Gravitational waves

Let us now look again at the weak-field EEs,

$$\partial_\mu \partial^\mu \bar{h}^{\alpha\beta} = -16\pi G T^{\alpha\beta}$$

But this time we will work in vacuum ($T^{\alpha\beta}=0$) and not make assumptions about the relative size of the different components of $\bar{h}^{\alpha\beta}$ and their derivatives.

In vacuum, $\Rightarrow \partial^\mu \partial_\mu \bar{h}^{\alpha\beta} = \Box \bar{h}^{\alpha\beta} = 0$

$$\Rightarrow \frac{\partial^2 \bar{h}^{\alpha\beta}}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\bar{h}^{\alpha\beta}$$ – wave equation

Solution ansatz: $\bar{h}^{\alpha\beta} = \mathrm{Re}\left[A^{\alpha\beta} e^{ik_\mu x^\mu}\right]$ – plane wave solution

complex constants

real constants

As discussed earlier, we treat $\bar{h}^{\alpha\beta}$ as a tensor in SR (& raise & lower indices with $\eta_{\alpha\beta}$); similarly we can treat $A^{\alpha\beta}$ and k_α as the components of a SR tensor & one-form respectively.

We will work with the complex solution in the square brackets & take the real part at the end. This ansatz for the solution implies $\partial^\mu \partial_\mu (e^{ik_\alpha x^\alpha}) = 0 \Rightarrow ik_\mu k^\mu e^{ik_\alpha x^\alpha} = 0$

$\Rightarrow k_\mu k^\mu = \eta^{\mu\nu} k_\mu k_\nu = 0$, i.e k_μ should be a null one-form.

k_μ describes the wavenumber of the wave, and we conventionally write $k^\mu = \{\omega, \vec{k}\}$

This means a 3-vector not a general vector

k_μ being null means

$\omega^2 - |\vec{k}|^2 = 0 \Rightarrow \omega = |\vec{k}|$ – dispersion relation of the wave, matches that of light

This suggests the wave travels at lightspeed. We can show this explicitly by considering a photon with momentum k^μ – since k^μ is null it is a viable momentum vector for a photon.

The photon's trajectory is $x^\mu(\lambda) = k^\mu \lambda + L^\mu$

Thus $k_\mu x^\mu = k_\mu L^\mu =$ constant along the photon's trajectory,

but $\Rightarrow \bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha}$ is also constant along the photon's trajectory - i.e the perturbation to the metric travels along the trajectory with the photon, at the same speed (the photon always "sees" the same $\bar{h}^{\alpha\beta}$ at its location). Thus GR predicts gravitational waves (GWs) that travel at lightspeed.

(This prediction has now been confirmed to high precision by the near-simultaneous detection of GWs & EM radiation from a neutron star merger 130 million light-years away; gamma-ray burst detected 1.7 s after GW merger signal.)

Recall we imposed a gauge condition to obtain the weak-gravity version of the EEs, $\bar{h}^{\mu\nu}{}_{,\nu} = 0$.

With our ansatz, $\bar{h}^{\mu\nu}{}_{,\nu} = \mathrm{Re}[A^{\mu\nu} \partial_\nu e^{ik_\alpha x^\alpha}]$

$$= \mathrm{Re}[i A^{\mu\nu} k_\nu e^{ik_\alpha x^\alpha}]$$

So to ensure the gauge condition is satisfied, we need $A^{\mu\nu} k_\nu = 0$, i.e $A^{\mu\nu}$ must be orthogonal to k_ν.

General solution is a linear combination of solutions satisfying all these criteria.

Transverse traceless gauge

The Lorentz gauge is not enough to fully remove the gauge freedom, as we discussed earlier. Under a gauge transformation $h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$

$\Rightarrow \bar{h}_{\mu\nu} \longmapsto \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi^\alpha{}_{,\alpha}$, & preserving the Lorentz gauge condition $\partial_\nu \bar{h}^{\mu\nu} = 0$ requires that

$$\partial^\nu \xi_{\mu,\nu} + \underbrace{\partial^\nu \xi_{\nu,\mu} - \eta_{\mu\nu}\partial^\nu \xi^\alpha{}_{,\alpha}}_{} = 0$$

Thus

$$\partial^\nu\partial_\nu \xi_\mu = \Box \xi_\mu = 0$$

$= \partial^\alpha \partial_\mu \xi_\alpha - \partial_\mu \partial_\alpha \xi^\alpha$

$= \partial_\mu (\partial_\alpha \xi^\alpha - \partial_\alpha \xi^\alpha)$ as we raise & lower indices with $\eta^{\alpha\beta}$ = constant

$= 0$

& we can modify $\bar{h}_{\mu\nu}$ by any ξ with zero d'Alembertian while preserving Lorentz gauge. It is convenient to choose

$\xi_\alpha = B_\alpha \exp(ik_\mu x^\mu)$, with k^μ the same null vector as in the plane-wave solution

Then
$$\bar{h}_{\alpha\beta} \rightarrow \bar{h}_{\alpha\beta} - \partial_\beta (B_\alpha \exp(ik_\mu x^\mu)) - \partial_\alpha (B_\beta \exp(ik_\mu x^\mu))$$
$$+ \eta_{\alpha\beta}\, \partial_\lambda (B^\lambda e^{i(k_\mu x^\mu)})$$

$$\Rightarrow A_{\alpha\beta} \rightarrow A_{\alpha\beta} - ik_\beta B_\alpha - ik_\alpha B_\beta + ik_\lambda B^\lambda \eta_{\alpha\beta}$$

In <u>transverse traceless</u> (TT) gauge we choose the B_α one-form to set $A^\alpha{}_\alpha = 0$, $A_{\alpha\beta} U^\beta = 0$ where U^β describes the components of a fixed timelike 4-velocity vector of our choice.

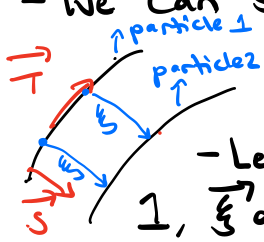
In TT gauge, $h^\alpha{}_\alpha = 0$ so $h^{\alpha\beta} = \bar{h}^{\alpha\beta}$.

Recall we are expanding around Minkowski space: let us choose the coordinates for that background spacetime (e.g via a background LT) so that $U^\beta = \delta^\beta{}_0$, i.e the 4-vector points in the time direction. Then we have $A_{\alpha 0} = 0$. Let us label the direction of propagation of the wave as the z-direction, $k^\mu : (\omega, 0, 0, \omega)$, so $A_{\alpha\beta} k^\beta = 0 \Rightarrow A_{\alpha 0} = A_{\alpha 3}$, & in TT gauge $\Rightarrow A_{\alpha 3} = 0$ (i.e the wave has components only in the directions <u>transverse</u> to its motion). Then the only non-zero elements of $A_{\alpha\beta}$ are $A_{11}, A_{22}, A_{12} = A_{21}$. Finally the trace condition gives $A^\alpha{}_\alpha = \eta^{\alpha\beta} A_{\alpha\beta} = A_{11} + A_{22} = 0$, $\Rightarrow A_{11} = -A_{22}$.

So in TT gauge any gravitational (plane) wave is characterized by its wavenumber k^μ + two constants A_{11}^{TT}, A_{12}^{TT}
– all the rest of the freedom in our original solution was just gauge freedom, non-physical.

How do gravitational waves affect particles?

- Consider two freely-falling test particles
- When a gravitational wave passes through, it slightly curves space between them $\longrightarrow$ leads to deviation of their geodesic trajectories
- We can study this using the equation of geodesic deviation,



$$\nabla_{\vec{T}} \nabla_{\vec{T}} S^\alpha = R^\alpha{}_{\sigma\rho\beta} S^\beta T^\sigma T^\rho$$

- Let $\vec{U} = \vec{T}$, $U^\alpha = \frac{dx^\alpha}{d\tau}$, be the 4-velocity vector of particle 1, $\vec{\xi} \propto \vec{S}$ is the displacement vector stretching from particle 1 to particle 2 at a fixed value of τ
- Let us work in the rest frame of particle 1 – since particle 1 is freely-falling this furnishes a series of locally inertial frames along the particle trajectory. The proper time τ of particle 1 provides a coordinate time t at each point.
- We will assume the 2nd particle is highly non-relativistic in this frame and sufficiently close to particle 1 that its proper time $\tau \simeq t$ & the spatial components of ξ give the proper distance between the particles (up to quadratic corrections)
- Because we are now adjusting our coordinate system as we move along the geodesic trajectory, the Christoffel symbols at the origin of $\vec{\xi}$ remain zero along the trajectory.
- Then from $U^\mu \nabla_\mu (U^\nu \nabla_\nu \xi^\alpha) = R^\alpha{}_{\sigma\rho\beta} \xi^\beta U^\sigma U^\rho$,

  we can rewrite the LHS as

$$\frac{dx^\mu}{d\tau}\left(\frac{\partial}{\partial x^\mu} (U^\nu \nabla_\nu \xi^\alpha) + \underbrace{\Gamma^\alpha{}_{\mu\gamma} (U^\nu \nabla_\nu \xi^\gamma)}_{\to 0 \text{ as all Christoffel symbols vanish}} \right.$$

$$= \frac{d}{d\tau}\left(U^\nu \nabla_\nu \xi^\alpha\right) = \frac{d}{d\tau}\left(\frac{dx^\nu}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\nu} + \Gamma^\alpha{}_{\beta\nu}\,\xi^\beta\right)\right)$$

$$= \frac{d^2}{d\tau^2}\xi^\alpha + \underbrace{\Gamma^\alpha{}_{\beta\nu}}_{=\,0 \text{ in local inertial frame}} \frac{d}{d\tau}\left(\frac{dx^\nu}{d\tau}\xi^\beta\right)$$

since $\Gamma^\alpha{}_{\beta\nu}$ remains zero as τ varies

$$= \frac{d^2\xi^\alpha}{d\tau^2},$$ and so

$$R^\alpha{}_{\sigma\rho\beta}\,\xi^\beta U^\sigma U^\rho = \frac{d^2\xi^\alpha}{d\tau^2} \approx \frac{d^2\xi^\alpha}{dt^2}.$$ Since $U^\sigma = (1,0,0,0)$

$$\Rightarrow \frac{d^2\xi^\alpha}{dt^2} \approx R^\alpha{}_{00\beta}\,\xi^\beta$$

Now for this section we have picked a specific coordinate system that may not satisfy the gauge conditions, but it is easy to show $R_{\mu\nu\alpha\beta}$ is gauge-invariant (check it!) We can also identify ξ^α as the proper distances along the x^α coordinate directions spanned by $\vec{\xi}$. Then shifting into the TT coordinate system yields:

- $R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right)$ in the weak-gravity limit, and with our gauge choice $h^{\alpha\beta} = \bar{h}^{\alpha\beta}$, we obtain

$$R_{\alpha 00\nu} = \frac{1}{2}\left(h^{TT}_{\alpha\nu,00} + h^{TT}_{00,\alpha\nu} - h^{TT}_{\alpha 0,0\nu} - h^{TT}_{0\nu,\alpha 0}\right)$$

But in TT gauge $h_{0\alpha} = 0$, so

$\Rightarrow R_{\alpha 00\nu} = \frac{1}{2}\frac{\partial^2}{\partial t^2} h^{TT}_{\alpha\nu}$, and so the equation of geodesic deviation becomes

$$\frac{\partial^2}{\partial t^2}\xi^\alpha \approx \frac{1}{2}\xi^\beta \frac{\partial^2}{\partial t^2} h^{TT\,\alpha}{}_\beta$$

Thus the effect of a gravitational wave on our test particles is to induce a small acceleration in their displacement, proportional to their initial separation, with a size controlled by $\frac{\partial^2}{\partial t^2} h^{TT\,\alpha}{}_\beta$.