The weak-field limit

- Einstein's equations are in general extremely difficult to solve.
- When dealing with weak gravitational fields, it is often helpful to expand $g_{\mu\nu}$ around the Minkowski metric.
- Specifically, suppose we can choose coordinates where $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$ everywhere in spacetime. We say that in this case spacetime is nearly flat, and we can expand in the small parameters $h_{\alpha\beta}$.
- Now $h_{\alpha\beta}$ on its own need not describe the components of a tensor, but it is still helpful to know how it transforms.
- We will be interested in particular in two types of transformations: background Lorentz transformations and gauge transformations.

Background LTs

In SR + Cartesian components, Lorentz transformations are a coordinate transformation with components:

$$\Lambda^{\bar{\alpha}}{}_{\beta} : \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for e.g. a Lorentz boost along the x-axis

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

A general LT is this boost combined with rotations.

Now in GR this is still a valid coordinate transformation - just in general it no longer leaves $g_{\mu\nu}$ unchanged.

But if $\Lambda^{\bar{\alpha}}{}_{\beta}$ does have the form of a LT in SR, we know

$\Lambda^{\alpha}{}_{\bar{\beta}} \Lambda^{\gamma}{}_{\bar{\delta}} \eta_{\alpha\gamma} = \eta_{\bar{\beta}\bar{\delta}}$. And so if $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$,

$$g_{\bar\alpha\bar\beta} = \Lambda^\gamma{}_{\bar\alpha}\Lambda^\delta{}_{\bar\beta}(\eta_{\gamma\delta}+h_{\gamma\delta}) = \eta_{\bar\alpha\bar\beta} + \Lambda^\gamma{}_{\bar\alpha}\Lambda^\delta{}_{\bar\beta}h_{\gamma\delta}$$

Let us define $h_{\bar\alpha\bar\beta} = \Lambda^\gamma{}_{\bar\alpha}\Lambda^\delta{}_{\bar\beta}h_{\gamma\delta}$, i.e. under this class of transformations, $h_{\alpha\beta}$ transforms like the components of a tensor in SR. Can treat spacetime like flat spacetime with an "extra tensor" $h_{\alpha\beta}$ defined on it (for some purposes - not all), & this transformation leaves the spacetime manifestly nearly-flat.

Gauge transformations

Another class of transformations that leaves the metric in the form $\eta_{\alpha\beta}+h_{\alpha\beta}$ is the gauge transformations.

Basic idea: let $x^{\alpha'} = x^\alpha + \xi^\alpha(\vec{x})$ ⟶ small parameter that depends on position

$$\Lambda^{\alpha'}{}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta} = \delta^\alpha{}_\beta + \frac{\partial \xi^\alpha}{\partial x^\beta}$$

$$\Lambda^\alpha{}_{\beta'} = \frac{\partial x^\alpha}{\partial x^{\beta'}} = \delta^\alpha{}_\beta - \frac{\partial \xi^\alpha}{\partial x^\beta} + O(|\xi^\alpha{}_{,\beta}|^2)$$

Applying this transformation law to $g_{\alpha\beta}$ & dropping all terms 2nd-order or higher in ξ, h, we have

$$g_{\mu'\nu'} = g_{\alpha\beta}\Lambda^\alpha{}_{\mu'}\Lambda^\beta{}_{\nu'} = (\eta_{\alpha\beta}+h_{\alpha\beta})(\delta^\alpha{}_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta{}_\nu - \xi^\beta{}_{,\nu})$$

$$\approx \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \quad \text{where } \xi_\mu \equiv \eta_{\mu\alpha}\xi^\alpha$$

$$\equiv \eta_{\mu'\nu'} + h_{\mu'\nu'}$$

So defining $\eta_{\mu\nu} = \eta_{\mu'\nu'}$, we have $h_{\mu'\nu'} = h_{\mu\nu} - \xi^\mu{}_{,\nu} - \xi^\nu{}_{,\mu}$

Gauge transformations shift h without changing (at 1st order in small parameters) the physics of the system or the division of g into η + a small correction.

We can choose ξ^α arbitrarily - this is "gauge freedom".

Principles of weak-field calculations

- Treat the spacetime as SR equipped with a tensor $h_{\mu\nu}$

- Expect equations constructed from $h_{\mu\nu}$ to be valid tensor equations in SR (transformations = Lorentz transforms) but not more broadly
- Use Minkowski metric to raise & lower indices
- Write curvature, other relevant quantities as an expansion in $h_{\mu\nu}$

e.g to 1st order, $R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$

- Use gauge transformations to simplify results

The Einstein tensor in weak gravity

Define $h^{\mu}{}_{\beta} \equiv \eta^{\mu\alpha} h_{\alpha\beta}$, $h^{\mu\nu} \equiv \eta^{\nu\beta} h^{\mu}{}_{\beta}$, $h \equiv h^{\alpha}{}_{\alpha}$

and the trace reverse $\bar{h}$ with $\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$.

Note $\bar{h} \equiv \bar{h}^{\alpha}{}_{\alpha} = h - 2h = -h$.

Also $h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}$

With these definitions, it follows by some algebra that $G_{\alpha\beta} \doteq -\frac{1}{2}\left[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu}\right]$ to 1st order

We can simplify this expression by selecting an appropriate gauge transformation to give $h_{\mu\nu}$ desired properties.

For example, the Lorentz gauge demands $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ $\forall\mu$. This requirement is 4 equations, & there are four unknown gauge functions ξ^{α} - guess that we can always adjust ξ^{α} to ensure $\bar{h}^{\mu\nu}{}_{,\nu} = 0$

(see Schutz p. 193 for proof - it is actually not even a unique solution)

In this case, all but the first term in $G_{\alpha\beta}$ vanishes,

$$G_{\alpha\beta} = -\frac{1}{2}\,\partial_\mu \partial^\mu \bar{h}_{\alpha\beta}$$

Thus in the weak field the Einstein equations become

$$\partial_\mu \partial^\mu \bar{h}^{\alpha\beta} = -16\pi G T^{\alpha\beta}$$

Newtonian gravity

Newtonian gravity can be characteized by $\nabla^2\phi = 4\pi G\rho$.
Sourced by stationary masses (at least dominantly)
$\rightarrow$ largest component in $T^{\alpha\beta}$ should be T^{00}, i.e energy/mass density ρ

$\Rightarrow$ EEs give $\nabla^2 \bar{h}^{00} = -16\pi G\rho = -4\nabla^2\phi$,
i.e we should identify $\bar{h}^{00} = -4\phi$, with the other components of $\bar{h}^{\alpha\beta}$ small
Approximate Newtonian solution: $\bar{h}^{\alpha\beta} : \left(\begin{array}{c|ccc} -4\phi & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & 0 & \\ 0 & & & \end{array}\right)$

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h} = \bar{h}^{\alpha\beta} - 2\phi\,\eta^{\alpha\beta}$$

$$: \begin{pmatrix} -2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{pmatrix}$$

$\Rightarrow$ $g_{\alpha\beta}$ has components $\left(\begin{array}{c|c} -(1+2\phi) & 0 \\ \hline 0 & 1-2\phi \end{array}\right)$

This is an approximate solution to the linearized EEs that reproduces $\nabla^2\phi = 4\pi G\rho$. Note if we had left in a free rescaling parameter, $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\cdot K$, we would replace $\phi \rightarrow \kappa\phi$.

Let us use this metric to solve for the motion of a freely-falling particle, check if we can reproduce Newtonian gravity.

Let us take ϕ to be time-independent & further a function of z only. We can solve the geodesic equation for the particle 4-velocity; we know U_t, U_x, U_y are all conserved (as the metric is independent of t, x, y).

Thus let us focus on $U_z = \frac{dz}{d\tau}$. We have

$$\frac{d^2 z}{d\tau^2} + \Gamma^3{}_{\mu\beta} \frac{dx^\mu}{d\tau} \frac{dx^\beta}{d\tau} = 0 .$$

Let us choose $U_x = U_y = 0$ initially, & by their conservation they will remain so, & $U^x = g^{xx} U_x = 0$

(same for U^y)

$$\Rightarrow \frac{d^2 z}{d\tau^2} + \Gamma^3{}_{00} (U^0)^2 + 2\Gamma^3{}_{03} U^0 \left(\frac{dz}{d\tau}\right) + \Gamma^3{}_{33} \left(\frac{dz}{d\tau}\right)^2 = 0$$

Now

$$\Gamma^3{}_{00} = \frac{1}{2} g^{33} (\partial_0 g_{03} + \partial_0 g_{30} - \partial_3 g_{00})$$

$$\doteq -\frac{1}{2}(-2\phi'(z))$$

$$= \phi'(z)$$

$$\Gamma^3{}_{03} = \frac{1}{2} g^{33} (\partial_0 g_{33} + \partial_3 g_{03} - \partial_3 g_{03})$$

$$= 0$$

$$\Gamma^3{}_{33} = \frac{1}{2} g^{33} (\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33})$$

$$\doteq \frac{1}{2} \partial_z (-2\phi)$$

$$= -\phi'(z)$$

Thus the equation of motion becomes

$$\frac{d^2 z}{d\tau^2} + (\phi'(z))\left(\frac{dt}{d\tau}\right)^2 - \phi'(z)\left(\frac{dz}{d\tau}\right)^2 \approx 0$$

In the NR limit, $\left|\frac{dt}{d\tau}\right| \approx 1 \gg \left|\frac{dz}{d\tau}\right|$, so we have

$\frac{d^2 z}{dt^2} \approx -\phi'(z)$, i.e the acceleration due to gravity is given by the first derivative of the potential.

If we rescaled the RHS of the EEs by K, so $\phi \to K\phi$, we would have $\frac{d^2 z}{dt^2} = -K\frac{d\phi}{dz}$, but we see $K=1$ is the correct choice.