The dual basis

Last time we said we could define one-form components $p_\alpha = \tilde{p}(\vec{e}_\alpha)$ given a basis $\{\vec{e}_\alpha\}$ for V

But $\tilde{p}$ is an element of a vector space; in principle we could choose any d-dimensional basis $\{\tilde{\omega}^\alpha\}$ for that space, & the coefficients of $\tilde{p}$ in that basis need not be what we have called p_α (i.e. for a general basis $\{\tilde{\omega}^\alpha\}$, $\tilde{p} \neq \sum_\alpha p_\alpha \tilde{\omega}^\alpha = \sum_\alpha \tilde{p}(\vec{e}_\alpha)\tilde{\omega}^\alpha$).

What if we did want p_α to give the coefficients of $\tilde{p}$ with respect to the basis $\{\tilde{\omega}^\alpha\}$? Then we would have:

$$\tilde{p} = \sum_\alpha \tilde{p}(\vec{e}_\alpha)\,\tilde{\omega}^\alpha \Rightarrow \tilde{p}(\vec{e}_\beta) = \sum_\alpha \tilde{p}(\vec{e}_\alpha)\,\tilde{\omega}^\alpha(\vec{e}_\beta) \text{ for all } \alpha, \beta$$

& to satisfy this requirement for all α, β & all one-forms $\tilde{p}$, we need $\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$. (And by definition, $\vec{e}_\beta(\tilde{\omega}^\alpha) = \delta^\alpha{}_\beta$.)

This condition defines a set of basis one-forms $\{\omega^\alpha\}$ that is the basis dual to $\{\vec{e}_\alpha\}$. So long as we use the dual basis, we can unambiguously talk about the "components of $\tilde{p}$" as either a function acting on V or an element of V^* – they are identical.

Note the components of a vector $v \in V$ then satisfy $v^\alpha = \tilde{\omega}^\alpha(v) = v(\tilde{\omega}^\alpha)$; to check this, note

$$\Rightarrow v(\tilde{\omega}^\alpha) = \sum_\beta v^\beta\, \vec{e}_\beta(\tilde{\omega}^\alpha) = \sum_\beta v^\beta \delta^\alpha{}_\beta = v^\alpha.$$

Now we can write a general $\binom{M}{N}$-tensor in terms of its components as:

$$T = T^{\alpha_1 \ldots \alpha_M}{}_{\beta_1 \ldots \beta_N}\, \vec{e}_{\alpha_1} \otimes \ldots \otimes \vec{e}_{\alpha_M} \otimes \tilde{\omega}^{\beta_1} \otimes \ldots \tilde{\omega}^{\beta_N}$$

where what this means is the basis vectors/one-forms on the RHS each act on the appropriate argument (e.g $\vec{e}_{\alpha_1}$ acts on the first (one-form) argument, $\vec{e}_{\alpha_M}$ acts on the Mth argument, etc) and the resulting scalars are multiplied together. We say that $\otimes$ is an outer product symbol, and similarly we can define the outer product of any two tensors by:

$$T\otimes S([\ \ ], \{\ \ \}) = T([\ \ ])\, S(\{\ \ \})$$

set of vector/one-form arguments for T

set of vector/one-form arguments for S

Note that $T\otimes S \neq S\otimes T$; they differ in the order of arguments that are associated with S, T (e.g for $T\otimes S$, T is applied to the first set of arguments; for $S\otimes T$, S is applied to those arguments instead).

The space of all $\binom{M}{N}$-tensors, i.e the space spanned by the basis $\vec{e}_{\alpha_1}\otimes\ldots\otimes\vec{e}_{\alpha_M}\otimes\tilde{\omega}^{\beta_1}\otimes\ldots\tilde{\omega}^{\beta_N}$, is a vector space denoted $V\otimes\ldots\otimes V\otimes V^*\otimes\ldots\otimes V^*$. We call this space the tensor product of the component vector spaces (copies of V & V^*).

The metric & index raising/lowering

We see there is a natural one-to-one relationship between V and its dual space V^*, demonstrated e.g by their dual bases. We can use the metric as a map between these two spaces.

The metric maps $V\times V\rightarrow\mathbb{R}$. So suppose we give the metric one vector input only; we write $g_{\vec{v}}(\vec{u})\equiv g(\vec{v},\vec{u})$, then $g_{\vec{v}}: V\rightarrow\mathbb{R}$. That is,

$g\vec{v}$ is a one-form, $g\vec{v} \in V^*$, and we can define $\tilde{v} \equiv g\vec{v} \in V^*$. How are the components of $\vec{v}$ & $\tilde{v}$ related (assuming we use the dual basis for V^*)?

$$v_\alpha = \tilde{v}(\vec{e}_\alpha) = g(\vec{v}, \vec{e}_\alpha) = g\left(\sum_\beta v^\beta \vec{e}_\beta, \vec{e}_\alpha\right) = \sum_\beta v^\beta g_{\beta\alpha}$$

Thus we see the metric provides a map between one-forms & vectors, & the components of $\vec{v}$ & its counterpart $\tilde{v}$ are related by the metric — this is just "index lowering".

(Note the relation between $\vec{v}$ & $\tilde{v}$ is independent of the basis choice; the basis just fixes the components $g_{\alpha\beta}$ & tells us how to relate the <u>components</u> of $\vec{v}$ & $\tilde{v}$.)

To go in the opposite direction, we want a $\binom{2}{0}$-tensor g^{-1}, such that we can define $g^{-1}_{\tilde{v}} : V^* \to \mathbb{R}$ as $g^{-1}_{\tilde{v}}(\tilde{u}) = g^{-1}(\tilde{v}, \tilde{u})$. Requiring that $\vec{v} = g^{-1}_{\tilde{v}}$ then implies that

$$v^\alpha = \vec{v}(\tilde{\omega}^\alpha) = g^{-1}(\tilde{v}, \tilde{\omega}^\alpha) = g^{-1}\left(\sum_\beta v_\beta \tilde{\omega}^\beta, \tilde{\omega}^\alpha\right)$$
$$= \sum_\beta v_\beta (g^{-1})^{\beta\alpha}$$
$$= \sum_{\beta,\gamma} v^\gamma g_{\gamma\beta} (g^{-1})^{\beta\alpha}$$

and so to ensure this we need $g_{\gamma\beta}(g^{-1})^{\beta\alpha} = \delta^\alpha{}_\gamma$, i.e if g^{-1} (written out in components) should be the matrix inverse of g. We usually simply write $(g^{-1})^{\alpha\beta} \longrightarrow g^{\alpha\beta}$ for the inverse metric, so

$v^\alpha = v_\beta g^{\beta\alpha}$ — this is "index raising".

Note we can write $g(\vec{u}, \vec{v}) = \tilde{u}(\vec{v}) = \vec{v}(\tilde{u})$

(in components, $g_{\alpha\beta} u^\alpha v^\beta = u_\alpha v^\alpha$)

Since g^{-1} is a $\binom{2}{0}$-tensor, it produces a scalar dot product when it acts on one-forms:

$g^{-1}(\tilde{u}, \tilde{v}) = \vec{u}(\tilde{v}) = \tilde{v}(\vec{u})$. Provided g is symmetric,

$g(\vec{u}, \vec{v}) = g(\vec{v}, \vec{u}) = \tilde{v}(\vec{u}) = g^{-1}(\tilde{u}, \tilde{v})$

i.e $g_{\alpha\beta} u^\alpha v^\beta = g^{\alpha\beta} u_\alpha v_\beta$.

The same approach generalizes to $\binom{M}{N}$ tensors: given a $\binom{M}{N}$ tensor $T: \underbrace{V^* \times \ldots \times V^*}_{M \text{ copies}} \times \underbrace{V \times \ldots \times V}_{N \text{ copies}} \longrightarrow \mathbb{R}$,

we can define a closely related $\binom{M-1}{N+1}$ tensor ("raising an index") by taking one of the V^* one-form arguments and swapping it with the associated vector, or a $\binom{M+1}{N-1}$ tensor by swapping a vector argument for the associated one-form. In terms of components, this is done by contracting the relevant index with $g^{\alpha\beta}$ or $g_{\alpha\beta}$ as appropriate.

Coordinate transformations

We can now look at how all the components transform under a change of basis, $\vec{e}_{\bar{\beta}} = \Lambda^\alpha{}_{\bar{\beta}}\, \vec{e}_\alpha$.

First, the dual basis must also transform, to satisfy $\tilde{\omega}^{\bar{\beta}}(\vec{e}_{\bar{\alpha}}) = \delta^{\bar{\beta}}{}_{\bar{\alpha}}$. Let us write $\tilde{\omega}^{\bar{\beta}} = \Gamma^{\bar{\beta}}{}_\alpha\, \tilde{\omega}^\alpha$. Then

$\tilde{\omega}^{\bar{\beta}}(\vec{e}_{\bar{\alpha}}) = \Lambda^\gamma{}_{\bar{\alpha}}\, \Gamma^{\bar{\beta}}{}_\delta\, \tilde{\omega}^\delta(\vec{e}_\gamma) = \Lambda^\gamma{}_{\bar{\alpha}}\, \Gamma^{\bar{\beta}}{}_\gamma$, and we

need $\Gamma^{\bar\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\bar\alpha} = \delta^{\bar\beta}{}_{\bar\alpha}$, i.e we need $\Gamma^{\bar\beta}{}_{\gamma}$ to yield the inverse transformation of $\Lambda^{\gamma}{}_{\bar\alpha}$, & we write it as $\Gamma^{\bar\beta}{}_{\gamma} = \Lambda^{\bar\beta}{}_{\gamma}$
(i.e the coordinate transform from barred to unbarred basis vectors
= " " " " unbarred to barred basis one-forms)

$\Rightarrow \tilde\omega^{\bar\beta} = \Lambda^{\bar\beta}{}_{\alpha}\tilde\omega^{\alpha}$

Now the components of an arbitrary $\binom{M}{N}$ tensor are given by

transformed tensor components

$$T^{\bar\alpha_1\ldots\bar\alpha_M}{}_{\bar\beta_1\ldots\bar\beta_N} = T(\tilde\omega^{\bar\alpha_1},\ldots\tilde\omega^{\bar\alpha_M}, \vec e_{\bar\beta_1},\ldots \vec e_{\bar\beta_N})$$

(using linearity) $$= \Lambda^{\bar\alpha_1}{}_{\gamma_1}\ldots\Lambda^{\bar\alpha_M}{}_{\gamma_M}\Lambda^{\delta_1}{}_{\bar\beta_1}\ldots\Lambda^{\delta_N}{}_{\bar\beta_N}\, T(\tilde\omega^{\gamma_1}\ldots\tilde\omega^{\gamma_M}\, \vec e_{\delta_1}\ldots\vec e_{\delta_N})$$

one coordinate transform matrix per index

original tensor components

i.e such components transform like M copies of the dual basis & N copies of the original vector-space basis.
Of course, the tensors themselves & all quantities built directly from them (without using basis vectors) are invariant. We will demand that transformations be invertible, requiring that $\det(\Lambda^{\alpha}{}_{\beta'}) \neq 0$.

<u>Tools for constructing tensors</u>

- Linear combinations of rank $\binom{M}{N}$-tensors are also rank $\binom{M}{N}$-tensors.
- As discussed earlier, the outer product of two tensors yields a (larger) tensor.

In this case the components are multiplied, e.g for S a $\binom{2}{0}$-tensor & T a $\binom{0}{2}$-tensor, $(S\otimes T)^{\alpha\beta}{}_{\gamma\delta} = S^{\alpha\beta}T_{\gamma\delta}$.

- We can produce a $\binom{M-1}{N-1}$-tensor from a $\binom{M}{N}$ tensor via <u>index contraction</u>. Let us start from

$$T = T^{\alpha_1\ldots\alpha_M}{}_{\beta_1\ldots\beta_N}\,\vec e_{\alpha_1}\otimes\ldots\otimes\vec e_{\alpha_M}\otimes\tilde\omega^{\beta_1}\otimes\ldots\tilde\omega^{\beta_N}$$

Now we define T' by having one of the $\vec{e}_{\alpha_i}$ basis vectors act on one of the $\tilde{\omega}^{\beta_j}$ basis one-forms (instead of on an argument), yielding

$$T' = T^{\alpha_1 \ldots \alpha_M}{}_{\beta_1 \ldots \beta_N} \, \vec{e}_{\alpha_1} \otimes \ldots \vec{e}_{\alpha_{i-1}} \otimes \vec{e}_{\alpha_{i+1}} \otimes \ldots \vec{e}_{\alpha_N} \otimes \tilde{\omega}^{\beta_1} \otimes \ldots \tilde{\omega}^{\beta_{j-1}} \otimes \tilde{\omega}^{\beta_{j+1}} \otimes \ldots \otimes \tilde{\omega}^{\beta_N} \cdot \delta^{\beta_j}{}_{\alpha_i}$$

For a (1,1)-tensor this contraction operation yields a scalar & is akin to taking a matrix trace. We can view the general case as a generalization of the trace. We can write $T' = \mathrm{contr}_{i,j}(T)$, where i & j label the contracted indices.

- An alternative notation for tensors (not used by our textbook, but used by Prof. Engelhardt in this year's 8.033) is to label the tensor itself (not its components) with Latin letters that indicate its rank, reserving Greek letters for components, e.g a (3,2)-tensor could be labeled $T^{ab c}{}_{de}$, meaning it takes 2 one-form arguments & 3 vector arguments.

This allows us to express contraction in particular more simply, e.g $T^{abc}{}_{ce}$ is the (2,1)-tensor obtained by contracting the 3rd raised index w/ the 1st lowered index, what the notation above would call $\mathrm{contr}_{3,1}(T)$.