Introduction

General relativity

- Gravity = geometry
- "Space tells matter how to move, matter tells space how to curve" (John Wheeler)
  - requires a mathematical description of curvature

From 8.033:

- Lorentz transformations of contra/covariant vectors & general tensors
- Describe transformation between coordinate systems defined by inertial observers.
- Index notation
- General idea that gravity = curved spacetime, equivalence principle

This class:

- Goal: teach you what you need to know to understand underpinnings of research in gravity, gravitational waves, cosmology

Schedule

Week 1: (~ entirely math) differentiable manifolds (to describe spacetime), vectors/one-forms/tensors (to describe particle properties, fields on spacetime, etc), general coordinate transformations

Week 2: (mostly math) calculus in curved spacetime, curvature, particle motion

Week 3: (physics!) stress-energy tensor, Einstein equations, recovering Newtonian gravity

Week 4: (experimental physics!) applications, focused on gravitational waves, cosmology

Assessment schedule

Pset 1 – Friday Jan 13 (this Friday!)

Pset 2 – Friday Jan 20

Pset 3 – Thursday Jan 26

Pset 4 – Wednesday Feb 1

Final exam (40%) – Friday Feb 3

note short gap

Be aware: 17 days to cover our leading/best theory of gravity (+ review + final)

Expected total hours ~ 6 units × 14 ~ 84

⇒ 21 hours / week (7 lecture/recitation, 14 psets/other study)

Pace is fast – no "study week" prior to final

No extensions (so solutions can be posted promptly) – psets can be excused in emergencies, talk to S3

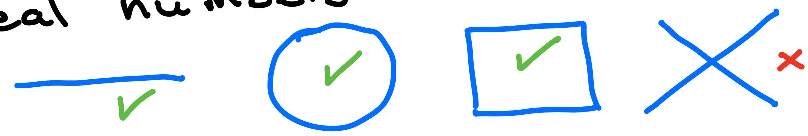
Note: For 1st week in particular, we will not follow textbook order closely, although all the material is there

Gravity as geometry

- We want to discuss curved spacetime
- The general language we use for describing curved spacetime is that of differential geometry

Differentiable manifolds

- n-dimensional manifold: (informal) "looks like" $\mathbb{R}^n$ locally
- $\mathbb{R}^n$ = n-dimensional Euclidean space = space of n-tuples of real numbers

Are these manifolds? 

- a set of points that can be continuously parametrized by a set of n real numbers ("coordinates"), at least locally

e.g. surfaces are 2D manifolds, unit sphere is parameterized by θ, ϕ; the 2D plane can be parametrized by $\{x, y\}$, $\{r, \theta\}$, etc; spacetime is 4D (e.g $\{t, x, y, z\}$)

- In GR context, think of "points in the manifold" as "events in spacetime" - coordinates tell us about space/time location of the events
- More formal definition: a <u>(coordinate) chart</u> is an invertible continuous map between an open subset of the manifold and an open subset of $\mathbb{R}^n$ - associates each point in (a subset of) the manifold with a unique n-tuple of numbers = its coordinates under the chart.
  (Think of flat maps of the Earth's surface, e.g using longitude & latitude coordinates.)

(Note: "Open" = no boundary, we won't need formal def

- In general, there may not be a single chart that covers the <u>whole</u> manifold (e.g North Pole does not have a well-defined longitude) - a set of charts is called an <u>atlas</u> when the union of their domains = the whole manifold.

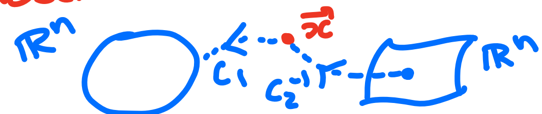
- A manifold is a set for which such an atlas exists.
- Atlas = coordinate system(s) - not unique, just gives one way to label the manifold / set of events using real numbers
- A <u>differentiable manifold</u> is one whose atlases allow us to do calculus. Given a point X on the manifold, there will be a chart $C_1 : M_1 \to \mathbb{R}^n$, where M_1 is a subset of the manifold containing X.

If the coordinates of X in C_1 are $\{x_1, x_2, \dots, x_n\}$, so $C_1(X) = \{x_1, x_2, \dots x_n\}$, and f is a scalar function $f: M_1 \to \mathbb{R}$, then we can define $f_{C_1}(\{x_1, \dots, x_n\}) = f(\vec{x})$ where $f_{C_1} : \mathbb{R}^n \to \mathbb{R}$, & then we can take partial derivatives of f_{C_1} with respect to the coordinates.

Now if $\vec{x}$ lies in the domain of two charts C_1 & C_2, i.e the atlas has two coordinate systems for the region around $\vec{x}$, we might ask which coordinates we should use to differentiate: it turns out that if $C_1 \circ C_2^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable for all choices of C_1, C_2, then the corresponding derivatives are related by the chain rule, e.g if the C_2 coordinates are $\{y_1, \dots, y_n\}$, then

$$\frac{\partial f_{C_1}}{\partial x_i} = \frac{\partial f_{C_2}}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

evaluated using $C_1 \circ C_2^{-1}$, which describes the coordinate transform



If there is an atlas satisfying this condition, we say the manifold is <u>differentiable.</u>

- We will assume for the rest of the class that we are working with differentiable manifolds.

Tangent spaces

We will want to define vector & tensor fields on our spacetime manifolds - let's start with vectors, which can be thought of as tangents to curves.

Given a point in the manifold, consider the set of paths/curves passing through that point.

- Path: a connected series of points in a manifold. Points along a path (like any other points in a manifold) can be labeled by their coordinates in a local chart.
- Curve: a continuous parameterization of a path, or equivalently, a continuous function that takes in real numbers in some interval and outputs points on the manifold,

  $\Gamma : [a,b] \to M, \; a<b, \; a,b \in \mathbb{R}$

  For example, for a 2D manifold with coordinates ξ, η in a chart C_1, we can write the coordinates of the path:

  $\{(\xi, \eta) = C_1(\Gamma(s)), \; s \in [a,b] \subset \mathbb{R}\}$

  Here Γ translates s into points on the path and the chart gives their coordinates
- Different parameterization for the same path (i.e different function Γ) = different curve.
- Given a curve Γ, parameterized by s, we can define the directional derivative $\frac{d}{ds}$, which when applied to a function on the manifold, tells us the rate of change of that function along the curve. e.g for a scalar function $\phi : M \to \mathbb{R}$, $\left.\frac{d\phi}{ds}\right|_{x=\Gamma(s)} = \frac{\phi(\Gamma(s+\varepsilon)) - \phi(\Gamma(s))}{\varepsilon}$
- We can associate $\frac{d}{ds}$ with the tangent vector to the curve
  - this is a coordinate-independent definition of the tangent

vector

- To see how this relates to older definitions you may have seen, note that with coordinates $\{x^1, \dots x^n\}$, $\frac{d}{ds} = \sum_\mu \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu}$ by the chain rule.
- We say $\frac{d}{ds}$ has components $\{\frac{dx^\mu}{ds}, \mu = 1 \dots n\}$
- These components tell me how the coordinates change as we move along the curve = the "direction" of the curve relative to the coordinate system = the components of the tangent vector
- The tangent space V to the manifold at a point is the space of all tangent vectors to curves passing through that point
- So we see our differentiable manifolds naturally/automatically have spaces of associated vectors at every point
- One important set of curves: timelike trajectories with parameter s = proper time τ.
  The tangent vectors to these curves have components $\frac{dx^\mu}{d\tau}$: components of 4-velocity
  i.e all 4-velocities for particles at X will be elements of the tangent space at X
  (& recall from 4-velocity we can build up 4-momentum, 4-acceleration, 4-force, etc by simple operations)

<u>Example</u>: tangents to a circle in 2D polar coords

Curve: $\theta = s$, $r = R$, $s \in [0, 2\pi]$

$\Rightarrow$ tangent vector has components $\left(\frac{dr}{ds}, \frac{d\theta}{ds}\right) = (0,1)$ in (r,θ) coordinate system;

Space of curves on the circle passing through a point P is 1D (e.g $\theta = \lambda s$, $r = R$, $s \in [0, 2\pi]$ $\longrightarrow$ $\frac{d}{ds}$ has components $(0,\lambda)$)

Tangent space to circle manifold at P is 1D - all other tangents at P are multiples of $(0,1)$