Research Prospectus

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Introduction

I am interested in Discrete Geometry. Specifically, I study discrete configurations of points, circles, and spheres. I am motivated by endeavors to characterize projective polyhedra up to Möbius transformations. Currently, I am exploring global rigidity (uniqueness); in the next phase of my research, I will focus on existence. I aim to use this as a path into constructing projective structures on manifolds. In this statement, I emphasize the connections between circle configurations and polyhedral geometry, detail my contributions, and outline future directions and projects suitable for undergraduate research in this topic.

Background & Motivation

My area of research stems from the study of *circle packings* (circle patterns on a surface with an underlying triangulation specifying circle overlaps) and is inspired by the Koebe-Andre'ev-Thurston (KAT) Circle Packing theorem [15]. For simplicity, a special case of this theorem, where all overlaps are tangencies, is stated here.

Theorem 1 (Koebe Circle Packing Theorem) *Given a triangulation $\mathcal{K}$ of a topological sphere, there exists a tangency circle packing $\mathcal{K}(\mathcal{C})$ on the Riemann sphere $\mathbb{S}^2$ with the combinatorics of $\mathcal{K}$. Circle packing $\mathcal{K}(\mathcal{C})$ is unique up to Möbius transformations of the sphere.*

The (KAT) Circle Packing theorem is used as a tool to approximate angle-preserving maps on surfaces by constructing a maximal packing of the surface [14]. In this way, circle packings are used in building *conformal tilings* – tilings of surfaces with specified angle patterns. These tilings have been used by a number of people in their effort to solve the Cannon Conjecture [4], [9], [10], [6] .

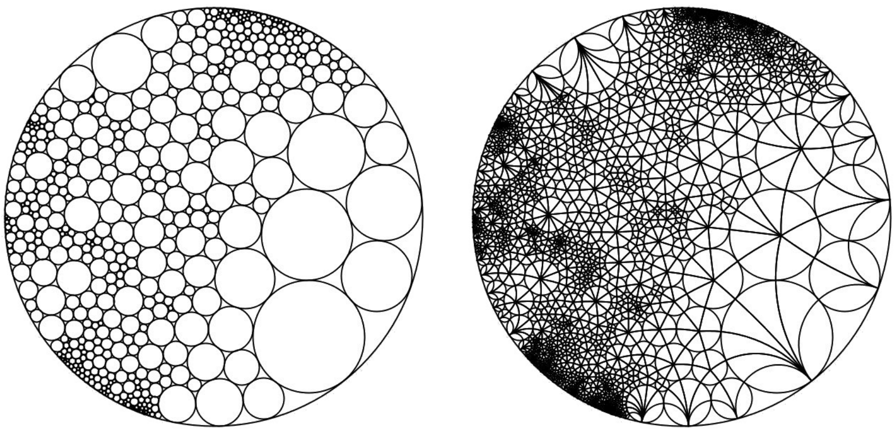

Fig 1: A tangency circle packing of the unit disk (left), and its underlying triangulation (right). Each vertex in the triangulation represents a circle. Each edge between vertices means the two circles are tangent. Made with CirclePack software.

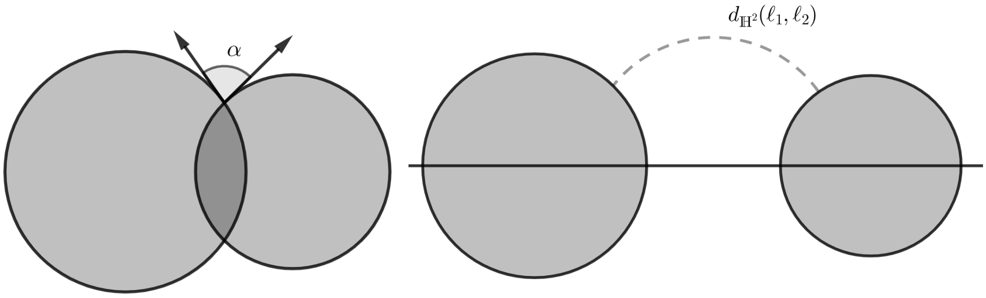

Fig 2: (Left) When oriented circles C, C' are intersecting, $-1 \leq$ InvDist$(C, C') \leq 1$. (Right) When C, C' are disjoint, $1 <$ InvDist$(C, C') < \infty$, or $\infty <$ InvDist$(C, C') < -1$

My field grew out of generalizing the (KAT) Circle Packing theorem from circle packings to more general circle patterns called *inversive distance circle packings (IDCPs)*. Here, circles may intersect at an angle, be tangent, or disjoint. The edges in the corresponding triangulation are equipped with a real number specifying the *inversive distance*, a measure of the separation between two circles.

Definition 1 *Let C_1 and C_2 be oriented circles in $\mathbb{S}^2$. When C_1 and C_2 are intersecting,*

$$\textit{InvDist}(C_1, C_2) = \cos \alpha,$$

where α is the oriented angle of intersection of C_1 and C_2. When C_1 and C_2 are disjoint,

$$\textit{InvDist}(C_1, C_2) = \cosh d_{\mathbb{H}^2}(\ell_1, \ell_2),$$

where $\ell_1 = D \cap C_1$, and $\ell_2 = D \cap C_2$, for a disc D mutually orthogonal to C_1 and C_2, used as a model of the hyperbolic plane. As such, $d_{\mathbb{H}^2}(\ell_1, \ell_2)$ is the hyperbolic distance between ℓ_1 and ℓ_2. Here, InvDist(C_1, C_2) denotes the **inversive distance** *between C_1 and C_2.*

While inversive distance is not a distance function, it serves as a Möbius invariant of the placement of two circles in a plane.

Broadening the view to IDCPs creates difficulties: (KAT) circle packing theorem doesn't generalize to IDCPs. Not all edge-labeled triangulations have an IDCP realization, and not all IDCPs are rigid; see [3] for the construction of an example. IDCPs are special cases of circle configurations with underlying edge-labeled polyhedral graphs (3-vertex-connected, planar graphs) – such configurations are called *circle polyhedra*. Under this lens, rigidity of IDCPs becomes a question related to the famous Cauchy's Rigidity Theorem, which states that two convex, combinatorially equivalent, bounded Euclidean polyhedra with corresponding congruent faces are themselves congruent. Convexity is a powerful notion in the discipline of polyhedral geometry (see [8]), so it is a natural step to introduce an analogous notion for circle polyhedra.

Circles have their own geometry under Möbius transformations acting on $\mathbb{S}^2$, where there is a notion of a *circle point* (a circle in $\mathbb{S}^2$), *circle line*, and *circle plane*. There is a strong connection between circle polyhedra in $\mathbb{S}^2$ and *projective polyhedra*. If we consider $\mathbb{RP}^3 = \mathbb{E}^3 \cup \mathbb{RP}^2$ as our model of real projective space, a projective polyhedron can always be transformed to look like a bounded Euclidean polyhedron. Within $\mathbb{E}^3$, the unit sphere $\mathbb{S}^2$ serves as the ideal boundary for the Klein model of Hyperbolic space $\mathbb{H}^3$. With this setup, the supporting planes of faces of a projective polyhedron intersect $\mathbb{S}^2$ as a circle, for instance. One of my interests is understanding conditions for rigidity and existence of circle polyhedra and their corresponding projective polyhedra.

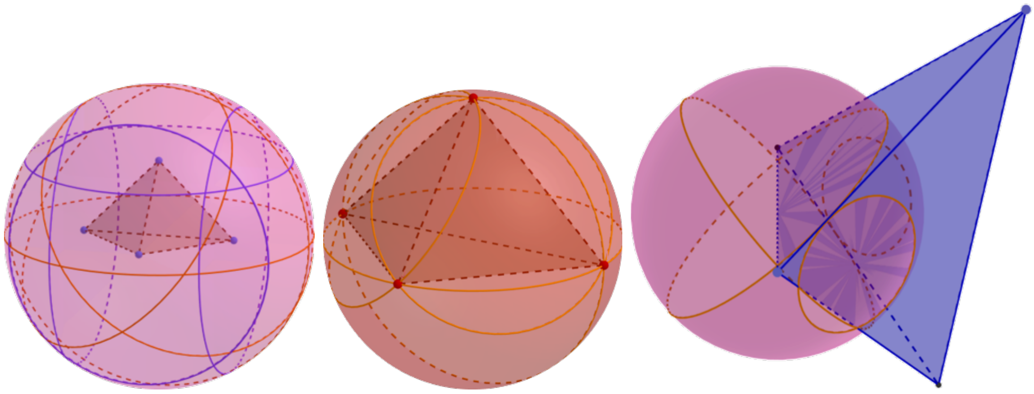

Fig 3: Projective polyhedra and their corresponding circle polyhedra. (Left) Hyperbolic polyhedron, (center) ideal polyhedron, (right) hyperideal polyhedron.

Several cases of projective polyhedra have been classified. In [13] Andre'ev classified hyperbolic convex polyhedra with acute dihedral angles. In [12] Rivin classified hyperbolic convex polyhedra with vertices at the ideal boundary, called *ideal polyhedra*. Bao and Bonahon in [1] push those vertices past the ideal boundary to classify *hyperideal polyhedra*, where every edge must intersect hyperbolic space. One primary reason topologists are interested in ideal polyhedra is because of their use as building blocks when constructing hyperbolic 3-manifolds [1]. Studying more general projective polyhedra could be useful to those attempting construction of projective structures on manifolds.

Most recently, Bowers, Bowers, and Pratt in [7] explored rigidity of more general hyperideal polyhedra. Their result is stated in terms of corresponding circle polyhedra in the 2-sphere.

Theorem 2 (BBP) *Any two convex and proper non-unitary circle polyhedra with Möbius-congruent circle faces that are based on the same oriented abstract spherical polyhedron and are consistently oriented are Möbius-congruent.*

One important ingredient in this proof is a generalization of Cauchy's arm lemma to polygonal paths in $\mathbb{H}^2$. It roughly states that convex circle polyhedra are rigid. They introduced a notion of convexity analogous to the definition for Euclidean polyhedra, in that all circle points must "lie on one side" of each circle face, bounded by a certain inversive distance. This result directly implies that all convex IDCPs are rigid.

In [7], only circle polyhedra corresponding to projective polyhedra with faces intersecting $\mathbb{H}^3$, and non-tangent edges are used. Bao and Bonahon imposed that their projective polyhedra must have (non-tangent) edges that intersect $\mathbb{H}^3$. There is no generalized rigidity statement for projective polyhedra up to Möbius transformation yet.

Results

I am working towards rigidity of general projective polyhedra up to Möbius transformations by relaxing the restrictions on circle patterns further. I study configurations of intermingled circles and points in the 2-sphere, and hyperbolic points in $\mathbb{H}^3$. Convexity and polyhedral geometry have been invaluable tools in this field; I am building up similarly helpful properties and tools in conformal geometry.

Analyzing these geometric objects together is a natural consequence of using the ambient *Lorentz Space*, $\mathbb{R}^4$ equipped with the *Lorentz inner product* $\langle \cdot, \cdot \rangle$. Here, the Lorentz inner product of vectors $v, w \in \mathbb{R}^4$ is $\langle v, w \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4$, and is used to divide Lorentz vectors into three different types. A vector v is *space-like* if $|v|^2 = \langle v, v \rangle > 0$, *time-like* if $|v|^2 < 0$, and *light-like* if $|v|^2 = 0$. See [11] for more information.

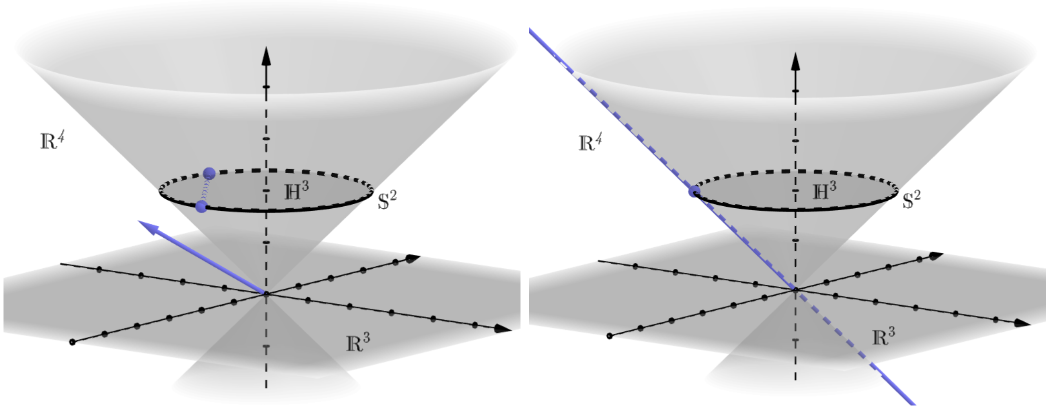

Fig 4: The light cone is a prominent feature of Lorentz space. Space-like vectors lie outside the light cone, time-like vectors lie inside the light cone, and light-like vectors lie on the light cone. For every space-like vector v, there is a time-like 3-dimensional subspace P_C such that for all $w \in P_C, \langle v, w\rangle = 0$, where P_C intersects $\mathbb{S}^2$ in a circle C.

When v and w are space-like unit vectors, they correspond to circles C_v, C_w in $\mathbb{S}^2$, and $\langle v, w\rangle = \mathsf{InvDist}(C_v, C_w)$. Likewise, when v and w are time-like unit vectors, they correspond to hyperbolic points in $\mathbb{H}^3$, and $\langle v, w\rangle$ corresponds to hyperbolic distance between the hyperbolic points. For points p, q, r, s in $\mathbb{S}^2$, if a corresponding light-like vector is chosen for each point, v_p, v_q, v_r, v_s, then the absolute value of the cross ratio of four points is

$$|p, q, r, s| = \frac{\langle v_p, v_q\rangle\langle v_r, v_s\rangle}{\langle v_p, v_r\rangle\langle v_q, v_s\rangle}.$$

Möbius transformations are positive Lorentz transformations restricted to $\mathbb{S}^2$; the geometry of hyperbolic space and its ideal boundary are extrinsically encoded in the geometry of Lorentz space.

Crane and Short use this connection in [5], where they state (in loose terms) that if you know the inversive distance between *every* pair of circles, then the collection is unique up to Möbius transformations. They also state that a collection of points in $\mathbb{S}^2$ is unique up to Möbius transformations if the absolute cross ratio is known between *every* 4-tuple of points. In both statements, a maximal amount of conformal invariant information is utilized.

In Euclidean space, a line can be uniquely determined by two distinct points; a plane is determined by three linearly independent points. In the geometry of circles, a circle line (coaxial family) can be determined by two distinct circles. Without going into detail, a circle plane is determined by three *independent* circles in $\mathbb{S}^2$. I introduce independence as a condition for collections of points and circles to reduce the amount of conformal invariant information needed for rigidity.

Definition 2 *A collection of four distinct circles $\{C_1, C_2, C_3, C_4\}$ in $\mathbb{S}^2$ is* **independent** *if C_1, C_2, C_3 and C_4 don't all belong to the same circle plane.*

Lemma 3 (Graham) *Let $\{C_1, C_2, C_3, C_4\}$ be a collection of independent circles in $\mathbb{S}^2$, and let $\{v_1, v_2, v_3, v_4\}$ be its corresponding collection of space-like unit vectors in $\mathbb{R}^4$. Then $\{v_1, v_2, v_3, v_4\}$ is linearly independent in $\mathbb{R}^4$.*

With this observation, rigidity of collections of circles becomes a more manageable question of rigidity of collections of Lorentz vectors.

Theorem 4 (Graham) *Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of distinct Lorentz vectors in $\mathbb{R}^4$, indexed by the same set, with at least 4 elements $\{v_i\}$ and $\{v'_i\}$, respectively, that form a basis. Then $\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle$ for $i = 1, 2, 3, 4$ where $v_i \neq v_\alpha$, $v'_i \neq v'_\alpha$, if and only if there is a positive Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$. for each $\alpha \in \mathcal{A}$.*

This result holds for any collection of Lorentz space vectors, yielding a different meaning depending on the collections it's used with.
Space-like unit vectors. This case is equivalent to uniquely placing a collection of circles in $\mathbb{S}^2$ based on their inversive distance to an independent subcollection of circles.
Time-like unit vectors. This case uniquely places a collection of hyperbolic points in $\mathbb{H}^3$ based on their hyperbolic distance to a linearly independent subcollection of points (an alternate version of Menger's theorem for hyperbolic points [2]).
Space-like and time-like unit vectors. In this case, the statement uniquely places hyperbolic points and the hyperbolic planes corresponding to the circles on the ideal boundary of hyperbolic space using hyperbolic distance.

Problems arise when vectors in the collection are light-like, because there is not an intrinsic conformal invariant attached to the Lorentz inner product of light-like vectors. If all vectors in the collection are light-like, one may replace the Lorentz inner product with the absolute cross ratio of corresponding points in $\mathbb{S}^2$. Expanding on this, I form a more general conformal invariant that can relate points in $\mathbb{S}^2$ to circles in $\mathbb{S}^2$ or to hyperbolic points in $\mathbb{H}^3$.

Definition 3 *Let v_1, v_2, v_3 be space-like vectors (or time-like vectors), and w a light-like vector. Then $|w, v_1, v_2, v_3| = \dfrac{\langle w, v_1\rangle\langle v_2, v_3\rangle}{\langle w, v_2\rangle\langle v_1, v_3\rangle}$ is called the* **Lorentz cross ratio** *of w, v_1, v_2, v_3.*

For a point p and three circles in $\mathbb{S}^2$ (or three hyperbolic points), the Lorentz ratio of corresponding w, v_1, v_2, v_3 is independent of the choice of w for p. The following Lemma shows its geometric significance.

Lemma 5 (Graham) *Let v_t be a sequence of space-like (or time-like) vectors limiting toward the light-like line ℓ, and let v_1, v_2, v_3 be fixed space-like (resp. time-like) vectors. For a vector w corresponding to ℓ, if $v_t \to \ell$, then $|v_t, v_1, v_2, v_3| \to |w, v_1, v_2, v_3|$.*

This lemma yields a conformal invariant that expands the consequences of the previous theorem, by providing the opportunity to consider points in the ideal boundary with hyperbolic points and circles bounding hyperbolic planes. For example, if v_t, v_1, v_2, v_3 are all space-like, then the Lorentz ratio $|w, v_1, v_2, v_3|$ is the limiting value of a ratio of inversive distances as one circle's radius limits to 0.

Theorem 6 (Graham) *Let $\{v_\alpha, v_\beta, v_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ and $\{v'_\alpha, v'_\beta, v'_\gamma : \alpha, \beta, \gamma \in \mathcal{A}\}$ be two collections of distinct space-like, time-like, and light-like vectors in $\mathbb{R}^4$, with at least 4 space-like (or time-like) vectors $\{v_i\}$ and $\{v'_i\}$, respectively, that form a basis. Then,*

$$\langle v_\alpha, v_i\rangle = \langle v'_\alpha, v'_i\rangle, \langle v_\beta, v_i\rangle = \langle v'_\beta, v'_i\rangle,$$

for $i = 1, 2, 3, 4$, for all space-like vectors v_α, v'_α, and for all time-like vectors v_β, v'_β, and

$$|v_\gamma, v_i, v_j, v_k| = |v'_\gamma, v'_i, v'_j, v'_k|,$$

for each triple $1 \leq i, j, k \leq 4$, and all light-like v_β, v'_β if and only if there is a positive Lorentz transformation Φ such that $\Phi(v_\alpha) = v'_\alpha$, $\Phi(v_\beta) = v'_\beta$, and $\Phi(v_\gamma) = v'_\gamma$ for each $\alpha, \beta, \gamma \in \mathcal{A}$.

As a result, one can determine when a collection of intermingled hyperbolic points, hyperbolic planes, and ideal points is globally unique up to hyperbolic isometries. Previously, each kind of object had to be considered separately.

The theorem holds in $\mathbb{R}^{n+1}$, and thus also applies to collections of hyperbolic points, $(n-2)$-spheres bounding $(n-1)$-planes in $\mathbb{H}^n$, and points in $\mathbb{S}^{n-1}$. With the added condition of independence, there is a reduction in needed conformal invariant information: for a collection of size m, originally $m(m-1)/2$ pieces of information were needed in [5]. With this improvement, $k(m-k)$ pieces of information are used, where k is the dimension of the ambient space.

Both improvements are important in analyzing projective polyhedra in dimension 3. The former is significant because tetrahedra are fundamental building blocks for all other polyhedra, and non-degenerate tetrahedra are the convex hull of linearly independent collections of vertices. The latter is important because the most general projective polyhedra have vertices corresponding to hyperbolic points, ideal points, and circles in the ideal boundary of hyperbolic 3-space.

Rigidity of convex IDCPs. I have used rigidity of independent collections of circles in answering "how much extra inversive distance information is needed to make a convex inversive distance circle packing in a topological disk D rigid?" If m is the number of circles on the boundary of D, then $m-3$ pieces of extra inversive distance information must be added. Note that when $m=3$, no extra inversive distances must be known, and adding in the boundary circle of D produces an inversive distance circle packing of $\mathbb{S}^2$. This yields an alternate proof that convex inversive distance circle packings are rigid in $\mathbb{S}^2$. No mechanics from Cauchy's Rigidity Theorem are used in this version.

Future Work

One favored aspect of my research is that it incorporates ideas from several classical fields of mathematics. This work has rich roots in hyperbolic geometry, rigidity theory, and complex analysis. There are abundant opportunities that arise from the interactions of these areas.

My main focus moving forward is exploring the rigidity of convex projective polyhedra up to Möbius transformations by relaxing the definition of a convex circle polyhedron to include configurations of points and circles. I plan to consider conditions under which convex projective polyhedra exist with the aim of forming a complete classification statement. From there, I will survey the role of structural rigidity theory in projective polyhedra by constructing flexible circle polyhedra (polyhedral graphs with a continuous family of non-Möbius-equivalent realizations). Eventually, I intend to study the global rigidity of complex projective structures on a surface via my work in circle packings.

Another exciting aspect of my field is that the level of mathematics is reasonably accessible to undergraduate mathematics students. Many of the concepts involved appeals to one's spatial intuition, making it ideal for undergraduate research: there is no shortage of visual examples to work from. Since inversive distance is an invaluable tool for my work, I would challenge students who work with me to discover new formulas for inversive distance. There are at least seven different inversive distance formulas currently, each with its own advantages. As another project, undergraduates could look at a given configuration of circles, all inversive distances known, and use underlying combinatorics to reduce as much inversive distance information as possible, while maintaining rigidity. I am also looking for students with a programming background to work with me in learning how to code visualizations and 3D-print examples together. There are free programs available online for this project: CirclePack and KoebeLib. Circle packing has been used at ORNL to 3D-print wind turbines that use less material with equivalent structural integrity; this kind of project could open doors to applying circle packing in real-world applications.

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