# The Einstein equations

- We now have all the tools we need to describe the curvature of a general spacetime manifold, geodesic trajectories, & tensors defined on the manifold that describe observables.
- So how do we explain gravity?
- Observationally, gravity is sourced by mass.
- Hypothesis: gravity is spacetime curvature

+ general covariance: the relation between curvature & its sources should not depend on the choice of coordinate system

i.e we are looking for an equation relating <u>tensors</u>

Attempt 1: Newtonian physics

$$\nabla^2 \phi = 4\pi G \rho_m$$

$\phi$: gravitational potential

$\rho_m$: mass density

$\nabla^2$: 2nd derivatives - something to do with curvature?

Hypothesis: need to replace LHS w/ tensor describing curvature & RHS with tensor describing $\rho$

RHS first: $\rho_m$ is not Lorentz-invariant on its own

Total energy density $\rho$ is the 00 component of the stress-energy tensor $T^{00}$ in flat space.

Proposal: RHS of our desired equation should be $T^{\mu\nu}$ (up to some prefactor).

Properties of $T^{\mu\nu}$: $\binom{2}{0}$-tensor, + satisfies local stress-energy conservation: $\nabla_\mu T^{\mu\nu} = 0$

Thus we hypothesize our theory of gravity will be:

$T^{\mu\nu} = \binom{2}{0}$ tensor describing curvature with zero covariant derivative

Suppose we write

$$T^{\mu\nu} = \alpha R^{\mu\nu} + \beta R g^{\mu\nu} + \gamma g^{\mu\nu}$$

Then $\nabla_\mu T^{\mu\nu} = \alpha \nabla_\mu R^{\mu\nu} + \beta (\nabla_\mu R) g^{\mu\nu}$, since $\nabla(g^{\mu\nu}) = 0$.

From the Bianchi identities, $\nabla_\mu R^{\mu\nu} = \frac{1}{2} g^{\nu\mu} \nabla_\mu R$

$\Rightarrow \nabla_\mu T^{\mu\nu} = \left(\frac{\alpha}{2} + \beta\right) g^{\mu\nu} (\nabla_\mu R)$, so for $\nabla_\mu T^{\mu\nu} = 0$ we need $\beta = -\alpha/2$

So our hypothetical theory of gravity that respects general covariance & stress-energy conservation is

$$T^{\mu\nu} = \alpha \underbrace{\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right)}_{G^{\mu\nu}} + \gamma g^{\mu\nu},$$

or using the conventional symbols,

$$\boxed{G^{\mu\nu} + \Lambda g^{\mu\nu} = k T^{\mu\nu}}$$

These are Einstein's field equations of general relativity (modulo the prefactors which we still need to determine)

- Still our best current theory of gravity (100+ years old)
- Have passed an enormous array of experimental tests
- Not the only possible theory of gravity - but currently explains observations extremely well
- To obtain $k$, we need to match onto Newtonian gravity
  - we will do that next and find $k = 8\pi G$.
- $\Lambda$ controls the cosmological constant we will discuss later - for the moment, let us set it to zero.
- Observationally $\Lambda$ appears to be extremely tiny but non-zero, & has (extremely) important effects on the large-scale expansion of the universe, but is generally unobservable in non-cosmological contexts
  - The puzzle of why $\Lambda$ is so small but non-zero remains open - "cosmological constant problem"

- Note we can fully contract both sides to get (if $\Lambda=0$): $R - \frac{1}{2} R \times 4 = k T^{\mu}{}_{\mu} \Rightarrow R = -k T^{\mu}{}_{\mu}$

Thus we can equivalently write the EEs as:

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} T^{\alpha}{}_{\alpha}\, g_{\mu\nu}\right).$$

In particular, if $T_{\mu\nu}=0$ (i.e in vacuum) at some point, then $R_{\mu\nu}=0$ at that point.

## Properties of Einstein's equations (EEs)

- If we view $T^{\alpha\beta}$ as given, the EEs tell us how to solve for the metric $g^{\alpha\beta}$
- $G^{\alpha\beta}$, $T^{\alpha\beta}$, $g^{\alpha\beta}$ are all symmetric: 10 independent components
- Thus (in a given coordinate system) EEs give 10 coupled 2nd-order differential equations for 10 unknowns
- But $g^{\alpha\beta}$ cannot be uniquely determined by $T^{\alpha\beta}$ as we can always perform (global) coordinate transformations
  - four coordinate degrees of freedom
  - six functions in the 10 $g_{\alpha\beta}$ that characterize geometry independent of coordinates
  - the constraint $\nabla_\mu T^{\mu\nu} = \nabla_\mu G^{\mu\nu} = 0$ places 4 constraints on the DE system
  - leads to 6 independent (but coupled) differential equations for these 6 functions describing the metric geometry

## A note on an alternate derivation/motivation for the EEs

- An alternative way to proceed is to construct a proposed Lagrangian/action for GR & then get the equations of motion (the EEs) as Euler-Lagrange equations from extremizing the action
- It turns out the only independent scalar

constructed from the metric & its derivatives up to 2nd order is the Ricci scalar

- The Ricci scalar is thus a natural possible choice for the Lagrangian of the theory, yielding the Hilbert action (for the gravity-only, vacuum theory):

$$S_H = \int \sqrt{-g}\, R \, d^n x$$

Ricci scalar

volume element discussed earlier

- To add matter/energy, we add extra matter terms to the action,

$$S = \frac{1}{16\pi G} S_H + S_M$$

again we get the prefactor by matching Newtonian gravity

matter action

- After some effort (see e.g Carroll, "Spacetime & Geometry", section 4.3 in the 1st edition) this action yields the equations of motion,

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = \underbrace{\frac{-1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}}$$

This turns out to be an alternate way to define the stress-energy tensor,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$