Vectors, one-forms, tensors

In this lecture we are temporarily going to talk about general, abstract vector spaces, and understand how to build up the structure of tensors in that context. Then we will apply this to tangent spaces for differentiable manifolds.

Vector spaces

- An abstract vector is any element of a vector space
- Vector space: set of objects (vectors) plus operations of addition & scalar multiplication
  - Addition takes 2 vectors as inputs & produces a 3rd vector, $\vec{u} \in V$, $\vec{v} \in V \Rightarrow \vec{u}+\vec{v}=\vec{w} \in V$
  - scalar multiplication takes a (real or complex) scalar & a vector, & yields another vector,

    $\vec{v} \in V$, $\alpha \in F \Rightarrow \alpha\vec{v} \in V$

    e.g $\mathbb{R}$ or $\mathbb{C}$, we say V is a vector space over the field F, in this class $F=\mathbb{R}$
  - These operations must satisfy a set of axioms:

    $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$, $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

    associativity commutativity

    $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$, $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ distributivity

    $1.\vec{v}=\vec{v}$, $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$, for $\vec{u},\vec{v},\vec{w} \in V$, $\alpha,\beta \in F$

    $0 \in V$, i.e V contains an identity element for addition

If $\vec{v} \in V$ then $-\vec{v} \in V$, i.e every vector has an additive inverse

- Classic example of a vector space: $\mathbb{R}^n$ (i.e n-dimensional space of real numbers)
- With our addition & scalar multiplication operations, any <u>linear combination</u> of vectors in V must also be a vector in V (e.g $\vec{w} = \sum_i \alpha^i \vec{v}_i \in V$, $\vec{v}_i \in V$ & $\alpha^i \in F$)
- Given a vector space of dimension d, we can describe <u>all</u> vectors in the space as linear combinations of d linearly-independent basis vectors
- Given a set of basis vectors $\{\vec{e}_i\}$ and any vector v, there is a unique decomposition $\vec{v} = \sum_i v^i \vec{e}_i$ where $v^i \in F$, & we say the v^i are the components of v in the basis $\{\vec{e}_i\}$. Convention: vector components are denoted with raised indices
- There are many possible choices of basis, corresponding to different coordinate systems.

Tensors

The coordinate-free definition of a $(0,N)$-tensor T is a multilinear map from N copies of a vector space V (written $\underbrace{V \times V \times \ldots \times V}_{N \text{ copies}}$) into the real numbers $\mathbb{R}$, $T: V \times \ldots \times V \longrightarrow \mathbb{R}$.

That is, for $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_N \in V$, $T(\vec{v}_1, \ldots \vec{v}_N) \in \mathbb{R}$ & T is a linear function in each argument separately. e.g $T(\vec{v}_1, \alpha\vec{v}_2 + \beta\vec{w}_2, \vec{v}_3, \ldots \vec{v}_N)$

$$= \alpha T(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_N) + \beta T(\vec{v}_1, \vec{w}_2, \vec{v}_3, \ldots \vec{v}_N)$$

Note this is basis-independent – the choice of basis vectors in V is completely irrelevant, cannot affect the output of T (this is the sense in which this definition is "coordinate-free").

Example: the dot product of two vectors in $\mathbb{R}^n$ is a function that takes 2 vectors as input & returns a real scalar (which is independent of the choice of coordinate system, e.g it can be written
$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| |\vec{v}_2| \cos\theta$ → angle between the two vectors),
i.e it is a $(0,2)$-tensor.

In more general spaces, the metric is a $(0,2)$-tensor that generalizes the notion of the dot product; i.e it again takes two vectors to a scalar; we can write $g(\vec{v}, \vec{w}) = g_{\alpha\beta} v^\alpha w^\beta$

Tensor components

Given a basis $\{e_i\}$, $i = 1..d$ for the vector space d, the components of a tensor just describe its action on the basis vectors:

$$T_{\alpha\beta\gamma\ldots} = T(\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_\gamma, \ldots)$$

Convention: components of $\binom{0}{N}$ tensors are labeled with lowered indices

Because of the multilinearity property, we can describe the action of the tensor on any set of vectors in terms of the tensor components + vector components. For example, for a $(0,3)$-tensor,

$$T(\vec{u},\vec{v},\vec{w}) = T(\sum_i u^i\vec{e}_i, \sum_j v^j\vec{e}_j, \sum w^k\vec{e}_k)$$

$$= \sum_i \sum_j \sum_k u^i v^j w^k \, T(\vec{e}_i, \vec{e}_j, \vec{e}_k)$$

$$= \sum_i \sum_j \sum_k u^i v^j w^k \, T_{ijk}$$

Note the RHS must be basis-independent

This is consistent with $g_{\alpha\beta}$ being the "components of the metric" in a given basis / coordinate system.

Represents contracting a tensor with vectors $\rightarrow$ invariant

One-forms

Now consider $\binom{0}{1}$-tensors, or "one-forms" (aka dual vectors, aka covariant vectors). Let us denote one-forms as $\tilde{u}, \tilde{v}, \tilde{w}$, etc (vs $\vec{u}, \vec{v}, \vec{w}$ for vectors).

By definition, one-forms map a vector into $\mathbb{R}$.

Given two one-forms $\tilde{p}, \tilde{q}$, we can define their sum as $(\tilde{p}+\tilde{q}) : \vec{v} \longrightarrow \tilde{p}(\vec{v}) + \tilde{q}(\vec{v}) \in \mathbb{R}$ for any $\vec{v} \in V$

i.e. $\tilde{p}+\tilde{q}$ is also a one-form. Likewise, given a one-form $\tilde{p}$ & a scalar α, we can define a new one-form $\alpha\tilde{p}$ by $\alpha\tilde{p} : \vec{v} \longrightarrow \alpha\tilde{p}(\vec{v}) \in \mathbb{R}$

Thus the one-forms also form a vector space V^*: we call it the <u>dual vector space</u> to V, & its elements are one-forms or dual vectors. Each one-form has d components, given by

$$p_\alpha = \tilde{p}(\vec{e}_\alpha), \quad \alpha = 1..d, \quad \text{so } \tilde{p}(\vec{v}) = \sum_{\alpha=1}^{d} v^\alpha p_\alpha$$

Note this means $\sum v^i p_i$ is basis / coordinate-independent for any $\vec{v} \in V$, $\tilde{p} \in V^*$, even though the individual components are basis-dependent.

To visualize one-forms: if a vector is an arrow, let a one-form be a stack of surfaces. When a one-form acts on a vector, the resulting scalar = the # of surfaces the vector pierces.

Larger one-forms have surfaces closer together. Like a vector holds information about direction & magnitude.

Analogy: contour lines on a map - closer together = steeper gradient.

Now note for each $v \in V$ we can now define a map from one-forms into real numbers, $\vec{v}: V^* \to \mathbb{R}$, as $\vec{v}(\tilde{p}) \equiv \tilde{p}(\vec{v})$, just as one-forms map vectors into real numbers.

We can now define $\binom{N}{0}$-tensors as multilinear maps from $\underbrace{V^* \times \dots \times V^*}_{N \text{ copies}} \longrightarrow \mathbb{R}$, and vectors as $\binom{1}{0}$-tensors.

Most generally, $\binom{M}{N}$-tensors are multilinear maps from $\underbrace{V^* \times \dots \times V^*}_{M \text{ copies}} \times \underbrace{V \times \dots \times V}_{N \text{ copies}} \longrightarrow \mathbb{R}$.

That is: given V (vector space)

$\updownarrow$ $\longmapsto$ (M,N) tensors = functions that take M dual vectors & N vectors as inputs, output real numbers

V^* (dual space)

Next question: how should we define the basis for V^*?

What about a basis for the full space of tensors?