Gravitational wave polarization

From last time we had the plane wave solution in TT gauge for a wave traveling in the z-direction,

$\bar{h}^{\alpha\beta} = \mathrm{Re}\left[A^{\alpha\beta} e^{ik_\mu x^\mu}\right]$ with $A_{11} = -A_{22}$ & $A_{12} = A_{21}$ being the only non-zero components of $A_{\alpha\beta}$, and $k_\mu x^\mu = -\omega t + \omega z$.

Then from $\frac{\partial^2}{\partial t^2} \xi^\alpha \approx \frac{1}{2} \xi^\beta \frac{\partial^2}{\partial t^2} h^{TT\alpha}{}_\beta$, the perturbations to the particle displacement vector are:

$$\frac{\partial^2}{\partial t^2} \xi^\alpha = \mathrm{Re}\left[\frac{1}{2} \xi^\beta A^\alpha{}_\beta \frac{\partial^2}{\partial t^2}\left(e^{i(\omega z - \omega t)}\right)\right]$$

$$= -\frac{\omega^2}{2} \xi^\beta \, \mathrm{Re}\left[A^\alpha{}_\beta \, e^{i(\omega z - \omega t)}\right]$$

with non-zero perturbations:

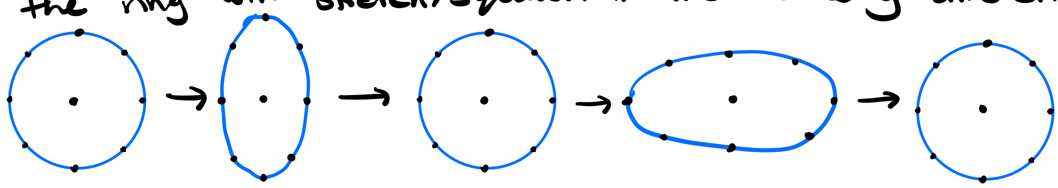
$$\frac{\partial^2}{\partial t^2} \xi^x = \frac{-\omega^2}{2}\Big[\xi^x \, \mathrm{Re}\left(A_{11} e^{i(\omega z - \omega t)}\right) + \xi^y \, \mathrm{Re}\left(A_{12} e^{i(\omega z - \omega t)}\right)\Big]$$

$$\frac{\partial^2}{\partial t^2} \xi^y = \frac{-\omega^2}{2}\Big[-\xi^y \, \mathrm{Re}\left(A_{11} e^{i(\omega z - \omega t)}\right) + \xi^x \, \mathrm{Re}\left(A_{12} e^{i(\omega z - \omega t)}\right)\Big]$$

So we expect oscillatory perturbations in the directions transverse to the wave; particles initially separated in the x-direction measure the A_{11} & A_{12} coefficients by oscillations in the x & y directions respectively, for particles separated in the y-direction it is the opposite.

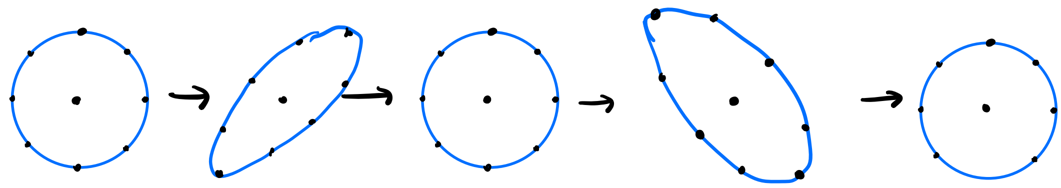
Suppose we consider only a wave with $A_{11} = -A_{22} \neq 0$, $A_{12} = A_{21} = 0$, encountering a ring of test particles in the x-y plane.

As the wave passes through, particles separated in the x direction will oscillate toward or away from each other in the x direction; the same for the y direction, except 180° out of phase, so the ring will stretch/squash in the x & y directions



For this reason, we often call this the "+" polarization, and write $h_{11} = -h_{22} = h_+$.

If we instead have $A_{21} = A_{12} \neq 0$, $A_{11} = A_{22} = 0$, particles separated in y will see their separation stretch in the x direction, & vice versa:



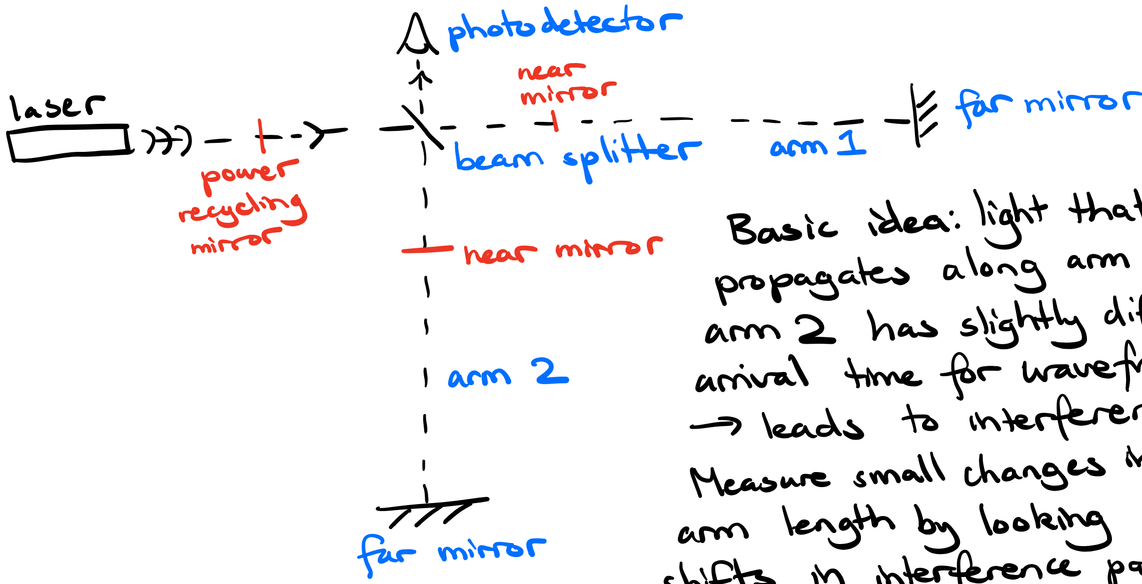
We call this the "x" polarization and write $h_{12} = h_\times$.

So a given plane wave is characterized by k^μ, $h_\times$, h_+, & $h_\times$ & h_+ describe 2 possible transverse polarizations, similar to the polarizations of photons.

Detecting gravitational waves

(Note: at the time the textbook was written, this was a theoretical discussion, but no longer; the first direct detection of gravitational waves was achieved by the LIGO experiment in 2016.)

There are multiple approaches to the measurement of gravitational waves, but let us focus here on interferometers.



Basic idea: light that propagates along arm 1 vs arm 2 has slightly different arrival time for wavefronts → leads to interference. Measure small changes in arm length by looking at shifts in interference pattern.

(More precisely, arms can be adjusted so in the absence of a signal there is perfect destructive interference.)

Consider a gravitational wave traveling into the page with + polarization - one arm will be compressed, the other elongated, in an oscillatory pattern

If GW arrives from right/left with + polarization, one arm will be expanded/compressed, the other unaffected - still detectable signal

No sensitivity to case where GW propagates exactly into/out of page with × polarization (arms stretch equally)

Overall: linearly polarized detector that responds to signals arriving from almost all directions.

What detector sensitivity is needed?

We will not explore the origin of GWs in detail (although the textbook has more in-depth information), but we can make some rough estimates of their size.

For a source of EM radiation, we can expand the far-field behavior of the radiation into dipole, quadrupole, etc terms, and find that for a stationary 2^L-pole source, the field falls off as $1/r^{L+2}$ where r = distance from the source. Thus the lowest-L terms dominate at large distances. There is no monopole (spherically symmetric) term, because of charge conservation - such a term would come from oscillations of the electric monopole moment, which is just the total charge of the system, which does not oscillate via charge conservation.

In EM the dominant dipole-radiation term comes from oscillations of the electric dipole moment $\vec{d} = \sum_i q_i \vec{r}_i$ of a system of charged particles,

$\frac{d^2}{dt^2}\vec{d} = \sum_i q_i \frac{d^2\vec{r}_i}{dt^2}$, which need not be zero in the presence of accelerating chages.

But in the Newtonian limit of GR, where gravity is sourced by mass, the analogous variation of the gravitational dipole moment would be

$\frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i\right)$ for a system of masses,

which is zero by momentum conservation.

In the full GR case, conservation of stress-energy ($\nabla_\mu T^{\mu\nu} = 0$) likewise forbids the dipole/monopole terms, and the leading contribution in the far field is the quadrupole ($L=2$) term; for non-relativistic sources,

$\bar{h}_{ij} \approx \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r)$, where $I_{ij}(t_r)$ describes the quadrupole moment of the energy density of the source at the time when the radiation is emitted t_r.

For a system of mass M & radius R, I_{ij} is of order $\sim MR^2$, and so we can estimate

$$\bar{h}_{ij} \sim O\left(\frac{GM}{r}\Omega^2 R^2\right)$$ where Ω is the relevant frequency for the oscillations that generate the wave

$$\sim O(GMv^2/r)$$

where $v = \Omega R$ is the characteristic velocity scale for components of the system.

For merging 10 solar mass BHs, 100 million light years away, optimistically taking v close to 1, this gives us

$$\bar{h}_{ij} \sim O(10^{-20})$$

$\Rightarrow$ We need to be able to probe fractional changes in length of this order to search for massive objects ($\gtrsim$ stars) colliding/oscillating with velocities close to the speed of light in distant galaxies - rare, very-high-energy events.

Current LIGO sensitivity is at the level of a few $\times 10^{-22}$ - has observed a total of 90 events, thought to arise from mergers of black holes (BHs) & neutron stars (NSs) (all of BH-BH, BH-NS & NS-NS events have been seen)