Return to manifolds

- Now we have built up all this technology, let's apply it to manifolds & their associated vector spaces = tangent spaces!
- Recap: given a curve $\Gamma : [a,b] \to \mathcal{M}$, where $\Gamma(s) = X$ is a point on the manifold, we can associate the tangent vector to the curve at X with the directional derivative $\frac{d}{ds}$.
- Tangent space at X = vector space containing all such tangents to curves passing through X.

$$\frac{d}{ds} = \underbrace{\frac{dx^\mu}{ds}}_{\text{components of tangent vector}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{basis vectors}}$$

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \vec{e}_\mu$$

What is the dual space? Dual basis $\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$

How do one-forms act on $\frac{d}{ds}$?

Given a scalar function $\phi : \mathcal{M} \to \mathbb{R}$,

$$\frac{d\phi}{ds} = \frac{dx^\mu}{ds} \frac{\partial \phi}{\partial x^\mu}$$

Define the <u>gradient one-form</u> $\tilde{d}\phi$ by its action on $\frac{d}{ds}$, (recalling one-forms are functions acting on vectors)

$$\tilde{d}\phi\left(\frac{d}{ds}\right) = \frac{d\phi}{ds} \in \mathbb{R}$$

Now choose $\phi = x^\alpha$, then $\tilde{d}\phi = \tilde{d}x^\alpha$ and satisfies

$\tilde{d}x^\alpha(\partial_\beta) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$, & $\{\tilde{d}x^\alpha\}$ forms the dual basis to $\{\partial_\beta\}$.

Note we can write $\tilde{d}\phi = \underbrace{\frac{\partial \phi}{\partial x^\mu}}_{\text{components}} \underbrace{\tilde{d}x^\mu}_{\text{basis one-forms}}$

and $\tilde{d}\phi\left(\frac{d}{ds}\right) = \underbrace{\frac{dx^\mu}{ds}}_{\text{components of tangent vector}} \underbrace{\frac{\partial\phi}{\partial x^\mu}}_{\text{components of gradient 1-form}} = \frac{d\phi}{ds}$

scalar obtained via contraction

The gradient one-form has components $\frac{\partial\phi}{\partial x^\mu}$ - i.e it tells us how ϕ changes with (spacetime) position, like the grad of a scalar field.

Normal one-forms

One-forms naturally describe gradients & also normals to a surface.

- A one-form is normal to a surface if it gives zero when applied to every vector tangent to the surface
- This corresponds to the vector associated with the one-form (via the metric) having zero dot product with all vectors tangent to the surface
- If the surface is closed and divides spacetime into an "inside" and "outside", a normal one-form is said to be outward if its value on vectors which point outward from the surface is positive.
- Given a 3D surface in 3+1D spacetime, the dot product of the normal one-form with itself characterizes whether the surface is timelike (positive dot product, i.e one-form is spacelike), spacelike (negative dot product, i.e one-form is timelike)

or null (zero dot product).

- Gradient one-form $\tilde{d}\phi$ is normal to surfaces of constant ϕ.

We call the basis we have discussed here—that is, $\vec{e}_\mu = \partial_\mu$, $\tilde{\omega}^\mu = \tilde{d}x^\mu$, a <u>coordinate basis</u> – under this choice coordinate transformations = basis transformations.

(For a discussion of non-coordinate bases, see Schutz 5.5)

By the chain rule, under a coordinate transformation $\{x^\alpha\} \to \{x^{\bar{\alpha}}\}$, $\partial_{\bar{\beta}} = \frac{\partial}{\partial x^{\bar{\beta}}} = \frac{\partial x^\alpha}{\partial x^{\bar{\beta}}} \frac{\partial}{\partial x^\alpha} = \frac{\partial x}{\partial x^{\bar{\beta}}} \partial_\alpha$

Comparing to our earlier general expression for basis transformations, $\vec{e}_{\bar{\beta}} = \Lambda^\alpha{}_{\bar{\beta}}\, \vec{e}_\alpha$, we can identify

$$\Lambda^\alpha{}_{\bar{\beta}} = \frac{\partial x^\alpha}{\partial x^{\bar{\beta}}}.$$

Now all our general technology can be applied to the tangent space + tensors built on it.

In particular, we will work with (pseudo-) Riemannian manifolds, which have metric tensors acting on their tangent spaces at each point.

- As previously we will denote the metric tensor by g.
- The metric in general must be a symmetric $\binom{0}{2}$ tensor field

It adds structure to the manifold, serves as a measure of "distance" between points (via the invariant interval), defines curvature

- As we discussed earlier, the metric provides a map between vectors & one-forms ($\sim$ raising & lowering indices) at every point on the manifold

Local flatness

- Given a metric $g(P)$ at a point P, with components $g_{\alpha\beta}(P)$ in some initial coordinate system, we can always find a transformation $\Lambda^{\alpha'}{}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}}$ such that $g_{\alpha'\beta'}(P) = \Lambda^{\gamma}{}_{\alpha'} \Lambda^{\delta}{}_{\beta'} g_{\gamma\delta}(P) = \eta_{\alpha'\beta'}$, where $\eta_{\alpha'\beta'}$ has components $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ - i.e the Minkowski space metric

This result assumes that $g_{\alpha\beta}$, viewed as a matrix, has 3 positive & one negative eigenvalues - it is a general theorem that a symmetric real matrix can always be diagonalized with each entry on the diagonal being $+1$, -1, or 0 (if the matrix is invertible, 0's on the diagonal are not allowed), & with the number of $+(-)1$ entries matching the number of positive (negative) eigenvalues.

Physically, this transformation corresponds to a transformation to a local inertial frame where special relativity can be (locally) applied - also known as a freely falling frame. Note this transformation is not unique: once we have one inertial frame, Lorentz boosts/rotations will generate others.

In order for $\Lambda^{\alpha'}{}_{\beta}(P)$ to represent a coordinate transformation, it will in general need to be different at different points P - if the same coordinate transformation $\Lambda^{\alpha'}{}_{\beta}$ works at all points, we will have found a global inertial frame, and we can just transform to that coordinate system and use SR.

But we might hope that $\Lambda^{\alpha'}{}_\beta(P)$ will be a good approximation to the desired transformation for points P' sufficiently close to P, and in fact this is true; the local flatness theorem states that we can always choose a coordinate system $\{x^\mu(P)\}$ such that

$$g_{\alpha\beta}(P) = \eta_{\alpha\beta} \quad \text{(as we argued already)}$$

$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(P) = 0$$

(i.e when we Taylor-expand $g_{\alpha\beta}$ around P, all the 1st-order terms vanish)

Thus information about the curvature of spacetime at a point P is stored in the 2nd-derivative terms, which cannot be made to vanish by a clever coordinate choice,

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} g_{\alpha\beta}(P) \neq 0$$

These are the same terms that give rise to tidal forces due to non-uniform gravity (recall the equivalence principle: a freely-falling observer in uniform gravity is indistinguishable from an inertial observer in SR; non-uniform gravity leads to observable tidal forces not found in SR, i.e curvature effects).

Proof sketch (see Schutz p.149 for more details):

- Consider an arbitrary transformation of the original metric, Taylor-expand both the metric & the transformation around P. This gives:

$$g_{\mu'\nu'}(\vec{x}) = \Lambda^\alpha{}_{\mu'}|_P \Lambda^\beta{}_{\nu'}|_P \, g_{\alpha\beta}(P) \leftarrow \text{0th order}$$

$$+ (x^{\gamma'} - x_0^{\gamma'}) \Big[\Lambda^\alpha{}_{\mu'}|_P \Lambda^\alpha{}_{\nu'}|_P \, g_{\alpha\beta,\gamma'}|_P \leftarrow \text{1st order}$$

$$+ \Lambda^\alpha{}_{\mu'}|_P \, g_{\alpha\beta}|_P \frac{\partial^2 x^\beta}{\partial x^{\gamma'} \partial x^{\nu'}}\Big|_P + \Lambda^\beta{}_{\nu'}|_P \, g_{\alpha\beta}|_P \frac{\partial^2 x^\alpha}{\partial x^{\gamma'} \partial x^{\mu'}}\Big|_P \Big]$$

$$+ \text{2nd-order terms}$$

At 0th order, there are 10 independent parameters in the metric we want to set to zero (10 not 16 since $g_{\alpha\beta} = g_{\beta\alpha}$)

and 16 free parameters in $\Lambda^{\alpha}{}_{\mu'}|_P$ – so we can always fix $g_{\mu'\nu'} = \eta_{\alpha\beta}$ at lowest order, with 6 free parameters left over – these capture Lorentz boosts & rotations, i.e transforms that leave $\eta_{\alpha\beta}$ fixed.

At 1st order, there are 40 (10×4) independent parameters to set to zero in $g_{\alpha\beta,\gamma}$. How many ways can we choose $\frac{\partial^2 x^\beta}{\partial x^{\gamma'}\partial x^{\nu'}}$, to zero out 1st-order contributions to $g_{\mu'\nu'}(\vec{x})$? Symmetric in $\gamma' \longleftrightarrow \nu'$, so $4\times 10 = 40$ possible choices – exactly enough, no parameters left over (so the only freedom in choosing the local inertial frame is that contained in LTs).

(At next order, we have 100 independent values in the 2nd derivatives of the metric, & 80 free params in $\frac{\partial^3 x^\alpha}{\partial x^{\gamma'}\partial x^{\nu'}\partial x^{\mu'}}$; thus we expect 20 parameters describing curvature that cannot be removed by a coordinate transformation.)

The knowledge that such a transformation <u>exists</u> is very helpful even without explicitly writing it out – it allows us to calculate invariants in the local inertial frame & then translate them directly to the desired coordinate system.

Lengths & volumes

The metric also allows us to define length & volume elements on the manifold. For the length of a curve $x^\mu(\lambda)$, we have $L = \int_{\text{curve}} \sqrt{|g_{\alpha\beta}\,dx^\alpha dx^\beta|} = \int_a^b \sqrt{\left|g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right|}\,d\lambda$

$$L = \int_a^b |\vec{V}\cdot\vec{V}|\,d\lambda$$

where $a \le \lambda \le b$ and $\vec{V}$ is the tangent vector to the curve

For the volume element, we know that in the local inertial frame (with $g_{\alpha\beta} = \eta_{\alpha\beta}$) the volume element is $dx^0 dx^1 dx^2 dx^3$;

in any other coordinate system, given by $x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^{\beta}$, we have $dx^0 dx^1 dx^2 dx^3 = \det(\Lambda^{\alpha}{}_{\beta'})\, dx^{0'} dx^{1'} dx^{2'} dx^{3'}$

We need to calculate the Jacobian, $\det(\Lambda^{\alpha}{}_{\beta'})$.
But we know that in matrix notation, $g = \Lambda \eta \Lambda^T$ (check this from $g_{\beta'\gamma'} = \Lambda^{\alpha}{}_{\beta'} \Lambda^{\delta}{}_{\gamma'} \eta_{\alpha\delta}$), and so $\det(g) = \det(\Lambda)\det(\eta)\det(\Lambda^T)$. Further, $\det(\Lambda) = \det(\Lambda^T)$, and $\det(\eta) = -1$, so $\det \Lambda = \sqrt{-\det(g)}$.
The standard notation is to write g for the determinant of the matrix with components $g_{\alpha\beta}$, so $\det \Lambda = \sqrt{-g}$, and the volume element $dV = \sqrt{-g}\, dx^{0'} dx^{1'} dx^{2'} dx^{3'}$

Example: flat spacetime in spherical polar coordinates:

$$g_{\alpha\beta} : \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad g = -r^4 \sin^2\theta$$

$\Rightarrow dV = r^2 \sin\theta \, dt\, dr\, d\theta\, d\phi$, which is as expected.