The Einstein equations & cosmology

We want to apply $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$ to the FLRW metric, $ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$

Since we demand spatial isotropy + homogeneity, we can argue that $T_{\mu\nu}$ should be diagonal, since off-diagonal terms in $T_{\mu\nu}$ represent the flux of the μth component of the 4-momentum in the direction ν. If we change the direction of <u>one</u> coordinate, $x^\mu \to -x^\mu$ for a specific μ, this will flip the sign of $T_{\mu\nu}$ for all $\nu \neq \mu$, but since there are no preferred directions this should also leave $T_{\mu\nu}$ unchanged.

Thus we expect only the diagonal equations to have non-zero RHS, and with some work we can compute the Einstein tensor for the FLRW metric as:

$$G_{\mu\nu} = \begin{cases} 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right), & \mu=\nu=0 \\ -\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) g_{ii}, & \mu=\nu=i \\ 0 & \text{otherwise} \end{cases}$$

where $\dot{a} \equiv \frac{da}{dt}$

So all the off-diagonal equations are trivial. Furthermore, if we rewrote the metric in Cartesian coordinates, isotropy/homogeneity means all the x-y-z components would need to be equal. So the EEs in this case boil down to only <u>two</u> equations.

If $T^\mu{}_\nu$ is diagonal with components (A, B, B, B), in Cartesian coordinates, this is characteristic of the stress-energy tensor of a perfect fluid in its rest frame, which in a general frame can be written,

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$$

where ρ = energy density, P = pressure, u^μ = bulk 4-velocity

Note that the $\Lambda g^{\mu\nu}$ term can be absorbed into $T^{\mu\nu}$

$T^{\mu\nu} \longrightarrow T^{\mu\nu} - \frac{\Lambda}{8\pi G} g^{\mu\nu}$, & this is equivalent to

mapping $\rho \rightarrow \rho + \frac{\Lambda}{8\pi G}$, $P \rightarrow P - \frac{\Lambda}{8\pi G}$

Thus we will consider $T^{\mu\nu}$ to include the Λ term from here on, with an effective $\rho = \frac{\Lambda}{8\pi G}$ & $P = -\rho$.

Furthermore, taking the comoving frame to be the rest frame of the fluid (since there are no preferred directions), $u^i = 0$ & $(dx_0)^2 g^{00} = -d\tau^2 \Rightarrow u_0 = \frac{dx_0}{d\tau} = -1$, and so

$$T_{\mu\nu} = \begin{cases} (\rho+P)(u_0)^2 + P g_{00} = \rho, & \mu=\nu=0 \\ (\rho+P)(u_i)^2 + P g_{ii} = P g_{ii}, & \mu=\nu=i \\ 0 & \text{otherwise} \end{cases}$$

Then the EEs become:

$$3\left(\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right) = 8\pi G T_{00} = 8\pi G \rho \Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

$$\left(2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right) g_{ii} = -P g_{ii} \Rightarrow P = -\left[2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right]$$

The 2nd of these just ensures stress-energy conservation: note that $\frac{d}{dt}(8\pi G \rho a^3) + 8\pi G P \frac{d}{dt}(a^3)$

$$= \frac{d}{dt}\left(3(\dot a)^2 a + 3ka\right) - 3a^2\dot a\left(2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right)$$

$$= 6\ddot a \dot a a + 3(\dot a)^3 + 3k\dot a - 6a\dot a\ddot a - 3(\dot a)^3 - 3\dot a k$$

$= 0$. Thus the effect of the 2nd equation is to ensure $\frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3)$. Writing $V = a^3$ as the volume element, $\Rightarrow \frac{d}{dt}(\rho V) = -P\frac{dV}{dt}$, which is schematically $dE = -PdV$, and can be shown to follow from $\nabla_\mu T^{\mu\nu} = 0$.

So the only new equation from the EEs, beyond energy conservation, is the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

We traditionally define $H \equiv \frac{\dot{a}}{a}$ so

$$\Rightarrow H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Energy content of the universe

In many cases we can approximate the dominant contribution to ρ as having a simple equation of state, $P = w\rho$, for some constant w. In this case the energy conservation/continuity equation becomes

$$\frac{d}{dt}(\rho a^3) = -w\rho \frac{d}{dt}(a^3) \Rightarrow \dot{\rho} a^3 + 3a^2 \dot{a} \rho = -w\rho(3a^2\dot{a})$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho \propto a^{-3(1+w)}$$

That is, w fixes how ρ scales with a.

Examples:

(1) $w=0$, $P=0$, $\rho \propto a^{-3}$ – "dust", pressureless matter

(2) $w=\frac{1}{3}$, $P=\frac{\rho}{3}$, $\rho \propto a^{-4}$ – radiation/relativistic particles

(3) $w=-1$, $P=-\rho$, $\rho \propto a^{0}$ – "dark energy", behavior given by cosmological constant

Suppose all three of these components contribute. Then we can write:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = \frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{rad,0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{\Lambda,0}\right] - \frac{k}{a_0^2}\left(\frac{a}{a_0}\right)^{-2}$$

where $a_0 = a(t_{today})$, & $\rho_{x,0}$ = density of X component today, where $X = m$ (matter), rad (radiation), Λ (dark energy)

Defining $H_0 \equiv H(t_{today})$, "critical density" $\rho_c \equiv \frac{3H_0^2}{8\pi G}$, and

density if $k=0$

$\rho > \rho_c \Rightarrow k>0$, $\rho < \rho_c \Rightarrow k<0$

$$\Omega_x \equiv \frac{\rho_{x,0}}{\rho_c}, \quad \Omega_k \equiv \frac{-k}{a_0^2 H_0^2}, \text{ we have}$$

$$\frac{H^2}{H_0^2} = \left(\frac{\rho_{m,0}}{\rho_c}\right)\left(\frac{a}{a_0}\right)^{-3} + \left(\frac{\rho_{rad,0}}{\rho_c}\right)\left(\frac{a}{a_0}\right)^{-4} + \frac{\rho_{\Lambda,0}}{\rho_c} - \frac{k}{a_0^2 H_0^2}\left(\frac{a}{a_0}\right)^{-2}$$

$$= \Omega_m \left(\frac{a}{a_0}\right)^{-3} + \Omega_{rad}\left(\frac{a}{a_0}\right)^{-4} + \Omega_k (a/a_0)^{-2} + \Omega_\Lambda$$

At earlier times, a was smaller & thus the Ω_m and (especially) Ω_{rad} terms were enhanced - we believe that (post-inflation) the universe started radiation-dominated, then became matter-dominated, then dark-energy dominated (one might have expected a k-dominated epoch too, but Ω_k appears very small - maybe zero - in our cosmos).

Measurements from Planck [arXiv: 1807.06209]

$\Omega_m = 0.3153 \pm 0.0073$

$\Omega_\Lambda = 0.6847 \pm 0.0073$

both huge outstanding problems puzzles of dark matter and dark energy respectively.

$\Omega_k = 0.0007 \pm 0.0019$

$\Omega_r \doteq 9 \times 10^{-5}$