The source of gravity

- We understand now how to describe curvature
- Riemann curvature tensor characterizes differences between actual spacetime & globally flat spacetime
  $\longrightarrow$ good candidate to describe gravity
- But we are missing the question of <u>why</u> the manifold is curved – what sources the curvature?
- Newtonian answer: gravitational field sourced by mass
- But in GR, mass is certainly not required for particles to feel gravity – light travels on geodesics – characterized by energy-momentum 4-vector
  (also, from SR: $E = m$, mass & kinetic energy are not separately conserved, if massless particles didn't experience gravity we would violate energy conservation)
- Hypothesis: need to work with p^μ, not m
- For source term, also want to go beyond point masses – consider <u>density</u> of p^μ, akin to charge density in E&M
- Charge $= Q =$ Lorentz invariant, $J^\mu =$ charge/current density 4-vector
- "Energy-momentum current density" = <u>stress-energy tensor</u>

To build up this tensor, let's start by thinking about number density

Consider a collection of particles at rest with number density n (particles / unit volume).

Now consider a frame where all these particles are moving with 4-velocity v^μ. Let us denote their 3-velocity as $\vec{u}$.

Define the 4-vector $\vec{N}$ with $N^\mu = n v^\mu$ → 4-velocity vector

rest density = scalar

Now $N^0 = \gamma n$ = physical number density in this frame, enhanced due to length contraction in direction of motion

$\gamma = \frac{1}{\sqrt{1-u^2}}$

$N^i = (\underbrace{\gamma n}_{\text{density}}) (\underbrace{\vec{u}}_{\text{3-velocity}})^i$ = number current density (particles/area/time)

So N^μ describes the number density + number current density in the specified frame.

Note that to compute the flux of particles through a surface, we can act with N^μ on the unit one-form normal to the surface.

e.g N^x = number current density in the x-direction
= number flux across constant-x surface

We can think of N^0 as a current density across a surface of constant time, rather than a surface of constant spatial coordinate.

We can sum N^μ vectors for multiple populations – interpretation of N^μ as "number flux across surface of constant x^μ" is preserved

But now we want to go beyond number density current to energy-momentum density current.

For a population of particles with equal 4-momenta p^μ and number current density 4-vector N^ν, we can define a tensor by its components:

$T^{\mu\nu} = p^\mu N^\nu$

= flux of μth component of momentum across a surface of constant x^ν

e.g T^{00} = flux of energy across a surface of fixed t in Minkowski space

= energy density

This definition is quite general - we can build up general $T^{\mu\nu}$ by summing the results for particle populations with differing energy/momentum/number density, but this description remains true.

For the example we began with, where $N^\mu = n v^\mu$, & assuming massive particles with $p^\mu = m v^\mu$, it is trivial that $T^{\alpha\beta} = nm v^\alpha v^\beta$ = symmetric.

It is actually more generally true that $T^{\mu\nu}$ is a symmetric tensor (see Schutz 4.5 for proof).

Conservation of stress-energy

With this interpretation of $T^{\mu\nu}$, we can write down a continuity or conservation law that says the net 4-momentum flowing into a region must match the increase in energy within that region. (Exactly analogous to continuity equation for charge in E&M.)

In flat spacetime, for charge/current density we have

$$\frac{\partial J^0}{\partial t} = \frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J} = -\sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i}$$

ordinary divergence → related to total flow out of small volume element via divergence theorem

In exact analogy, replacing J^μ by $T^{\lambda\mu}$ for fixed λ, we have

$$\partial_\mu T^{\lambda\mu} = 0, \quad \text{or} \quad T^{\lambda\mu}{}_{,\mu} = 0$$

Now in curved spacetime, this is not a coordinate-independent equation – but it is a <u>local</u> statement of energy conservation, & so at each point in the manifold, it must hold in this form in local inertial coordinates.

But in local inertial coordinates, the ordinary partial derivative of a tensor is the same as its covariant derivative (by definition), so $\Rightarrow T^{\lambda\mu}{}_{;\mu} = T^{\lambda\mu}{}_{,\mu} = 0$ in local inertial coordinates

$\Rightarrow T^{\lambda\mu}{}_{;\mu} = 0$ in <u>all</u> frames, as $T^{\lambda\mu}{}_{;\mu}$ gives the components of a tensor.

<u>Perfect fluids</u>

Finally, when we discuss cosmology, there will be a particular class of stress-energy tensors that will be very important, representing idealized "perfect fluids".

Let us consider perfect fluids in their rest frame (no net spatial momentum) & in local inertial coordinates at a point.

In this frame, perfect fluids are defined by a stress-energy tensor with components of the form:

$$T^{\mu\nu}: \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

ρ = energy density
P = pressure

To facilitate coordinate transformations, it is helpful to write this as the tensor equation

$$T^{\alpha\beta} = (\rho + P) U^\alpha U^\beta + P g^{\alpha\beta}$$

where U^α is a 4-velocity for the bulk motion of the fluid, & can be defined as a 4-vector with components $U^0 = 1$, $U^i = 0$ in the specified frame. Because we are in inertial coordinates, $g^{\alpha\beta} = \eta^{\alpha\beta}$, recovering the desired expression.

But as a tensor equation, the highlighted result is valid in all frames; ρ & P are taken to be the rest-frame energy density and pressure.