Hansen Lecture Notes:
Additive and Multiplicative Processes, Martingales, and the Billingsley CLT

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1 Introduction

- The Central Limit Theorem (CLT) is a fundamental statistical result that describes the asymptotic distribution of a sample average. It is used with econometric data to conduct inference on estimates of unknown model parameters.
- The standard CLT applies to independent and identically distributed (iid) observations. But we believe that elements of a time-series are generated in a dependent fashion, that today's events shape tomorrow's. And the standard CLT does not hold under this alternative assumption.
- Extending our study of tochastic process theory allows us to state conditions under which a version of the CLT applies to our observations so we can conduct inference testing as in an iid setting.
- The martingale property is the concept that allows us to revive are key concepts in stochastic process theory. Like stationarity and ergodicity, it is abstract and unfamiliar, but resurrects properties of observations obtained from iid distributions:
  - In an iid setting, the best guess for each subsequent observation differs from the realization of that observation, on average, by the variance of the sampling distribution. The martingale the variance of each subsequent observation is the variance of the sampling distribution (???)
- Additive and multiplicative functionals are important classes of stochastic processes. Although these processes need not be martingales, we can decompose them into a martingale component, a stationary component, and a trend component. We can then use the Billingsley CLT to derive the asymptotic distribution of their sample average.

2 Martingales, Additive Functionals, and Multiplicative Functionals

- (Def.) A stochastic process, $(\{X_t\}_{t=0}^{\infty}, S, X, Pr)$, is a martingale iff its present value is the conditional expectation of all future values, $\forall j > 0\ E[X_{t+j} \mid \mathcal{F}_t] = X_t$.
  - (Claim) A stochastic process, $(\{X_t\}_{t=0}^{\infty}, S, X, Pr)$, is a martingale iff its present value is the conditional expectation of its next value, $E[X_{t+1} \mid \mathcal{F}_t] = X_t$.
  - ($\Rightarrow$) $E[X_{t+j} \mid \mathcal{F}_t] \overset{LIE}{=} E\big[E[X_{t+j} \mid \mathcal{F}_{t+j-1}, \mathcal{F}_t]\big] \overset{\mathcal{F}_{t+j-1} \in \mathcal{F}_t}{=} E\big[E[X_{t+j} \mid \mathcal{F}_{t+j-1}] \mid \mathcal{F}_t\big] \overset{X mart.}{=} E[X_{t+j-1} \mid \mathcal{F}_t]$ and induction.
  - ($\Leftarrow$) $E[X_{t+j} \mid \mathcal{F}_t] = X_t\ \forall j > 0 \Rightarrow E[X_{t+1} \mid \mathcal{F}_t] = X_t$
- (Def.) An additive functional, $(\{Y_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$, is a sequence of random variables, $\{Y_t\}_{t=0}^{\infty}$, derived from a stationary Markov process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, a sequence of iid rondom variables, $\{W_t\}_{t=0}^{\infty}$, and a function, $\kappa : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, according to $Y_{t+1} - Y_t = \kappa(X_t, W_{t+1})$.
  - (Claim) An additive functional, $(\{Y_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$, is a martingale, $\forall j\ E[Y_{t+j} \mid \mathcal{F}_t] = Y_t$, iff the expectation of the increment given previous observations is zero, $E[\kappa(X_t, W_{t+1}) \mid \mathcal{F}_t] = 0\ \forall t$.
  - ($\Rightarrow$) $E[Y_{t+j} \mid \mathcal{F}_t] \overset{LIE}{=} E[Y_{t+1} \mid \mathcal{F}_t] = E[Y_t + \kappa(X_t, W_{t+1}) \mid \mathcal{F}_t] \overset{Y_t \in \mathcal{F}_t}{=} Y_t + E[\kappa(X_t, W_{t+1}) \mid \mathcal{F}_t] \overset{E[\kappa(X_t, W_{t+1}) \mid \mathcal{F}_t] = 0}{=} Y_t$.
- (Def.) A multiplicative functional, $(\{M_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$, is a sequence of random variables, $\{M_t\}_{t=0}^{\infty}$, derived from a stationary Markov process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, a sequence of iid random variables, $\{W_t\}_{t=0}^{\infty}$, and a function, $\kappa : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, according to $\frac{M_{t+1}}{M_t} = e^{\kappa(X_t, W_{t+1})}$.
  - (Claim) A multiplicative functional, $(\{M_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$, is a martingale, $\forall j > 0\ E[M_{t+j} \mid \mathcal{F}_t] = M_t$, iff the expectation of the exponentiated increment given previous observations is one, $E[e^{\kappa(X_t, W_{t+1})} \mid \mathcal{F}_t] = 1\ \forall t$.
  - (Pf.) $E[M_{t+j} \mid \mathcal{F}_t] \overset{LIE}{=} E[M_{t+1} \mid \mathcal{F}_t] = E[e^{\kappa(X_t, W_{t+1})} M_t \mid \mathcal{F}_{t+1}] \overset{M_t \in \mathcal{F}_t}{=} M_t E[e^{\kappa(X_t, W_{t+1})} \mid \mathcal{F}_t] \overset{E[e^{\kappa(X_t, W_{t+1})} \mid \mathcal{F}_t] = 1}{=} M_t$

3 The Conditional Expectation Operator and the 'Summed Future Expectations' Operator

- (Def.) A stationary, ergodic Markov process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, defines a conditional expectation operator, $\mathbb{T}$, where $\mathbb{T} f(X_t) = E[f(X_{t+1}) \mid X_t] = \int f(x^*) P(dx^* \mid X_t)$.

- (Claim) $\mathbb{T}^j f(X_t) = E[f(X_{t+j}) \mid X_t]$
- (Pf.) $\mathbb{T}^j f(X_t) \overset{\mathbb{T}^j def.}{=} \mathbb{T}\mathbb{T}^{j-1} f(X_t) \overset{\mathbb{T} def.}{=} E[\mathbb{T}^{j-1} f(X_{t+1}) \mid X_t] \overset{\mathbb{T}^j def.}{=} E[\mathbb{T}\mathbb{T}^{j-2} f(X_{t+1}) \mid X_t] \overset{\mathbb{T} def.}{=} E[E[\mathbb{T}^{j-2} f(X_{t+2}) \mid X_{t+1}] \mid X_t] = E[E[\mathbb{T}^{j-2} f(X_{t+2}) \mid X_{t+1}, X_t] \mid X_t] \overset{LIE}{=} E[\mathbb{T}^{j-2} f(X_{t+2}) \mid X_t]$ and induction.

- (Def.) A stationary, ergodic Markov process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, defines an 'summed future expectations' (my term) operator, $(\mathbb{I} - \mathbb{T})^{-1}$, where $(\mathbb{I} - \mathbb{T})^{-1} f(X_t) = \sum_{j=0}^{\infty} \mathbb{T}^j f(X_t)$.
  - (Claim) $(\mathbb{I} - \mathbb{T})(\mathbb{I} - \mathbb{T})^{-1} f(X_t) = f(X_t)$
  - N.B. This gives a motivation for representing the 'summed future expectations' operator as $(\mathbb{I} - \mathbb{T})^{-1}$
  - (Pf.) $(\mathbb{I}-\mathbb{T})(\mathbb{I}-\mathbb{T})^{-1} f(X_t) \overset{(\mathbb{I}-\mathbb{T})^{-1} def.}{=} (\mathbb{I}-\mathbb{T}) \sum_{j=0}^{\infty} \mathbb{T}^j f(X_t) = \sum_{j=0}^{\infty} \mathbb{T}^j f(X_t) - \sum_{j=1}^{\infty} \mathbb{T}^j f(X_t) \overset{\mathbb{T}^j f(X_t) = E[f(X_{t+j}) \mid X_t]}{=} \sum_{j=0}^{\infty} E[f(X_{t+j}) \mid X_t] - \sum_{j=1}^{\infty} E[f(X_{t+j}) \mid X_t] \overset{E[f(X_t) \mid X_t] = f(X_t)}{=} f(X_t)$

4 Contraction Properties of the Conditional Expectation Operator

- (Def.) A stationary, ergodic Markov Process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, defines a set of scalar-valued functions with finite second moment under the stationary distribution, $\mathcal{L}^2 = \{f : \mathbb{R}^n \to \mathbb{R} \mid E[f(X_t)^2] = \int f(x)^2 Q(dx) < +\infty\}$
  - (Claim) Given a stationary, ergodic Markov process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, and the norm $\|f\| = \left(\int f(x)^2 Q(dx)\right)^{\frac{1}{2}}$, $\mathbb{T}$ is a weak contraction on $\mathcal{L}^2$, $\|\mathbb{T}f\| \leq \|f\| \ \forall f \in \mathcal{L}^2$.
  - (Pf.) $E[e_{t+1} \mid X_t] \overset{e_{t+1} := f(X_{t+1}) - \mathbb{T}f(X_t)}{=} E[f(X_{t+1}) - \mathbb{T}f(X_t) \mid X_t] = E[f(X_{t+1}) \mid X_t] - \mathbb{T}f(X_t) \overset{\mathbb{T} def.}{=} \mathbb{T}f(X_t) - \mathbb{T}f(X_t) = 0$
    $\Rightarrow \int f(x)^2 Q(dx) \overset{E[.] def.}{=} E[f(X_{t+1})^2] \overset{e_{t+1} := f(X_{t+1}) - \mathbb{T}f(X_t)}{=} E[(\mathbb{T}f(X_t) + e_{t+1})^2] \overset{E[e_{t+1} \mid X_t] = 0, LIE, alg.}{=} E[(\mathbb{T}f(X_t))^2] + E[e_{t+1}^2] \overset{E[e_{t+1}^2] \geq 0, f \in \mathcal{L}^2}{\geq} E[(\mathbb{T}f(X_t))^2] \overset{E[.] def.}{=} \int (\mathbb{T}f(x))^2 Q(dx)$
    $\Rightarrow \|f\| \overset{\|.\| def.}{=} \left(\int f(x)^2 Q(dx)\right)^{\frac{1}{2}} \overset{\int f(x)^2 Q(dx) \geq \int (\mathbb{T}f(x))^2 Q(dx)}{\geq} \left(\int (\mathbb{T}f(x))^2 Q(dx)\right)^{\frac{1}{2}} \overset{\|.\| def.}{=} \|\mathbb{T}f\|$

- (Def.) A stationary, ergodic Markov Process, $(\{X_t\}_{t=0}^{\infty}, Q, P)$, defines a set of scalar-valued functions with zero first moment and finite second moment under the stationary distribution, $\mathcal{N} = \{f : \mathbb{R}^n \to \mathbb{R} \mid E[f(X_t)] = \int f(x) Q(dx) = 0, E[f(X_t)^2] = \int f(x)^2 Q(dx) < +\infty\} \subset \mathcal{L}^2$

- (Claim) Given a stationary, ergodic Markov process, ($\{X_t\}_{t=0}^{\infty}$, Q, P), and the norm $\|f\| = \left(\int f(x)^2 Q(dx)\right)^{\frac{1}{2}}$, $\mathbb{T}$ is a strong contraction on $\mathcal{N}$, $\exists \rho \in (0,1)$ s.t. $\|\mathbb{T}f\| \leq \rho\|f\| \ \forall f \in \mathcal{N}$.
- (Pf.) Claimed but not proven in Hansen's notes.

- (Def.) A stationary, ergodic Markov process, ($\{X_t\}_{t=0}^{\infty}$, Q, P), and a function with zero expectation and finite squared expectation under its distribution, $f \in \mathcal{N}$, define a function, $g = (\mathbb{I}-\mathbb{T})^{-1}f$. (N.B. g is introduced only for notational convenience below.)
  - (Claim) g is well-defined (non-infinite).
  - (Pf.) Informal. This comes from the fact that $\mathbb{T}$ is a strong contraction on $\mathcal{N}$ and so terms of $\mathbb{T}^j$ in $(\mathbb{I}-\mathbb{T})^{-1}$ will converge.
  - (Claim) $g(X_t) - \mathbb{T}g(X_t) = f(X_t)$
  - (Pf.) $g(X_t) - \mathbb{T}g(X_t) \overset{g def.}{=} (\mathbb{I}-\mathbb{T})^{-1}f(X_t) - \mathbb{T}(\mathbb{I}-\mathbb{T})^{-1}f(X_t) \overset{alg.}{=} (\mathbb{I}-\mathbb{T})(\mathbb{I}-\mathbb{T})^{-1}f(X_t) = f(X_t)$
  - (Claim) $E[g(X_{t+1}) - \mathbb{T}g(X_t) \mid X_t] = 0$.
  - (Pf.) $E[g(X_{t+1}) - \mathbb{T}g(X_t) \mid X_t] = E[g(X_{t+1}) \mid X_t] - \mathbb{T}g(X_t) \overset{\mathbb{T} def.}{=} \mathbb{T}g(X_t) - \mathbb{T}g(X_t) = 0$

5 Martingale Decomposition of Additive Functionals

- (Claim) The increment, $Y_{t+1} - Y_t$, to an additive functional, ($\{Y_t\}_{t=0}^{\infty}$, κ, ($\{X_t\}_{t=0}^{\infty}$, Q, P), $\{W_t\}_{t=0}^{\infty}$), can be decomposed into the sum of a constant, ν, the increment to a stationary distribution, $g(X_{t+1}) - g(X_t)$, and the increment to an additive functional martingale, $\kappa_a(X_t, W_{t+1})$.
  - (Pf.) $Y_{t+1} - Y_t \overset{\{Y_t\} def.}{=} \kappa(X_t, W_{t+1})$
    $\overset{alg.}{=} \Big[\kappa(X_t, W_{t+1}) - E[\kappa(X_t, W_{t+1}) \mid X_t]\Big]$
    $+ \Big[E[\kappa(X_t, W_{t+1}) \mid X_t] - E[\kappa(X_t, W_{t+1})]\Big] + \Big[E[\kappa(X_t, W_{t+1})]\Big]$
    $\overset{:=}{=} \kappa_1(X_t, W_{t+1}) + f(X_t) + \nu$
    $\overset{f \in \mathcal{L}^2}{=} \kappa_1(X_t, W_{t+1}) + g(X_t) - \mathbb{T}g(X_t) + \nu$
    $\overset{alg.}{=} \kappa_1(X_t, W_{t+1}) + g(X_t) - g(X_{t+1}) + \Big(g(X_{t+1}) - \mathbb{T}g(X_t)\Big) + \nu$
    $\overset{:=}{=} \kappa_1(X_t, W_{t+1}) + g(X_t) - g(X_{t+1}) + \kappa_2(X_t, W_{t+1}) + \nu$
    $= \kappa_a(X_t, W_{t+1}) + g(X_t) - g(X_{t+1}) + \nu$

    And $E[\kappa_1(X_t, W_{t+1}) \mid X_t] = E[\kappa_2(X_t, W_{t+1}) \mid X_t] = 0$

- (Claim) An additive functional, $(\{Y_t\}_{t=0}^{\infty},\ \kappa,\ (\{X_t\}_{t=0}^{\infty},\ Q,\ P),\ \{W_t\}_{t=0}^{\infty})$, can be decomposed into the sum of a constant, $Y_0 + g(X_0)$, a determinate trend component, νt, the realization of a stationary distribution, $g(X_t)$, and an additive functional martingale.

  - (Pf.) $Y_t \overset{alg.}{=} Y_0 + \sum_{j=0}^{t-1}(Y_{j+1} - Y_j)$
    $= Y_0 + \sum_{j=0}^{t-1}\Big(\kappa_1(X_j, W_{j+1}) + \kappa_2(X_j, W_{j+1})\Big) + \sum_{j=0}^{t-1}\Big(g(X_j) - g(X_{j+1})\Big) + \nu t$
    $\overset{alg.}{=} Y_0 + \sum_{j=0}^{t-1}\kappa_a(X_j, W_{j+1}) + g(X_0) - g(X_t) + \nu t$

    Because $(\{X_t\}_{t=0}^{\infty},\ Q,\ P)$ is stationary, $(\{g(X_t)\}_{t=0}^{\infty},\ Q,\ P)$ is also stationary.

    Because $E[\kappa_a(X_t, W_{t+1}) \mid X_t] \overset{\kappa_a def.}{=} 0$, our additive functional, $(\{\sum_{j=0}^{t-1}\kappa_a(X_j, W_{j+1})\}_{t=1}^{\infty},\ \kappa_a,\ (\{X_t\}_{t=0}^{\infty},\ Q,\ P),\ \{W_t\}_{t=0}^{\infty})$, is a martingale.

6 Examples of Additive Functional Martingale Decomposition

- VAR (example)
- VAR with discrete regime change (example)

7 Martingale Decomposition of Multiplicative Functionals

8 Examples of Multiplicative Functional Martingale Decomposition

9 Multiplicative Functional Martingales and their Corresponding Additive Functional Supermartingales

- (Def.) Given a multiplicative functional, $(\{M_t\}_{t=0}^{\infty},\ \kappa,\ (\{X_t\}_{t=0}^{\infty},\ Q,\ P),\ \{W_t\}_{t=0}^{\infty})$, the corresponding additive functional, $(\{Y_t\}_{t=0}^{\infty},\ \kappa,\ (\{X_t\}_{t=0}^{\infty},\ Q,\ P),\ \{W_t\}_{t=0}^{\infty})$ is defined by $Y_t = \ln M_t$.
- (Def.) A supermartingale, $(\{X_t\}_{t=0}^{\infty},\ S,\ X,\ Pr)$, is a stochastic process with the property that $E[X_{t+1} \mid \mathcal{F}_t] < X_t$.
- (Claim) Given a multiplicative functional that is a martingale, $(\{M_t\}_{t=0}^{\infty},\ \kappa,\ (\{X_t\}_{t=0}^{\infty},\ Q,\ P),\ \{W_t\}_{t=0}^{\infty})$ s.t. $E[M_{t+1} \mid \mathcal{F}_t] = M_t$, the corresponding

additive functional, $(\{Y_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$ s.t. $Y_t = \ln M_t$, is a super-martingale, $E[Y_{t+1} \mid \mathcal{F}_t] < Y_t \ \forall t$.

- (Pf.) $E[Y_{t+1} \mid \mathcal{F}_t] - Y_t \overset{Y_t \in \mathcal{F}_t}{=} E[Y_{t+1} - Y_t \mid \mathcal{F}_t] \overset{\kappa def.}{=} E[\kappa(X_t, W_{t+1}) \mid \mathcal{F}_t] = E[\ln e^{\kappa(X_t,W_{t+1})} \mid \mathcal{F}_t] \overset{Jensen}{\leq} \ln E[e^{\kappa(X_t,W_{t+1})} \mid \mathcal{F}_t] \overset{martingale}{=} ln1 = 0$, with strict inequality whenever $\kappa(X_t, W_{t+1})$ is non-degenerate.

- (Claim) A multiplicative functional that is a martingale, $(\{M_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$ s.t. $E[M_{t+1} \mid \mathcal{F}_t] = M_t$, converges in probability to 0, $M_t \underset{t\to\infty}{\overset{p}{\to}} 0$.
  - N.B. This result may seem surprising since $E[M_{t+j} \mid \mathcal{F}_t] = M_t \ \forall j > 0$.
  - (Pf.) $\frac{1}{t} ln M_t \overset{M_t = e^{Y_t}}{=} \frac{1}{t} Y_t = \frac{1}{t} Y_0 + \frac{1}{t} \sum_{t=0}^{\infty} \kappa(X_t, W_{t+1}) \underset{t\to\infty}{\overset{\{(X_t,W_{t+1})\}_{t=0}^{\infty} sta., erg., LLN}{\to}} E[\kappa(X_t, W_{t+1})] \overset{LIE}{=} E\big[E[\kappa(X_t, W_{t+1}) \mid \mathcal{F}_t]\big] \underset{E[X|X<0]<0}{\overset{supermartingale}{<}} 0 \overset{M_t \nrightarrow 0 \Rightarrow \Leftarrow}{\Rightarrow} M_t \to 0$

10 The Billingsley CLT

- (Claim) For an (additive) martingale, $\{Y_t\}_{t=0}^{\infty}$ s.t. $E[Y_{t+1} - Y_t \mid \mathcal{F}_t] = 0$,
  - (Pf.) Due to Billingsley
- (Claim) For an additive functional, $(\{Y_t\}_{t=0}^{\infty}, \kappa, (\{X_t\}_{t=0}^{\infty}, Q, P), \{W_t\}_{t=0}^{\infty})$, $\frac{1}{\sqrt{N}} Y_N \overset{d}{\to} \mathcal{N}(0, ?)$.
  - (Pf.) $\frac{1}{\sqrt{N}} Y_N \overset{mart.ext.}{=} \frac{1}{\sqrt{N}} Y_0 + \frac{1}{\sqrt{N}} \sum_{t=0}^{N} \kappa_\alpha(X_t, W_{t+1}) + \frac{1}{\sqrt{N}} g(X_0) - \frac{1}{\sqrt{N}} g(X_{N+1})$