Cryptography - *Day 3*

Defining Security

Review

XOR Operation

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- Suppose $A = a_1 \ldots a_n$ and $B = b_1 \ldots b_n$ then A $\oplus$ B = C where C$= c_1 \ldots c_n$ such that

  c_i = 0 if $a_i = b_i$ and

  $c_i = 1$ if $a_i \neq b_i$.

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$$\oplus\ 0000\ 1110$$

XOR Operation

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- A $\oplus$ B = $1001\ 0010$

$$
\begin{array}{r}
1001\ 0010 \\
\oplus\ 0000\ 1110 \\
1001\ 1100
\end{array}
$$

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- $\mathrm{Enc}_k(m_1 \ldots m_t)$: output $c_1 \ldots c_t$, where
  $$c_i := m_i \oplus k$$
- $\mathrm{Dec}_k(c_1 \ldots c_t)$: output $m_1 \ldots m_t$, where
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- “Hi” = 0x48 69 = 0100 1000 0110 1001
- XOR with “Hi” with the key
- 0100 1000 0110 1001 $\oplus$
  1111 0001 1111 0001
  = 1011 1001 1001 1000=0xB9 98=unprintable

Byte-wise Vigenère cipher

- The key is a string of bytes
- The plaintext is a string of bytes
- To encrypt, XOR each character in the plaintext with the next character of the key
  - Wrap around in the key as needed
- Decryption just reverses the process

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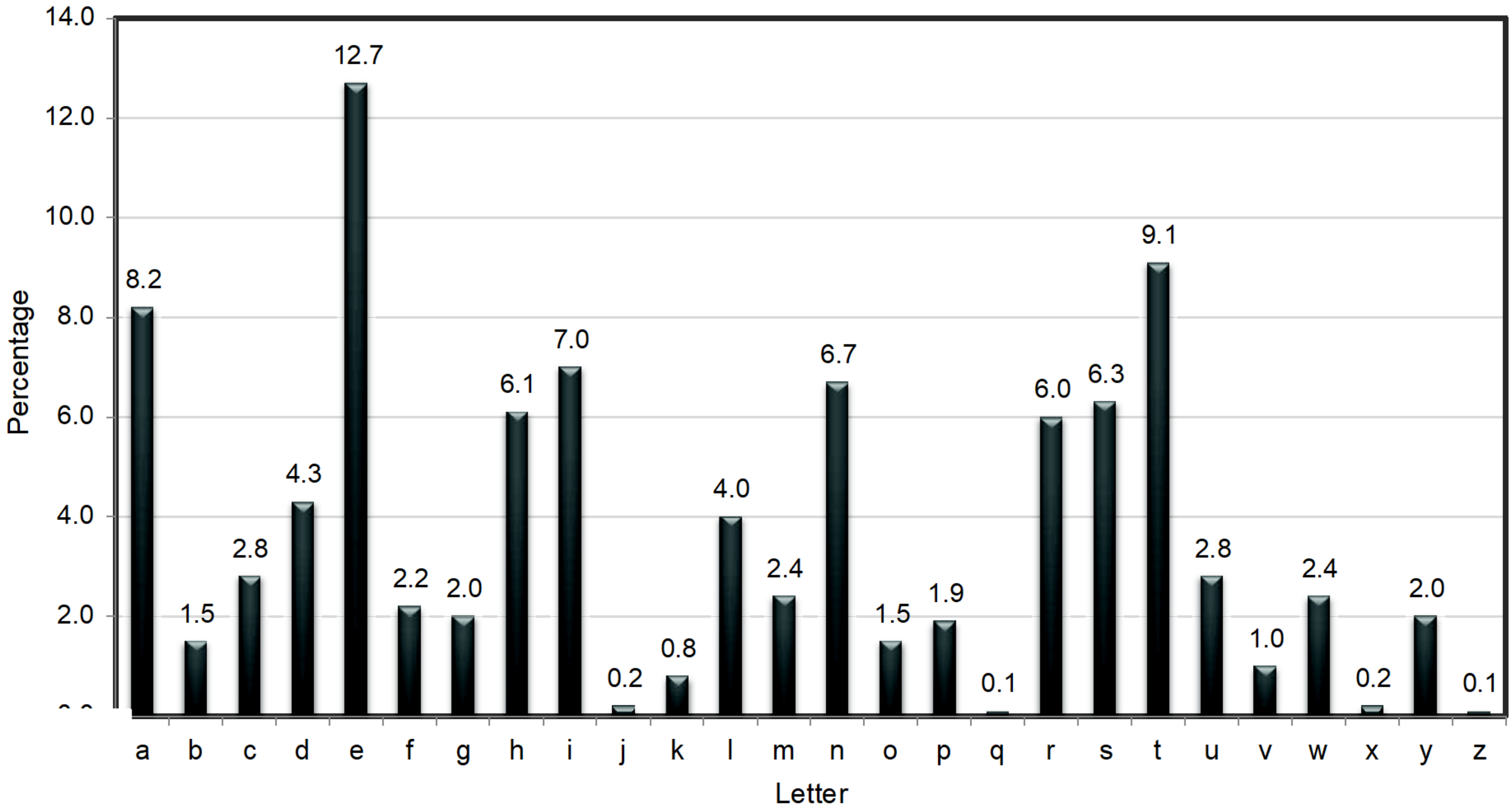
- “Hello!” = 0x48 65 6C 6C 6F 21
- XOR with 0xA1 2F A1 2F A1 2F
- 0x48 $\oplus$ 0xA1
  - 0100 1000 $\oplus$ 1010 0001 = 1110 1001 = 0xE9

- Ciphertext: 0xE9 4A CD 43 CE 0E

Attacking the (variant) Vigenère cipher

- Two steps:
  - Determine the key length
  - Determine each byte of the key

Using plaintext letter frequencies



Useful observations

- Only 128 valid ASCII chars (128 bytes invalid)
- 0x20-0x7E printable
- 0x41-0x7a includes upper/lowercase letters
  - Uppercase letters begin with 0x4 or 0x5
  - Lowercase letters begin with 0x6 or 0x7

Determining the key length

- Let p_i (for $0 \leq i \leq 255$) be the frequency of **byte** i in general English text
  - I.e., $p_i = 0$ for $i < 32$ or $i > 127$
  - I.e., p_{97} = frequency of ‘a’
  - The distribution is far from uniform

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  - If we take every N^{th} character and calculate frequencies, we should get the p_i's in permuted order
  - If we take every M^{th} character (M not a multiple of N) and calculate frequencies, we should get something close to uniform

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  - If a permutation of p_i, then $\Sigma\, q_i^2 \approx \Sigma\, p_i^2$
    - Could compute $\Sigma\, p_i^2$ (but somewhat difficult)
    - Key point: will be much larger than 1/256

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- Compute $\Sigma\, q_i^2$ for each possible key length, and look for maximum value

Determining the i^{th} byte of the key

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Determining the i^{th} byte of the key

- Assume the key length N is known
- Look at every N^{th} character of the ciphertext, starting with the i^{th} character
  - Call this the i^{th} ciphertext “stream”
  - Note that all bytes in this stream were generated by XORing plaintext with the same byte of the key

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  - Call this the i^{th} ciphertext "stream"
  - Note that all bytes in this stream were generated by XORing plaintext with the same byte of the key
- Try decrypting the stream using every possible byte value B
  - Get a candidate plaintext stream for each value

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- When the guess B is correct:
  - All bytes in the plaintext stream will be between 32 and 127

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- Could use $\{p_i\}$ as before, but not easy to find
- When the guess B is correct:
  - Frequencies of lowercase letters (as a fraction of all lowercase letters) should be close to known English-letter frequencies
    - Tabulate observed letter frequencies $q'_0, \ldots, q'_{25}$ (as fraction of all lowercase letters)
    - Should find $\Sigma\, q'_i\, p'_i \approx \Sigma\, p'^2_i \approx 0.065$, where p'_i corresponds to English-letter frequencies
    - In practice, take B that maximizes $\Sigma\, q'_i\, p'_i$

Defining secure encryption

Crypto definitions (generally)

- Security guarantee/goal
  - What we want to achieve and/or what we want to prevent the attacker from achieving

- Threat model
  - What (real-world) capabilities the attacker is assumed to have

Recall

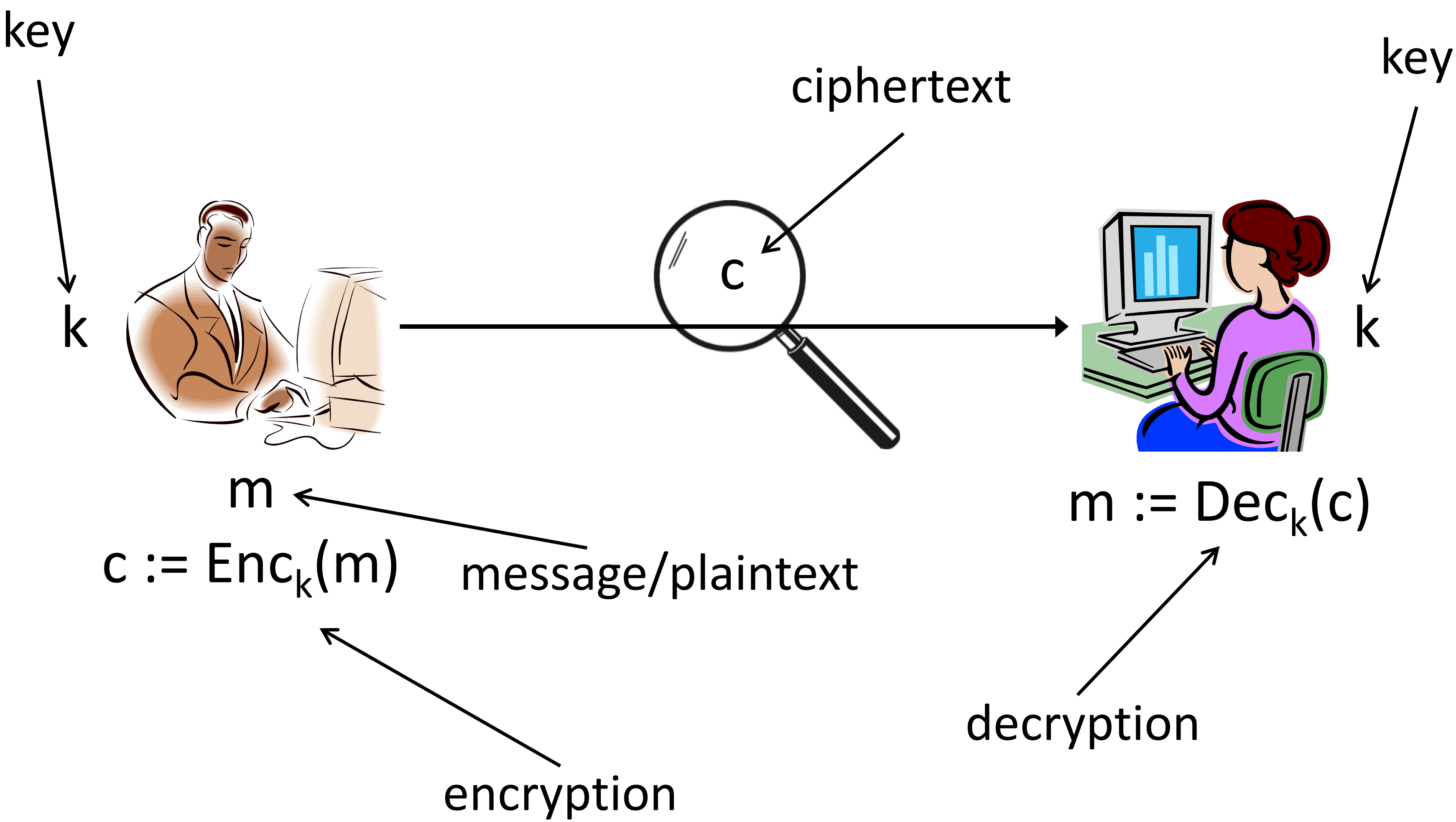
- A *private-key encryption scheme* is defined by a message space $\mathcal{M}$ and algorithms (Gen, Enc, Dec):
  - Gen (key-generation algorithm): generates k
  - Enc (encryption algorithm): takes key k and message $m \in \mathcal{M}$ as input; outputs ciphertext c

    $$c \leftarrow \mathrm{Enc}_k(m)$$

  - Dec (decryption algorithm): takes key k and ciphertext c as input; outputs m.

    $$m := \mathrm{Dec}_k(c)$$

Private-key encryption



Threat models for encryption

- Ciphertext-only attack - obtain only ciphertext
- Known-plaintext attack – obtain ciphertext with some knowledge of the message
- Chosen-plaintext attack - obtain encryptions of chosen messages
- Chosen-ciphertext attack – obtain decryptions of chosen ciphertext

Goal of secure encryption?

- How would you define what it means for encryption scheme (Gen, Enc, Dec) over message space $\mathcal{M}$ to be secure?
  - Against a (single) ciphertext-only attack

Secure encryption?

- “Impossible for the attacker to learn the plaintext from the ciphertext”

Secure encryption?

- “Impossible for the attacker to learn the plaintext from the ciphertext”
  – What if the attacker learns 90% of the plaintext?

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Secure encryption?

- “Impossible for the attacker to learn any character of the plaintext from the ciphertext”
  - What if the attacker is able to learn (other) partial information about the plaintext?
    - E.g., salary is greater than $75K

Perfect secrecy

- “Regardless of any *prior* information the attacker has about the plaintext, the ciphertext should leak no *additional* information about the plaintext”

Example 1

- Consider the shift cipher
  - So for all $k \in \{0, \ldots, 25\}$, $\Pr[K = k] = 1/26$

- Say $\Pr[M = \text{‘a’}] = 0.7$, $\Pr[M = \text{‘z’}] = 0.3$

- What is $\Pr[C = \text{‘b’}]$?
  - Either $M = \text{‘a’}$ and $K = 1$, or $M = \text{‘z’}$ and $K = 2$
  - $\Pr[C=\text{‘b’}] = \Pr[M=\text{‘a’}] \cdot \Pr[K=1] + \Pr[M=\text{‘z’}] \cdot \Pr[K=2]$

    $= 0.7 \cdot (1/26) + 0.3 \cdot (1/26)$

    $= 1/26$

Example 2

- Consider the shift cipher, and the distribution $\Pr[M = \text{'one'}] = \frac{1}{2}$, $\Pr[M = \text{'ten'}] = \frac{1}{2}$

- $\Pr[C = \text{'rqh'}] = ?$

$$= \Pr[C = \text{'rqh'} \mid M = \text{'one'}] \cdot \Pr[M = \text{'one'}] + \Pr[C = \text{'rqh'} \mid M = \text{'ten'}] \cdot \Pr[M = \text{'ten'}]$$

$$= 1/26 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{2} = 1/52$$

Example 3

- Consider the shift cipher, and the distribution $\Pr[M = \text{'one'}] = ½$, $\Pr[M = \text{'ten'}] = ½$
- Take m = 'ten' and c = 'rqh'

- $\Pr[M = \text{'ten'} \mid C = \text{'rqh'}] = ?$

  $= 0$

  $\neq \Pr[M = \text{'ten'}]$

Example 4

- Shift cipher;
  Pr[M='hi'] = 0.3,
  Pr[M='no'] = 0.2,
  Pr[M='in']= 0.5

- Pr[M = 'hi' | C = 'xy'] = ?
  = Pr[C = 'xy' | M = 'hi'] · Pr[M = 'hi']/Pr[C = 'xy']

Example 4, continued

- $\Pr[C = \text{'xy'} \mid M = \text{'hi'}] = 1/26$

- $\Pr[C = \text{'xy'}]$

  $= \Pr[C = \text{'xy'} \mid M = \text{'hi'}] \cdot 0.3 + \Pr[C = \text{'xy'} \mid M = \text{'no'}] \cdot 0.2$
  $\quad + \Pr[C=\text{'xy'} \mid M=\text{'in'}] \cdot 0.5$
  $= (1/26) \cdot 0.3 + (1/26) \cdot 0.2 + 0 \cdot 0.5$
  $= 1/52$

Example 4, continued

- $\Pr[M = \text{'hi'} \mid C = \text{'xy'}] = ?$

  $= \Pr[C = \text{'xy'} \mid M = \text{'hi'}] \cdot \Pr[M = \text{'hi'}]/\Pr[C = \text{'xy'}]$

  $= (1/26) \cdot 0.3/(1/52)$

  $= 0.6$

  $\neq \Pr[M = \text{'hi'}]$

Conclusion

- The shift cipher is not perfectly secret!
  - At least not for 2-character messages

- How to construct a perfectly secret scheme?

One-time pad

- Patented in 1917 by Vernam
  - Recent historical research indicates it was invented (at least) 35 years earlier

- Proven perfectly secret by Shannon (1949)

One-time pad

- Let $\mathcal{M} = \{0,1\}^n$
- Gen: choose a uniform key $k \in \{0,1\}^n$
- $\mathrm{Enc}_k(m) = k \oplus m$
- $\mathrm{Dec}_k(c) = k \oplus c$

- Correctness:
$$\begin{aligned}\mathrm{Dec}_k(\, \mathrm{Enc}_k(m)\,) &= k \oplus (k \oplus m) \\ &= (k \oplus k) \oplus m = m\end{aligned}$$

One-time pad