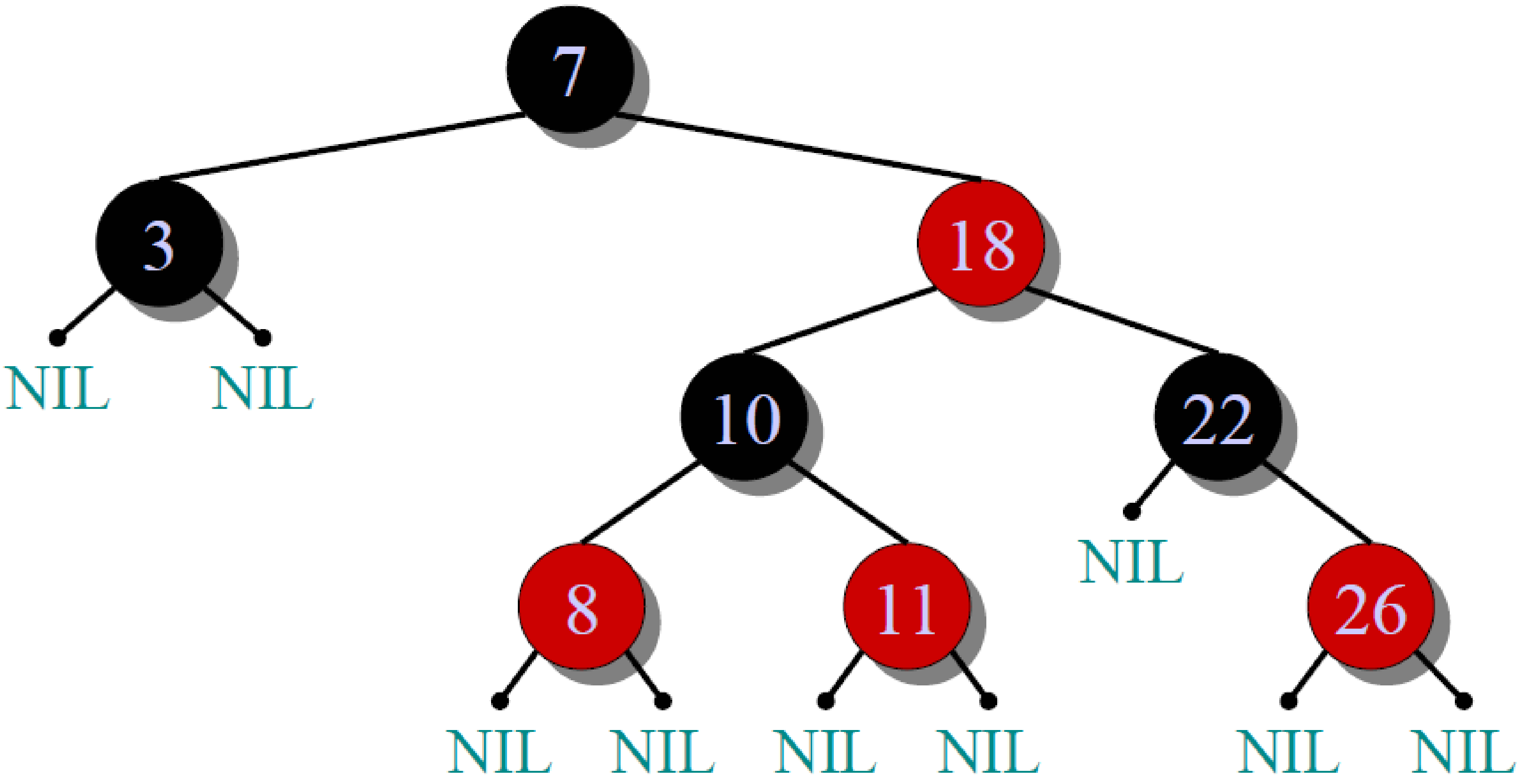
# Red-Black Tree

SWE2016-44

# Example of a Red-Black Tree



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2) **Root of tree is always black**
3) **There are no two adjacent red nodes (A red node cannot have a red parent or red child)**
4) **Every path from root to a NULL node has same number of black nodes**

# Why Red-Black Trees?

**Most of the BST operations (e.g., search, max, min, insert, delete.. etc) take O(h) time where h is the height of the BST.**

**→ O(n) for a skewed Binary tree.**

# Why Red-Black Trees?

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**→ O(n) for a skewed Binary tree.**

**→ Since a Red-Black Tree ensures almost balanced, the height of the tree remains O(Log n) after every insertion and deletion.**

**→ Red-Black Tree is not always a balanced tree.**

# How does a Red-Black Tree ensure balance?

**We can try any combination of colors and see all of them violate Red-Black tree property.**

```
A chain of 3 nodes is nodes is not possible in Red-Black Trees.
Following are NOT Red-Black Trees
        30             30               30
       / \            /  \             /  \
     20  NIL         20   NIL         20   NIL
    / \             / \              /  \
  10  NIL          10  NIL          10  NIL
Violates          Violates         Violates
Property 4.      Property 4       Property 3
```

# How does a Red-Black Tree ensure balance?

**We can try any combination of colors and see all of them violate Red-Black tree property.**

```
Following are different possible Red-Black Trees with above 3 keys
         20                          20
        /  \                        /  \
      10    30                    10    30
     / \   / \                   / \   / \
  NIL  NIL NIL NIL            NIL  NIL NIL NIL
```

# Black Height of a Red-Black Tree

**Black height is number of black nodes on a path from root to a leaf. Leaf nodes are also counted black nodes.**

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**From properties 3 (no two adjacent red nodes) and 4 (same number of black nodes),**

**Black-height >= h/2.**

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**Proof)**

1) **For a general Binary Tree, let k be the minimum number of nodes on all root to NULL paths, then n >= 2k – 1 (Ex. If k is 3, then n is at least 7). That is, k <= $Log_2$ (n+1).**

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2) **From 1) and property 4, there is a root to leaf path with at most $Log_2 (n+1)$ black nodes: h' <= $Log_2 (n+1)$**
3) **Black-height is at least h/2: h' >= h/2**
4) **From 2) and 3), h <= $2Log_2 (n+1)$**

# Insertion

**The goal of the insert operation is to insert key K into tree T, maintaining T's red-black tree properties**

# Insertion

**If T is a non-empty tree, then we do the following:**

1. **Use the BST insert algorithm to add K to the tree**

2. **Color the node containing K red**

3. **Restore red-black tree properties (if necessary)**

# Insertion

**To restore the violated property, we use:**

1. **Recoloring**

2. **Rotation (Left, Right, Double)**

**We try recoloring first, if recoloring doesn’t work, then we go for rotation.**

# Insertion

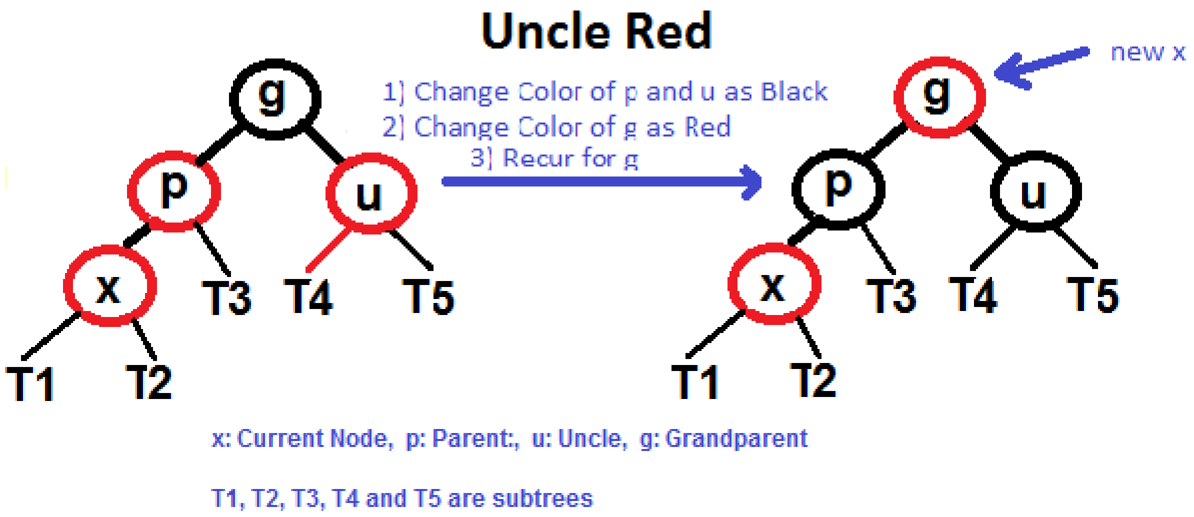
1) **Perform standard BST insertion and make the color of newly inserted nodes as RED.**

2) **If x is root, change color of x as BLACK (Black height of complete tree increases by 1).**

3) **Do following if color of x's parent is not BLACK and x is not root.**

# Insertion

a. If x’s uncle is RED (Grand parent must have been black from property 4)
   I. Change color of parent and uncle as BLACK.
   II. color of grand parent as RED.
   III. Change x = x’s grandparent, repeat steps 2 and 3 for new x.

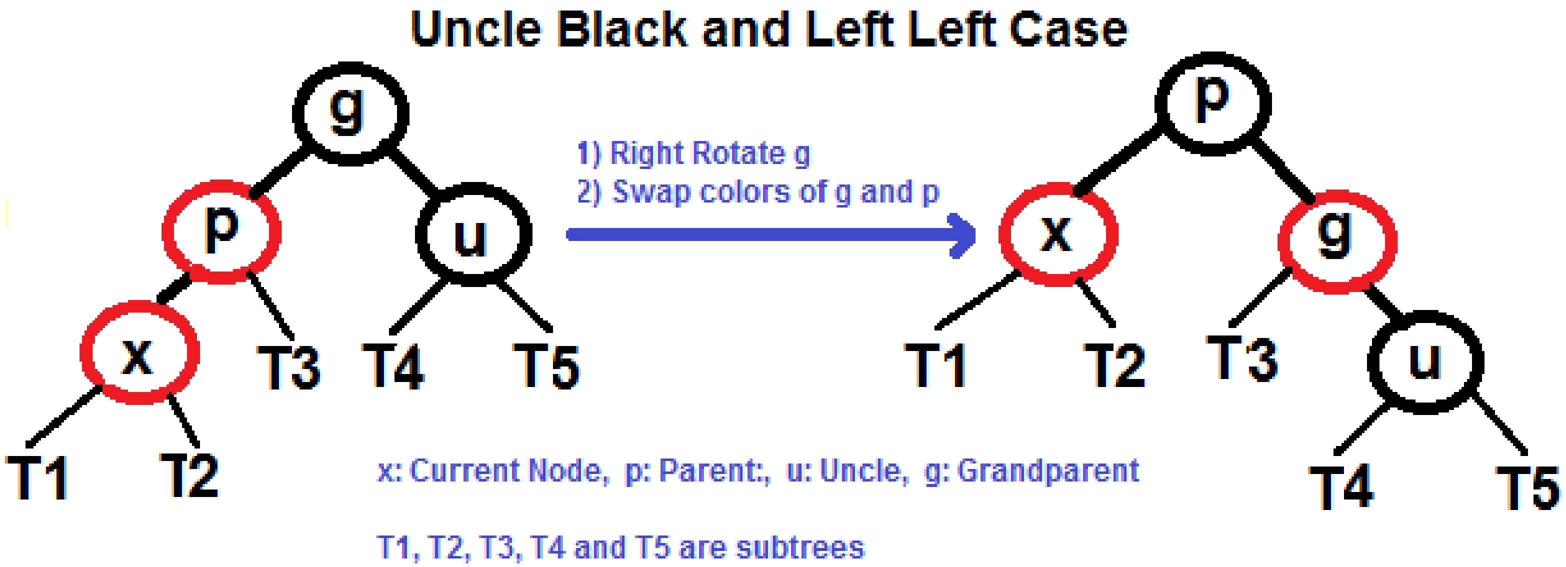


# Insertion

b. If x’s uncle is BLACK, then there can be four configurations for x, x’s parent (p) and x’s grandparent (g)
   I. Left Left Case (p is left child of g and x is left child of p)
   II. Left Right Case (p is left child of g and x is right child of p)
   III. Right Right Case (Mirror of case i)
   IV. Right Left Case (Mirror of case ii)

# Insertion

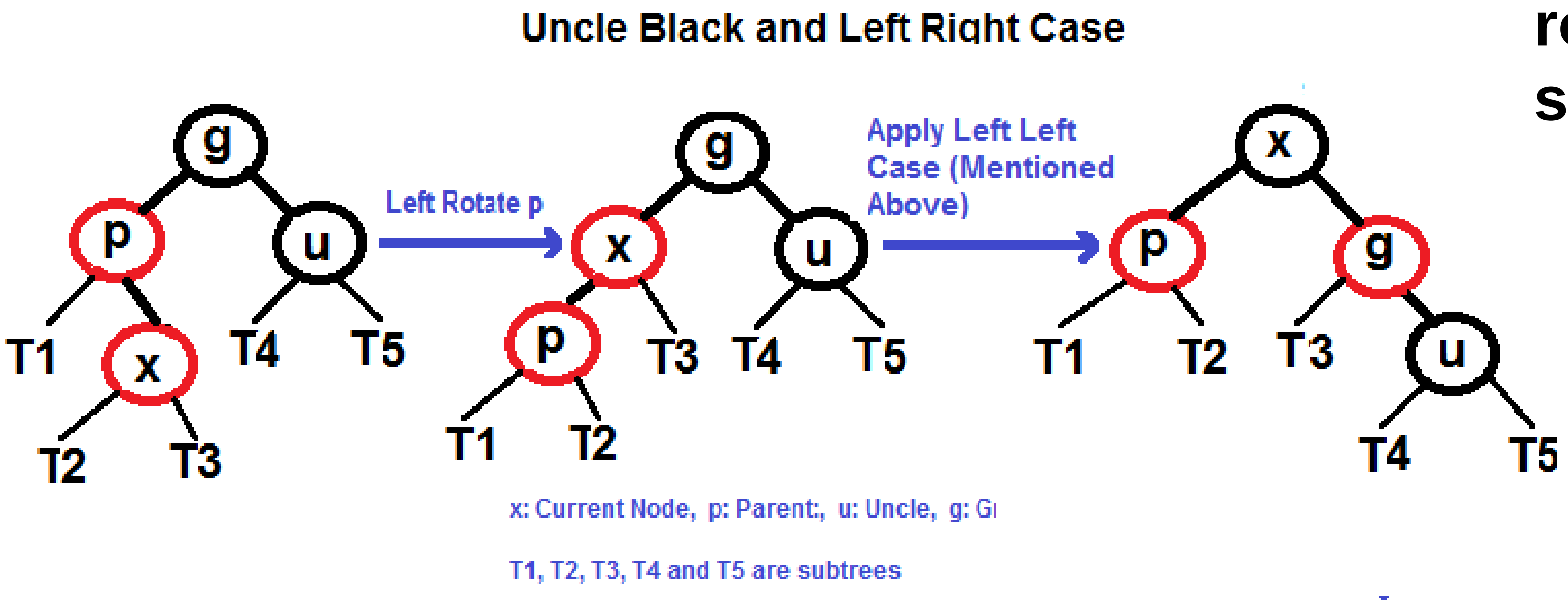
I. Left Left Case (p is left child of g and x is left child of p)




rotateRight(root, g)
swap(p→color, g→color)

# Insertion

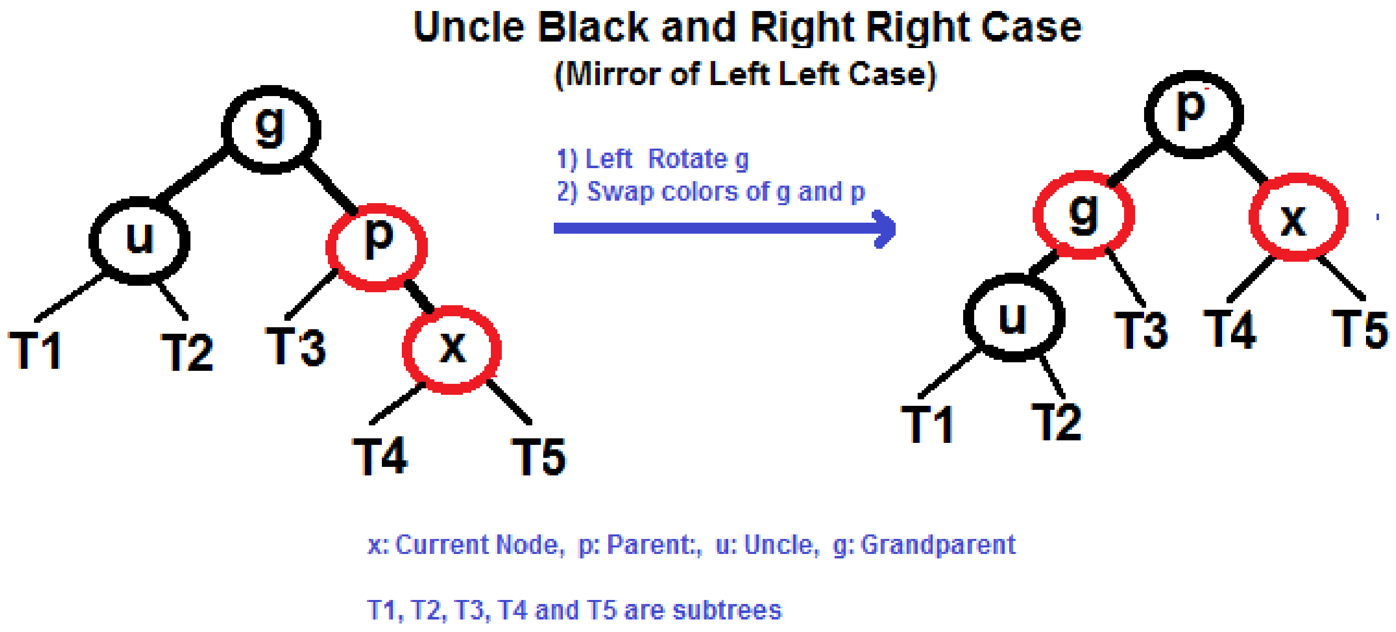
II. Left Right Case (p is left child of g and x is right child of p)




rotateLeft(root, p)
rotateRight(root, g)
swap(x→color, g→color)

# Insertion

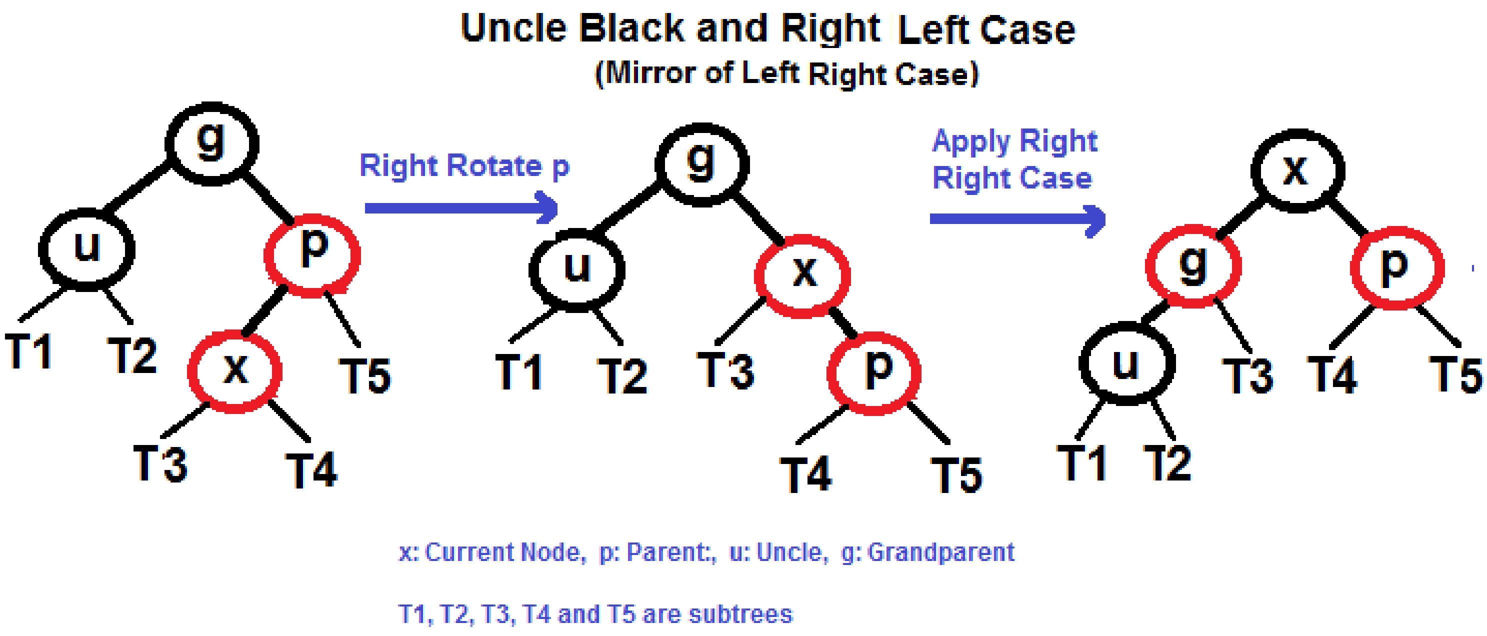
### III. Right Right Case (Mirror of case i)




**rotateLeft(root, g)**
**swap(p→color, g→color)**

# Insertion

## IV. Right Left Case (Mirror of case ii)




rotateRight(root, p)
rotateLeft(root, g)
swap(x→color, g→color)

# Insertion Analysis

- Go up the tree performing Case 3-a), which only recolors nodes.
- If Case 3-b) occurs, perform 1 or 2 rotations, and terminate.

→ Running time: O(log n) with O(1) rotations.

# Example of Insertion

11
1
14
2
7
15

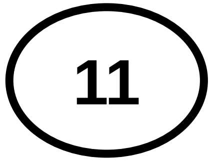
# Example of Insertion

11

**Insert 11**

**11**
**1**
**14**
**2**
**7**
**15**

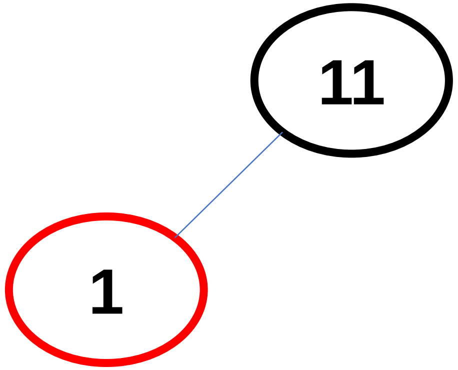
# Example of Insertion




**Insert 11**

**11**
**1**
**14**
**2**
**7**
**15**

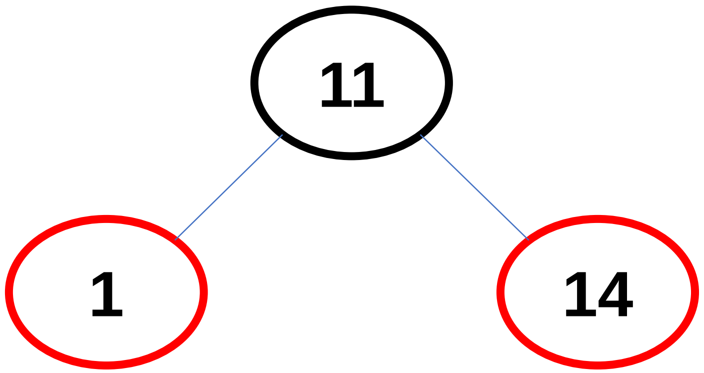
# Example of Insertion




**Insert 1**

**11**
**1**
**14**
**2**
**7**
**15**

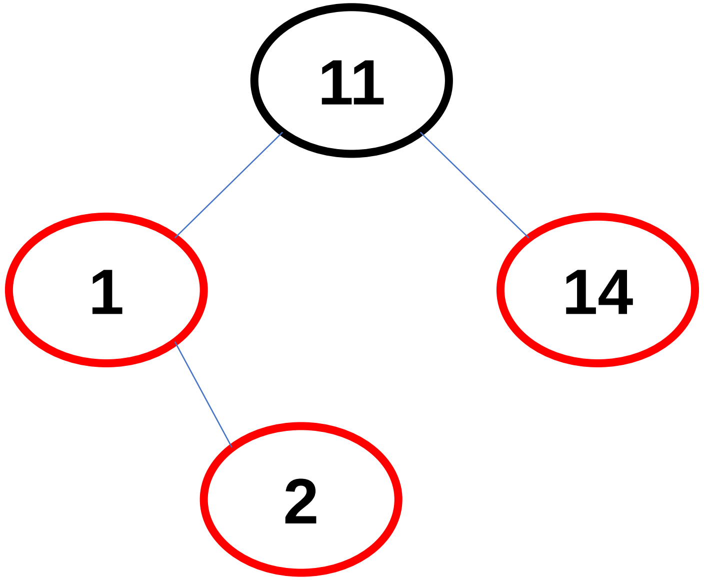
# Example of Insertion




**Insert 14**

**11**
**1**
**14**
**2**
**7**
**15**

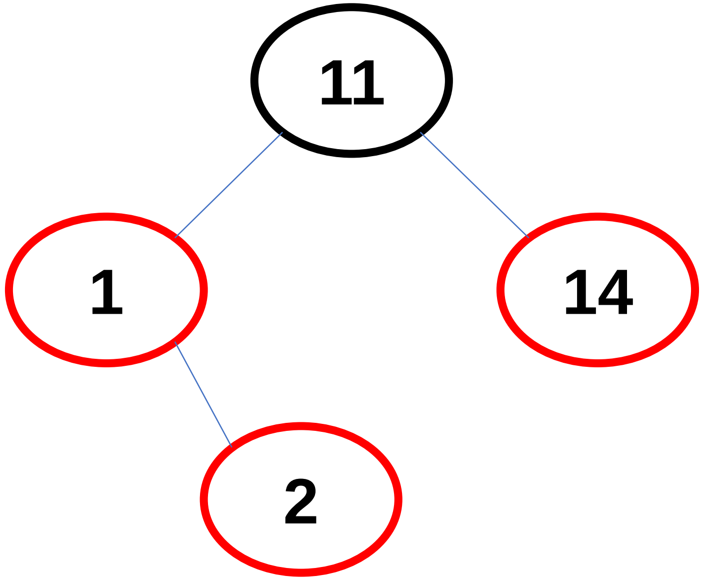
# Example of Insertion




**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

# Example of Insertion

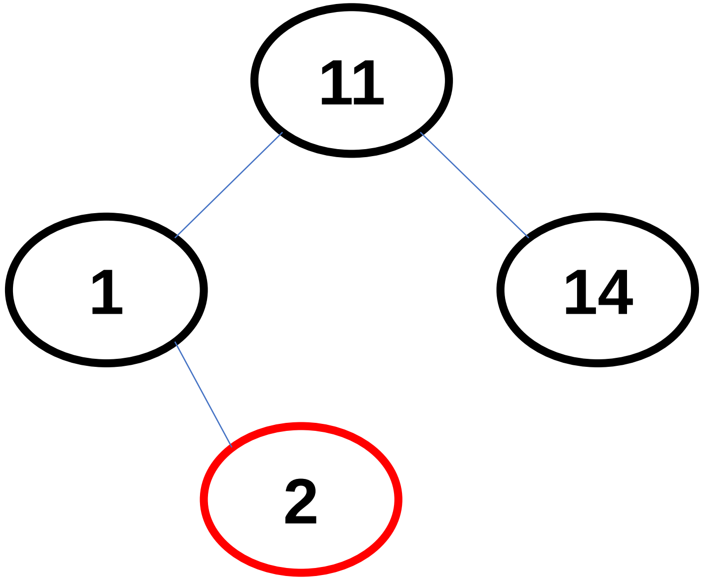



**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

**If X's uncle is RED and X's parent is not BLACK, change color of parent and uncle as BLACK**

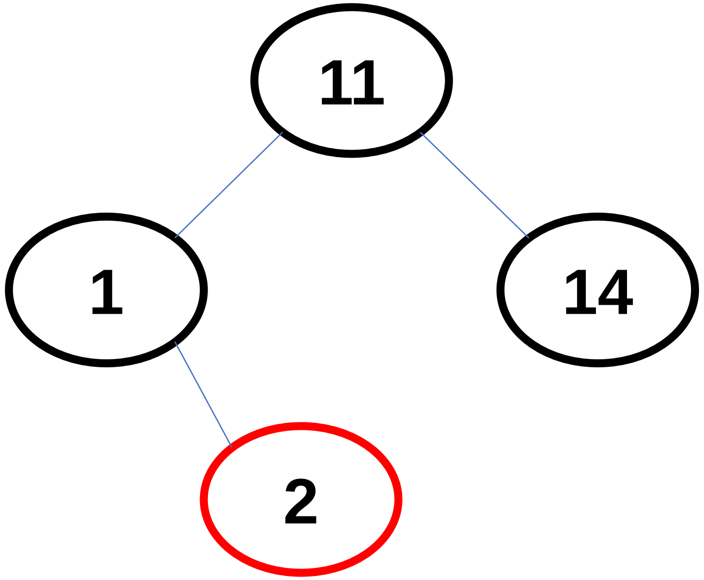
# Example of Insertion




**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

# Example of Insertion

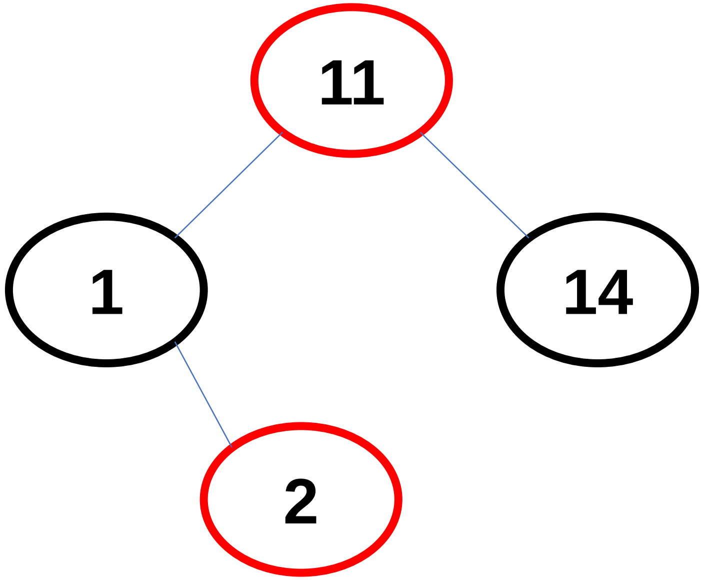



Insert 2

11
1
14
2
7
15

Color of Grand parent as RED

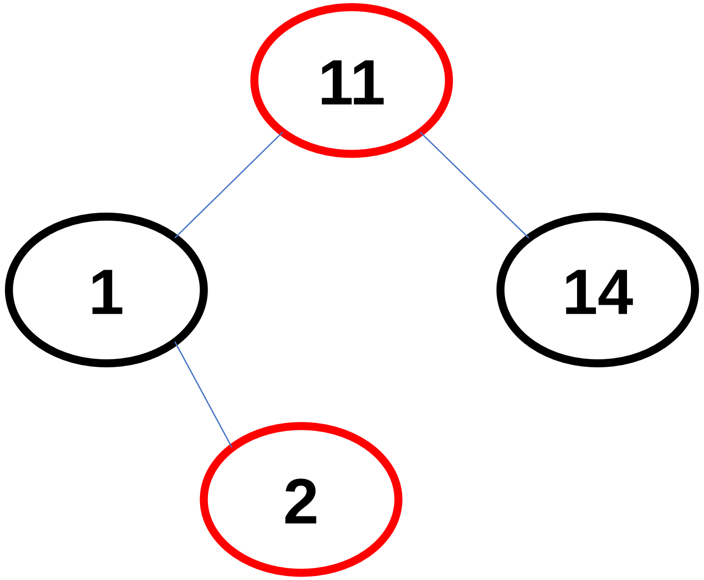
# Example of Insertion




**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

# Example of Insertion

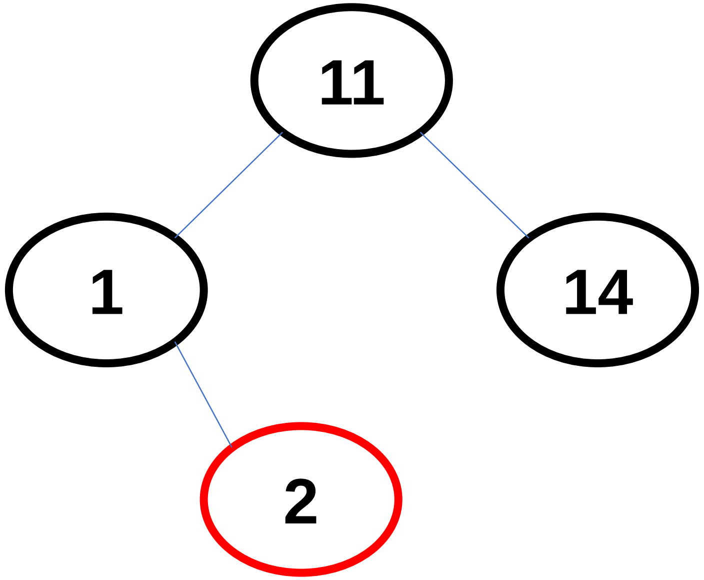



**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

**As 11 is a root node, change its color to black**

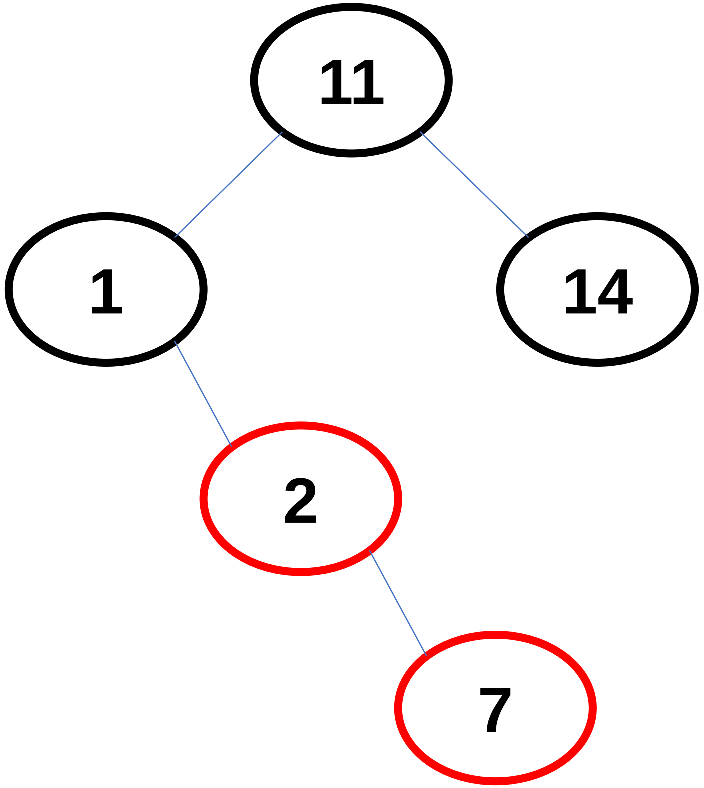
# Example of Insertion




**Insert 2**

**11**
**1**
**14**
**2**
**7**
**15**

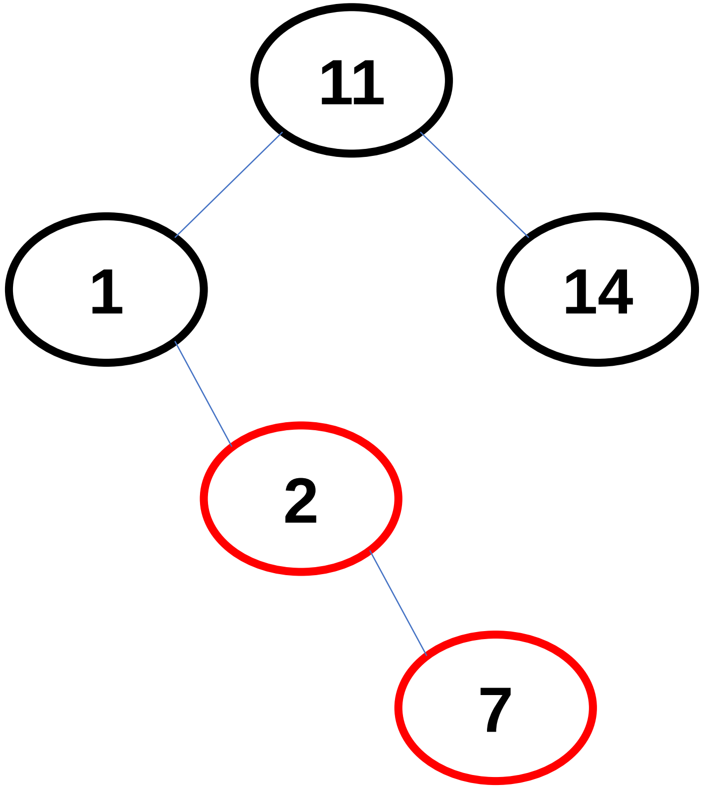
# Example of Insertion




**Insert 7**

**11**
**1**
**14**
**2**
**7**
**15**

# Example of Insertion

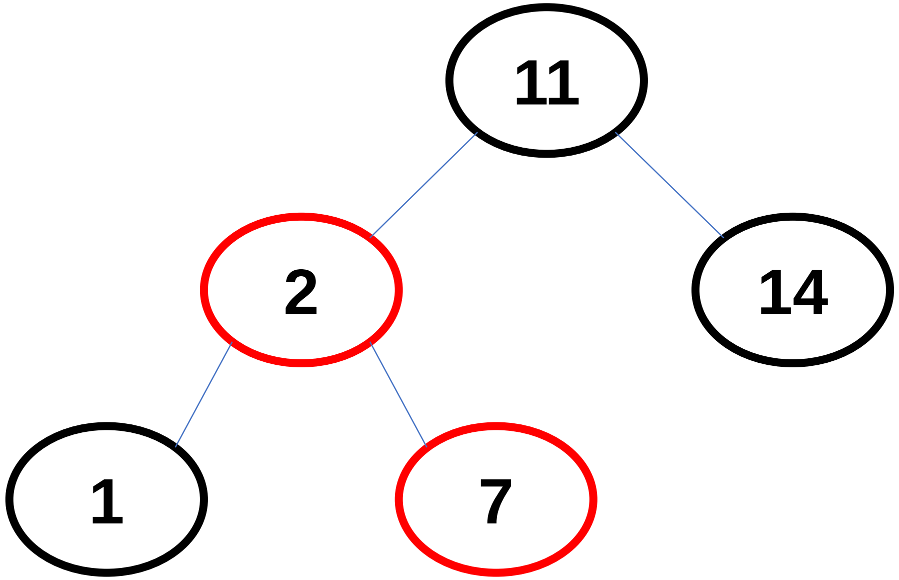



Insert 7

11
1
14
2
7
15

Left rotate(2) and recolor nodes

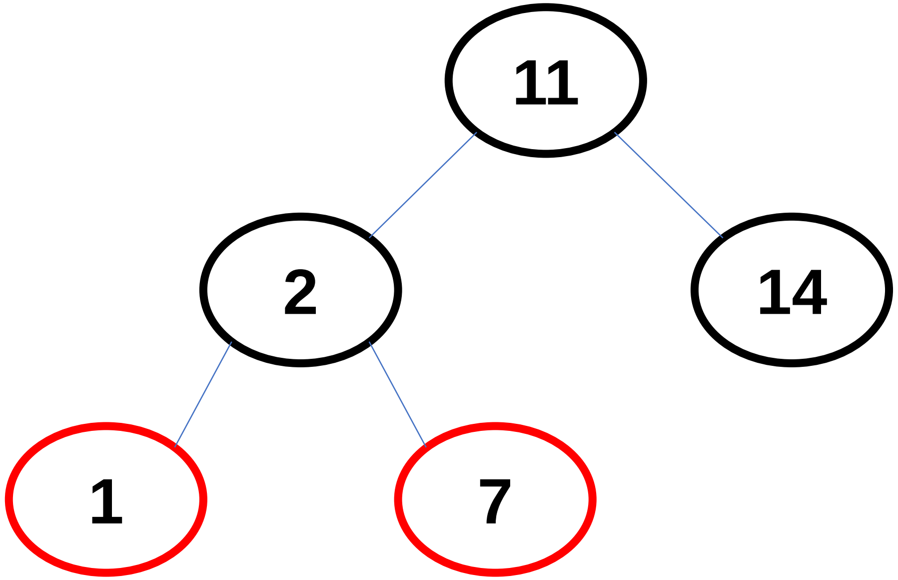
# Example of Insertion




**Insert 7**

**11**
**1**
**14**
**2**
**7**
**15**

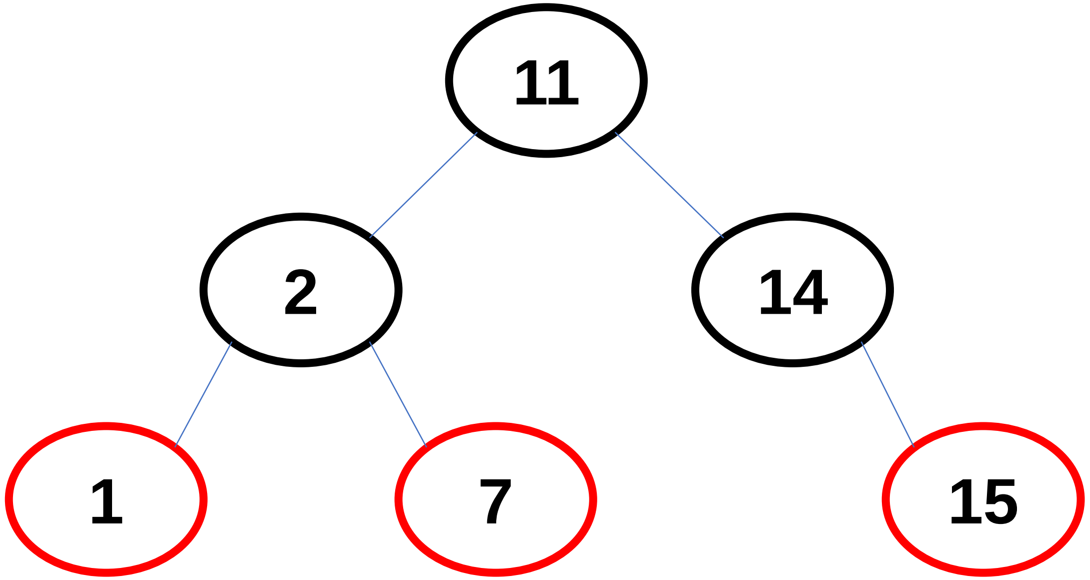
# Example of Insertion




**Insert 7**

**11**
**1**
**14**
**2**
**7**
**15**

# Example of Insertion




**Insert 15**

**11**
**1**
**14**
**2**
**7**
**15**

# Exercise

**Insert 3, 1, 5, 7, 6, 8, 9, 10**

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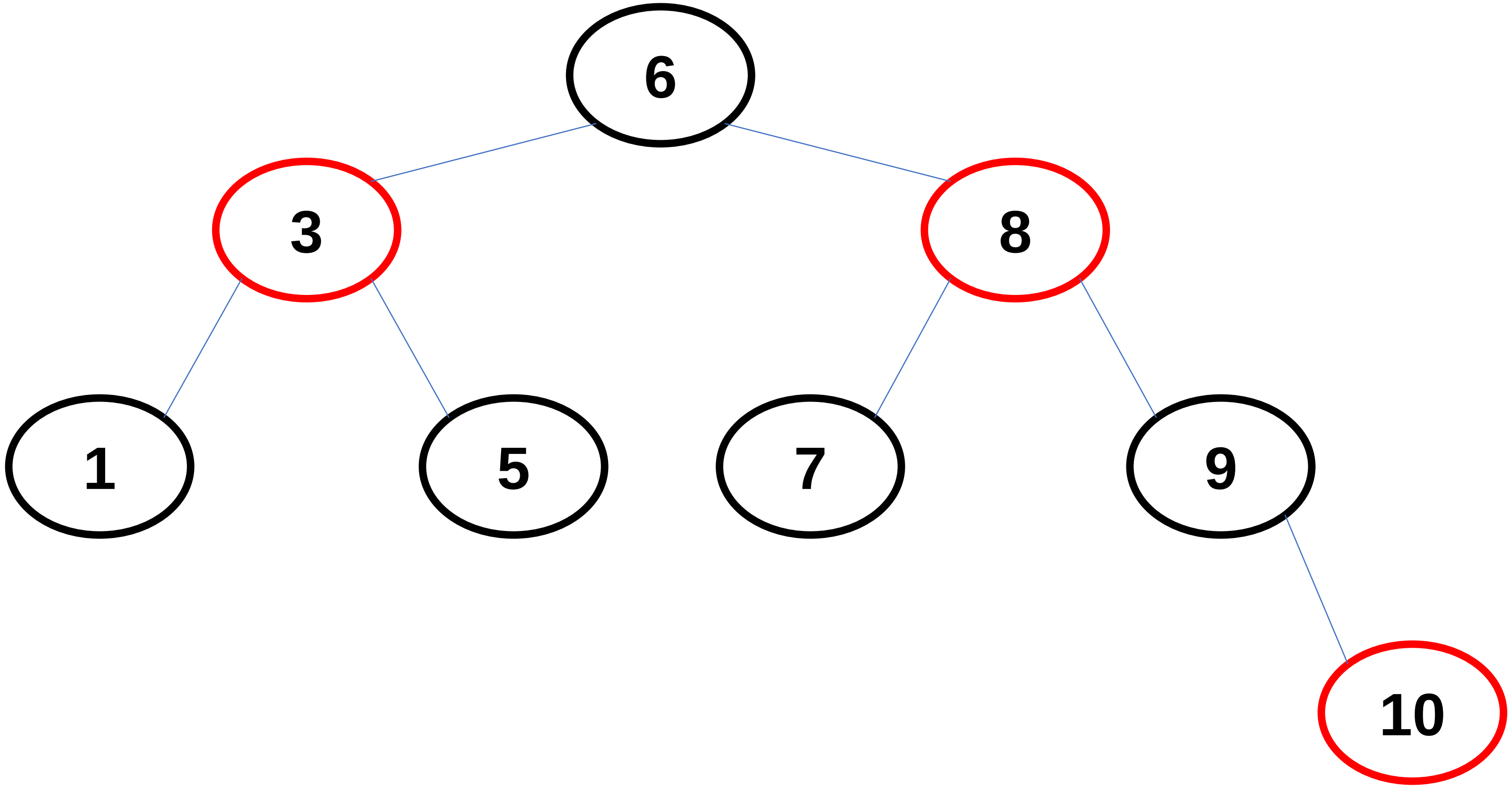

# Deletion

**Like Insertion, recoloring and rotations are used to maintain the Red-Black properties.**

**In delete operation, we check color of sibling to decide the appropriate case.**

# Deletion Algorithm

1. **Perform standard BST delete**
   - **A node which is either leaf or has only one child is deleted**
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   - **For an internal node: Copy the successor → <u>call delete for successor</u>**

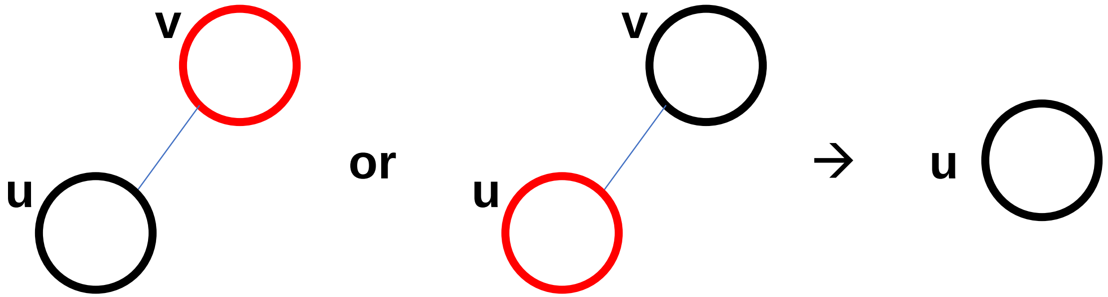
2. **Handle cases where a node is leaf or has one child**

3. **Let v be the node to be deleted and u be the child that replaces v**
   - **Note that u is NULL when v is a leaf and color of NULL is considered as Black**

# Deletion Algorithm

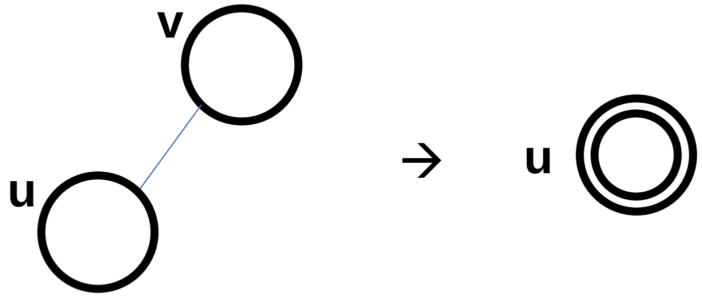
## a) If Either u or v is RED

- Mark the replaced child as black




## b) If Both u and v are BLACK

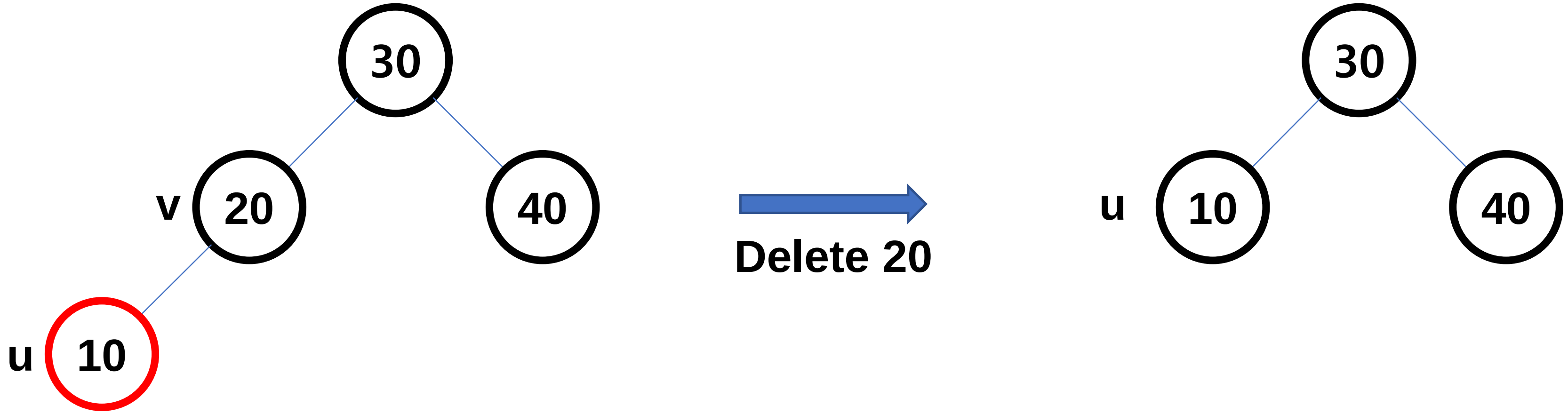
- Color u as a double black



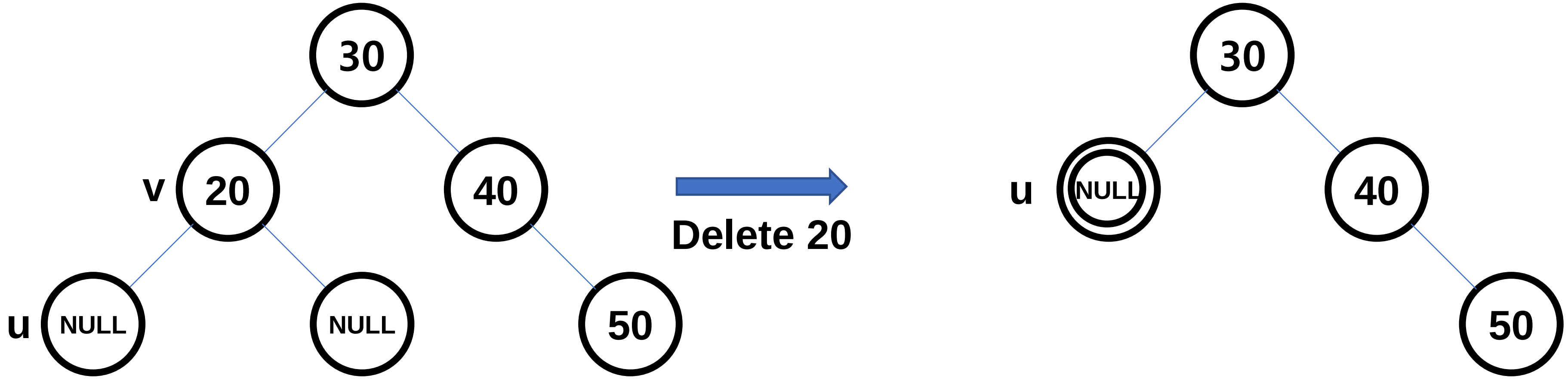

- If u is root, make it single black and return

# Deletion Algorithm

## a) If Either u or v is RED




## b) If Both u and v are BLACK



# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 1: If s is black and at least one of s's children is RED

- Case 2: If s is black and its both children are BLACK

- Case 3: If s is RED

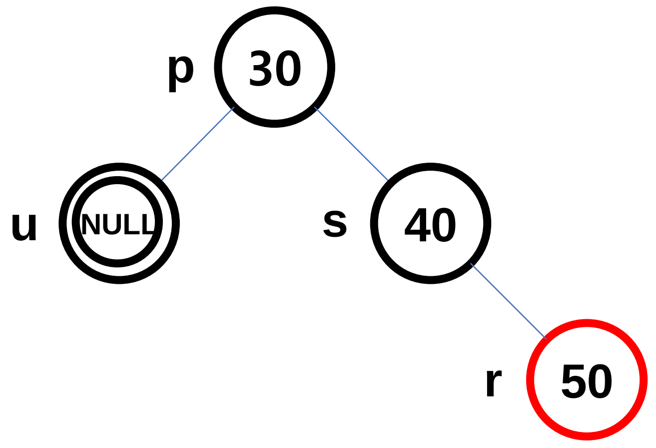
# Deletion Algorithm

**b) If Both u and v are BLACK**

- **Let sibling of node be s**

- **Case 1: If s is black and at least one of s’s children is RED**
- **Let the red child of s be r**

  I. **Left Left Case (Mirror of III)**
  II. **Left Right Case (Mirror of IV)**
  III. **Right Right Case**
  IV. **Right Left Case**



# Deletion Algorithm

## b) If Both u and v are BLACK

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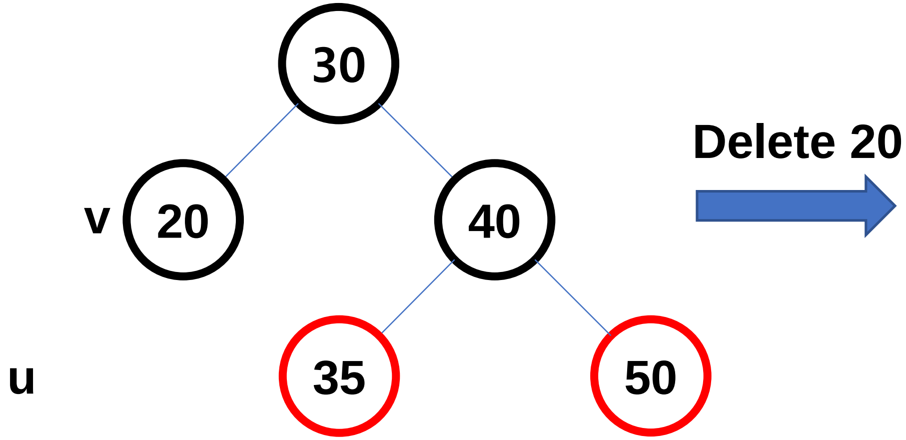
### III. Right Right Case (r → outer node)



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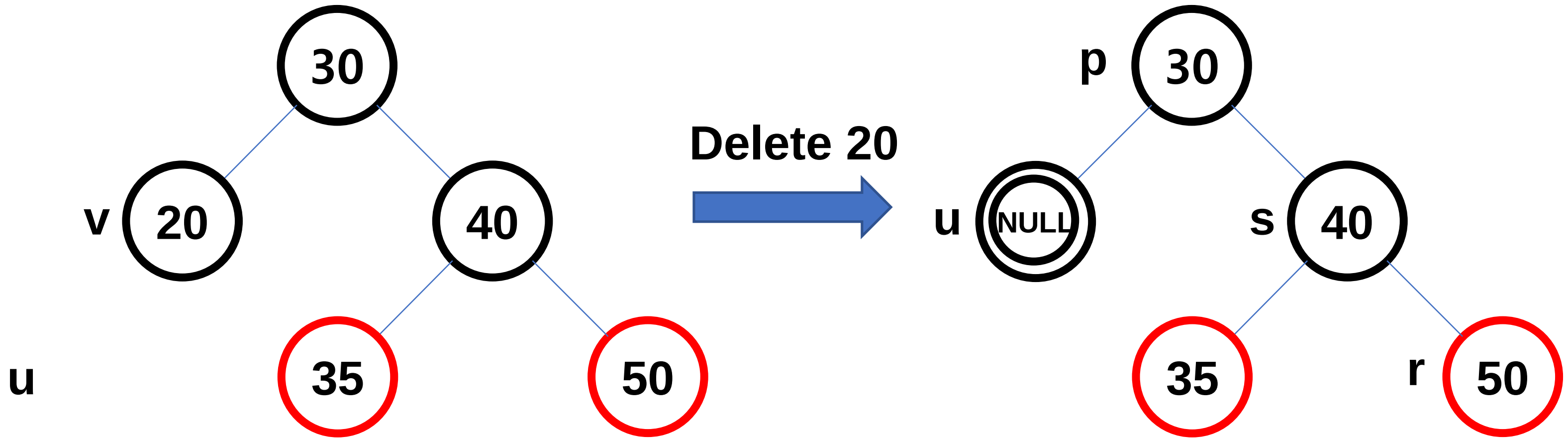

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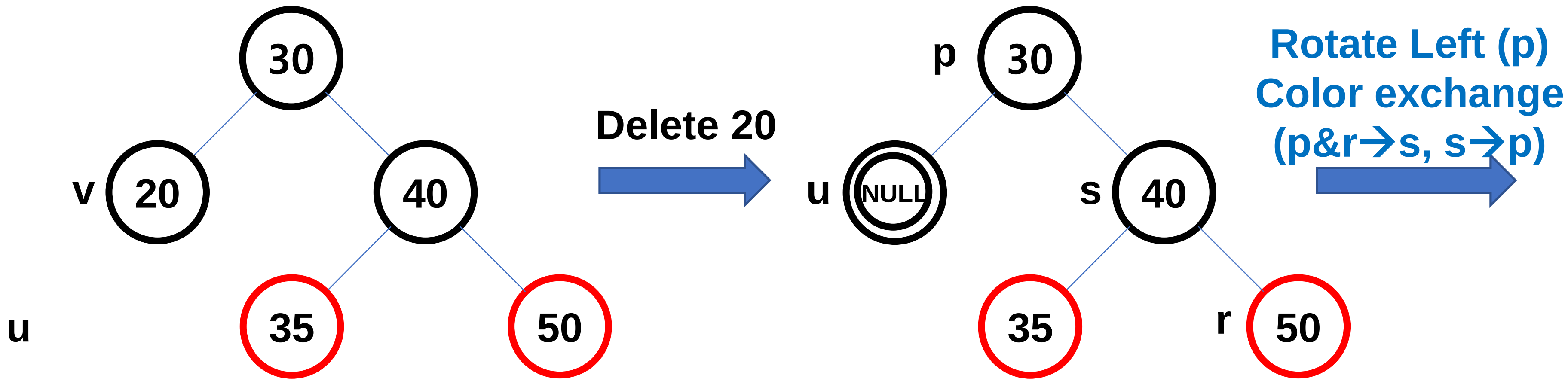

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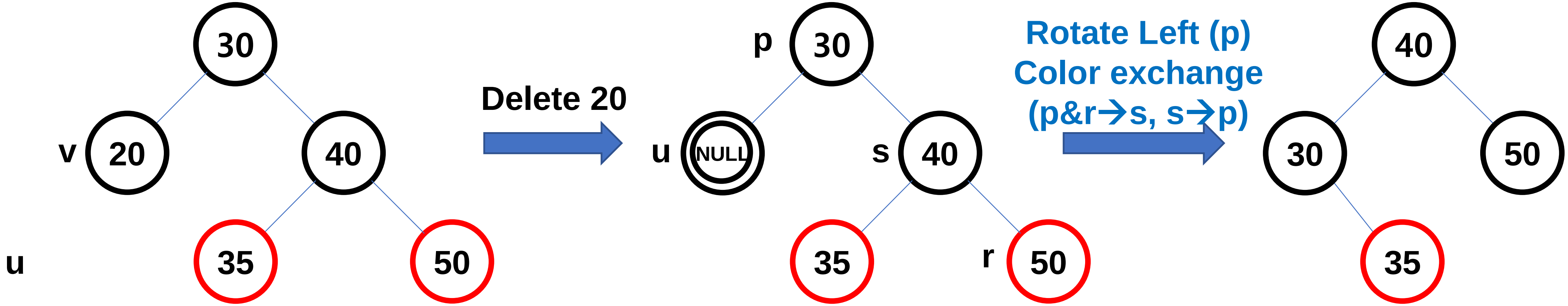

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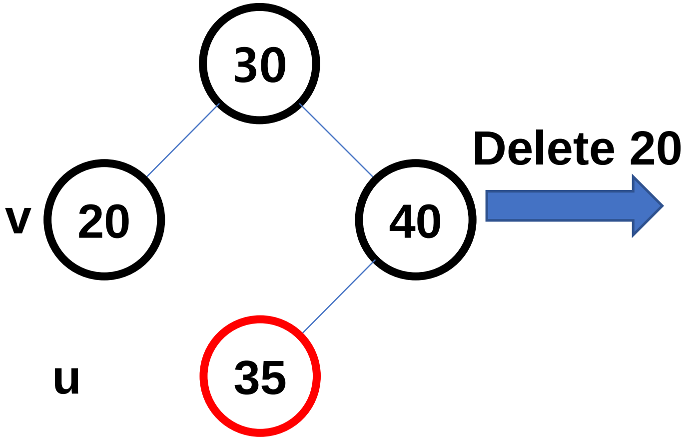
### IV. Right Left Case



# Deletion Algorithm

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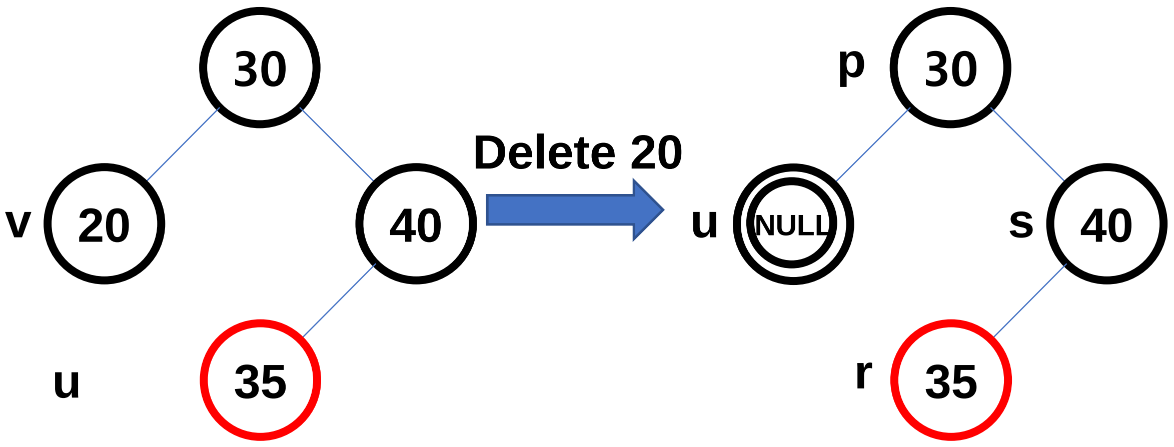

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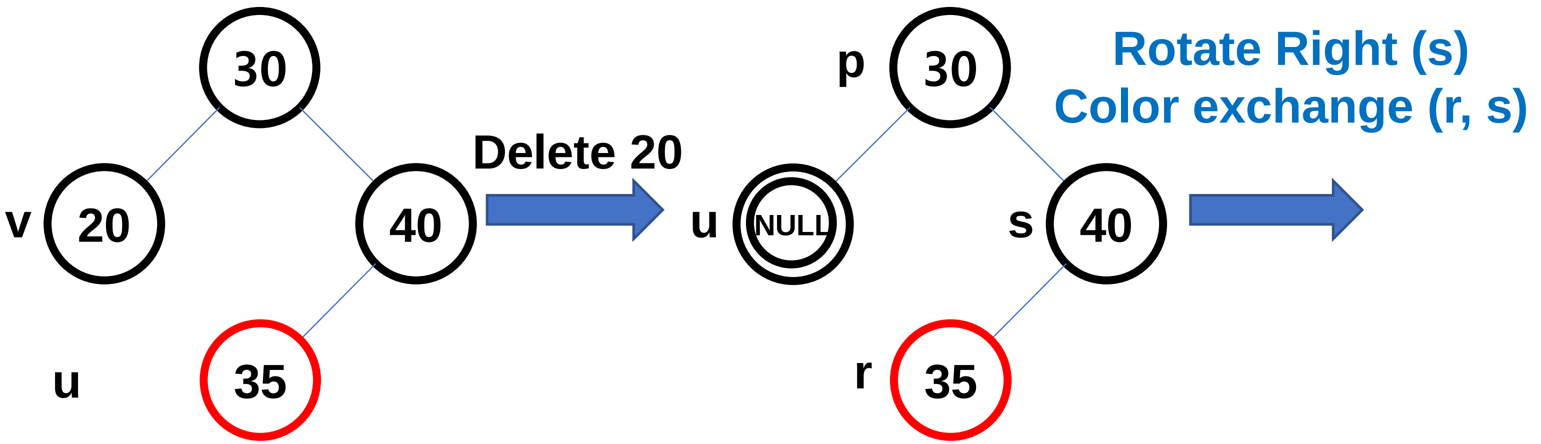

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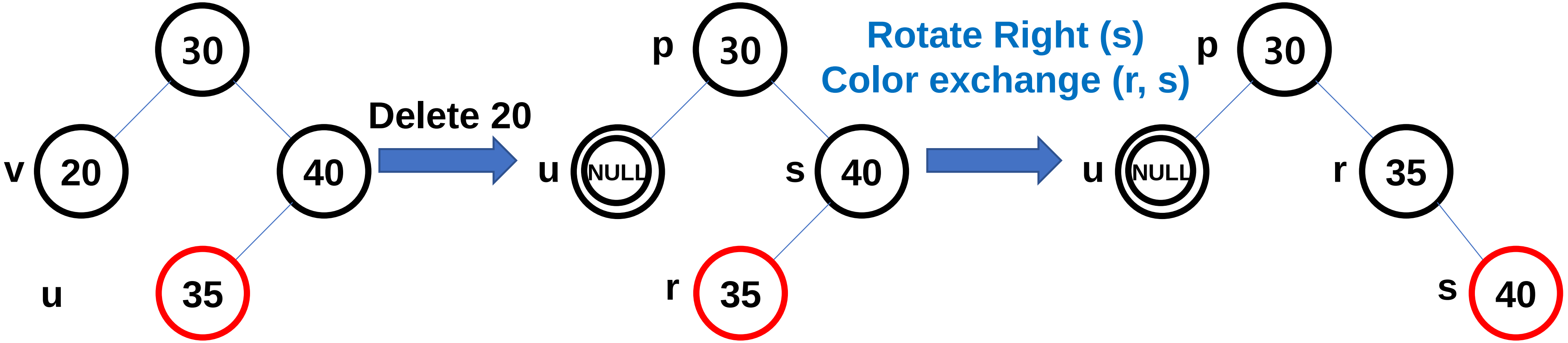
### IV. Right Left Case



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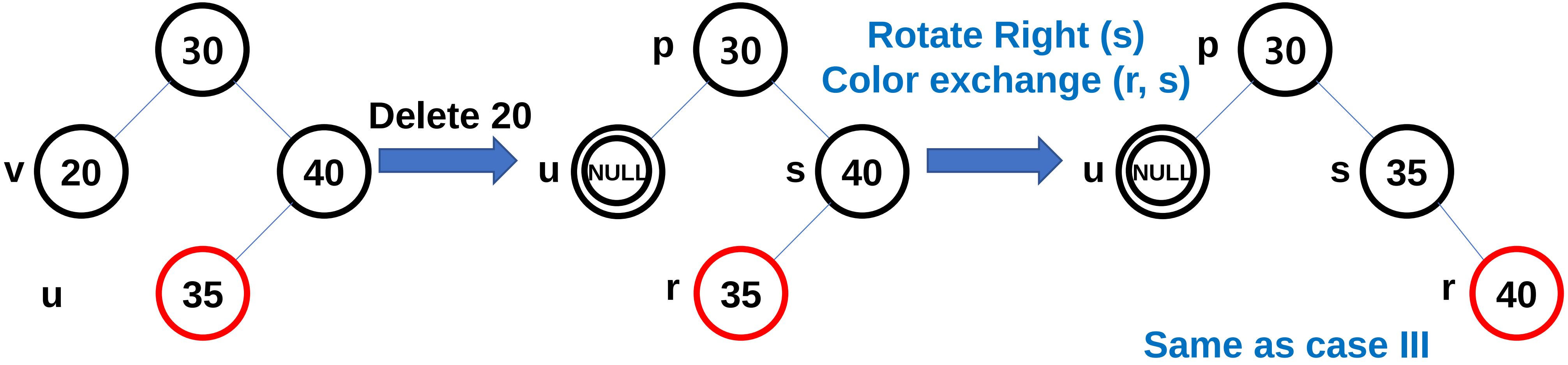

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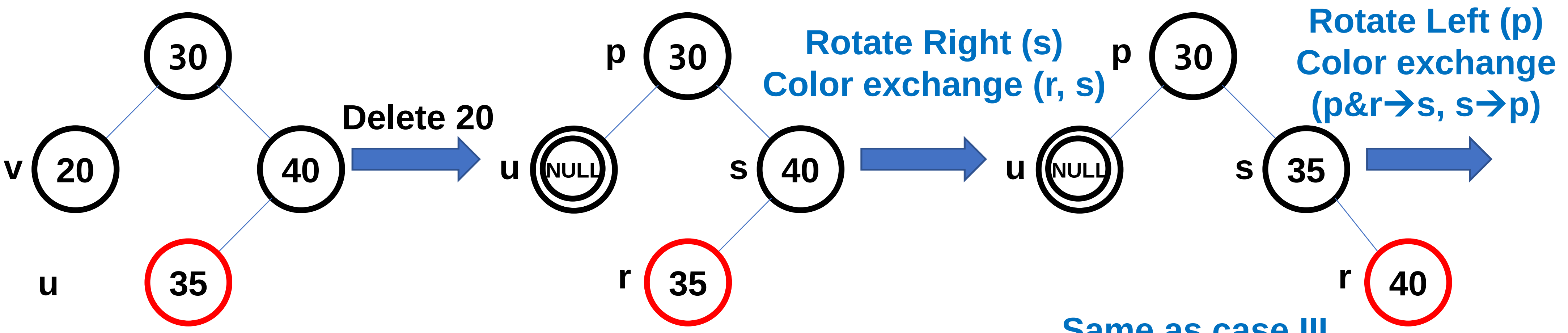
### IV. Right Left Case



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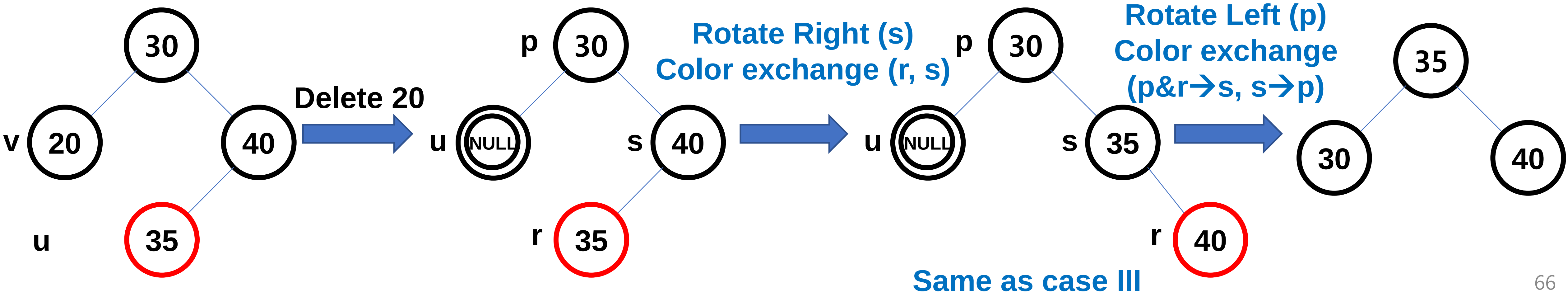
- Case 1: If s is black and at least one of s’s children is RED

### IV. Right Left Case



# Deletion Algorithm

## b) If Both u and v are BLACK

- **Let sibling of node be s**

- **Case 2: If s is black and its both children are BLACK**
  1) **Recolor p to BLACK and s to RED**
  2) **If parent is BLACK, recur for parent**

# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 2: If s is black and its both children are BLACK
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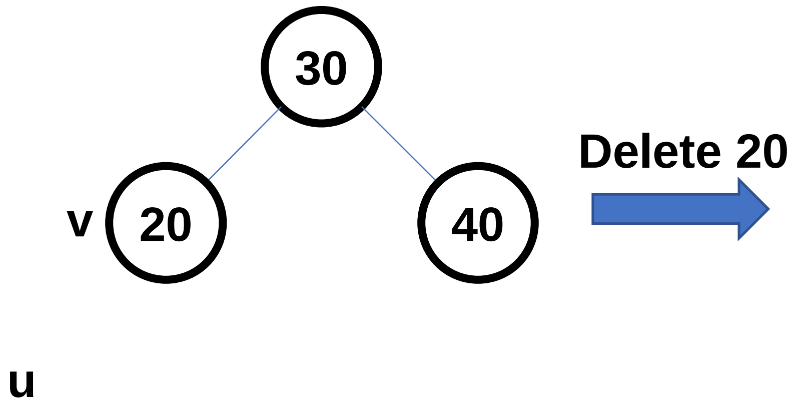
  2) If parent is BLACK, recur for parent



# Deletion Algorithm

## b) If Both u and v are BLACK

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- **Case 2: If s is black and its both children are BLACK**
  1) **Recolor p to BLACK and s to RED**
  2) **If parent is BLACK, recur for parent**


u

# Deletion Algorithm

## b) If Both u and v are BLACK

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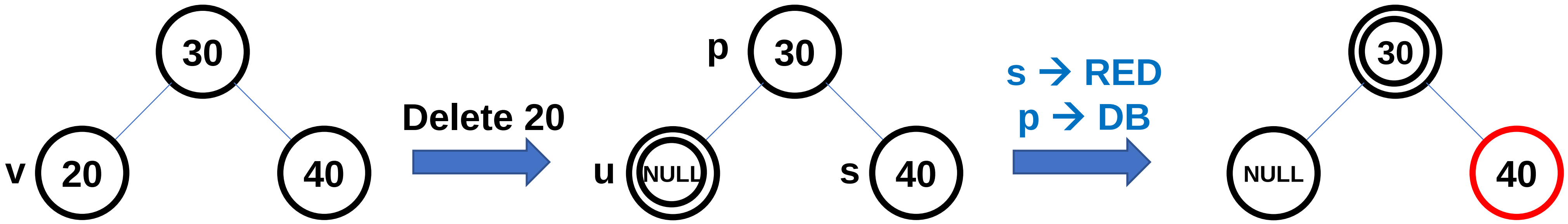
- Case 2: If s is black and its both children are BLACK
  1) Recolor p to BLACK and s to RED
  2) If parent is BLACK, recur for parent



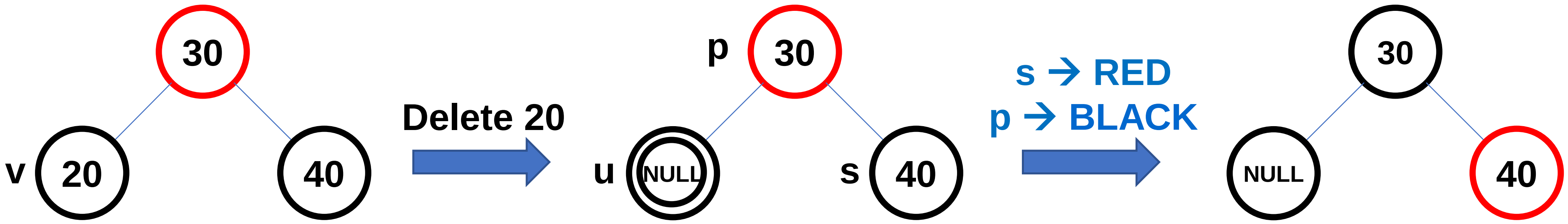

u

# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 2: If s is black and its both children are BLACK
  1) Recolor p to BLACK and s to RED
  2) If parent is BLACK, recur for parent




u

# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

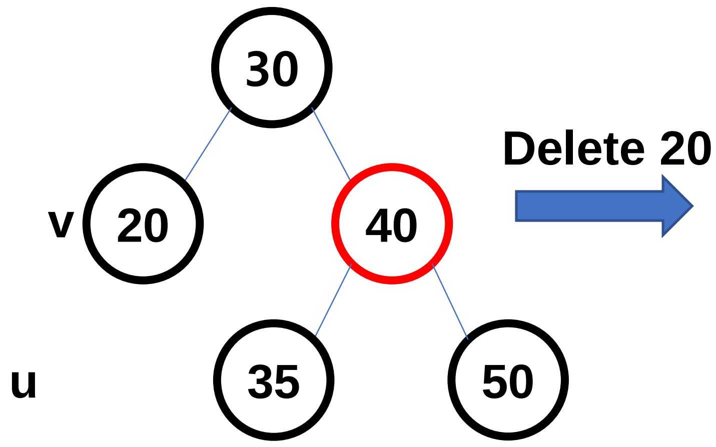
- Case 3: If s is RED
    1) Rotate p, Recolor p to RED and s to BLACK
    2) Case I or Case II

# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 3: If s is RED
    1) Rotate p, Recolor p to RED and s to BLACK
    2) Case I or Case II

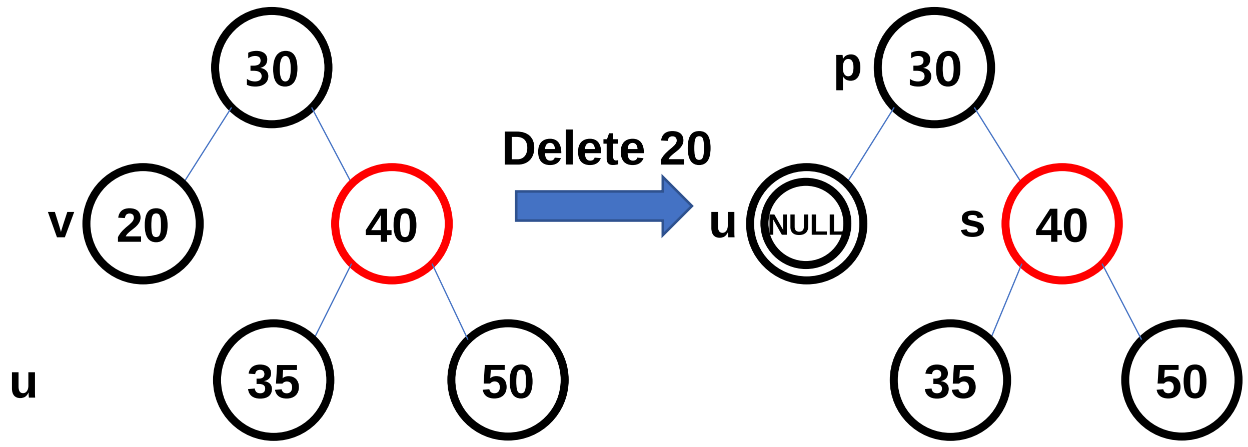


# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 3: If s is RED
  1) Rotate p, Recolor p to RED and s to BLACK
  2) Case I or Case II

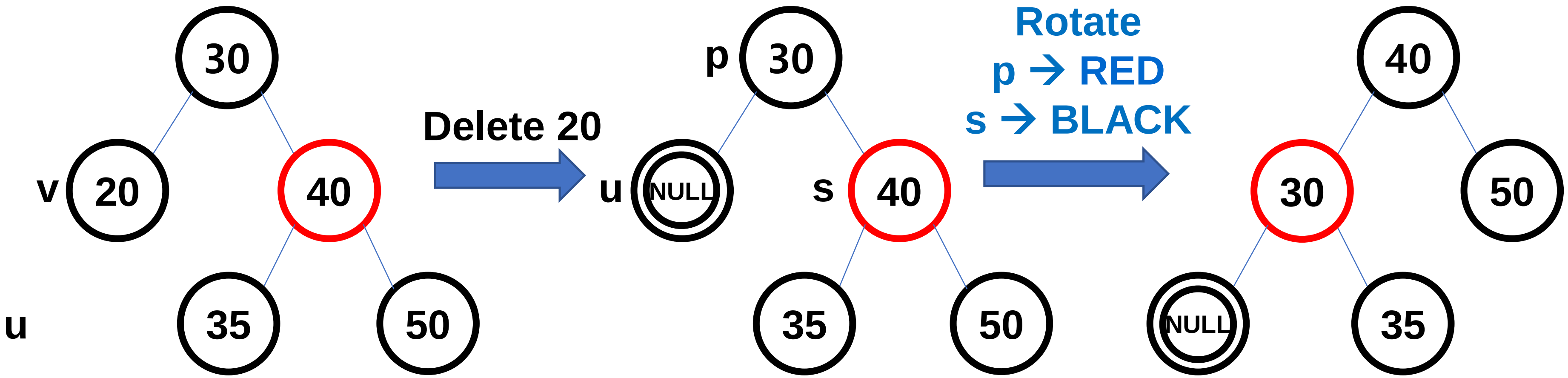


# Deletion Algorithm

## b) If Both u and v are BLACK

- **Let sibling of node be s**

- **Case 3: If s is RED**
  1) **Rotate p, Recolor p to RED and s to BLACK**
  2) **Case I or Case II**



# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 3: If s is RED
  1) Rotate p, Recolor p to RED and s to BLACK
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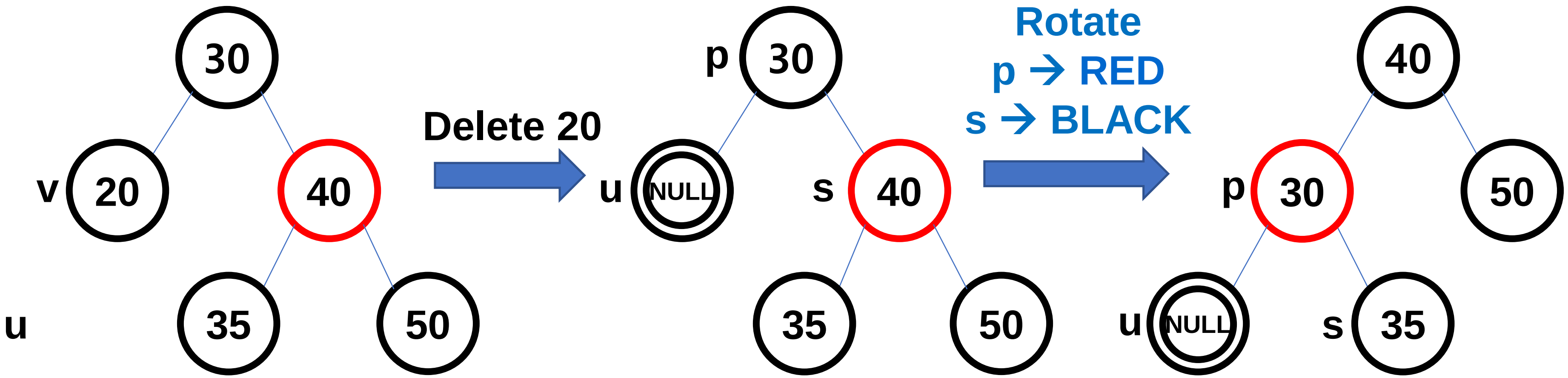
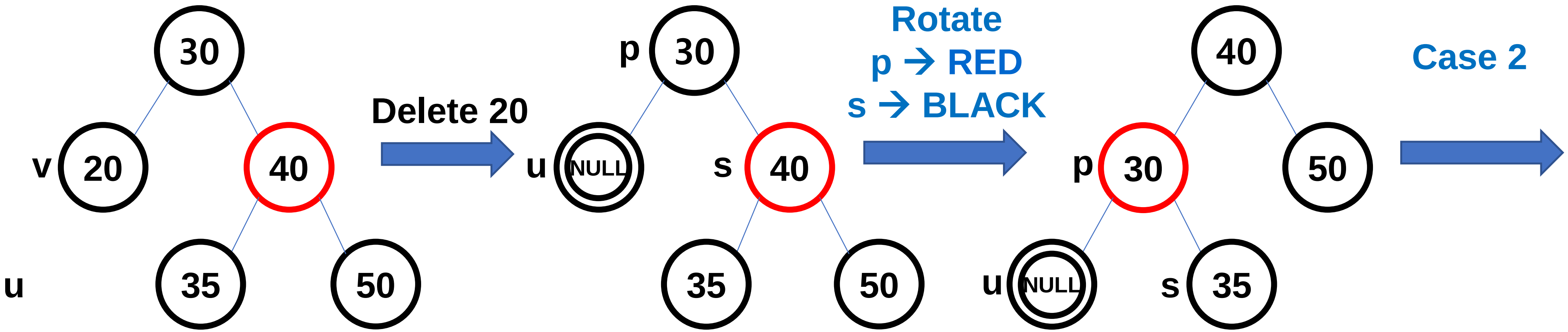


# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 3: If s is RED
  1) Rotate p, Recolor p to RED and s to BLACK
  2) Case I or Case II



# Deletion Algorithm

## b) If Both u and v are BLACK

- Let sibling of node be s

- Case 3: If s is RED
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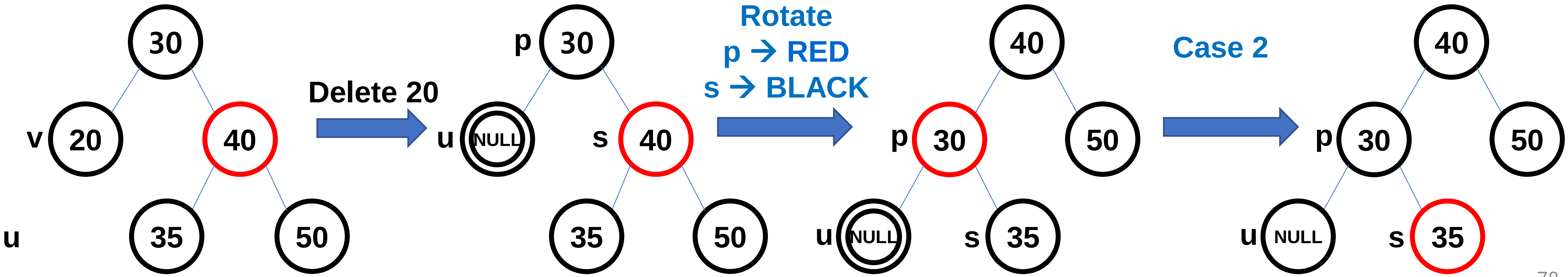


# Deletion Analysis

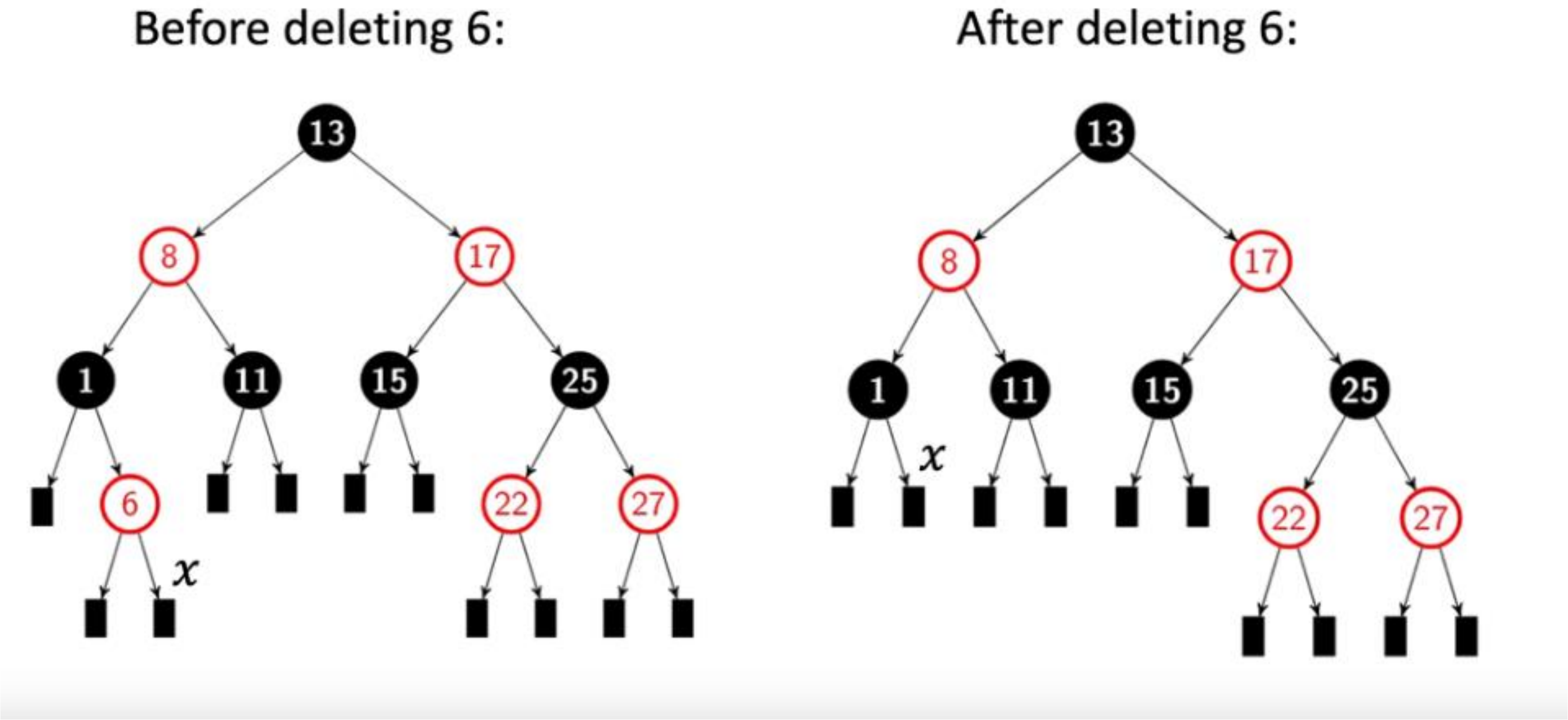
- **Case 2 is the only case in which more iterations occur.**

**→ u moves up 1 level. Hence, O(log n) iterations.**

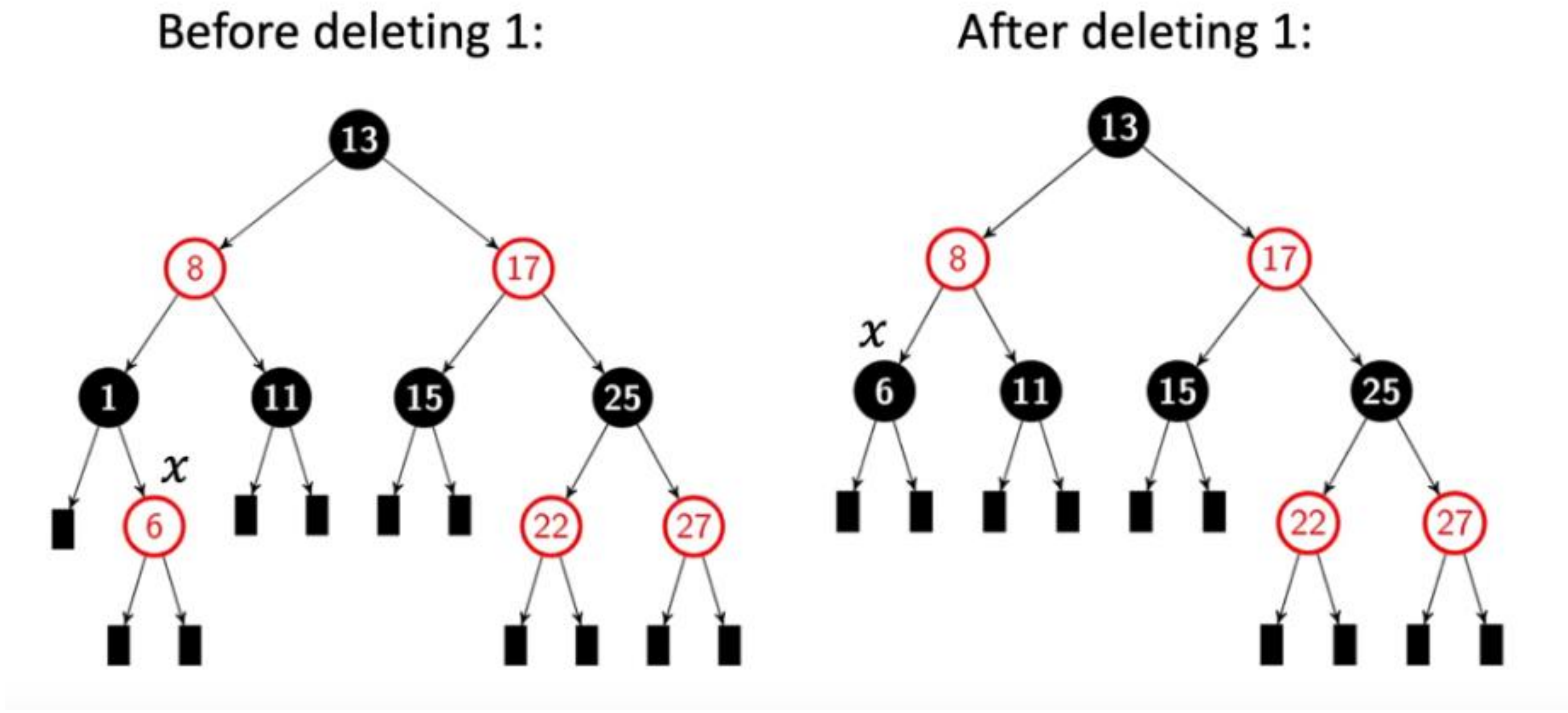
- **Each of cases 1 and 3 has at most ≤ 3 rotations in all**

**→ Running time: O(log n)**

# Example of Deletion (1)

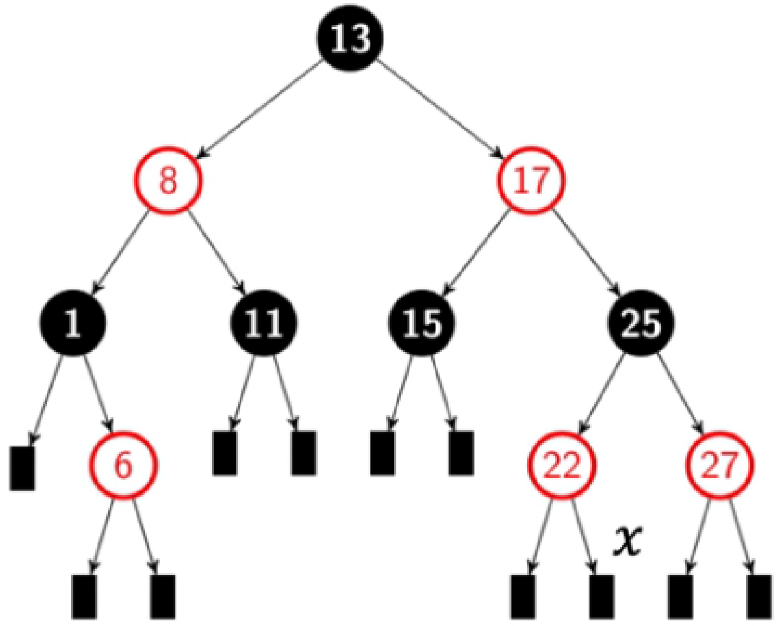


# Example of Deletion (2)

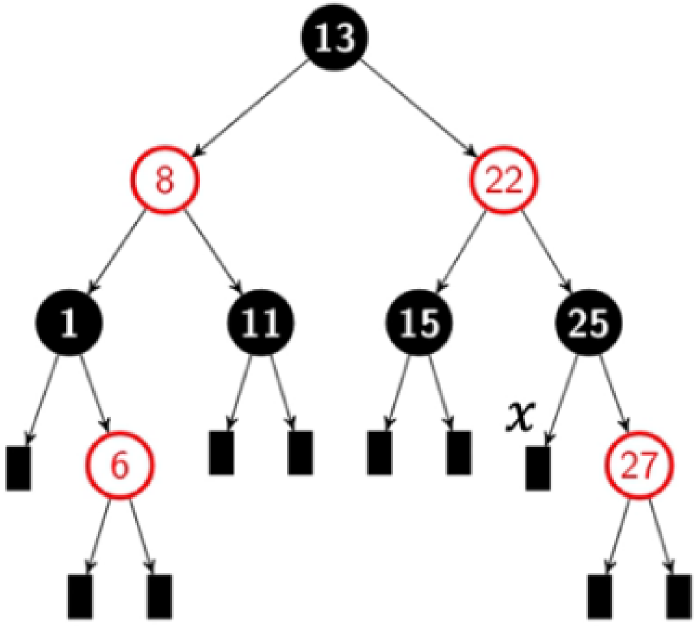


# Example of Deletion (3)

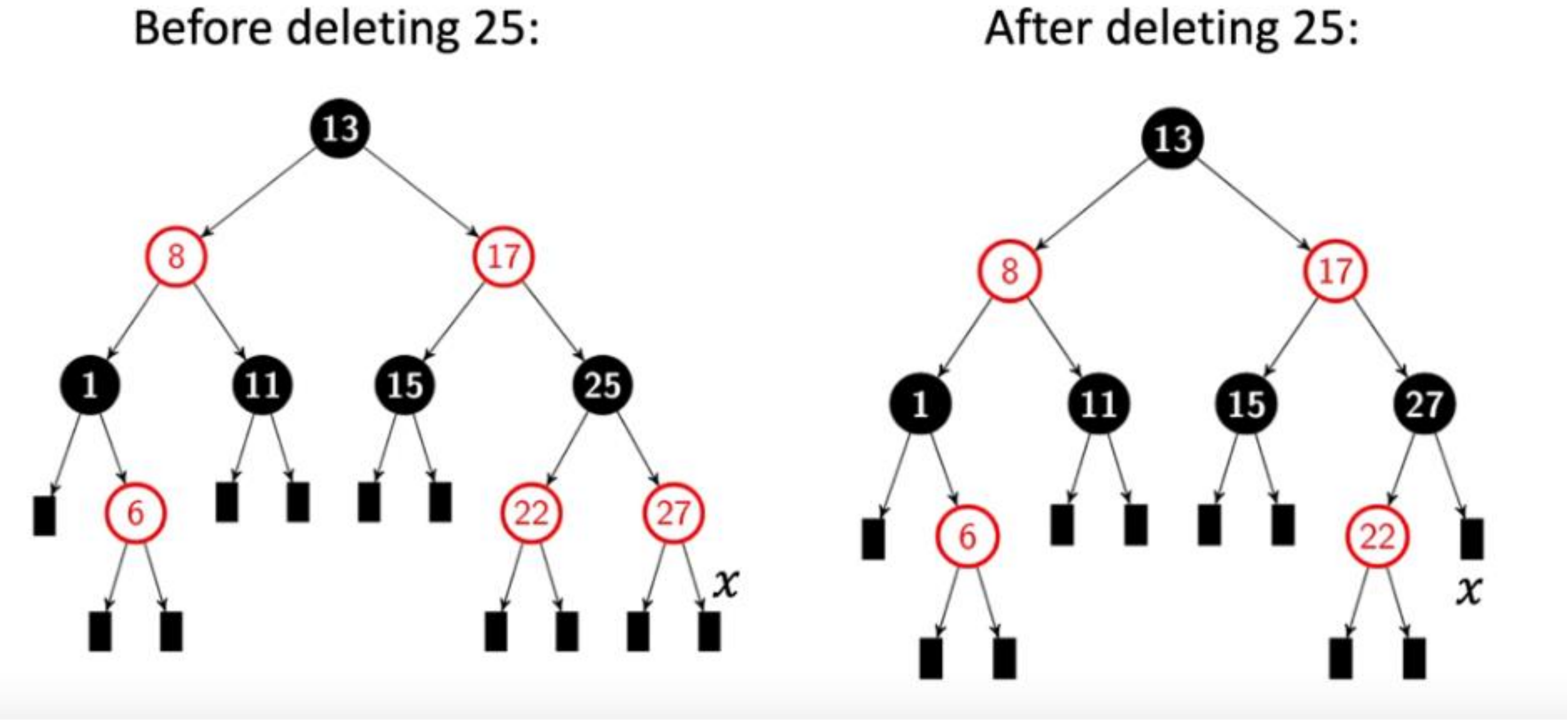
Before deleting 17:



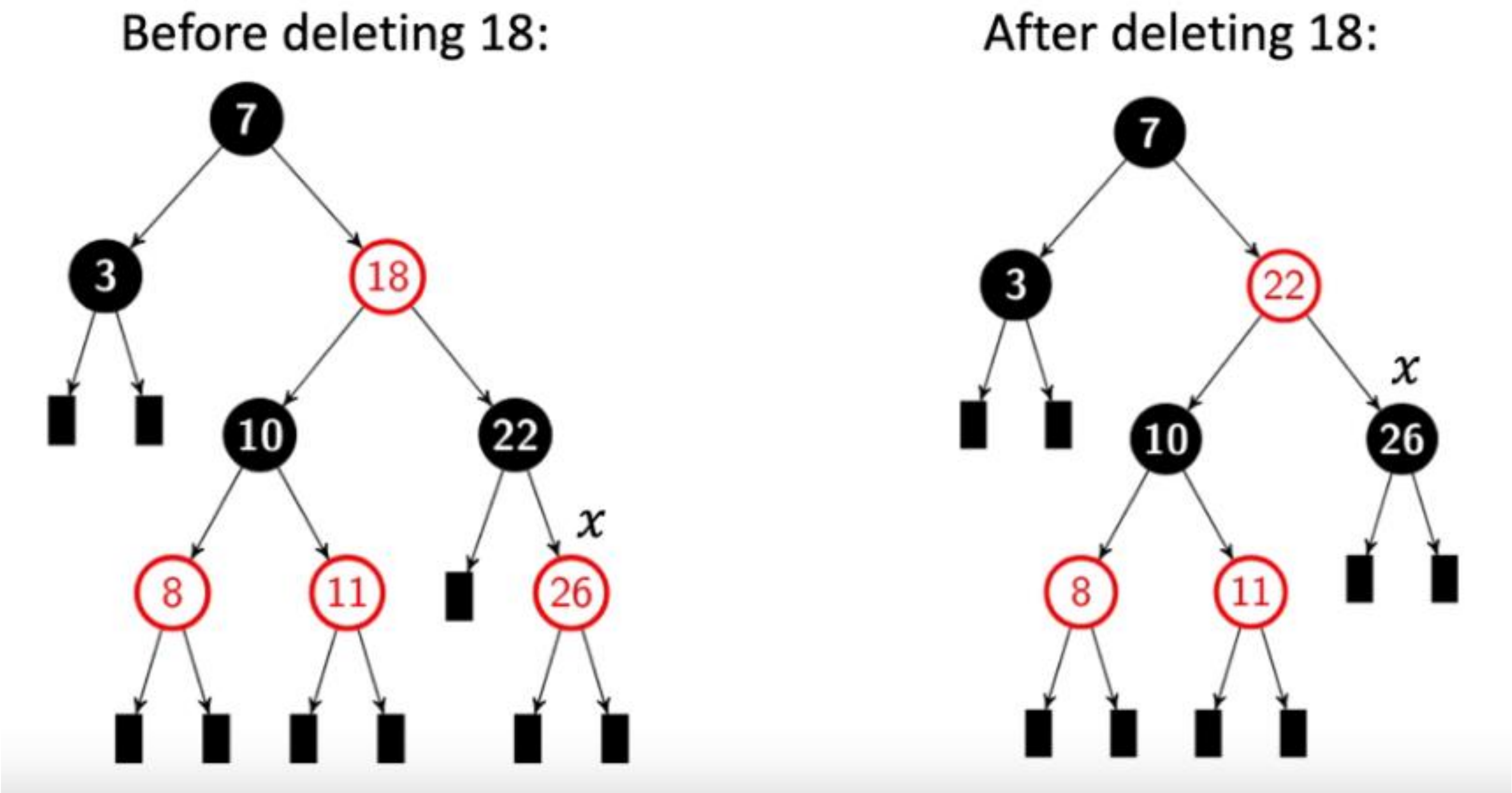

After deleting 17:



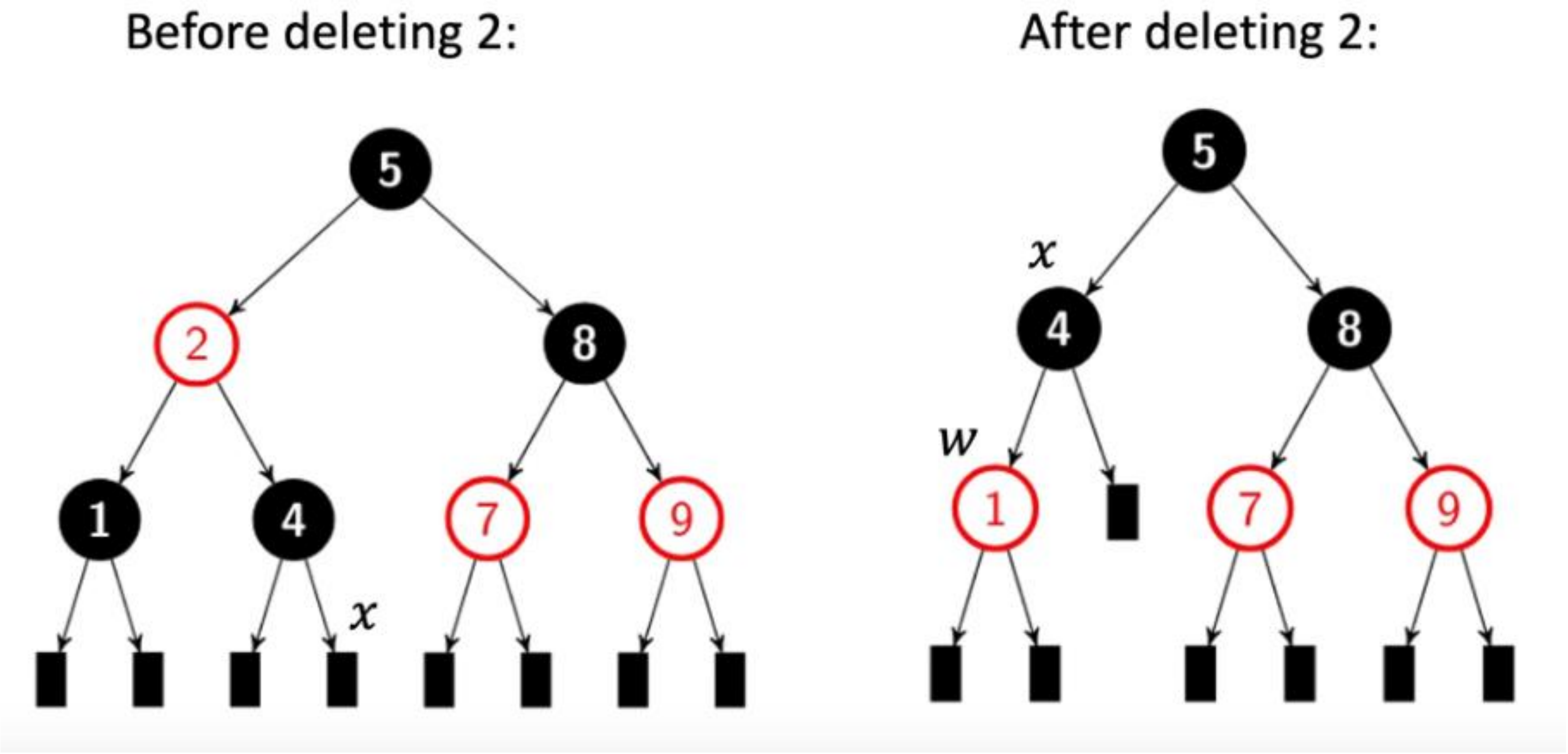
# Example of Deletion (4)



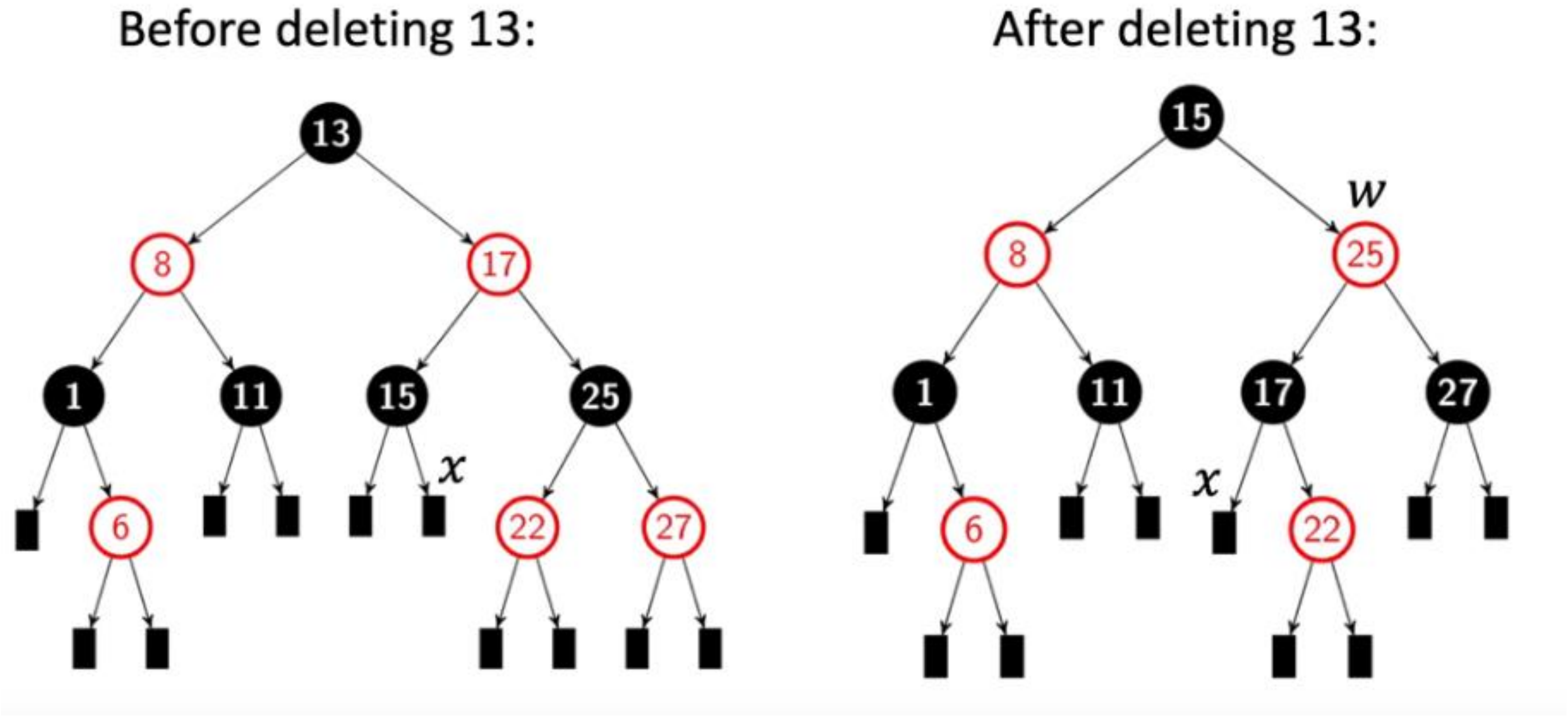
# Example of Deletion (5)



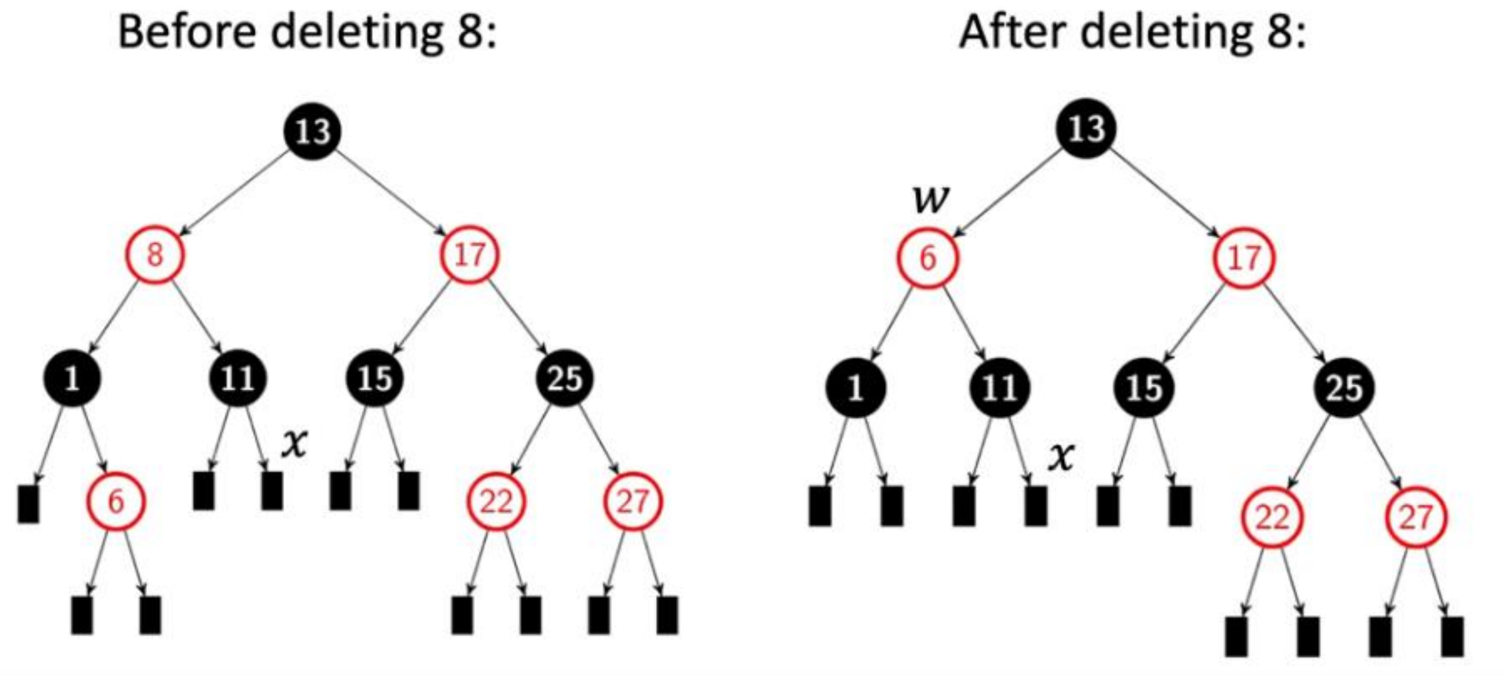
# Example of Deletion (6)



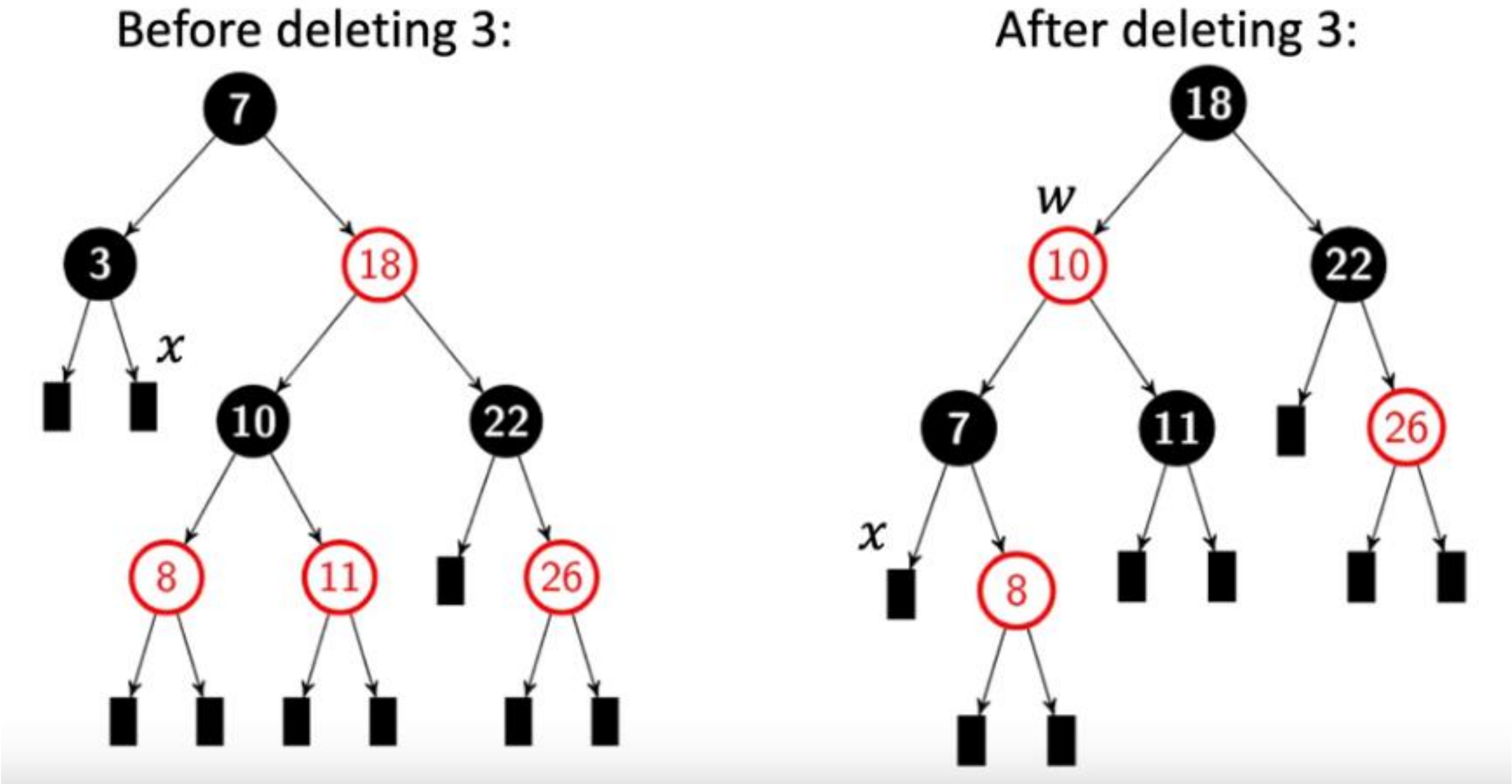
# Example of Deletion (7)



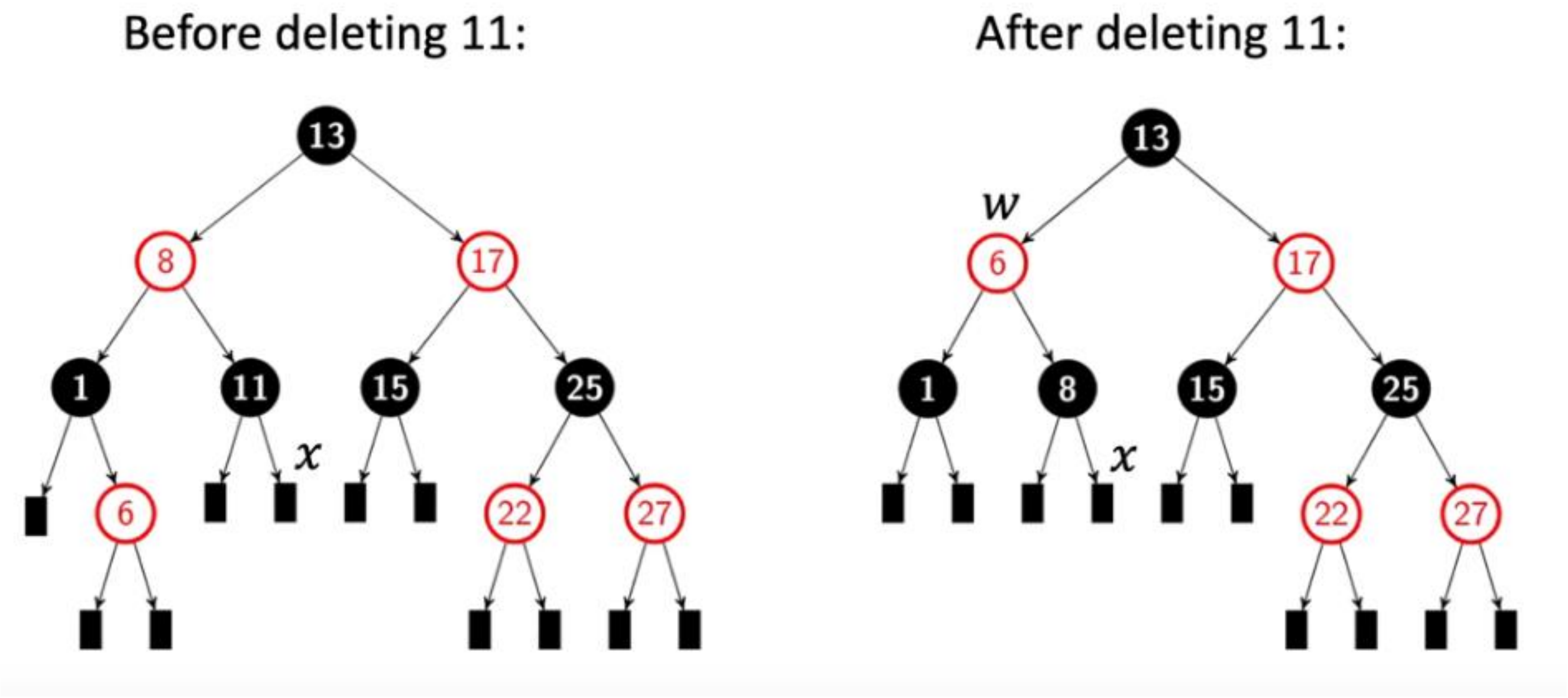
# Example of Deletion (8)



# Example of Deletion (9)



# Example of Deletion (10)



# Reference


- Charles Leiserson and Piotr Indyk, “*Introduction to Algorithms*”, September 29, 2004

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