Metrics for an expanding universe

To this point we have focused on solving the EEs via a weak-field approximation. But we can find exact solutions to the full EEs as well – today we will derive a metric to describe how our universe works on large scales.

Observationally, the cosmos is very inhomogeneous at small scales (e.g Earth vs interplanetary space) but quite homogeneous on the largest scales – the cosmic microwave background radiation, sometimes called the afterglow of the Big Bang, is uniform to about one part in 10^5 in all directions (once we factor out a dipole piece due to the velocity of the Earth).

However, it is not homogeneous in time; we have good data indicating the universe is expanding, and furthermore its expansion is accelerating.

Thus we will look for a solution that is:

(1) time-dependent

(2) spatially homogeneous & isotropic.

To develop a solution ansatz, let's consider familiar 2D surfaces – which are isotropic & homogeneous? (i.e no special points, no special directions)

- Flat plane (zero curvature), e.g $ds^2 = dx^2 + dy^2$
- Sphere (constant positive curvature), e.g $x^2+y^2+z^2=R^2$; $ds^2 = R^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$
- Pseudosphere / hyperboloid (constant negative curvature)

How do these generalize to 3D spatial surfaces in 4D spacetime? Consider the embeddings of 3D surfaces in 4D space, coordinates (x, y, z, w). What are the metrics on the analogues of the plane/sphere/pseudosphere?

Hyperplane: $ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$

Hypersphere: define by $x^2 + y^2 + z^2 + w^2 = R^2$

Introduce 4D "spherical polar" coords:

$w = r\cos\chi$
$z = r\sin\chi\cos\theta$
$x = r\sin\chi\sin\theta\cos\phi$
$y = r\sin\chi\sin\theta\sin\phi$

Then $x^2+y^2+z^2+w^2 = r^2$, so our hypersphere has $r = R$ & a surface described by coordinates χ, θ, ϕ, with metric $ds^2 = R^2[d\chi^2 + \sin^2\chi \, d\theta^2 + \sin^2\chi \sin^2\theta \, d\phi^2]$

We can define $\bar{r} \equiv \sin\chi$ $(-1 \leq \bar{r} \leq 1)$ & write this as:

$ds^2 = R^2 [\, d\bar{r}^2/(1-\bar{r}^2) + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2]$

(since $d\bar{r} = \cos\chi \, d\chi$)

This closely resembles the result for flat space, but with $dr^2 \rightarrow dr^2/(1-r^2)$ (& consequently r becomes bounded)

This surface has constant positive curvature, as expected (i.e Ricci scalar R = same everywhere)

The negative-curvature analogue is obtained by taking $dr^2 \rightarrow dr^2/(1+r^2)$ instead, so overall we have

$$ds^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \quad k = \underset{\text{flat}}{0}, \underset{\text{-ve curvature}}{+1}, \underset{\text{+ve curvature}}{-1}$$

We can show that these are the only homogeneous / isotropic solutions for the spatial part of the metric

To obtain an ansatz for the full time-dependent metric, we write $ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \right]$

time-dependent scale factor

(Friedman - Robertson-Lemaitre - Walker metric)

Thus as desired, slices of fixed t will be spatially homogeneous/isotropic. We will need to solve the EEs to obtain $a(t)$.

<u>Particle propagation in an expanding universe</u>

However, even without solving for $a(t)$, we can explore how particles propagate in this metric.

(1) Because the metric is spatially isotropic + homogeneous, free particles initially at rest in these coordinates do not move (as there is no preferred direction) – we say such particles are <u>comoving</u> & the coordinates we have chosen are <u>comoving coordinates</u>.

(2) Consequently, the proper distance between two comoving particles at a fixed time t is set by $\int \sqrt{ds^2} = a(t) \int \sqrt{dr^2/(1-kr^2) + r^2 d\Omega^2} \propto a(t)$ – if $a(t)$ grows with time then the distance between all comoving objects expands proportionally. This is what we mean when we say the universe is expanding.

(3) For light, $ds^2 = 0$, so we have

$$dt^2 = a(t)^2 \left(dr^2/(1-kr^2) + r^2 d\Omega^2 \right)$$

Without loss of generality, let us set the origin to be the emission point of the light, so its trajectory is radial: $d\Omega = 0$ (via spherical symmetry)

$$\Rightarrow dt = a(t)\, dr/\sqrt{1-kr^2} \Rightarrow \frac{dr}{dt} = \sqrt{1-kr^2}\, \frac{1}{a(t)}$$

The coordinate speed of light is not 1 in this cosmos, but that is OK. Note for $k=1$, the coordinate speed of light drops to zero as $r \to 1$, & light cannot cross this boundary: this is consistent with requiring $r<1$ in this case.

Cosmological horizons

Suppose the light is emitted at time t_e & received by an observer at coordinates (t, r).

Then $$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_{t_e}^t \frac{1}{a(t')} dt'$$

- If we take t_e to be the origin of the universe, $t_e = t_{init}$, then the r defined by this equation gives the most distant point light emitted from the origin can have reached.
- At time t, the proper distance from O to this r is the horizon distance

$$d_H = \int \sqrt{ds^2} = \int_0^r \frac{dr' a(t)}{\sqrt{1-kr'^2}} = a(t) \int_{t_{init}}^t \frac{1}{a(t')} dt'$$

Points separated by larger distances are not causally connected – light emitted from one of them has never yet had a chance to reach the other. This radius around us defines the observable cosmos.

- There is a related concept called the cosmological event horizon – the proper distance (at time t) to the furthest comoving observer such that light they emit at t will ever reach us in the future.

For this distance we require $\int_t^{t_{final}} \frac{dt'}{a(t')} = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}}$,

so $d_{EH} = \int \sqrt{ds^2} = a(t) \int_t^{t_{final}} \frac{dt'}{a(t')}$

In practice, in our universe, $d_H \approx 47$ billion lyr (neglecting early inflation - see pset!)
and $d_{EH} \approx 16$ billion lyr.

Cosmological redshifting

- We can also look at the energy of the light.
- Consider a train of wavefronts, the first emitted from the origin at t_e, the 2nd at $t_e + \delta t_e$, etc.
- Suppose they are received at $t_0, t_0 + \delta t_0, \ldots$ by a comoving observer at radius r.

- Now from above,

$$\int_{t_e}^{t_0} \frac{1}{a(t')} dt' = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{1}{a(t')} dt'$$

$$\approx \int_{t_e}^{t_0} \frac{1}{a(t')} dt' + \underbrace{\delta t_0 \frac{1}{a(t_0)} - \frac{\delta t_e}{a(t_e)}}_{=0}$$

$\Rightarrow \frac{\delta t_0}{\delta t_e} = \frac{a(t_0)}{a(t_e)}$ But the arrival frequency is $\frac{1}{\delta t_0}$ & the emission frequency is $1/\delta t_e$,

so $\Rightarrow \frac{\nu_e}{\nu_0} = \frac{a(t_0)}{a(t_e)}$

i.e the measured frequency of light scales inversely with the scale of the universe; expansion stretches wavelength / redshifts light & contraction would compress its wavelength / blueshift it.
Powerful tool for measuring expansion!