Parallel transport

- What if we now want to compare vectors at very different points on the manifold? They live in entirely different vector spaces. To compare them, we need a way to translate a vector $\vec{V} \in V(P_1)$ into a counterpart vector in $V(P_2)$ (for example — the same issue applies to general tensors).
- In principle there are multiple valid prescriptions to do this — they are called "connections" — but the one based on the covariant derivative we defined above has particularly nice properties & we will use it exclusively (it is sometimes called the Levi-Civita, Christoffel or Riemannian connection).
- The idea here is to choose a curve from P_1 to P_2 and then incrementally move along that path, adjusting $\vec{V}$ minimally to lie in the new tangent space at each step. To do this minimal adjustment, we pick the local inertial frame at P & require the components of $\vec{V}$ to remain constant as we slightly perturb P along the curve.

i.e if λ parameterizes the curve, we require

$$\frac{dV^\alpha}{d\lambda} = 0$$ at P, in the locally inertial coordinates at P

At P, $0 = \frac{dV^\alpha}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial V^\alpha}{\partial x^\mu} = U^\mu V^\alpha{}_{;\mu}$, since $V^\alpha{}_{,\mu} = V^\alpha{}_{;\mu}$ in the locally inertial coordinate system

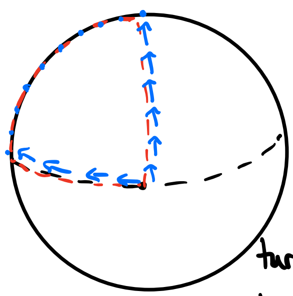
So as a tensor equation,

$U^\mu V^\alpha{}_{;\mu} = 0$ everywhere along the path,

i.e $U^\mu (\nabla \vec{V})^\alpha{}_\mu = U^\mu \nabla_\mu \vec{V} = 0$; writing $U^\mu \nabla_\mu \equiv \nabla_{\vec{U}}$ = the derivative along $\vec{U}$, the parallel transport condition is $\nabla_{\vec{U}} \vec{V} = 0$.

- Note that on a curved manifold, parallel transport is path-dependent – it is <u>not</u> true that transporting a vector in $V(P_1)$ to P_2 will give the same final vector in $V(P_2)$, if two different curves are used for the transport.

- We can actually use this as a measure of curvature. To see how it works, consider the sphere:



Start with an arrow pointing due west at the equator. Travel westward, keeping the vector pointed W, tangent to the Earth's surface. 1/4 of the way around, turn due N; your arrow is now at $90°$ to its original direction. Continue to the N pole, keeping the arrow $\perp$ to your line of travel (into the page/due W). At the pole, you will turn right & head south toward your starting point, keeping the arrow pointing due N. Upon arriving at your starting point, you have made three $90°$ turns & the arrow has rotated $90°$.

- That is, the curvature of the sphere induces a net rotation of the arrow around a closed path; equivalently, two parallel-transport paths to the north pole lead to different final orientations for the arrow.

<u>Geodesics</u>

"geodesic equation"

<u>Definition</u>: a geodesic is a curve that parallel-transports its own tangent vector; i.e the tangent vector satisfies $\nabla_{\vec{U}} \vec{U} = 0$.

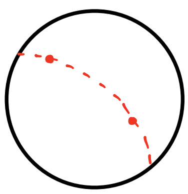
Heuristically, the tangent vector stays "as constant as possible" along a geodesic, only adapting to the change in the tangent space as one moves along the curve.

For particle trajectories in flat space, taking $\vec{U} = \frac{dx^\mu}{d\tau}$, this equation becomes $\frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{dx^\nu}{d\tau}\right) \underset{\text{via chain rule}}{=} \frac{d^2 x^\nu}{d\tau^2} = 0$

i.e geodesic particle trajectories are exactly the <u>straight-line, unaccelerated</u> trajectories, which we know are the trajectories followed by fee particles.

This makes sense intuitively: the tangent vector of a straight line points along the line at all points.

In curved spacetime, geodesics describe trajectories "as straight as possible" – e.g great-circle arcs on the Earth.



If we consider a free particle trajectoy, in each local inertial coordinate system it should satisfy the geodesic equation (i.e $\frac{d^2x^\mu}{d\tau^2} = 0$) by the equivalence principle (local inertial frame ~ flat spacetime) – but since $\nabla_{\vec{U}} \vec{U} = 0$ is a tensor equation, it must hold in all coordinate systems, at all points.

Thus fee-particle trajectories (parameterized by proper time τ) are geodesics.

Geodesics are also curves of extremal <u>proper length</u> once the endpoints are fixed (this is an alternate way to derive the geodesic equation).

In components, in a general coordinate system, where λ is the parameter of the curve, we have

$U^\mu = \frac{dx^\mu}{d\lambda}$, $U^\mu \frac{\partial}{\partial x^\mu} = \frac{d}{d\lambda}$, & the geodesic eqn becomes

$$U^\mu \left(\partial_\mu U^\alpha + \Gamma^\alpha{}_{\beta\mu} U^\beta\right) = 0$$

$$\Rightarrow \frac{d}{d\lambda} U^\alpha + \Gamma^\alpha{}_{\beta\mu} U^\mu U^\beta = 0$$

$$\Rightarrow \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\mu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\beta}{d\lambda}\right) = 0$$

This is the component form of the geodesic equation

This is a 2nd-order differential equation:
it has a unique solution if both the location of a point on the curve & a tangent vector at that point are given.

Note the geodesic eqn applies to curves, not just paths. If we reparameterize the curve in terms of $\lambda'(\lambda)$, the geodesic equation will not be satisfied in general, i.e

$$\frac{d^2x^\alpha}{d\lambda'^2} + \Gamma^\alpha{}_{\beta\mu}\left(\frac{dx^\mu}{d\lambda'}\right)\left(\frac{dx^\beta}{d\lambda'}\right) \neq 0.$$

However, if $\lambda' = a\lambda + b$ for some constants a, b, then the reparameterized curve will be a geodesic. We call such a parameter an affine parameter.

Conserved quantities along geodesics

We can rewrite the geodesic equation in an illustrative form:

$$\frac{d}{d\lambda}\left[g_{\rho\sigma}\frac{dx^\sigma}{d\lambda}\right] = \frac{1}{2}(\partial_\rho g_{\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

To prove this, let us expand this expression:

$$g_{\rho\sigma}\frac{d^2x^\sigma}{d\lambda^2} + \frac{dx^\sigma}{d\lambda}\frac{dx^\alpha}{d\lambda}\partial_\alpha g_{\rho\sigma} = \frac{1}{2}(\partial_\rho g_{\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

Now let us compare to our original formula, contracted with $g_{\rho\sigma}$:

$$g_{\rho\sigma}\frac{d^2x^\sigma}{d\lambda^2} = -g_{\rho\sigma}\Gamma^\sigma{}_{\beta\mu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\beta}{d\lambda}\right)$$

$$= -g_{\rho\sigma}\times\frac{1}{2}g^{\sigma\alpha}(\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} - \partial_\alpha g_{\beta\mu}) \times \left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\beta}{d\lambda}\right)$$

relabeling $\beta \to \nu$

$$= -\frac{1}{2}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu})\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)$$

$\hookrightarrow$ relabel $\mu \leftrightarrow \nu$

$$= -\underbrace{(\partial_\mu g_{\rho\nu})\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}_{\text{relabel } \mu\to\alpha,\ \nu\to\sigma} + \frac{1}{2}(\partial_\rho g_{\nu\mu})\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)$$

This agrees with the desired expression.

In particular, this means that if $\partial_\rho g_{\mu\nu} = 0$ for some fixed ρ, then $g_{\rho\sigma}\frac{dx^\sigma}{d\lambda}$ is constant along the geodesic.

So for example if the x^0 coordinate is time & the metric is time-independent, $\partial_t g_{\mu\nu} = 0$, then $g_{t\sigma}\frac{dx^\sigma}{d\lambda}$ is conserved. If the geodesic is the timelike trajectory of a massive particle with $\lambda = \tau$, then $\Rightarrow g_{t\sigma}U^\sigma = U_t$ is conserved, where U^μ is the 4-velocity. This gives a notion of conserved energy arising from time-independence of the metric (since in an inertial frame, $E = \gamma m = -mU_t$).

Similarly, if a metric is translationally invariant, we will have a conserved (vector) quantity similar to 3-momentum; if it is axially symmetric there will be a conserved quantity akin to angular momentum.