Beyond Newtonian gravity: tests of GR

Let us use the Newtonian metric we derived last time,

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$$

where ϕ is the Newtonian gravitational potential.

Now let us consider the path of a photon through this geometry. We will choose the parameter λ for its trajectory so $\frac{dx^\mu}{d\lambda} = p^\mu$.

The path should be a Minkowski-space geodesic (i.e a straight line) + a perturbation depending on ϕ,

$$x^\mu(\lambda) = \underbrace{x^{(0)\mu}(\lambda)}_{\text{background path}} + \underbrace{x^{(1)\mu}(\lambda)}_{\text{perturbation}}$$

Subtlety: we want to evaluate quantities along the background path, & solve for $x^{(1)\mu}(\lambda)$

But $\Rightarrow$ need to keep true path close to background path, so ϕ is similar on both

Approach: split path into short segments, approximate each one by a straight line (background path) when integrating along true path

Let $k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda}$, $l^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda}$

Photons follow null trajectories, $g_{\mu\nu}(k^\mu + l^\mu)(k^\nu + l^\nu) = 0$

Expand in h, l.

0th order: $\eta_{\mu\nu} k^\mu k^\nu = 0$. Let $k \equiv k^0 = \sqrt{|\vec{k}|^2}$

1st order: $2\eta_{\mu\nu} k^\mu l^\nu + h_{\mu\nu} k^\mu k^\nu = 0$

$$\Rightarrow -k\,l^0 + \vec{l}\cdot\vec{k} = -\tfrac{1}{2} h_{\mu\nu} k^\mu k^\nu$$

$$= -\tfrac{1}{2}\left[(-2\phi)k^2 + (-2\phi)|\vec{k}|^2\right]$$

$$= 2k^2\phi$$

Arrows now indicate 3-vectors, not general vectors

The Christoffel symbols for our metric are

$$\Gamma^0{}_{0i} = \Gamma^i{}_{00} = \partial_i \phi$$

$$\Gamma^i{}_{jk} = \delta_{jk} \partial_i \phi - \delta_{ik} \partial_j \phi - \delta_{ij} \partial_k \phi$$

Expanding the geodesic equation:

$$\frac{d}{d\lambda}(k^\mu + L^\mu) + \Gamma^\mu{}_{\rho\sigma}(k^\rho + L^\rho)(k^\sigma + L^\sigma) = 0$$

0th order: $\frac{dk^\mu}{d\lambda} = 0$, i.e $k^\mu =$ constant

1st order: $$\frac{dL^\mu}{d\lambda} = -\Gamma^\mu{}_{\rho\sigma} k^\rho k^\sigma$$

$$\Rightarrow \frac{dL^0}{d\lambda} = -\Gamma^0{}_{0i} k^0 k^i - \Gamma^0{}_{i0} k^0 k^i$$
$$= -2k(\partial_i \phi) k^i$$

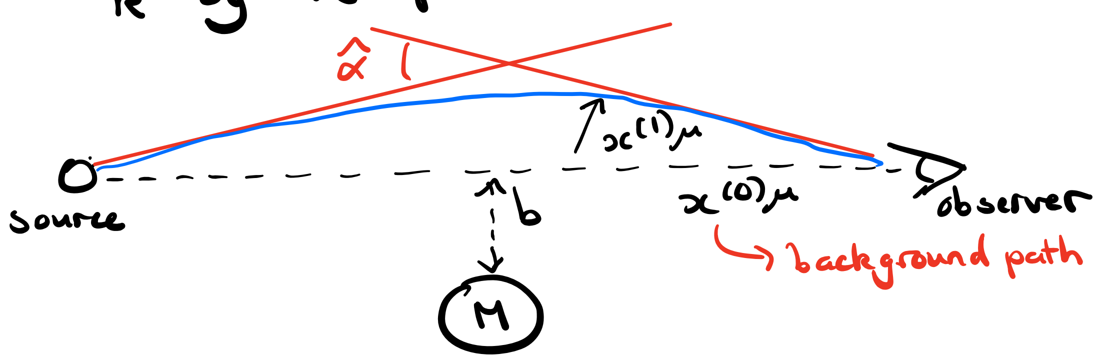
$$\frac{dL^i}{d\lambda} = -\Gamma^i{}_{00} k^2 - \Gamma^i{}_{jk} k^j k^k$$
$$= -\partial_i \phi\, k^2 - (k^2 \partial_i \phi - \vec{k}.\nabla\phi\, k^i - \vec{k}.\nabla\phi\, k^i)$$

$$\frac{d\vec{L}}{d\lambda} = -2k^2 \underbrace{\left[\nabla\phi - \vec{k}.\nabla\phi \frac{\vec{k}}{k^2}\right]}_{\text{This is the gradient transverse to the path, } \nabla_\perp \phi}$$

∇ here means the 3D grad operator, not covariant derivative

Note $(\nabla_\perp \phi) . \vec{k} = \vec{k}.\nabla\phi - \vec{k}.\nabla\phi = 0$.

Thus $\vec{L}$ is perturbed in a direction perpendicular to $\vec{k}$ by the potential ϕ.



We define the $\underbrace{\text{deflection angle}}$ as

$$\hat{\alpha} \equiv -\frac{\Delta \vec{L}}{k}, \text{ with } \Delta\vec{L} = \int \frac{d\vec{L}}{d\lambda}\, d\lambda$$

$$\Rightarrow \Delta\vec{L} = -2k^2 \int \nabla_{\perp} \phi \; d\lambda$$

If the source of ϕ is a point mass,

$$\phi = -\frac{GM}{r} = \frac{-GM}{\sqrt{x^2+b^2}}$$ for $\vec{k}$ in the x-direction, & placing the mass at $x=0$

$$\Rightarrow \nabla_{\perp}\phi = \nabla\phi - (\nabla\phi)_x \hat{x}$$

$$= \frac{-GM.b}{(b^2+x^2)^{3/2}}\hat{b}$$

$$= \frac{+GM\vec{b}}{(b^2+x^2)^{3/2}}$$

$$\Rightarrow \hat{\alpha} = 2\int \left(\frac{+GM\vec{b}}{(b^2+x^2)^{3/2}}\right) \underbrace{k\, d\lambda}_{= dx,\text{ so } \frac{dx}{d\lambda} = k}$$

$$= 2GM\vec{b} \int_{-\infty}^{\infty} \frac{dx}{(b^2+x^2)^{3/2}}$$ for path length $\gg b$

$$= \frac{4GM}{b^2}\vec{b}$$

$$|\hat{\alpha}| = 4GM/b$$

Measured in 1919 using the Sun as the mass, & observing the positions of background stars during a solar eclipse.

For the Sun, $GM/c^2 = 1.48\times10^5$ cm, &

$b >$ Solar radius $R \simeq 6.96 \times 10^{10}$ cm,

so $|\hat{\alpha}| \lesssim \dfrac{4 \times 1.48 \times 10^{5}}{6.96 \times 10^{10}} \approx 1.75$ arcsec

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Other classic tests of GR are gravitational redshift and the perihelion precession of Mercury: the gravitational redshift follows directly from the equivalence principle & occurs even in uniform gravity, but the perihelion precession probes spacetime curvature and can be derived from the Schwarzschild (black hole) metric.