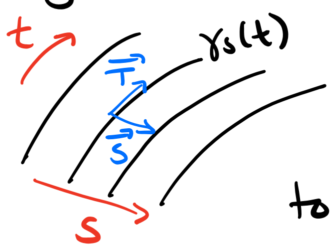
Geodesic deviation

Another signature of curvature is that "straight" parallel lines (i.e geodesics) do not remain parallel; we can characterize this geodesic deviation in terms of the Riemann tensor.

Suppose we have a family of geodesics $\gamma_s(t)$, where s labels which geodesic we are on (as a continuous parameter) & t is the affine parameter for each individual geodesic. We can treat the resulting family of geodesics as a surface $x^\mu(s,t)$, & s,t as coordinates on that surface.



We can now define the tangent vectors to the geodesics as normal, $T^\mu = \frac{\partial x^\mu}{\partial t}$, but also tangent vectors to the surface in the s-direction, $S^\mu = \frac{\partial x^\mu}{\partial s}$. These vectors tell us about the shape of the surface & how geodesics converge/diverge.

In particular, we can define a "relative velocity of geodesics" $V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$ (tells us how S changes in the direction of T, i.e how the "spacing" changes in the direction of the geodesic "flow") and a "relative acceleration of geodesics" $A^\mu = (\nabla_T V)^\mu$.

Now note $T^\rho \nabla_\rho S^\alpha = \frac{\partial x^\rho}{\partial t}\left[\frac{\partial}{\partial x^\rho}\frac{\partial x^\alpha}{\partial s} + \Gamma^\alpha{}_{\rho\beta}\frac{\partial x^\beta}{\partial s}\right]$

$$= \frac{\partial^2 x^\alpha}{\partial t\,\partial s} + \Gamma^\alpha{}_{\rho\beta}\frac{\partial x^\beta}{\partial s}\frac{\partial x^\rho}{\partial t}$$

$$= \frac{\partial x^\rho}{\partial s}\left[\frac{\partial}{\partial x^\rho}\frac{\partial x^\alpha}{\partial t} + \Gamma^\alpha{}_{\rho\beta}\frac{\partial x^\beta}{\partial t}\right]$$

(using commutativity of partial derivatives + $\Gamma^\alpha{}_{\rho\beta} = \Gamma^\alpha{}_{\beta\rho}$)

$$= S^\rho \nabla_\rho T^\alpha$$

Then we can compute A^α as:

$$A^\alpha = T^\rho \nabla_\rho V^\alpha = T^\rho \nabla_\rho (T^\beta \nabla_\beta S^\alpha)$$
$$= T^\rho \nabla_\rho (S^\beta \nabla_\beta T^\alpha)$$
$$= T^\rho (\nabla_\rho S^\beta \nabla_\beta T^\alpha + S^\beta \nabla_\rho \nabla_\beta T^\alpha)$$
$$= T^\rho (\nabla_\rho S^\beta)(\nabla_\beta T^\alpha)$$
$$+ S^\beta T^\rho (\nabla_\beta \nabla_\rho T^\alpha + R^\alpha{}_{\sigma\rho\beta} T^\sigma)$$

(the covariant derivative inherits the product rule from the ordinary partial derivative)

↳ using commutator

$$= R^\alpha{}_{\sigma\rho\beta} S^\beta T^\sigma T^\rho$$
$$+ S^\rho T^\beta \nabla_\rho \nabla_\beta T^\alpha + T^\rho (\nabla_\rho S^\beta)(\nabla_\beta T^\alpha)$$
$$= R^\alpha{}_{\sigma\rho\beta} S^\beta T^\sigma T^\rho + S^\rho \nabla_\rho (T^\beta \nabla_\beta T^\alpha)$$
$$- S^\rho (\nabla_\rho T^\beta) \nabla_\beta T^\alpha + T^\rho (\nabla_\rho S^\beta)(\nabla_\beta T^\alpha)$$

But as proved above, $S^\rho \nabla_\rho T^\beta = T^\rho \nabla_\rho S^\beta$, so the last 2 terms cancel

and $T^\beta \nabla_\beta T^\alpha$ vanishes everywhere by the geodesic equation, since T^α is the tangent vector to a geodesic.

Thus finally we have

$A^\alpha = R^\alpha{}_{\sigma\rho\beta} S^\beta T^\sigma T^\rho$, or equivalently

$$\nabla_T \nabla_T S^\alpha = R^\alpha{}_{\sigma\rho\beta} S^\beta T^\sigma T^\rho$$

This is the geodesic deviation equation (analogous to Eq. 6.87 in Schutz, although via a different derivation – describes how geodesic trajectories become non-parallel/accelerate relative to each other in curved spacetime. In flat spacetime, the RHS of the equation is zero.

The Riemann tensor fully describes the spacetime curvature,* but we will often find it useful to work with closely related tensors.

(* Note that using the identities we have derived, you can prove the Riemann tensor has 20 independent components, and if you count the degrees of freedom in the 2nd-order Taylor expansion of the metric that cannot be absorbed by choosing local inertial coordinates, which characterize curvature, you will find there are also 20 – see p. 150 Schutz.)

Specifically, the Ricci tensor and Ricci scalar are obtained by performing index contractions on the Riemann tensor:

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta} \quad \text{(Ricci tensor)}$$

$$R = g^{\mu\nu} R_{\mu\nu} \quad \text{(Ricci scalar)}$$

Note: because $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$,

$\Rightarrow R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} = g^{\mu\nu} R_{\nu\beta\mu\alpha} = R_{\beta\alpha}$, i.e the Ricci tensor is symmetric. The other identities for the Riemann tensor imply all other contractions of 2 indices either vanish or reduce to $\pm R_{\alpha\beta}$.

We can also apply the Bianchi identities to $R_{\alpha\beta}$,

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0.$$

Now first note that $\nabla_\mu g_{\alpha\beta} = \nabla^\mu g_{\alpha\beta} = 0$, as we discussed earlier – this means we can raise & lower indices inside covariant derivatives, i.e

$$\nabla_\lambda (g^{\alpha\mu} R_{\alpha\beta\mu\nu}) + \nabla_\nu (g^{\alpha\mu} R_{\alpha\beta\lambda\mu}) + \nabla_\mu (g^{\alpha\mu} R_{\alpha\beta\nu\lambda}) = 0$$

swap, introducing $-$sign

$$\Rightarrow \nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} + \nabla_\mu (R^\mu{}_{\beta\nu\lambda}) = 0$$

Now contracting the β & ν indices, we have

$$\nabla_\lambda (g^{\beta\nu} R_{\beta\nu}) - \nabla_\nu (g^{\beta\nu} R_{\beta\lambda}) + \nabla_\mu \underbrace{(g^{\beta\nu} g^{\mu\alpha} R_{\alpha\beta\nu\lambda})}_{= -g^{\beta\nu} g^{\mu\alpha} R_{\beta\alpha\nu\lambda}} = 0$$

$$\Rightarrow \nabla_\lambda R - \nabla_\nu R^\nu{}_\lambda - \nabla_\mu (g^{\mu\alpha} R_{\alpha\lambda}) = 0$$

Relabeling dummy indices, the last 2 terms are identical, & the first term is the covariant derivative of a scalar, i.e its partial derivative.

So $2 \nabla_\mu R^\mu{}_\lambda = \partial_\lambda R$, or equivalently,

$$\nabla_\mu (2 R^\mu{}_\lambda - \delta^\mu{}_\lambda R) = 0.$$

Raising the λ index gives $g^{\lambda\alpha} \nabla_\mu (2 R^\mu{}_\lambda - \partial^\mu_\lambda R) = 0$

$$\Rightarrow \nabla_\mu (2 R^{\mu\alpha} - R g^{\mu\alpha}) = 0.$$

This argument tells us there is a rank-2 curvature tensor with zero covariant derivative,

$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$. We call this the Einstein tensor.