# The Riemann tensor

At last we come to curvature! We will now study the parallel transport of an arbitrary vector around a closed loop, as a measure of the local curvature of a manifold.

Before we jump into the calculation, let us understand the general structure of what we want to evaluate.

Inputs: a vector to be transported + a loop in spacetime

We will consider for our loop a very tiny (since we are interested in local curvature) parallelogram, so our inputs include two vectors defining its sides.

We want to be able to calculate:

Output: the change to the original vector that results after parallel transport around the loop.

i.e we want a map from $V \times V \times V \mapsto V$.

But since elements of $V$ are maps from $V^* \mapsto \mathbb{R}$, we can regard what we want as a map from $V \times V \times V \times V^* \mapsto \mathbb{R}$.

Furthermore these maps should be multilinear (as e.g doubling the length of the side vectors, or the initial vector, should double the output $\Delta V$ - we will show this explicitly next).

Thus the required map is a (1,3)-tensor, which we will call $R$, and we expect the relation between components:

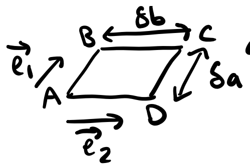
$$\Delta V^\alpha = R^\alpha{}_{\beta\gamma\delta} V^\beta b^\gamma a^\delta$$

change in $\vec{V}$ from parallel transport

original vector $\vec{V}$

vectors describing sides of parallelogram

Let us start by fixing $\vec{a} = \delta a . \vec{e_1}$, $\vec{b} = \delta b . \vec{e_2}$, i.e the side vectors point along coordinate directions. Let us start at a point A with a vector $\vec{V}$, and consider parallel-transporting $\vec{V}$ clockwise




around a loop with corners (in order) B, C & D. Then $V^\alpha(B) = V^\alpha(A) + \int_A^B \frac{\partial V^\alpha}{\partial x^1} dx^1$

$$V^\alpha(C) = V^\alpha(B) + \int_B^C \frac{\partial V^\alpha}{\partial x^2} dx^2, \quad V^\alpha(D) = V^\alpha(C) - \int_D^C \frac{\partial V^\alpha}{\partial x^1} dx^1,$$

and when we return to A (denoted Afinal),

$$V^\alpha(A_{final}) = V^\alpha(D) - \int_A^D \frac{\partial V^\alpha}{\partial x^2} dx^2$$

Thus $\Delta V^\alpha = V^\alpha(A_{final}) - V^\alpha(A_{initial})$

$$= \int_A^B \frac{\partial V^\alpha}{\partial x^1} dx^1 - \int_D^C \frac{\partial V^\alpha}{\partial x^1} dx^1 + \int_B^C \frac{\partial V^\alpha}{\partial x^2} dx^2 - \int_A^D \frac{\partial V^\alpha}{\partial x^2} dx^2$$

If the 1st & 2nd coordinates of A are $(a_0, b_0)$, then the 1st & 2nd integrals are over $x^1 \in [a_0, a_0 + \delta a]$, but one is evaluated at $x^2 = b_0$ & one at $x^2 = b_0 + \delta b$. Similarly for the 2nd pair (integral evaluated at $x^1 = a^0$ vs $x^1 = a_0 + \delta a$, integrating from $x^2 = b_0$ to $b_0 + \delta b$). Thus we can write:

$$\Delta V^\alpha = \int_{a_0}^{a_0+\delta a} \left[ \frac{\partial V^\alpha}{\partial x^1}(x^2 = b_0) - \frac{\partial V^\alpha}{\partial x^1}(x^2 = b_0 + \delta b) \right] dx^1$$
$$- \int_{b_0}^{b_0+\delta b} \left[ \frac{\partial V^\alpha}{\partial x^2}(x^1 = a_0) - \frac{\partial V^\alpha}{\partial x^2}(x^1 = a_0 + \delta a) \right] dx^2$$

But since we are performing parallel transport, along each segment we have $\nabla_{\vec{U}} \vec{V} = 0$, & since the tangent vectors to the sides are just $\delta a\, \vec{e}_1$, $\delta b\, \vec{e}_2$, this gives us

$$U^\mu (\partial_\mu V^\alpha + \Gamma^\alpha{}_{\beta\mu} V^\beta) = 0 \Rightarrow \partial_1 V^\alpha = -\Gamma^\alpha{}_{\beta 1} V^\beta$$

along paths of fixed $x^2$ (tangent in $\vec{e}_1$ direction) and $\partial_2 V^\alpha = -\Gamma^\alpha{}_{\beta 2} V^\beta$ along paths of fixed $x_1$. Thus

$$\Delta V^\alpha = \int_{a_0}^{a_0+\delta a} \left[ \Gamma^\alpha{}_{\beta 1} V^\beta (x^2 = b_0 + \delta b) - \Gamma^\alpha_{\beta 1} V^\beta (x^2 = b_0) \right] dx^1$$
$$- \int_{b_0}^{b_0+\delta b} \left[ \Gamma^\alpha{}_{\beta 2} V^\beta (x^1 = a_0 + \delta a) - \Gamma^\alpha_{\beta 2} V^\beta (x^1 = a_0) \right] dx^2$$

Since $\delta a$, $\delta b$ are small, we can Taylor expand, so e.g

$$(\Gamma^\alpha{}_{\beta 2}\, v^\beta)(x^1 = a_0 + \delta a) \approx (\Gamma^\alpha{}_{\beta 2}\, v^\beta)(a_0) + \delta a\, \partial_1 (\Gamma^\alpha{}_{\beta 2}\, v^\beta)$$

& we obtain:

$$\Delta V^\alpha \approx \int_{a_0}^{a_0+\delta a} \delta b\, \partial_2 (\Gamma^\alpha{}_{\beta 1}\, v^\beta)\, dx^1 - \int_{b_0}^{b_0+\delta b} \delta a\, \partial_1 (\Gamma^\alpha{}_{\beta 2}\, v^\beta)\, dx^2$$

$$\approx \delta a\, \delta b \left[ \partial_2 (\Gamma^\alpha{}_{\beta 1}\, v^\beta) - \partial_1 (\Gamma^\alpha{}_{\beta 2}\, v^\beta) \right]$$

Now we would like to pull out a factor of $v^\beta$; to do this, we again use parallel transport to write

First term: $\partial_2 (\Gamma^\alpha{}_{\beta 1}\, v^\beta) = v^\beta\, \partial_2 \Gamma^\alpha{}_{\beta 1} + \Gamma^\alpha{}_{\beta 1} (-\Gamma^\beta{}_{\gamma 2}\, v^\gamma)$

2nd term: $\partial_1 (\Gamma^\alpha{}_{\beta 2}\, v^\beta) = v^\beta\, \partial_1 \Gamma^\alpha{}_{\beta 2} + \Gamma^\alpha{}_{\beta 2} (-\Gamma^\beta{}_{\gamma 1}\, v^\gamma)$

Relabeling dummy indices gets us:

$$\Delta V^\alpha \approx \delta a\, \delta b\, v^\beta \left[ \partial_2 \Gamma^\alpha{}_{\beta 1} - \partial_1 \Gamma^\alpha{}_{\beta 2} - \Gamma^\alpha{}_{\gamma 1} \Gamma^\gamma{}_{\beta 2} + \Gamma^\alpha{}_{\gamma 2} \Gamma^\gamma{}_{\beta 1} \right]$$

We see that $\Delta V^\alpha$ is linear in the side lengths and in $v^\alpha$, as expected, and we can read off

$$R^\alpha{}_{\beta 2 1} = \Gamma^\alpha{}_{\beta 1, 2} - \Gamma^\alpha{}_{\beta 2, 1} + \Gamma^\alpha{}_{\gamma 2} \Gamma^\gamma{}_{\beta 1} - \Gamma^\alpha{}_{\gamma 1} \Gamma^\gamma{}_{\beta 2}$$

Choosing the sides in the (1,2) directions was arbitrary, so in general we have:

$$R^\alpha{}_{\beta \mu \nu} = \Gamma^\alpha{}_{\beta \nu, \mu} - \Gamma^\alpha{}_{\beta \mu, \nu} + \Gamma^\alpha{}_{\sigma \mu} \Gamma^\sigma{}_{\beta \nu} - \Gamma^\alpha{}_{\sigma \nu} \Gamma^\sigma{}_{\beta \mu}$$

Note: this sign choice is a convention which differs between references – check the overall sign in the reference you are using!

<u>Properties of the Riemann tensor</u>

Swapping $\mu \leftrightarrow \nu$ ~ changing which edge vector comes "first", i.e which way you go around the loop

$\Rightarrow$ leads to an overall sign flip

Note that $R^\alpha{}_{\beta\mu\nu}$ depends on derivatives of the Christoffel symbols, i.e 2nd derivatives of the metric, so it is suitable to describe curvature.

In the local inertial frame at a point P, the Christoffel symbols vanish (but not their derivatives), and we find

$$R^\alpha{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\beta,\nu\mu} + g_{\sigma\nu,\beta\mu} - g_{\beta\nu,\sigma\mu} - g_{\sigma\beta,\mu\nu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu}\right)$$

↳ these terms are all 2nd derivatives of the metric at P

Since partial derivatives commute & the metric is symmetric, & lowering the $\alpha$ index, we have (in the local inertial frame)

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\right)$$

This in turn tells us that

$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$, i.e $R_{\alpha\beta\mu\nu}$ is antisymmetric under exchange of the first two indices as well as the 2nd two indices.

$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$, i.e $R_{\alpha\beta\mu\nu}$ is symmetric under exchange of the two index pairs.

& $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$

These are tensor equations valid in all coordinate systems.

<u>Bianchi identities</u>

If we differentiate (with ordinary partial derivatives) the Riemann tensor in local inertial coordinates we find

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right)$$

It then follows that

$$\begin{aligned} &R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} \\ &= \frac{1}{2}\left(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right) \\ &\quad + \frac{1}{2}\left(g_{\alpha\lambda,\beta\nu\mu} - g_{\alpha\nu,\beta\lambda\mu} + g_{\beta\nu,\alpha\lambda\mu} - g_{\beta\lambda,\alpha\nu\mu}\right) \\ &\quad + \frac{1}{2}\left(g_{\alpha\mu,\beta\lambda\nu} - g_{\alpha\lambda,\beta\mu\nu} + g_{\beta\lambda,\alpha\mu\nu} - g_{\beta\mu,\alpha\lambda\nu}\right) \end{aligned}$$

$= 0$ using symmetry of the metric & commutativity of partial derivatives

But in this coordinate system, since the Christoffel symbols are zero at P, partial & covariant derivatives are equivalent, & this result implies

$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$ (Bianchi identities) – this is a tensor equation so true in all frames

## General comments

A flat manifold is one with a global definition of parallelism, where $\Delta\vec{V} = 0$ on parallel transport around any closed loop. This is equivalent to requiring $R^{\alpha}{}_{\beta\mu\nu} = 0$ at all points on the manifold.

We can also express the Riemann tensor in terms of the commutator of covariant derivatives;

$$[\nabla_\alpha, \nabla_\beta]V^\mu \equiv (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)V^\mu = R^{\mu}{}_{\nu\alpha\beta} V^\nu$$

You can check this explicitly by working out the components; more intuitively, $R^{\mu}{}_{\nu\alpha\beta}$ describes the difference in how $\vec{V}$ is parallel-transported from $A \to C$ along the two paths $A \to B \to C$ & $A \to D \to C$; the change in $\vec{V}$ along each of these paths is obtained by applying two covariant derivatives (corresponding to different coordinate directions), but in opposite order. The difference is thus determined by the commutator.