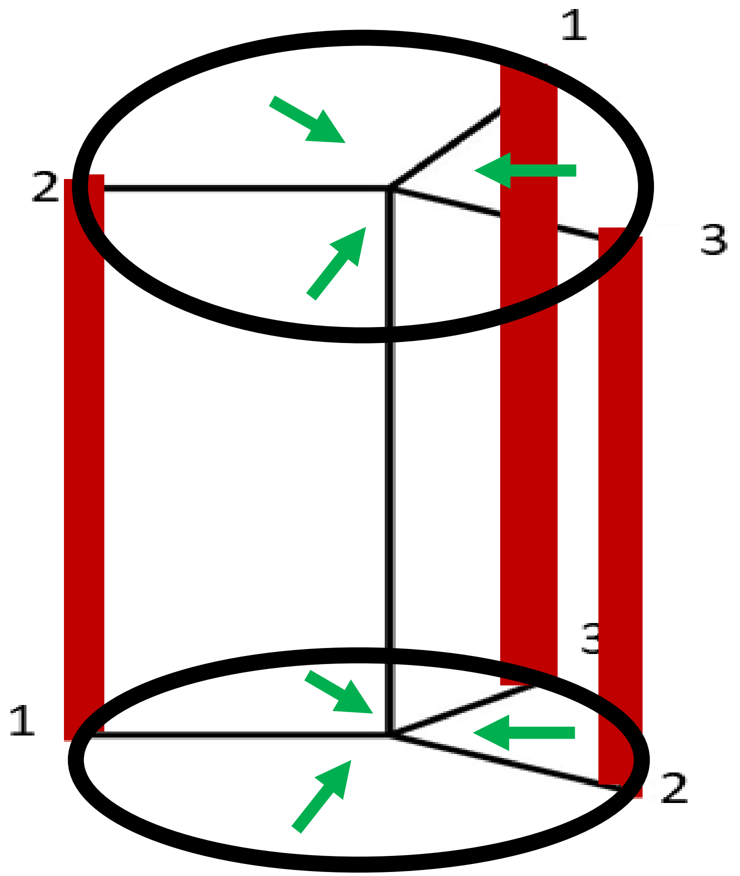

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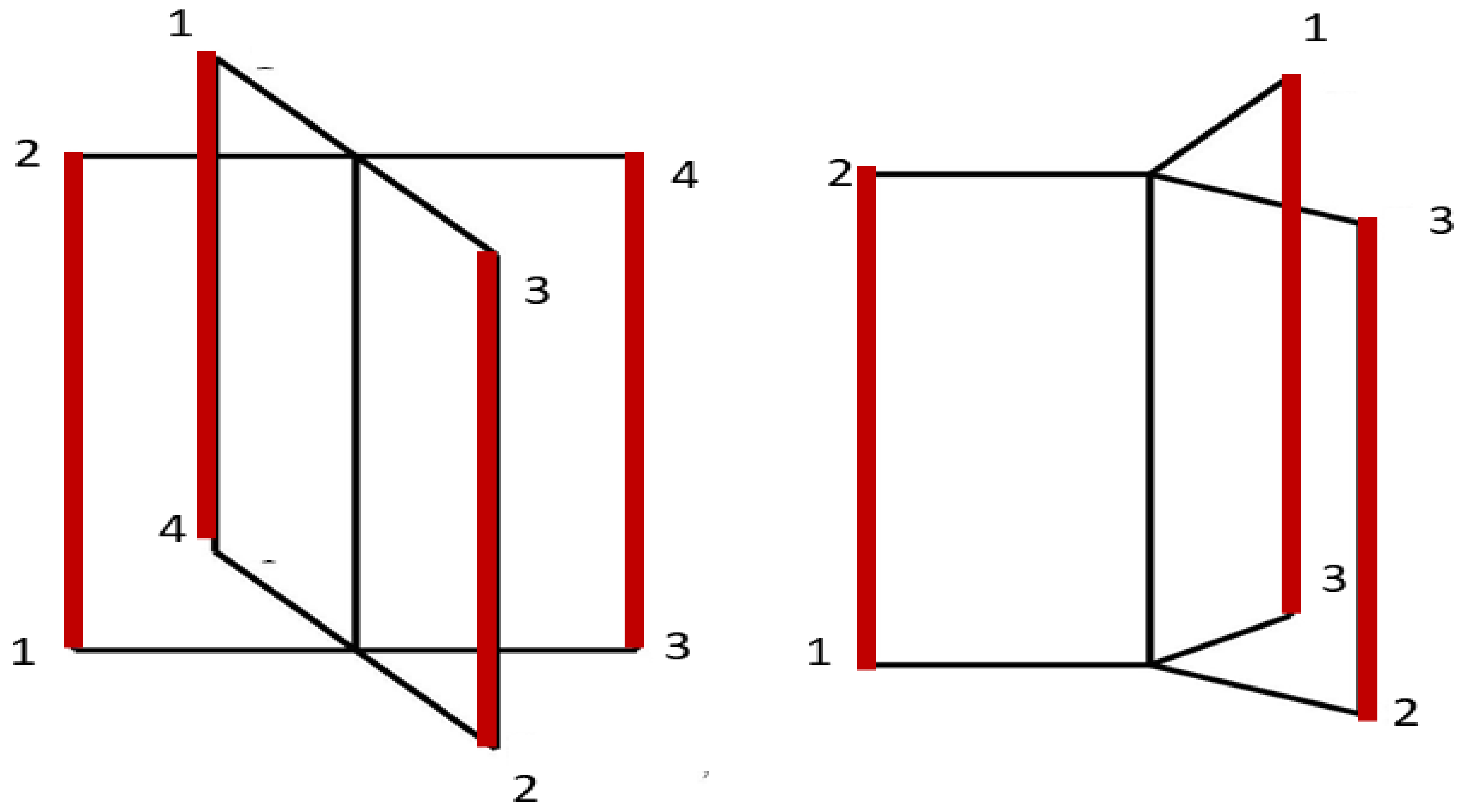

2-Stratifold

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A closed 2-stratifold ***X*** is a 2-complex where

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- ***X*** contains a collection ***S*** of finitely many s.c.c, such that the closure of ***X- S*** is a compact surface,

2-Stratifold

A closed 2-stratifold ***X*** is a 2-complex where

- ***X*** contains a collection ***S*** of finitely many s.c.c, such that the closure of ***X- S*** is a compact surface,

- and a neighborhood of each component in ***S*** consists of more than 2 sheets.

Branch Neighborhoods

Branch Neighborhoods

A simple closed curve in ***S***

Branch Neighborhoods

A simple closed curve in ***S***

- is called a singular curve

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A simple closed curve in ***S***

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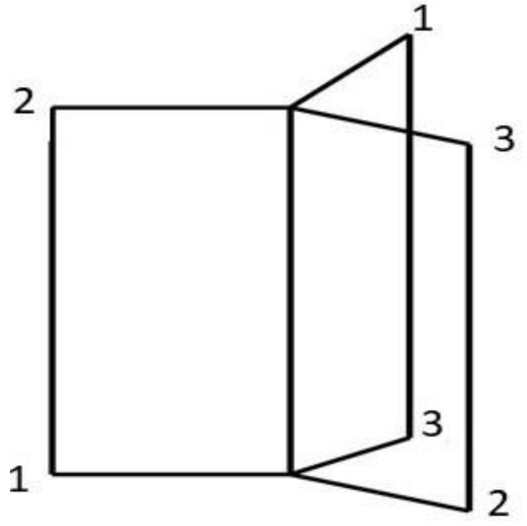
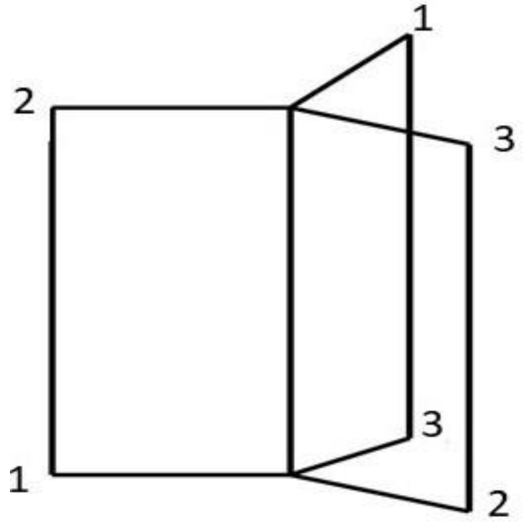
- and a regular neighborhood of a singular curve is called a branch neighborhood.

Trivalent Branch Neighborhoods

Trivalent Branch Neighborhoods

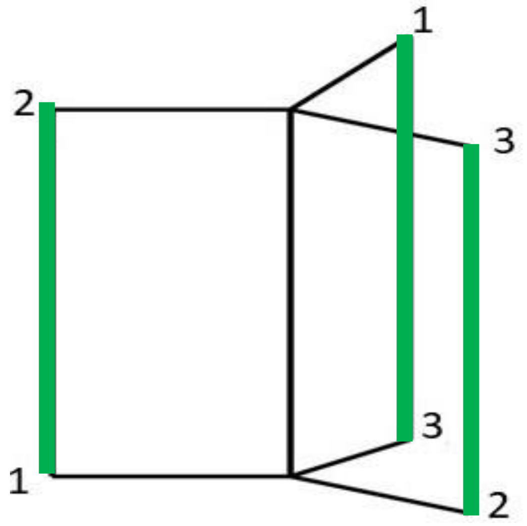


Trivalent Branch Neighborhoods



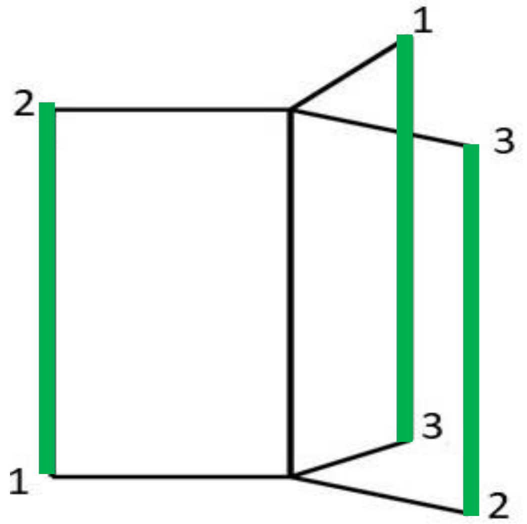
Gluing Action : (123)

Trivalent Branch Neighborhoods



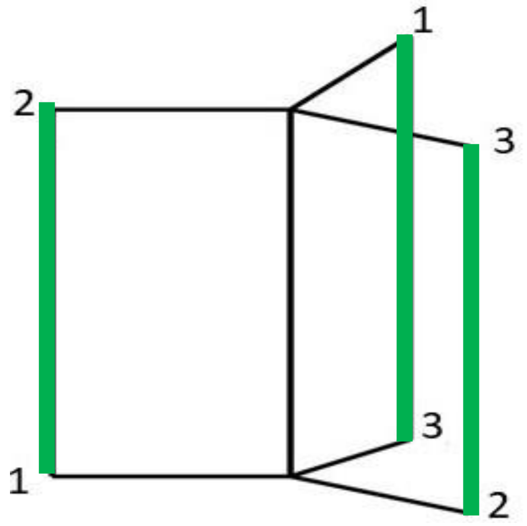
Gluing Action : (123)
Boundary Components : 1

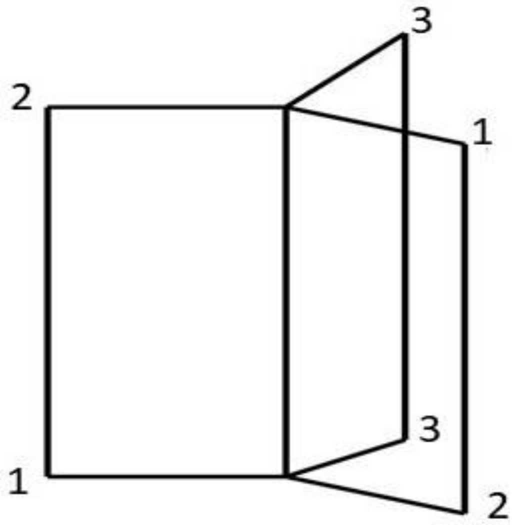
Trivalent Branch Neighborhoods



Gluing Action : (123)
Boundary Components : 1
Boundary Words : 3a

Trivalent Branch Neighborhoods





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Boundary Components : 1	
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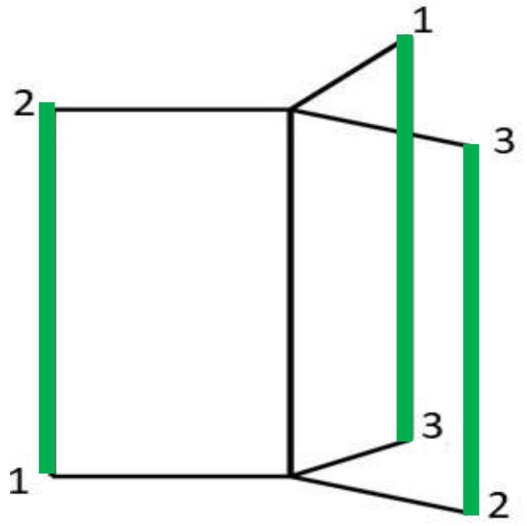
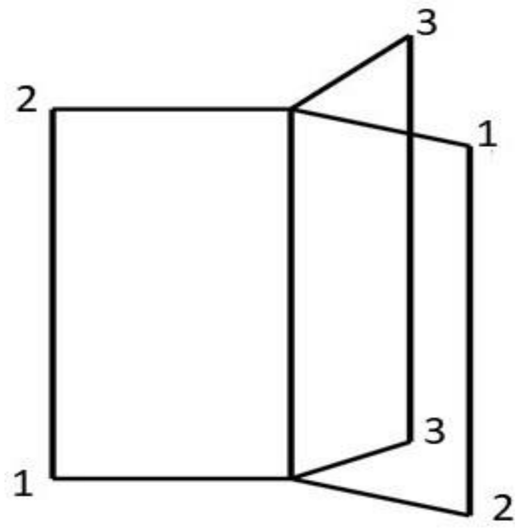
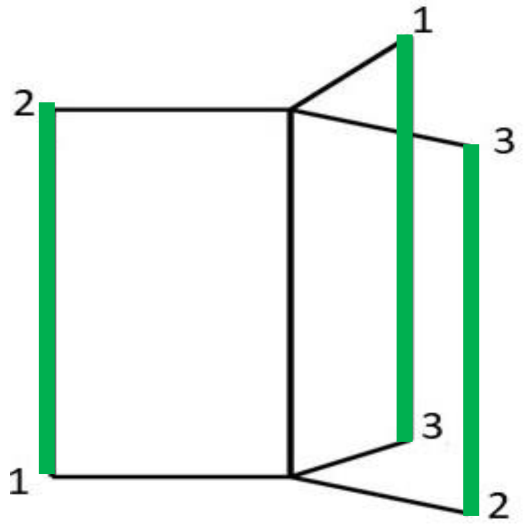
Trivalent Branch Neighborhoods

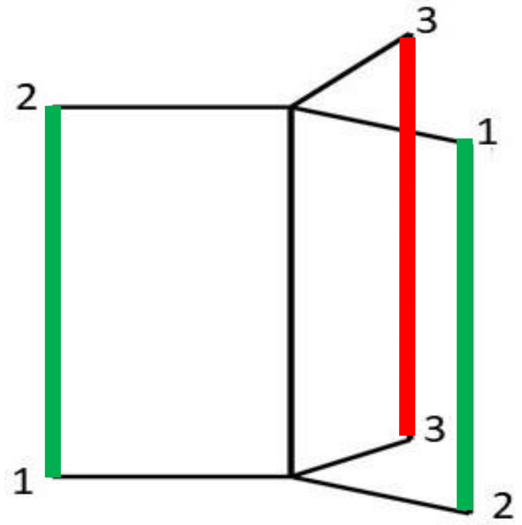




Gluing Action : (123)	Gluing Action : (12)(3)
Boundary Components : 1	
Boundary Words : 3a	

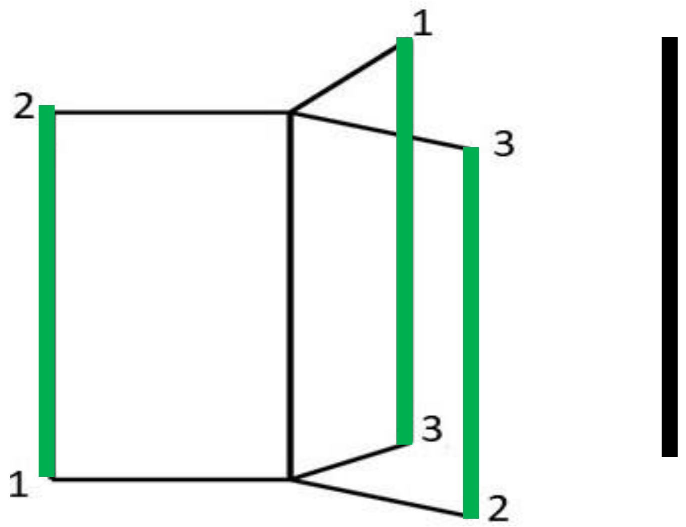
Trivalent Branch Neighborhoods

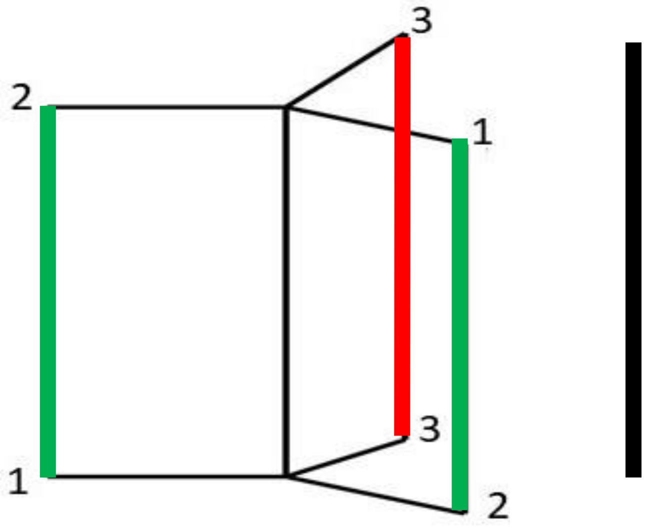




Gluing Action : (123)	Gluing Action : (12)(3)
Boundary Components : 1	Boundary Components : 2
Boundary Words : 3a	

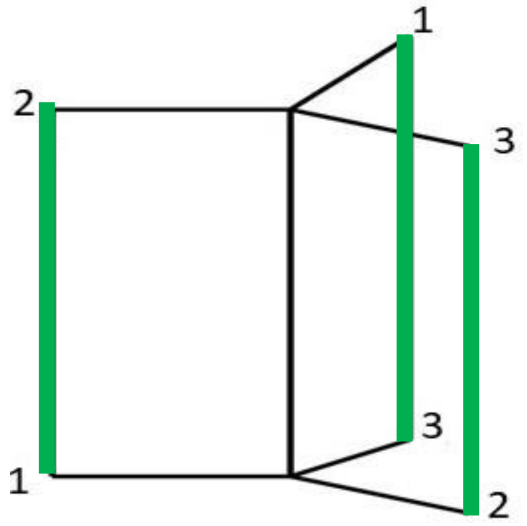
Trivalent Branch Neighborhoods

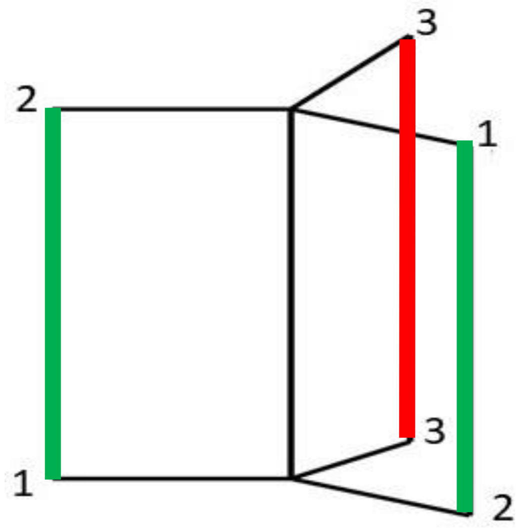


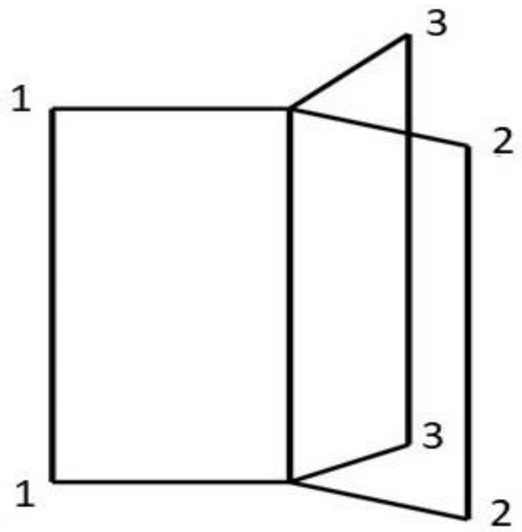


Gluing Action : (123)	Gluing Action : (12)(3)
Boundary Components : 1	Boundary Components : 2
Boundary Words : 3a	Boundary Words : 2a, a

Trivalent Branch Neighborhoods

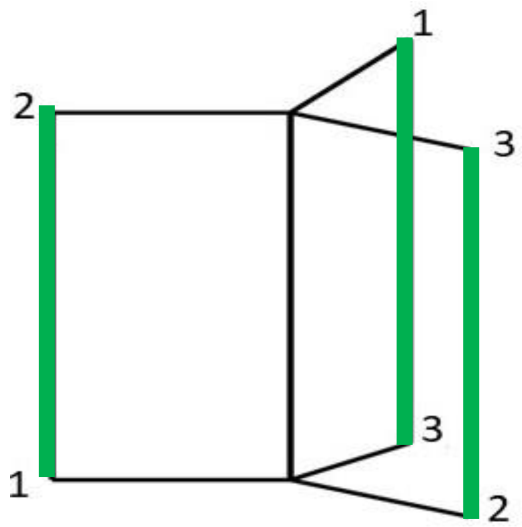


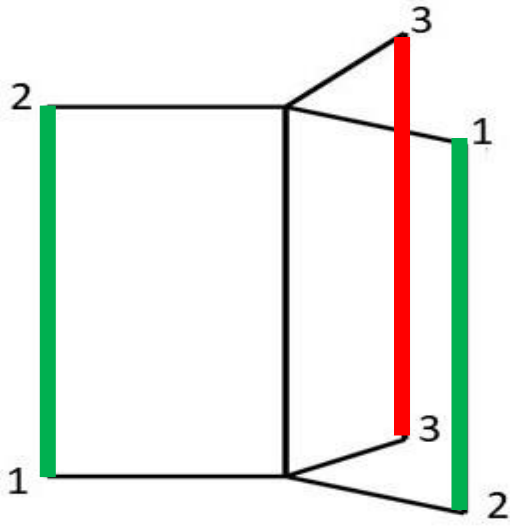


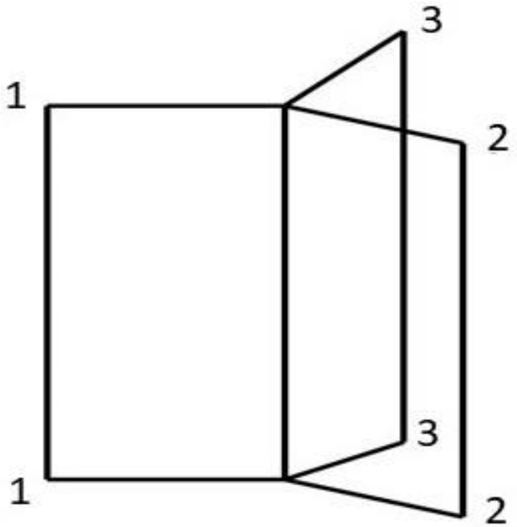


Gluing Action : (123)	Gluing Action : (12)(3)	
Boundary Components : 1	Boundary Components : 2	
Boundary Words : 3a	Boundary Words : 2a, a	

Trivalent Branch Neighborhoods

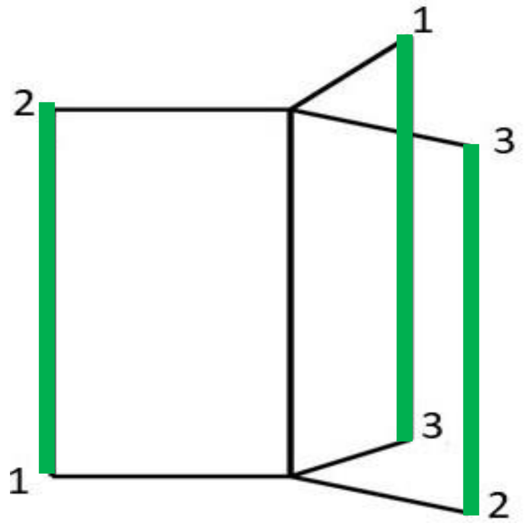


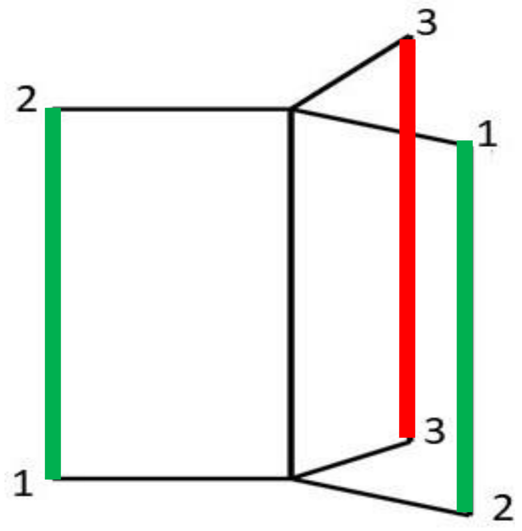


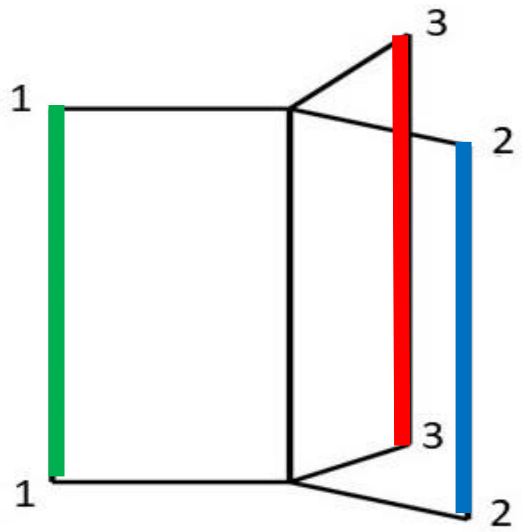


Gluing Action : (123)	**Gluing Action : (12)(3)**	**Gluing Action : (1)(2)(3)**
Boundary Components : 1	Boundary Components : 2	
Boundary Words : 3a	Boundary Words : 2a, a	

Trivalent Branch Neighborhoods







Gluing Action : (123)	Gluing Action : (12)(3)	Gluing Action : (1)(2)(3)
Boundary Components : 1	Boundary Components : 2	Boundary Components : 3
Boundary Words : 3a	Boundary Words : 2a, a	

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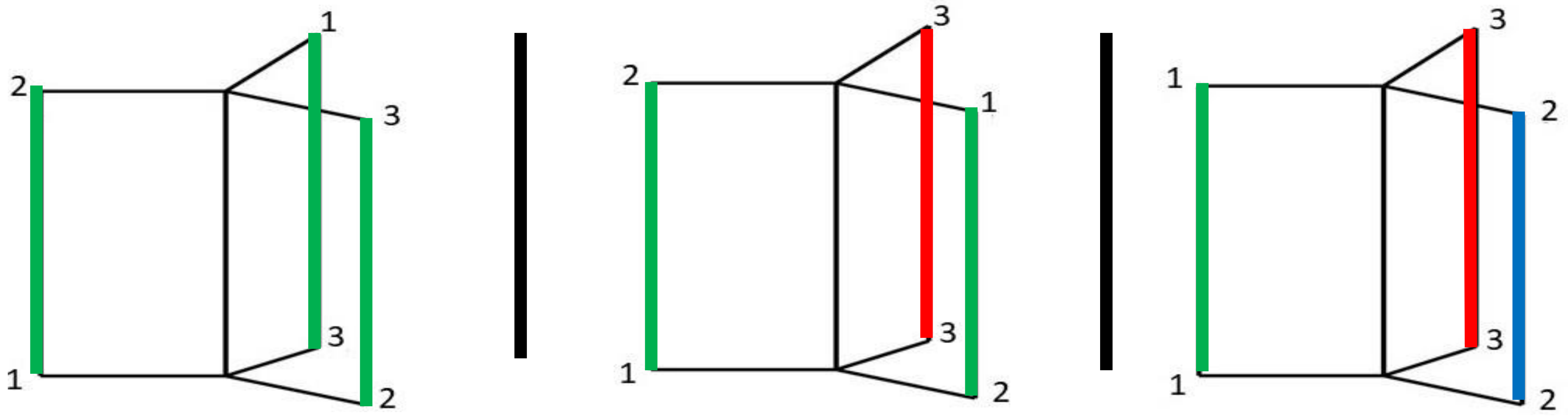
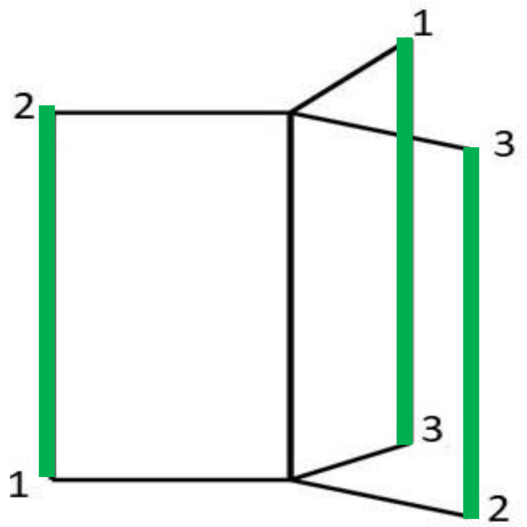
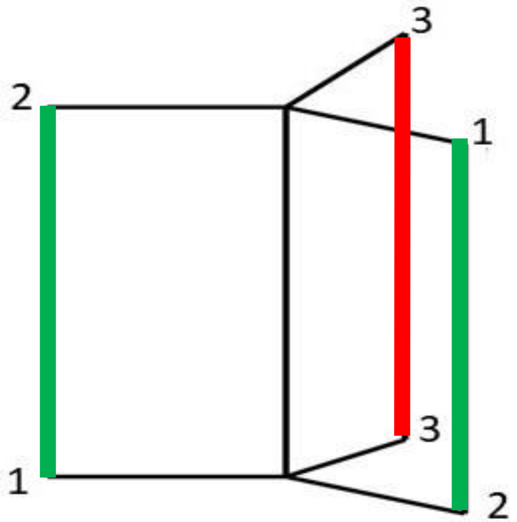
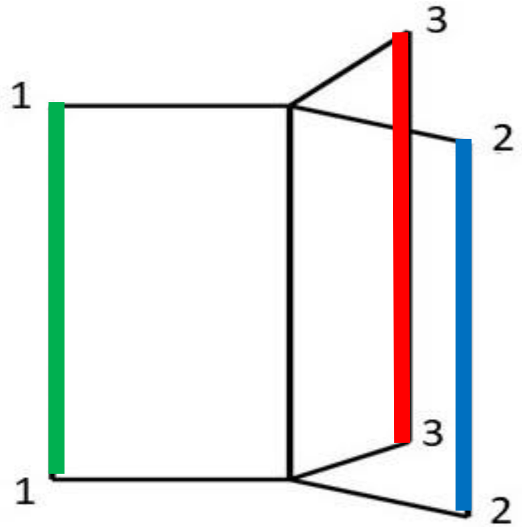


Gluing Action : (123)	Gluing Action : (12)(3)	Gluing Action : (1)(2)(3)
Boundary Components : 1	Boundary Components : 2	Boundary Components : 3
Boundary Words : 3a	Boundary Words : 2a, a	Boundary Words : a, a, a

Trivalent Branch Neighborhoods







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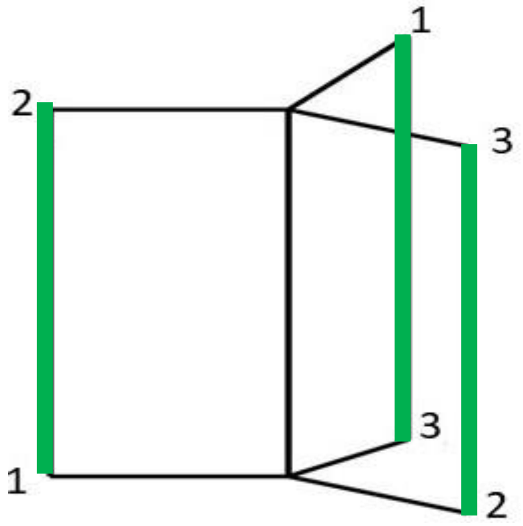

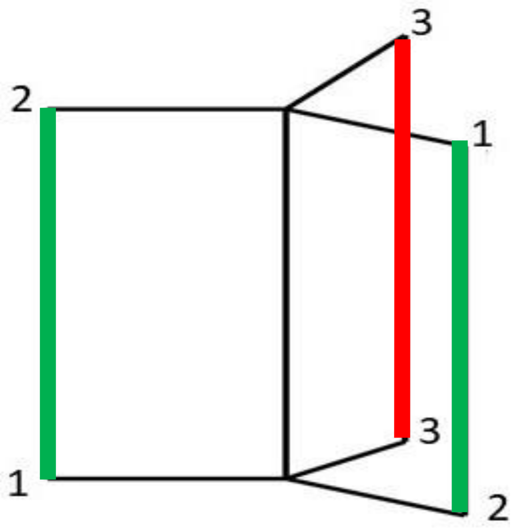

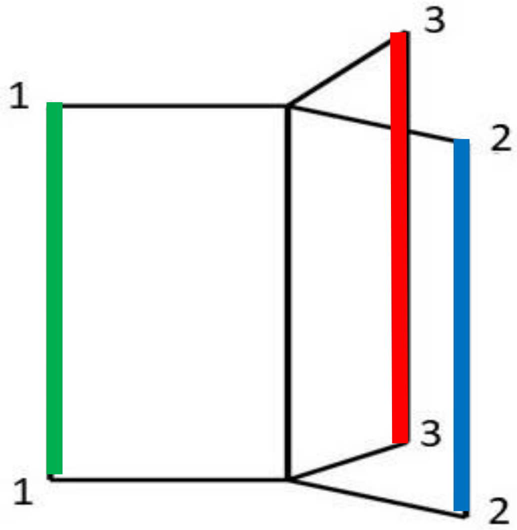

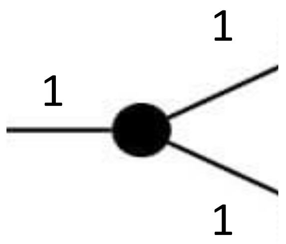

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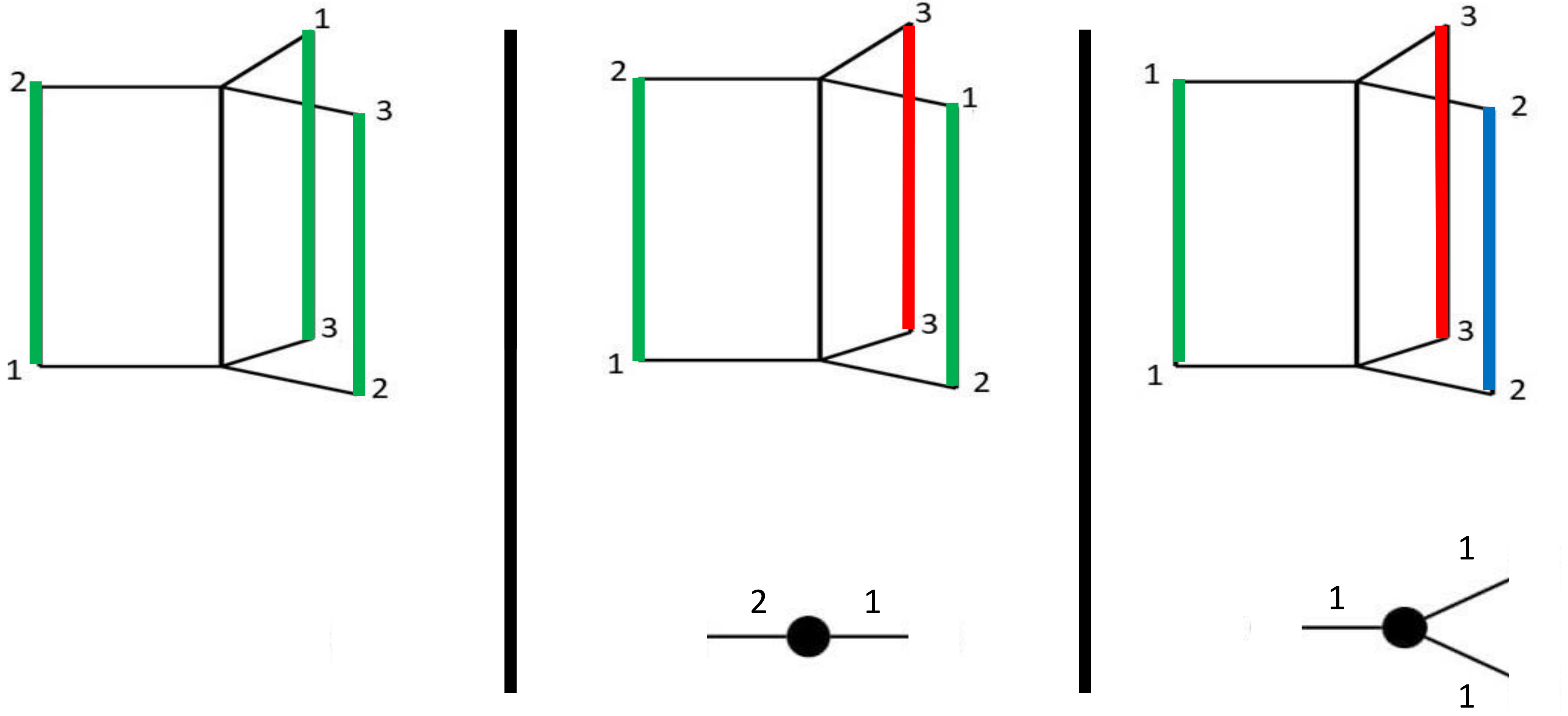

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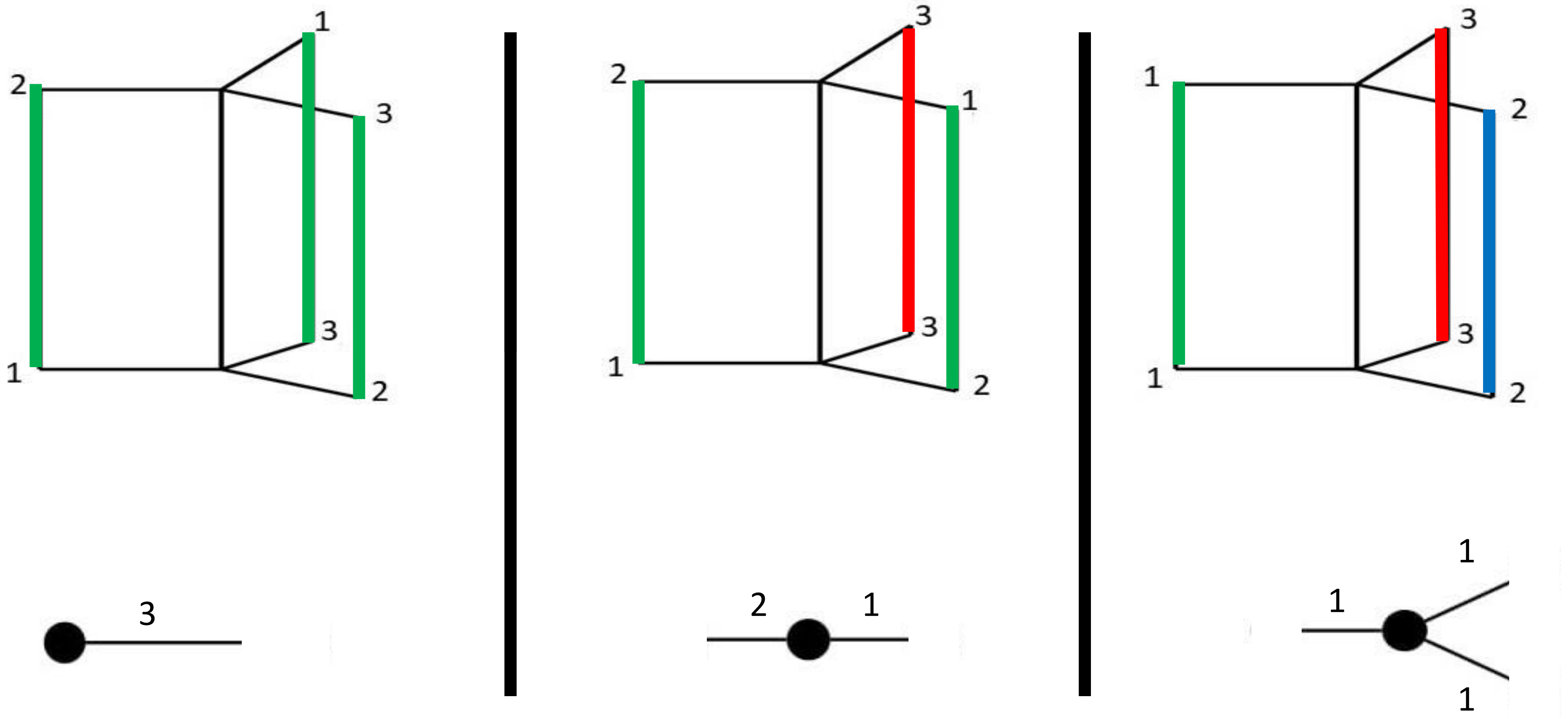

Embedding 2-Stratifolds into Manifolds

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- Every 2-stratifold embeds into 4-space.

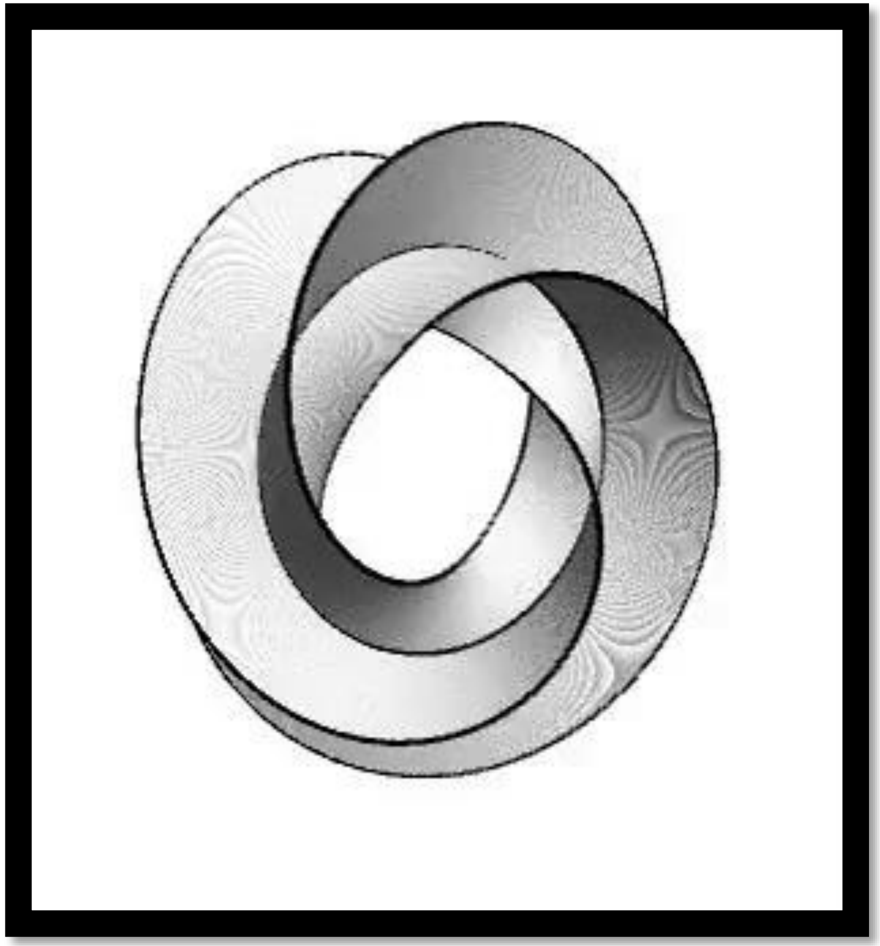
  S. Matsuzaki, M. Ozawa

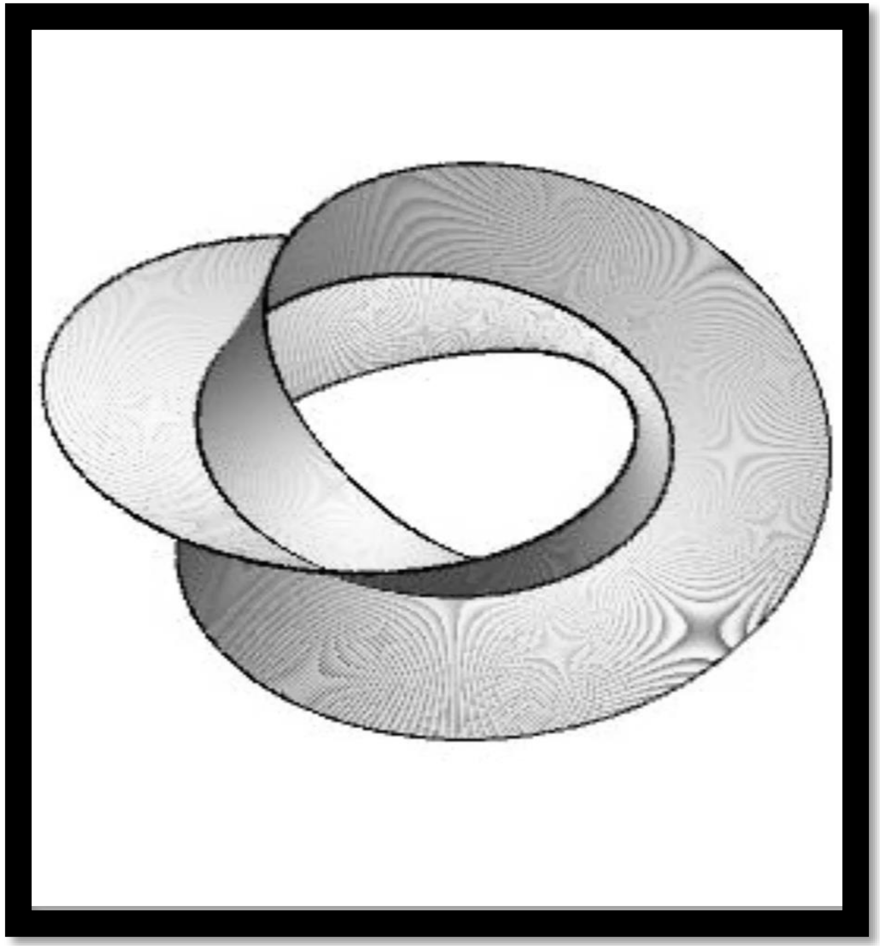
Embedding 2-Stratifolds into Manifolds

- Every 2-stratifold embeds into 4-space.
  S. Matsuzaki, M. Ozawa

- For a 2-stratifold to be embeddable in a 3-manifold it is necessary that for each branch neighborhood the boundary words are the same.

Embeddings of Branch Neighborhoods





Associated Graph

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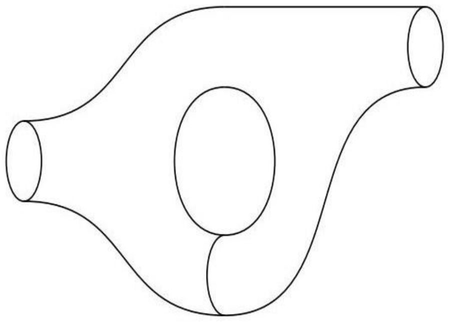

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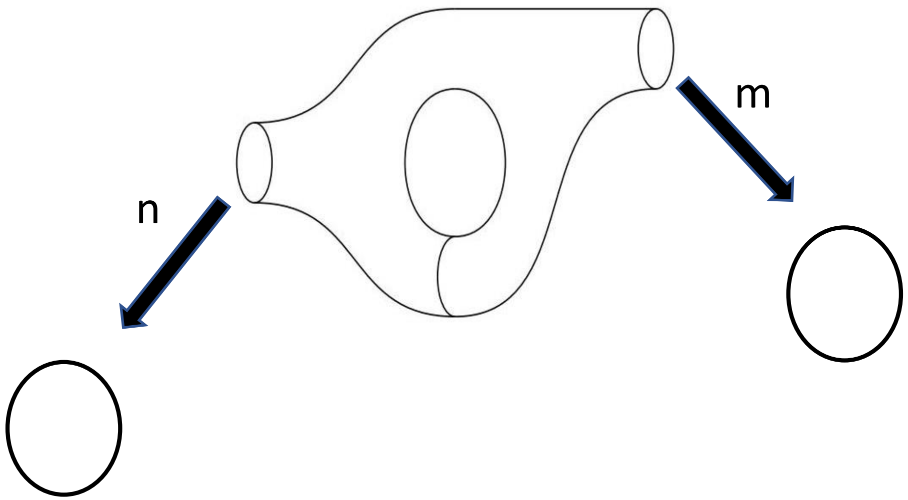

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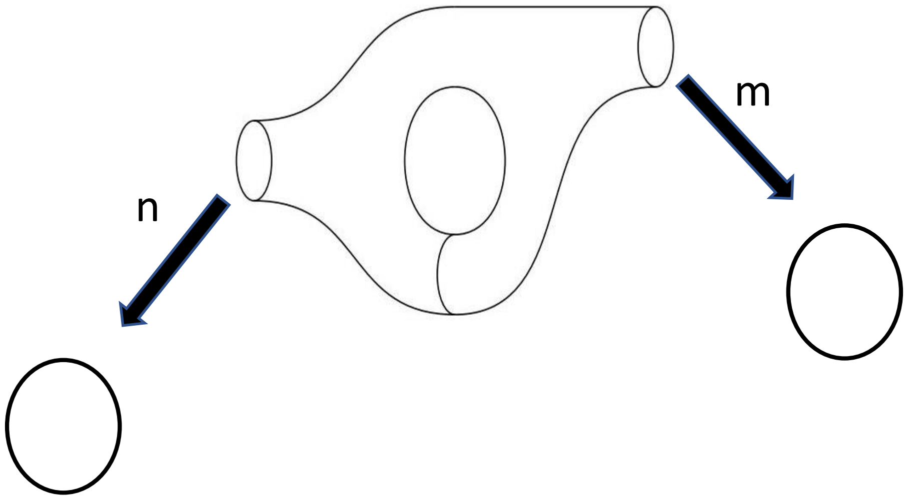

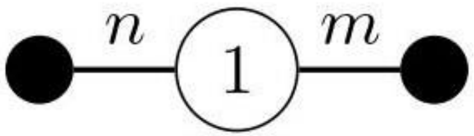

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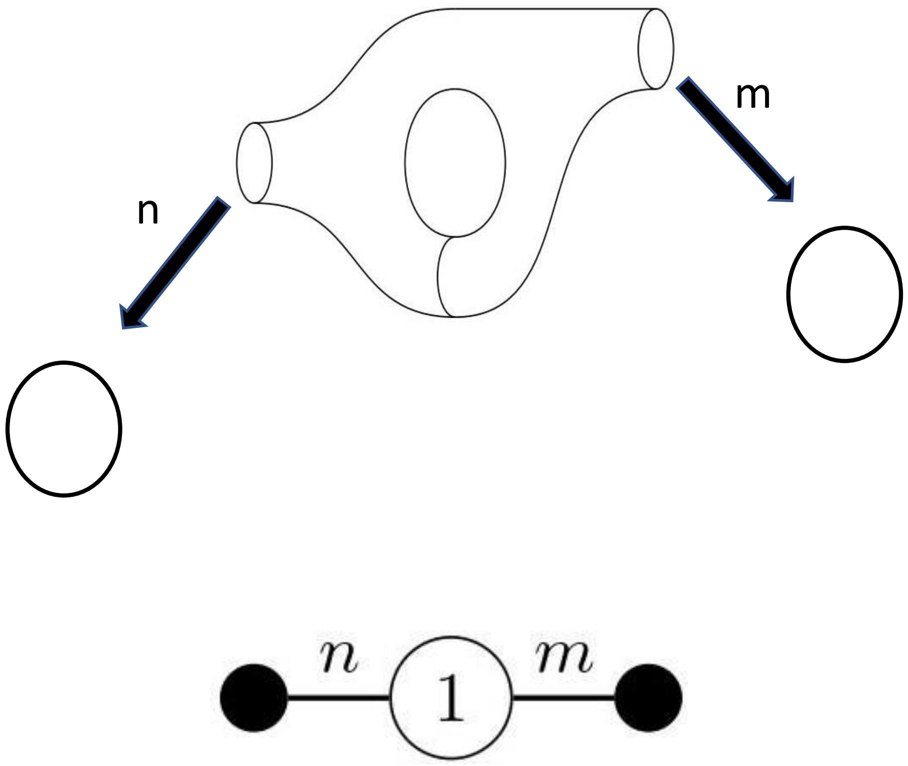
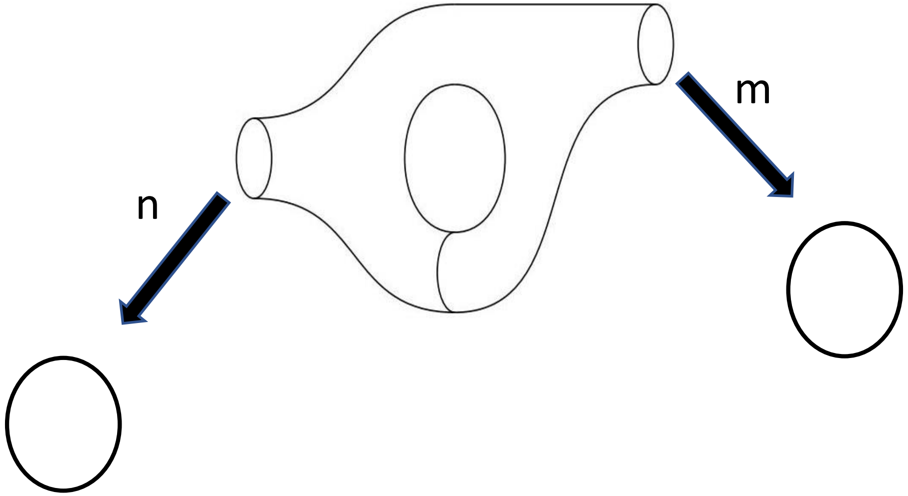
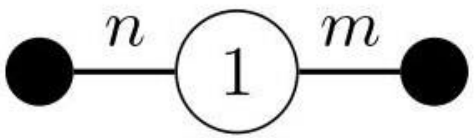


Generators

$$C, D, E, F, b_1, b_2$$

Associated Graph



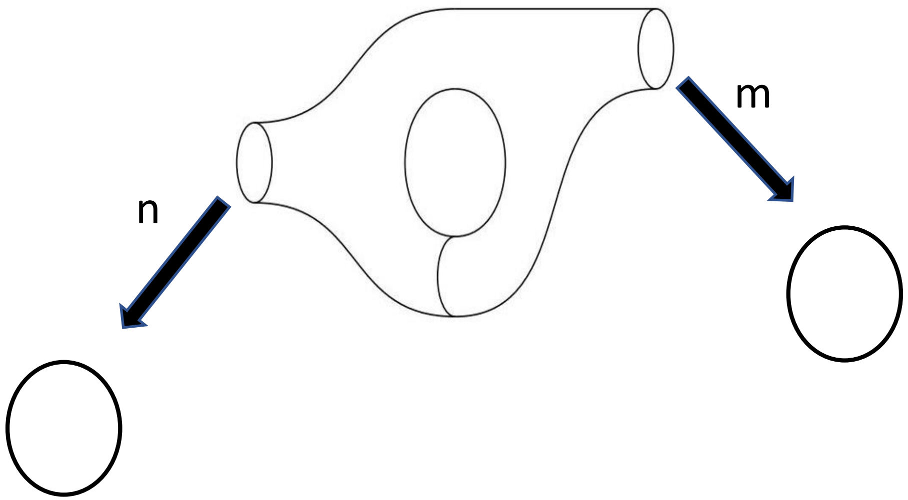


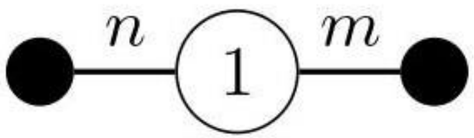
Generators

$$C, D, E, F, b_1, b_2$$

Generators from Torus

Associated Graph





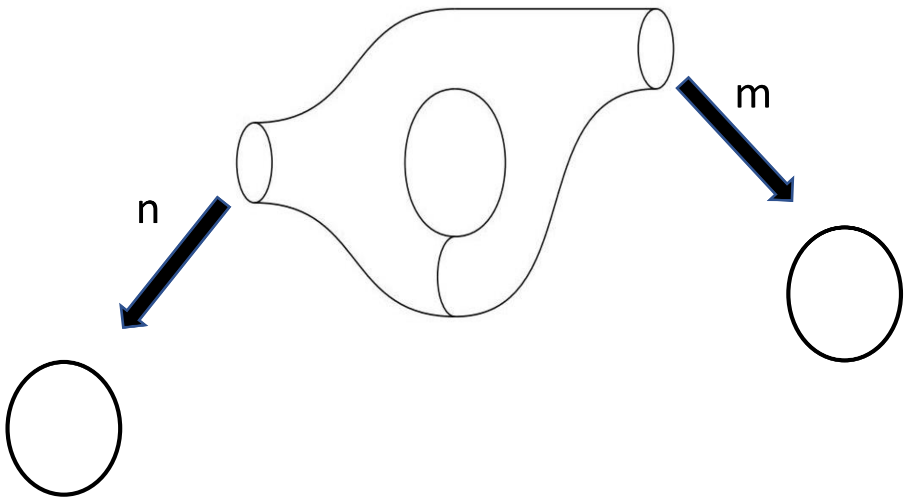
Generators

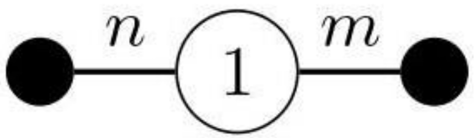
Generators from singular curves

$$\underbrace{C, D, E, F,}_{\text{Generators from Torus}} \overline{b_1}, b_2$$

Generators from Torus

Associated Graph





Generators

Generators from singular curves

$$C, D, E, F, b_1, b_2$$

Generators from Torus

Relations

$$b_1^n = E, b_2^m = F, [C, D]EF = 1$$

Associated Graph

.

- 2-Stratifolds are determined almost uniquely by their associated labeled graph

Part 2 : Theorems and Classifying 2-Stratifolds

Classifying 2-stratifolds

Classifying 2-stratifolds

Classification of closed surfaces groups

Classification of closed surfaces

Classifying 2-stratifolds

Classification of closed surfaces groups

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Classification of closed surfaces

Classification of closed 2-stratifold groups

Classifying 2-stratifolds

Classification of closed surfaces groups $\longleftrightarrow$ Classification of closed surfaces

Classification of closed 2-stratifold groups $\longrightarrow$ Classification of closed 2-stratifolds

Some Goals

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Main Goal.

Some Goals

Main Goal. Given a 2-stratifold group G, enumerate all possible 2-stratifolds whose fundamental group is G.

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Secondary Goal.

Some Goals

Main Goal. Given a 2-stratifold group G, enumerate all possible 2-stratifolds whose fundamental group is G.

Secondary Goal. Given a bipartite labeled graph representing a 2-stratifold, determine if the fundamental group is of a given type.

Initial Questions

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Question 1. What are the finite 2-stratifolds groups?

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Question 2. What is the graph type of a 2-stratifold with finite fundamental group?

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Answer 1. Finite Fuchsian Groups.

Question 2. What is the graph type of a 2-stratifold with finite fundamental group?

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Answer 1. Finite Fuchsian Groups.

Question 2. What is the graph type of a 2-stratifold with finite fundamental group?

Answer 2. Tree where almost all white vertices have genus zero and at most one black terminal vertex.

Trivalent Classification

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2-stratifolds where at most 3 sheets meet are trivalent.

Trivalent Classification

2-stratifolds where at most 3 sheets meet are trivalent.

- Trivalent 2-stratifold with trivial or infinite cyclic fundamental group.

  J.C. Gómez-Larrañaga, F. González-Acuña, and W. Heil

Trivalent Classification

2-stratifolds where at most 3 sheets meet are trivalent.

- Trivalent 2-stratifold with trivial or infinite cyclic fundamental group.

  J.C. Gómez-Larrañaga, F. González-Acuña, and W. Heil

- Trivalent 2-stratifolds with finite fundamental group.

  B.

Trivalent Algorithm

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Given a trivalent 2-stratifold it can be determined if

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Given a trivalent 2-stratifold it can be determined if

- It is has trivial or infinite cyclic fundamental group.

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Trivalent Algorithm

Given a trivalent 2-stratifold it can be determined if

- It is has trivial or infinite cyclic fundamental group.

  J.C. Gómez-Larrañaga, F. González-Acuña, and W. Heil

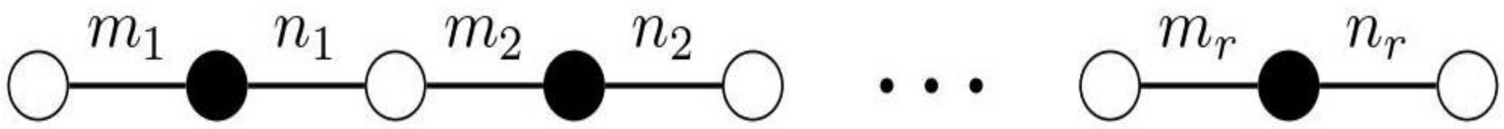
- It has finite fundamental group.

  B.

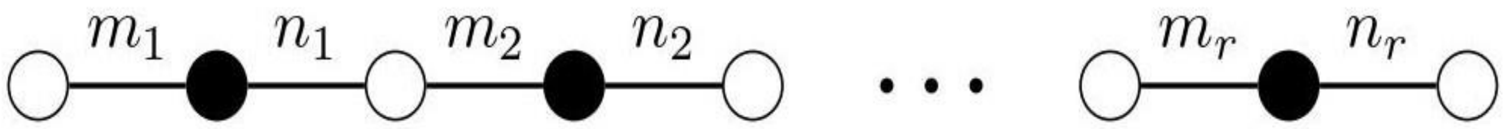
Part 3 : Classifying Trivalent 2-Stratifolds with Finite Fundamental group

Linear Stratifolds

Linear Stratifolds



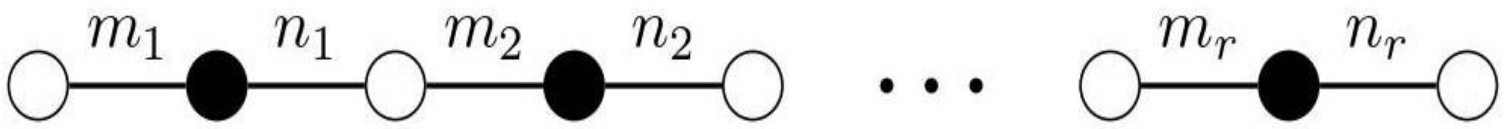
Linear Stratifolds



$$X_1^{n_1} = X_2^{m_2} \cdots X_{r-1}^{n_{r-1}} = X_r^{m_r}$$

$$X_1^{m_1} = 1 \qquad 1 = X_r^{n_r}$$

Linear Stratifolds



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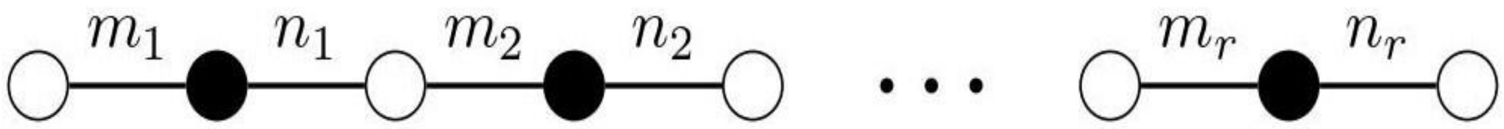

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Finite π_1

$\updownarrow$

Finite Cyclic

Star Stratifolds

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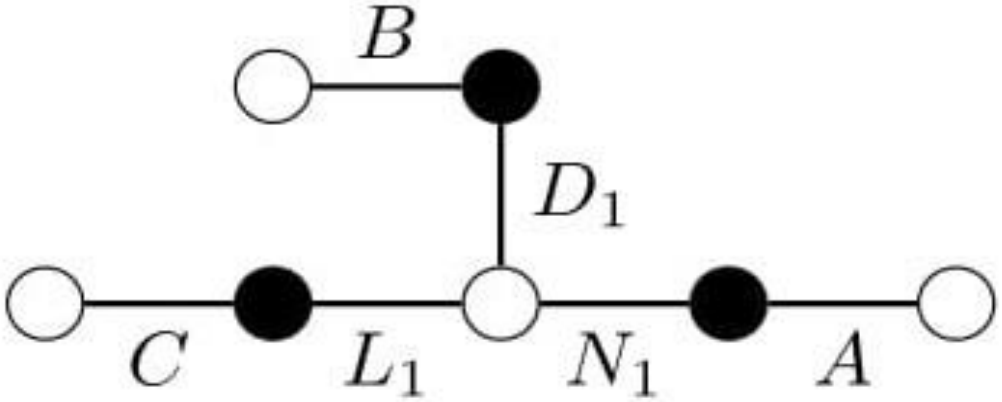

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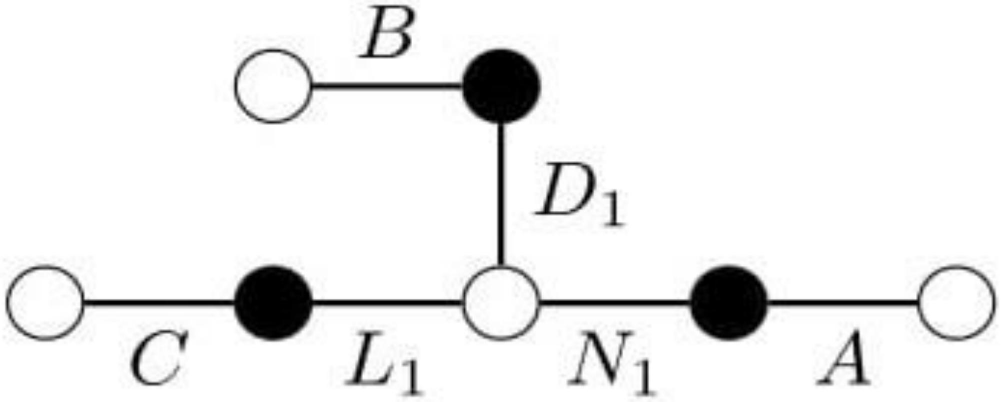

$X_1^A = 1 \qquad X_1^{N_1} X_2^{D_1} X_3^{L_1} = 1$

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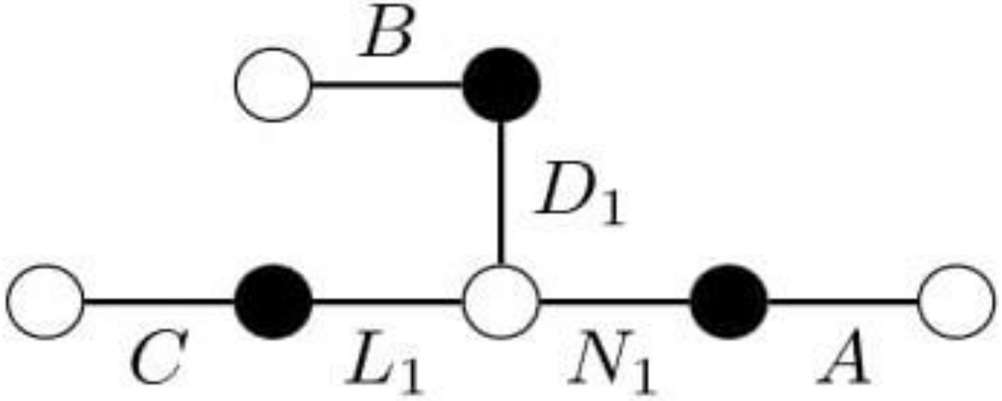
Star Stratifolds



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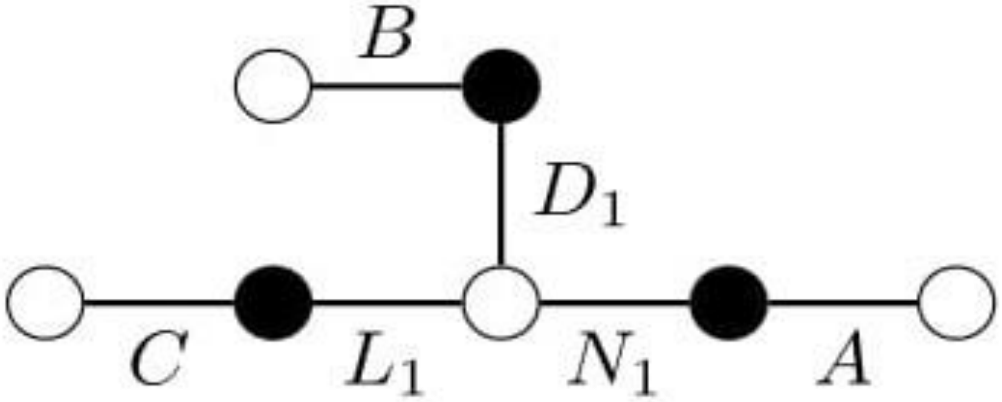
Star Stratifolds



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Finite π_1

$\updownarrow$

Dihedral,
Tetrahedral,
Octahedral, or
Dodecahedral

Pruned Trivalent Stratifolds

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- No terminal edges have label 1

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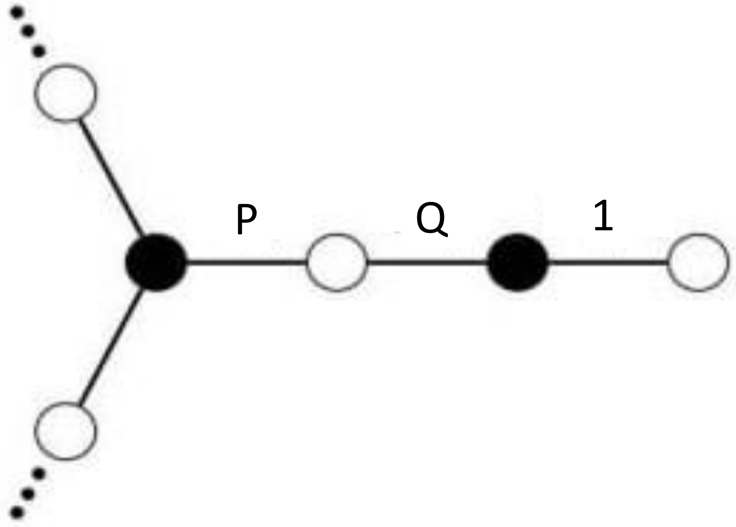
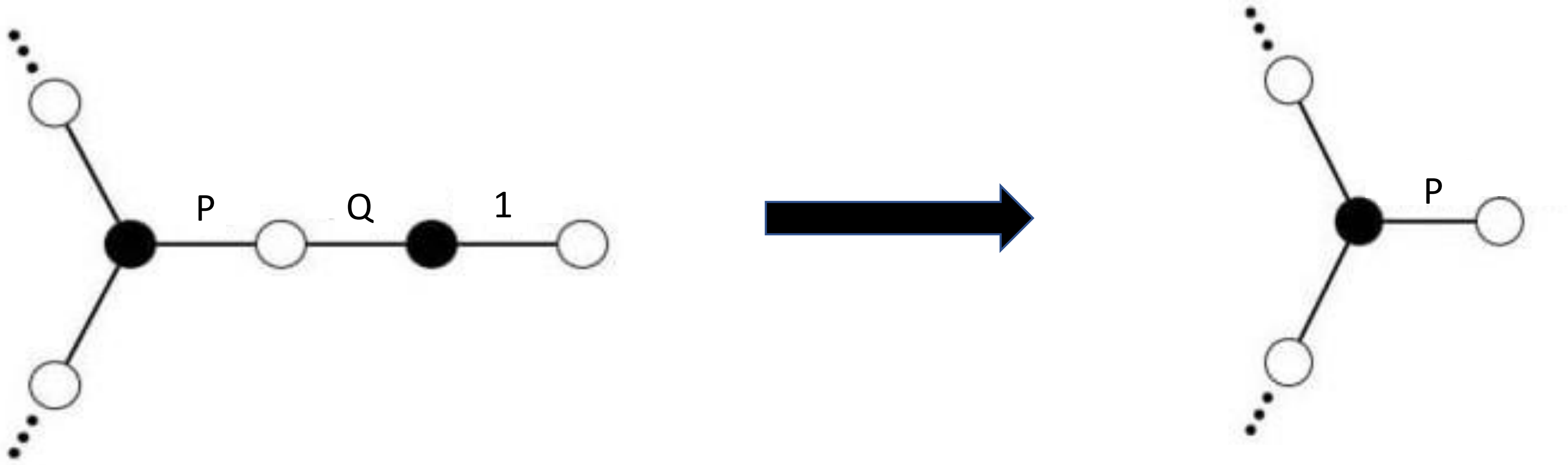


Pruned Trivalent Stratifolds

- No terminal edges have label 1



Some Facts

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Lemma. If a trivalent 2-stratifold has finite fundamental group then

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- almost all white vertices are genus zero,

Some Facts

Lemma. If a trivalent 2-stratifold has finite fundamental group then

- the associated graph is a tree,
- almost all white vertices are genus zero,
- and there is at most one of either a white vertex of degree 3, a black terminal vertex, or a terminal white vertex of genus -1.

Some Facts

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Corollary. The trivalent finite 2-stratifold groups are

Some Facts

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- Cyclic of order 2^n

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Trivalent Branch Neighborhoods

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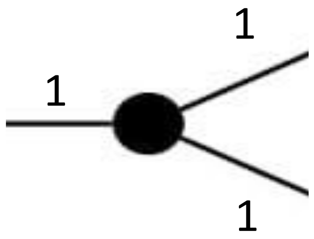


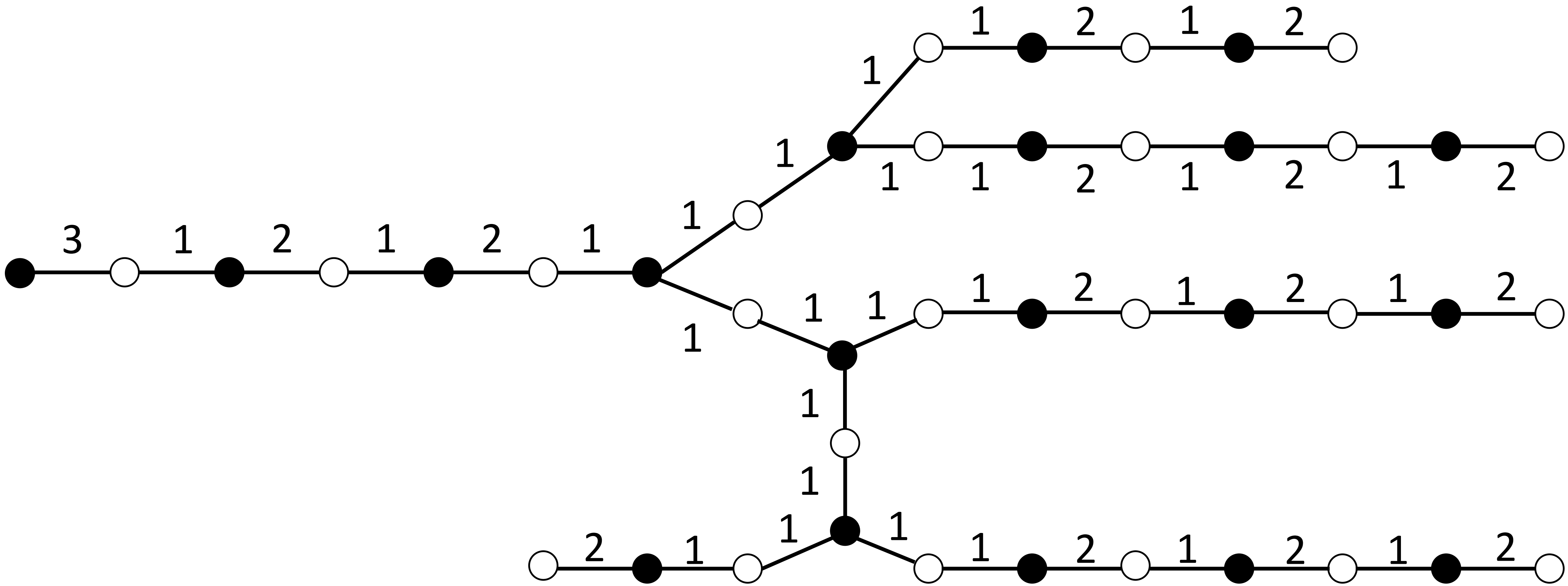
Cyclic Fundamental Group

Let X be a trivalent pruned 2-stratifold with cyclic fundamental group. If the fundamental group of X is cyclic of order $(3)2^n$ then the associated graph has

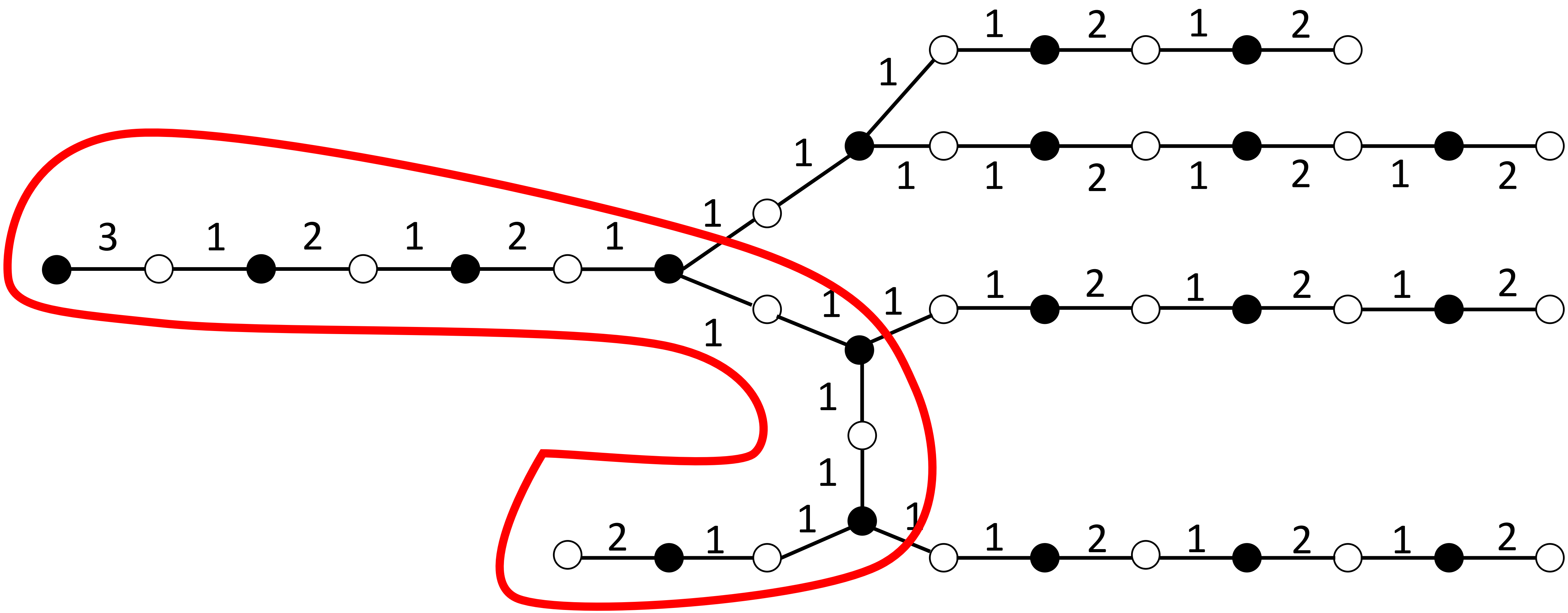
1. only white vertices of degree < 3 and
2. exactly one terminal black vertex.

Furthermore, all linear subgraphs L(V,T) starting at the black terminal vertex V and ending at a terminal vertex T has the labeling (12)(11)…(11)(12)(11)…(12) with at least n labels being 2 and there is at least one linear subgraph L having exactly n copies of 2 in their labeling.

Cyclic Fundamental Group



Cyclic Fundamental Group



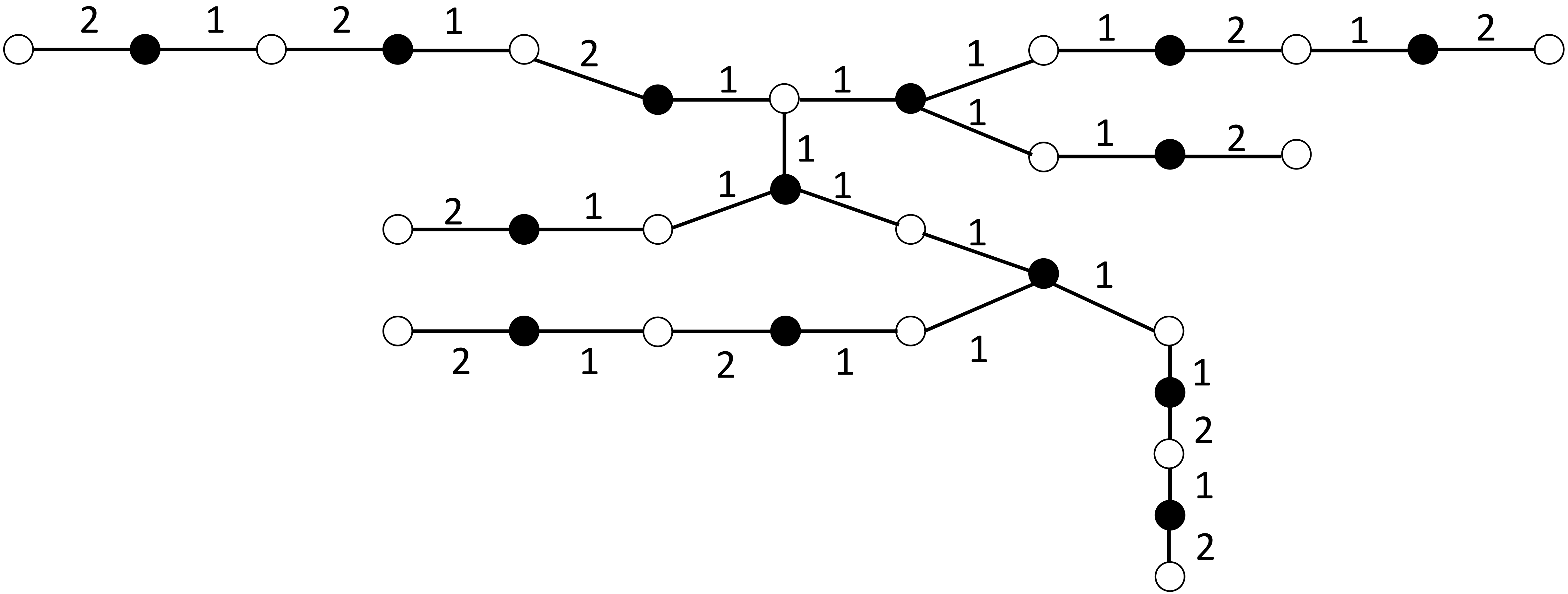
Dihedral Group

Let X be a trivalent pruned 2-stratifold with dihedral fundamental group. If the fundamental group of X is of order 2^n then the associated graph has

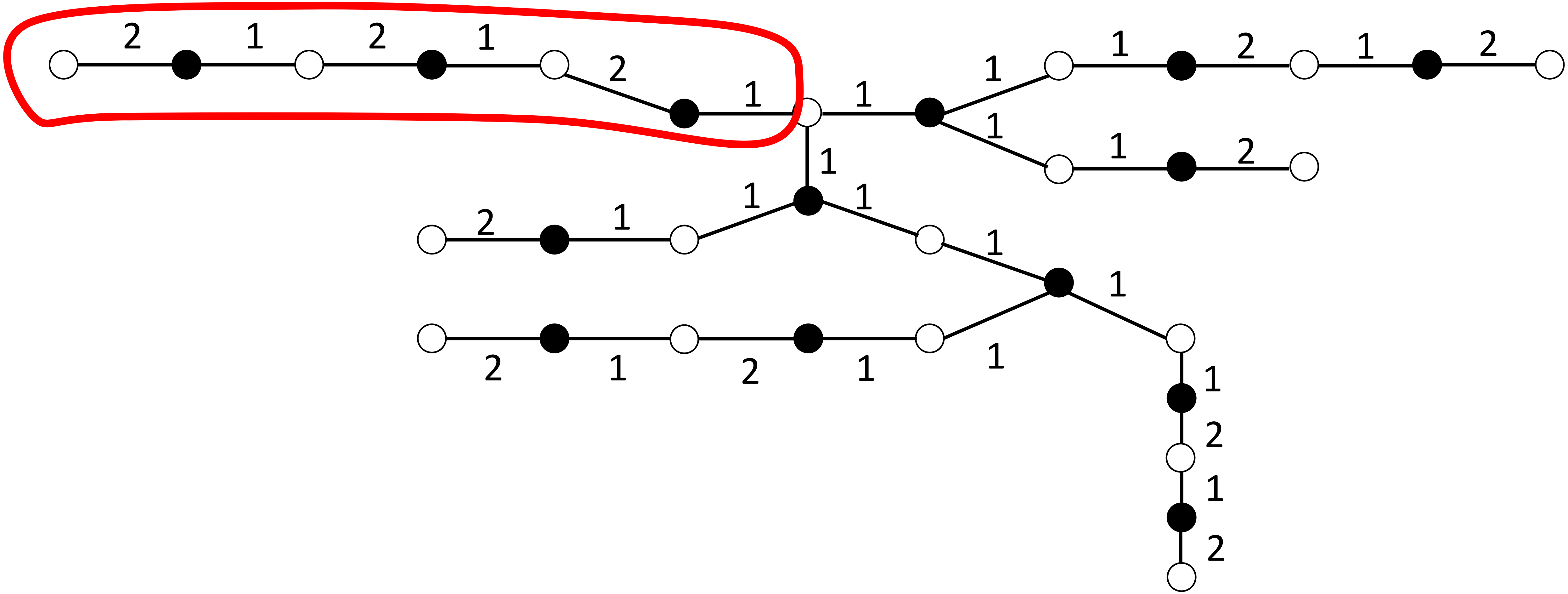
1. exactly one white vertex V of degree = 3,
2. all white terminal vertices,
3. and all other white vertices are of degree < 3.

Furthermore, all linear subgraphs L(V,T) starting at the white vertex V and ending at a terminal vertex T has the labeling (12)(11)…(11)(12)(11)…(12). Two of the subtrees must contain a linear subgraph H that starts at V and ends at terminal white vertex and contains only a single 2 label.

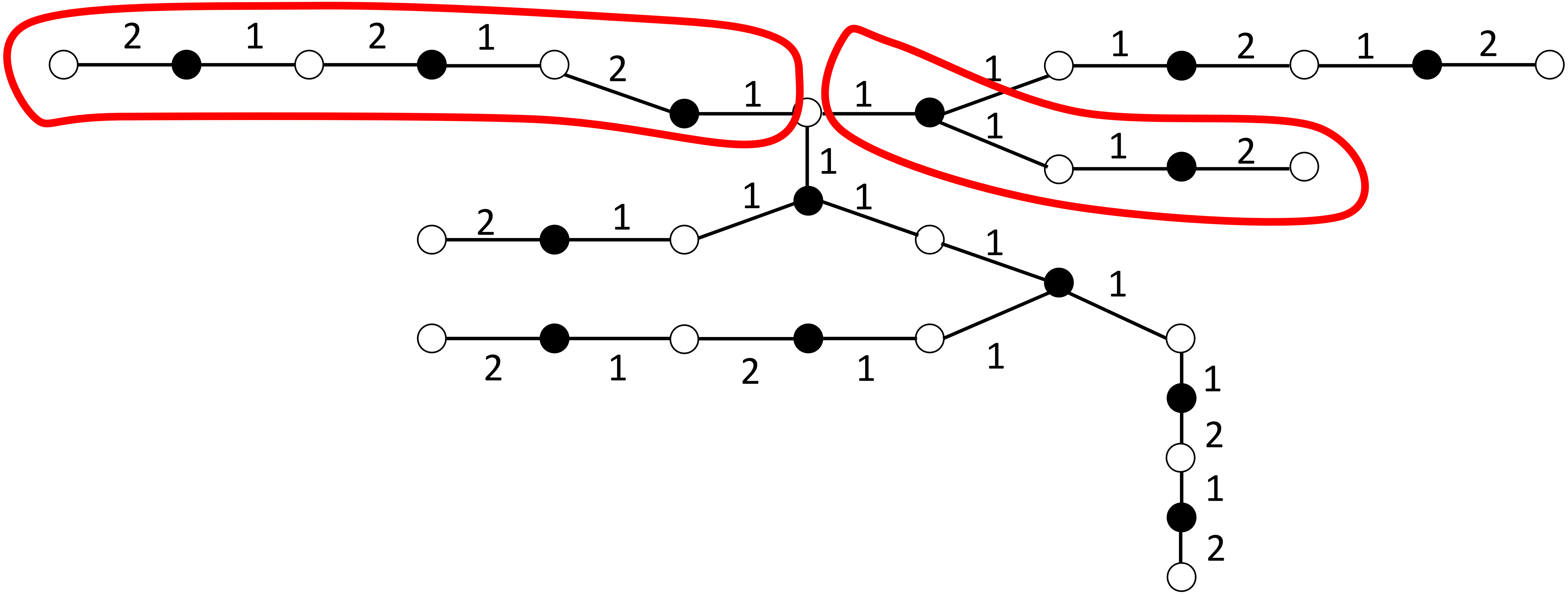
Dihedral Group



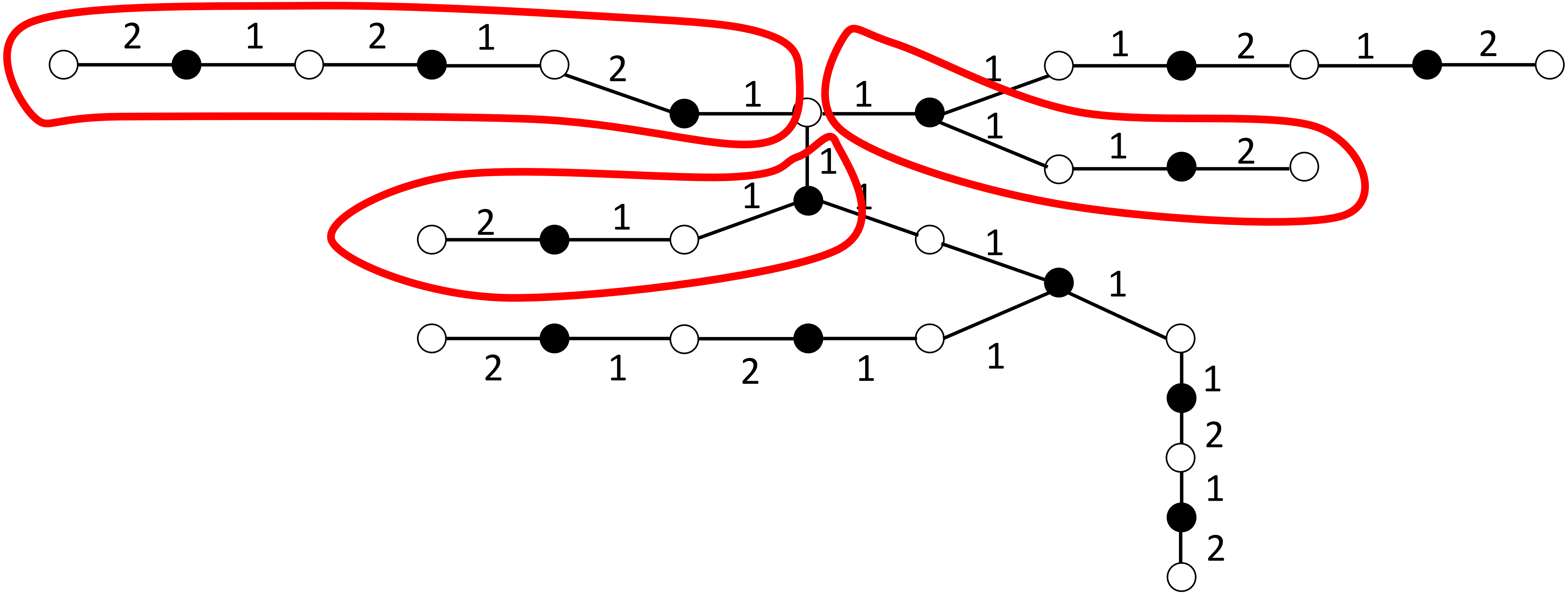
Dihedral Group



Dihedral Group



Dihedral Group



Thank You!

FLORIDA STATE UNIVERSITY
Mathematics