Derivatives of tensors

We have explored partial & directional derivatives of scalar functions as we move around a manifold
- what about derivatives of functions that return vectors?

Basic issue: Derivatives involve comparing vectors at different points – but on a general manifold, the tangent space V at each point could be quite different! No obvious well-defined way to take differences in general.
- Fortunately, the derivative only involves infinitesimally-separated points – can use notion that differentiable manifolds are locally like Euclidean space.
- We can of course just take partial derivatives of <u>components</u> (which are real numbers), but this doesn't have a coordinate-independent meaning
- This is apparent already for tensors in flat space-time, where the tangent space <u>is</u> the same at all points. Here we <u>can</u> differentiate the full tensor

$$\frac{\partial}{\partial x^\mu}\left(T^{\alpha_1\ldots\alpha_M}{}_{\beta_1\ldots\beta_N}\vec{e}_{\alpha_1}\otimes\ldots\otimes\vec{e}_{\alpha_M}\otimes\tilde{\omega}^{\beta_1}\otimes\ldots\tilde{\omega}^{\beta_N}\right)$$

But in general the result will <u>not</u> be solely determined by the derivatives of the components, but also derivatives of the basis vectors.

Polar coord example with a vector:

$$\frac{\partial}{\partial r}\vec{v} = \frac{\partial}{\partial r}\left(v^r\vec{e}_r + v^\theta\vec{e}_\theta\right)$$
$$= \underbrace{\left(\frac{\partial}{\partial r}v^r\right)\vec{e}_r + \left(\frac{\partial}{\partial r}v^\theta\right)\vec{e}_\theta}_{\text{terms from component derivatives}} + \underbrace{v^r\left(\frac{\partial}{\partial r}\vec{e}_r\right) + v^\theta\left(\frac{\partial}{\partial r}\vec{e}_\theta\right)}_{\text{terms from derivatives of the basis vectors}}$$

So in this case we need to know how the basis vectors change from point to point, and more generally we need to capture how the full tangent space changes.

Goal: define a derivative operation that acts on a tensor to produce another tensor associated with the same point on the manifold.

Strategy: go to the local inertial frame where $g_{\mu\nu} \to \eta_{\mu\nu}$ at the point in question, take derivative there, use tensor transformation to get components in original coordinate system.

Let us begin by differentiating vectors.

Given a vector $\vec{V}$ in the tangent space at point P, let $V^\alpha{}_{,\beta} \equiv \frac{\partial V^\alpha}{\partial x^\beta}$, and let us define the covariant derivative of $\vec{V}$ to be a $(1,1)$-tensor with components $V^\alpha{}_{;\beta}$ that satisfy

$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta}$ in locally inertial coordinates at P (i.e where $g_{\mu\nu} = \eta_{\mu\nu}$ & its 1st derivatives vanish at P)

This is a definition/construction of what we mean by the derivative operator, acting on vectors defined on our general Riemannian manifold.

We denote this $(1,1)$-tensor as $\nabla\vec{V}$, i.e $(\nabla\vec{V})^\alpha{}_\beta = V^\alpha{}_{;\beta}$. If we choose a coordinate system that is locally inertial at P_1 but not P_2, $V^\alpha{}_{;\beta}$ will equal $V^\alpha{}_{,\beta}$ at P_1 but not P_2.

Since for general manifolds there is no <u>global</u> inertial coordinate system, the components of $\nabla\vec{V}$ will generally differ from $V^\alpha{}_\beta$ - but by transforming to local inertial coordinates, we can evaluate the components of ∇T at any point for any coordinate system.

- We can use the same approach for (m,n)-tensors, defining ∇T to be the $(m,n+1)$-tensor whose components match the partial derivatives of components of T in local inertial coordinates.
- In particular, ∇g has components $g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} = 0$ in the local inertial frame (by the local flatness theorem). but a tensor with all zero components is zero in all frames, so $\nabla g = 0$.
- For a general vector $\vec{V}$ and coordinates $\{x^\alpha\}$, at a point P with local inertial coordinates $\{y^{\bar\alpha}\}$, we have the components

$$(\nabla\vec{V})^\alpha{}_\beta = \Lambda^\alpha{}_{\bar\gamma}\Lambda^{\bar\delta}{}_\beta (\nabla\vec{V})^{\bar\gamma}{}_{\bar\delta} = \Lambda^\alpha{}_{\bar\gamma}\Lambda^{\bar\delta}{}_\beta\, \partial_{\bar\delta}V^{\bar\gamma}$$

$$= \frac{\partial x^\alpha}{\partial y^{\bar\gamma}} \frac{\partial y^{\bar\delta}}{\partial x^\beta}\, \partial_{\bar\delta}V^{\bar\gamma}$$

Now note $\dfrac{\partial V^{\bar\gamma}}{\partial y^{\bar\delta}} = \dfrac{\partial x^\lambda}{\partial y^{\bar\delta}} \dfrac{\partial}{\partial x^\lambda}\left[\dfrac{\partial y^{\bar\gamma}}{\partial x^\tau} V^\tau\right]$

$$= \frac{\partial x^\lambda}{\partial y^{\bar\delta}} \frac{\partial y^{\bar\gamma}}{\partial x^\tau} \partial_\lambda V^\tau + \frac{\partial x^\lambda}{\partial y^{\bar\delta}} \frac{\partial^2 y^{\bar\gamma}}{\partial x^\lambda \partial x^\tau} V^\tau$$

and so we can write

$$(\nabla \vec{V})^\alpha{}_\beta = \Bigg[\underbrace{\frac{\partial x^\alpha}{\partial y^{\bar\gamma}} \frac{\partial y^{\bar\gamma}}{\partial x^\tau}}_{\delta^\alpha_\tau} \underbrace{\frac{\partial y^{\bar\delta}}{\partial x^\beta} \frac{\partial x^\lambda}{\partial y^{\bar\delta}}}_{\delta^\lambda_\beta} \Bigg] \partial_\lambda V^\tau$$

$$+ \Bigg[\frac{\partial x^\alpha}{\partial y^{\bar\gamma}} \underbrace{\frac{\partial x^\lambda}{\partial y^{\bar\delta}} \frac{\partial y^{\bar\delta}}{\partial x^\beta}}_{\delta^\lambda_\beta} \frac{\partial^2 y^{\bar\gamma}}{\partial x^\lambda \partial x^\tau} \Bigg] V^\tau$$

$$= \underbrace{\partial_\beta V^\alpha}_{\text{partial derivative term}} + \underbrace{(\partial_{\bar\gamma} x^\alpha)(\partial_\beta \partial_\tau y^{\bar\gamma})}_{\Gamma^\alpha{}_{\beta\tau}} V^\tau$$

correction term

The $\Gamma^\alpha{}_{\beta\tau}$ coefficients are <u>not</u> tensor components, they are coordinate-dependent Christoffel symbols.

Note that $\Gamma^\alpha{}_{\beta\tau} \equiv (\partial_{\bar\gamma} x^\alpha)(\partial_\beta \partial_\tau y^{\bar\gamma}) = \Gamma^\alpha{}_{\tau\beta}$ (as the partial derivatives commute)

We can similarly write the covariant derivatives of (m,n)-tensors in terms of Christoffel symbols:

$$(\nabla \vec{v})^\alpha{}_\beta = \partial_\beta v^\alpha + v^\mu \Gamma^\alpha{}_{\mu\beta} \quad \text{(vectors)}$$
$$(\nabla \tilde{p})_{\alpha\beta} = \partial_\beta p_\alpha - p_\mu \Gamma^\mu{}_{\alpha\beta} \quad \text{(one-forms)}$$

For higher-rank tensors, add one vector-like Christoffel symbol term for each raised index, one one-form-like term for each lowered index, e.g

$$\nabla_\beta B^\mu{}_\nu = B^\mu{}_{\nu,\beta} + B^\alpha{}_\nu \Gamma^\mu{}_{\alpha\beta} - B^\mu{}_\alpha \Gamma^\alpha{}_{\nu\beta}$$

We also use the notation $v^\beta \nabla_\beta T \equiv \nabla_{\vec{v}} T$, denoting the directional covariant derivative of T in the direction $\vec{v}$.

Now applying this rule to the metric, we find

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} - \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} = 0$$

$\Rightarrow g_{\alpha\beta,\mu} = \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu}$. Now relabeling indices,

$$g_{\alpha\mu,\beta} = \Gamma^\nu{}_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu{}_{\mu\beta} g_{\alpha\nu}$$

$$-g_{\beta\mu,\alpha} = -\Gamma^\nu{}_{\beta\alpha} g_{\nu\mu} - \Gamma^\nu{}_{\mu\alpha} g_{\beta\nu}$$

Adding the lines together & using $g_{\alpha\beta} = g_{\beta\alpha}$, $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$

$$\Rightarrow g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = g_{\beta\nu}(\Gamma^\nu{}_{\alpha\mu} - \Gamma^\nu{}_{\mu\alpha}) \rightarrow 0$$
$$+ g_{\nu\mu}(\Gamma^\nu{}_{\alpha\beta} - \Gamma^\nu{}_{\beta\alpha}) \rightarrow 0$$
$$+ g_{\alpha\nu}(\Gamma^\nu{}_{\beta\mu} + \Gamma^\nu{}_{\mu\beta})$$

$$\Rightarrow \Gamma^\nu{}_{\mu\beta} = \frac{1}{2} g^{\nu\alpha}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

(This should look familiar from 8.033!)

This very useful formula allows us to write down Christoffel symbols & hence covariant derivative components directly from the metric.